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Note on algebraic irregular Riemann–Hilbert correspondence

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Abstract – The subject of this paper is an algebraic version of the irregular Riemann–Hilbert correspondence which was mentioned in [Tsukuba J. Math. 44 (2020), 155–201]. In particular, we prove an equivalence of categories between the triangulated category $\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ of holonomic $\mathcal{D}$-modules on a smooth algebraic variety X over $\mathbb{C}$ and the triangulated category $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty})$ of algebraic $\mathbb{C}$-constructible enhanced ind-sheaves on a bordered space X^{an}_{∞}. Moreover, we show that there exists a t-structure on the triangulated category $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty})$ whose heart is equivalent to the abelian category of holonomic $\mathcal{D}$-modules on X. Furthermore, we shall consider simple objects of its heart and minimal extensions of objects of its heart.

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1. Introduction

After the appearance of the regular Riemann–Hilbert correspondence of Kashiwara [10], Beilinson and Bernstein developed systematically a theory of regular holonomic $\mathcal{D}$-modules on smooth algebraic varieties over the complex number field $\mathbb{C}$ and obtained an algebraic version of the Riemann–Hilbert correspondence stated as follows: Let X be a smooth algebraic variety over $\mathbb{C}$. We denote by X^{an} the underling complex

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analytic manifold of X, by $\mathbf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)$ the triangulated category of regular holonomic $\mathcal{D}_X$-modules on X and by $\mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$ the triangulated category of algebraic $\mathbb{C}$-constructible sheaves on X^{an}. Then there exists an equivalence of triangulated categories

$$\mathrm{Sol}_X\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)^{\mathrm{op}} \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X), \quad \mathcal{M} \mapsto \mathrm{Sol}_{X^{\mathrm{an}}}(\mathcal{M}^{\mathrm{an}}),$$

see [2, 4] and also [28] for the details. Here

$$\mathrm{Sol}_{X^{\mathrm{an}}}(\cdot) := \mathbf{R}\mathcal{H}om_{\mathcal{D}_{X^{\mathrm{an}}}}(\,\cdot\,, \mathcal{O}_{X^{\mathrm{an}}})\colon \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X^{\mathrm{an}}}) \to \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{X^{\mathrm{an}}})$$

is the solution functor on the complex analytic manifold X^{an} and $\mathcal{M}^{\mathrm{an}}$ is the analytification of $\mathcal{M}$ (see Section 2.5 for the definition). The triangulated category $\mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$ has a t-structure $({}^p\mathbf{D}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathbb{C}_X), {}^p\mathbf{D}^{\geq 0}_{\mathbb{C}\text{-c}}(\mathbb{C}_X))$ which is called the perverse t-structure by [1], see also [8, Thm. 8.1.27]. Let us denote by

$$\mathrm{Perv}(\mathbb{C}_X) := {}^p\mathbf{D}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathbb{C}_X) \cap {}^p\mathbf{D}^{\geq 0}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$$

its heart and call an object of $\mathrm{Perv}(\mathbb{C}_X)$ an algebraic perverse sheaf on X^{an}. The above equivalence induces an equivalence of categories between the abelian category $\mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X)$ of regular holonomic $\mathcal{D}$-modules on X and the one $\mathrm{Perv}(\mathbb{C}_X)$.

On the other hand, the problem of extending the analytic regular Riemann–Hilbert correspondence of Kashiwara to cover the case of analytic holonomic $\mathcal{D}$-modules with irregular singularities had been open for 30 years. After a groundbreaking development in the theory of irregular meromorphic connections by Kedlaya [16, 17] and Mochizuki [23, 24], D'Agnolo and Kashiwara established the Riemann–Hilbert correspondence for analytic irregular holonomic $\mathcal{D}$-modules in [5]. For this purpose, they introduced enhanced ind-sheaves extending the classical notion of ind-sheaves introduced by Kashiwara and Schapira in [12]. Let $\mathbf{X}$ be a complex analytic manifold. (In this paper, we use bold letters for complex manifolds to avoid confusion with algebraic varieties.) We denote by $\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{\mathbf{X}})$ the triangulated category of holonomic $\mathcal{D}_{\mathbf{X}}$-modules on $\mathbf{X}$ and by $\mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_{\mathbf{X}})$ the one of $\mathbb{R}$-constructible enhanced ind-sheaves on $\mathbf{X}$ (see [6, Def. 3.3.1]). We set $\mathrm{Sol}^{\mathrm{E}}_{\mathbf{X}}(\mathcal{M}) := \mathbf{R}\mathcal{I}hom_{\mathcal{D}_{\mathbf{X}}}(\mathcal{M}, \mathcal{O}^{\mathrm{E}}_{\mathbf{X}})$. Here $\mathcal{O}^{\mathrm{E}}_{\mathbf{X}}$ is the enhanced ind-sheaf of tempered holomorphic functions, see [5, Def. 8.2.1]. Then D'Agnolo and Kashiwara proved that the enhanced solution functor $\mathrm{Sol}^{\mathrm{E}}_{\mathbf{X}}$ induces a fully faithful embedding

$$\mathrm{Sol}^{\mathrm{E}}_{\mathbf{X}}\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{\mathbf{X}})^{\mathrm{op}} \hookrightarrow \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_{\mathbf{X}}).$$

Moreover, in [6] they gave a generalized t-structure $({}^{\frac{1}{2}}\mathbf{E}^{\leq c}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_{\mathbf{X}}), {}^{\frac{1}{2}}\mathbf{E}^{\geq c}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_{\mathbf{X}}))_{c\in\mathbb{R}}$ on $\mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_{\mathbf{X}})$ and proved that the enhanced solution functor induces a fully faithful embedding of the abelian category $\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{\mathbf{X}})$ of holonomic $\mathcal{D}$-modules on $\mathbf{X}$ into its heart ${}^{\frac{1}{2}}\mathbf{E}^{\leq 0}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_{\mathbf{X}}) \cap {}^{\frac{1}{2}}\mathbf{E}^{\geq 0}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_{\mathbf{X}})$. On the other hand, Mochizuki proved in [25] that the image of $\mathrm{Sol}^{\mathrm{E}}_{\mathbf{X}}$ can be characterized by the curve test. In [9], the author defined $\mathbb{C}$-constructability for enhanced ind-sheaves on $\mathbf{X}$ and proved that they are nothing but objects of the image of $\mathrm{Sol}^{\mathrm{E}}_{\mathbf{X}}$. Namely, we obtain an equivalence of categories between the triangulated category $\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{\mathbf{X}})$ of holonomic $\mathcal{D}$-modules on $\mathbf{X}$ and the one $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{\mathbf{X}})$ of $\mathbb{C}$-constructible enhanced ind-sheaves on $\mathbf{X}$:

$$\mathrm{Sol}^{\mathrm{E}}_{\mathbf{X}}\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{\mathbf{X}})^{\mathrm{op}} \xrightarrow{\sim} \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{\mathbf{X}}).$$

Remark that Kuwagaki introduced another approach to the irregular Riemann–Hilbert correspondence via irregular constructible sheaves which are defined by $\mathbb{C}$-constructible sheaves with coefficients in a finite version of the Novikov ring and special gradings in [18].

Therefore, it seems to be important to establish an algebraic irregular Riemann–Hilbert correspondence on a smooth algebraic variety. Although it may be known by experts, it is not in the literature to our knowledge. Thus we want to prove the algebraic irregular Riemann–Hilbert correspondence in this paper. Let X be a smooth algebraic variety over $\mathbb{C}$ and denote by $\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ the triangulated category of holonomic $\mathcal{D}_X$-modules. The following result is the main theorem of this paper:

Theorem 3.11. *There exists an equivalence of triangulated categories:*

$$\mathrm{Sol}^{\mathrm{E}}_{X_\infty}\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} \xrightarrow{\sim} \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty}).$$

See Section 3.2 for the definition of $\mathrm{Sol}^{\mathrm{E}}_{X_\infty}$ and Definition 3.10 for the definition of $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty})$.

Remark 1.1. In the case of quasi-projective variety, Kuwagaki [18] established an algebraic version of the irregular Riemann–Hilbert correspondence.

2. Preliminary notions and results

In this section, we briefly recall some basic notions and results which will be used.

2.1 – *Ind-sheaves on bordered spaces*

Let us recall some basic notions of ind-sheaves on bordered spaces. For the details, we refer to D'Agnolo–Kashiwara [5, §§2.3, 2.4, 2.5]. We also refer to Kashiwara–Schapira [12, 13] for ind-sheaves on a topological space.

Let us denote by $\mathrm{I}\mathbb{C}_{M_\infty}$ the abelian category of ind-sheaves on a bordered spaces $M_\infty = (M, \check{M})$ and denote by $\mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty})$ the triangulated category of them. Note that there exists the standard t-structure $(\mathbf{D}^{\leq 0}(\mathrm{I}\mathbb{C}_{M_\infty}), \mathbf{D}^{\geq 0}(\mathrm{I}\mathbb{C}_{M_\infty}))$ on $\mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty})$ which is induced by the standard t-structure on $\mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\check{M}})$. For a morphism $f_\infty\colon M_\infty \to N_\infty$ of bordered spaces, we have the Grothendieck operations $\otimes$, $\mathbf{R}\mathcal{I}hom$, $\mathbf{R}f_{\infty *}$, $\mathbf{R}f_{\infty !!}$, f_∞^{-1}, $f_\infty^{!}$. Note that there exists an embedding functor $\iota_{M_\infty}\colon \mathbf{D}^{\mathrm{b}}(\mathbb{C}_M) \hookrightarrow \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty})$. We sometimes write $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_{M_\infty})$ for $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_M)$, when considered as a full subcategory of $\mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty})$. Note also that the embedding functor ι_{M_∞} has a left adjoint functor $\alpha_{M_\infty}\colon \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty}) \to \mathbf{D}^{\mathrm{b}}(\mathbb{C}_M)$.

2.2 – *Enhanced ind-sheaves on bordered spaces I*

We shall recall some basic notions of enhanced ind-sheaves on bordered spaces and results on it. Reference are made to [14] and [6]. Moreover, we also refer to D'Agnolo–Kashiwara [5] and Kashiwara–Schapira [15] for the notions of enhanced ind-sheaves on good topological spaces.

Let $M_\infty = (M, \check{M})$ be a bordered space. We set $\mathbb{R}_\infty := (\mathbb{R}, \overline{\mathbb{R}})$ for $\overline{\mathbb{R}} := \mathbb{R} \sqcup \{-\infty, +\infty\}$, and let $t \in \mathbb{R}$ be the affine coordinate. The triangulated category of enhanced ind-sheaves on M_∞ is defined by

$$\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty}) := \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty \times \mathbb{R}_\infty})/\pi^{-1}\mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty}),$$

where $\pi\colon M_\infty \times \mathbb{R}_\infty \to M_\infty$ is the morphism of bordered spaces given by the first projection $M \times \mathbb{R} \to M$. Note that the quotient functor $\mathbf{Q}_{M_\infty}\colon \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty \times \mathbb{R}_\infty}) \to \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty})$ has fully faithful left and right adjoints

$$\mathbf{L}^{\mathrm{E}}_{M_\infty}, \mathbf{R}^{\mathrm{E}}_{M_\infty}\colon \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty}) \to \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty \times \mathbb{R}_\infty}).$$

Note also that $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty})$ has the standard t-structure $(\mathbf{E}^{\leq 0}(\mathrm{I}\mathbb{C}_{M_\infty}), \mathbf{E}^{\geq 0}(\mathrm{I}\mathbb{C}_{M_\infty}))$ which is induced by the standard t-structure on $\mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty \times \mathbb{R}_\infty})$. We denote by $\mathcal{H}^n\colon \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty}) \to \mathbf{E}^0(\mathrm{I}\mathbb{C}_{M_\infty})$ the n-th cohomology functor, where we set

$$\mathbf{E}^0(\mathrm{I}\mathbb{C}_{M_\infty}) := \mathbf{E}^{\leq 0}(\mathrm{I}\mathbb{C}_{M_\infty}) \cap \mathbf{E}^{\geq 0}(\mathrm{I}\mathbb{C}_{M_\infty}).$$

For a morphism $f_\infty\colon M_\infty \to N_\infty$ of bordered spaces, we have the six operations $\otimes^+$, $\mathbf{R}\mathcal{I}hom^+$, $\mathbf{E}f_\infty^{-1}$, $\mathbf{E}f_{\infty *}$, $\mathbf{E}f_\infty^{!}$, $\mathbf{E}f_{\infty !!}$ for enhanced ind-sheaves on bordered spaces. Note that there exists a morphism $\mathbf{E}f_{\infty !!} \to \mathbf{E}f_{\infty *}$ of functors $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty}) \to \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{N_\infty})$ and it is an isomorphism if f_∞ is proper. Moreover, we have outer-hom functors

$$\begin{aligned}
&\mathbf{R}\mathcal{I}hom^{\mathrm{E}}(K_1, K_2),\\
&\mathbf{R}\mathcal{H}om^{\mathrm{E}}(K_1, K_2) := \alpha_{M_\infty}\mathbf{R}\mathcal{I}hom^{\mathrm{E}}(K_1, K_2),\\
&\mathbf{R}\mathrm{Hom}^{\mathrm{E}}(K_1, K_2) := \mathbf{R}\Gamma(M; \mathbf{R}\mathcal{H}om^{\mathrm{E}}(K_1, K_2))
\end{aligned}$$

with values in $\mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty})$, $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_M)$ and $\mathbf{D}^{\mathrm{b}}(\mathbb{C})$, respectively. Here, $\mathbf{D}^{\mathrm{b}}(\mathbb{C})$ is the derived category of $\mathbb{C}$-vector spaces.

For $F \in \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty})$ and $K \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty})$, the objects

$$\begin{aligned}\pi^{-1}F \otimes K &:= \mathbf{Q}_{M_\infty}(\pi^{-1}F \otimes \mathbf{L}^{\mathrm{E}}_{M_\infty}K),\\ \mathbf{R}\mathcal{I}hom(\pi^{-1}F, K) &:= \mathbf{Q}_{M_\infty}(\mathbf{R}\mathcal{I}hom(\pi^{-1}F, \mathbf{R}^{\mathrm{E}}_{M_\infty}K))\end{aligned}$$

in $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty})$ are well defined. We set

$$\mathbb{C}^{\mathrm{E}}_{M_\infty} := \mathbf{Q}_{M_\infty}\big(\text{“}\varinjlim_{a\to+\infty}\text{”}\,\mathbb{C}_{\{t\geq a\}}\big) \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty}).$$

Note that there exists an isomorphism $\mathbb{C}^{\mathrm{E}}_{M_\infty} \simeq \mathbf{E}j^{-1}\mathbb{C}^{\mathrm{E}}_{\check{M}}$ in $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty})$. Then we have a natural embedding

$$e_{M_\infty}\colon \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty}) \to \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty}), \quad F \mapsto e_{M_\infty}(F) := \mathbb{C}^{\mathrm{E}}_{M_\infty} \otimes \pi^{-1}F.$$

By using [14, Prop. 2.18], for a morphism $f_\infty\colon M_\infty \to N_\infty$ of bordered spaces and objects $F \in \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty})$, $G \in \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{N_\infty})$ we obtain

$$\begin{aligned}\mathbf{E}f_{\infty!!}(e_{M_\infty}F) &\simeq e_{N_\infty}(\mathbf{R}f_{\infty!!}F),\\ \mathbf{E}f_\infty^{-1}(e_{N_\infty}G) &\simeq e_{M_\infty}(f_\infty^{-1}G),\\ \mathbf{E}f_\infty^{!}(e_{N_\infty}G) &\simeq e_{M_\infty}(f_\infty^{!}G).\end{aligned}$$

Let us define

$$\omega^{\mathrm{E}}_{M_\infty} := e_{M_\infty}(\omega_M) \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty})$$

where $\omega_M \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{M_\infty})$ ($= \mathbf{D}^{\mathrm{b}}(\mathbb{C}_M)$) is the dualizing complex, see [11, Def. 3.1.16] for the details. Then we have the Verdier duality functor $\mathrm{D}^{\mathrm{E}}_{M_\infty}\colon \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty})^{\mathrm{op}} \to \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty})$ for enhanced ind-sheaves on bordered spaces which is defined by $\mathrm{D}^{\mathrm{E}}_{M_\infty}(K) := \mathbf{R}\mathcal{I}hom^+(K, \omega^{\mathrm{E}}_{M_\infty})$. Note that for any $K \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\check{M}})$ we have an isomorphism

$$\mathrm{D}^{\mathrm{E}}_{M_\infty}(\mathbf{E}j^{-1}K) \simeq \mathbf{E}j^{-1}(\mathrm{D}^{\mathrm{E}}_{\check{M}}(K))$$

in $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty})$. Note also that for any $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_M)$ there exists an isomorphism

$$\mathrm{D}^{\mathrm{E}}_{M_\infty}(e_{M_\infty}\mathcal{F}) \simeq e_{M_\infty}(\mathrm{D}_M\mathcal{F})$$

in $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty})$.

Let $i_0\colon M_\infty \to M_\infty \times \mathbb{R}_\infty$ be the inclusion map of bordered spaces induced by $x \mapsto (x, 0)$. We set $\mathrm{sh}_{M_\infty} := \alpha_{M_\infty} \circ i_0^! \circ \mathbf{R}^{\mathrm{E}}_{M_\infty}\colon \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty}) \to \mathbf{D}^{\mathrm{b}}(\mathbb{C}_M)$ and call it the sheafification functor for enhanced ind-sheaves on bordered spaces. Note that for

$K \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\check{M}})$ we have an isomorphism

$$\mathrm{sh}_{M_\infty}(\mathbf{E}j^{-1}K) \simeq j^{-1}(\mathrm{sh}_{\check{M}}(K))$$

in $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_M)$. Note also that there exists an isomorphism $\mathcal{F} \xrightarrow{\sim} \mathrm{sh}_{M_\infty}(e_{M_\infty}(\mathcal{F}))$ for $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_M)$.

For a continuous function $\varphi\colon U \to \mathbb{R}$ defined on an open subset $U \subset M$, we set the exponential enhanced ind-sheaf by

$$\mathbb{E}^{\varphi}_{U|M_\infty} := \mathbb{C}^{\mathrm{E}}_{M_\infty} \otimes^{+} \mathbf{Q}_{M_\infty}(\mathbb{C}_{\{t+\varphi\geq 0\}}),$$

where $\{t+\varphi \geq 0\}$ stands for $\{(x,t) \in \check{M} \times \overline{\mathbb{R}} \mid t \in \mathbb{R},\ x \in U,\ t+\varphi(x) \geq 0\}$.

2.3 – *Enhanced ind-sheaves on bordered spaces II*

The aim of this subsection is to prepare some auxiliary results on enhanced ind-sheaves on bordered spaces which will be used in Section 3. In particular, we will prove – although this is known by experts – that for any smooth algebraic variety X the triangulated category $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{(X^{\mathrm{an}},\widetilde{X}^{\mathrm{an}})})$ does not depend on the choice of $\widetilde{X}$.

Let $M_\infty = (M, \check{M})$ and $N_\infty = (N, \check{N})$ be two bordered spaces.

Sublemma 2.1. *Let $f\colon M \to N$ be a continuous map and assume that $\check{M}$ and $\check{N}$ are compact. Then the map f induces a semi-proper morphism from M_∞ to N_∞.*

This sublemma is clear. Moreover, Lemma 2.2 below follows from this sublemma and [5, Lem. 3.2.3].

Lemma 2.2. *In the situation of Sublemma 2.1, we assume that the continuous map f is an isomorphism. Then the morphism induced by the map f is also an isomorphism between M_∞ and N_∞.*

By using [6, Lem. 2.7.6], for any $K \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty})$ we have

$$\pi^{-1}\mathbb{C}_M \otimes K \simeq \mathbf{E}j_{!!}\mathbf{E}j^{-1}K,$$

where $j\colon M_\infty \to \check{M}$ is the morphism of bordered spaces given by the embedding $M \hookrightarrow \check{M}$. Hence we have an equivalence of triangulated categories

$$\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty}) \underset{\mathbf{E}j^{-1}}{\overset{\mathbf{E}j_{!!}}{\underset{\longleftarrow}{\overset{\sim}{\longrightarrow}}}} \left\{K \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\check{M}}) \mid \pi^{-1}\mathbb{C}_M \otimes K \xrightarrow{\sim} K\right\}.$$

Sublemma 2.3 below follows from [5, Lem. 3.3.12].

Sublemma 2.3. *Let $f_\infty\colon M_\infty \to N_\infty$ be the morphism of bordered spaces associated with a continuous map $\check{f}\colon \check{M} \to \check{N}$ such that $\check{f}(M) \subset N$.*

(1) *For any $K \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty})$, there exist isomorphisms in $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{N_\infty})$:*

$$\mathbf{E}f_{\infty!!}K \simeq \mathbf{E}j_{N_\infty}^{-1}\mathbf{E}\check{f}_{\infty!!}\mathbf{E}j_{M_\infty!!}K,$$
$$\mathbf{E}f_{\infty*}K \simeq \mathbf{E}j_{N_\infty}^{-1}\mathbf{E}\check{f}_{\infty*}\mathbf{E}j_{M_\infty*}K.$$

(2) *For any $L \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{N_\infty})$, there exist isomorphisms in $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty})$:*

$$\mathbf{E}f_\infty^{-1}L \simeq \mathbf{E}j_{M_\infty}^{-1}\mathbf{E}\check{f}_\infty^{-1}\mathbf{E}j_{N_\infty!!}L \simeq \mathbf{E}j_{M_\infty}^{-1}\mathbf{E}\check{f}_\infty^{-1}\mathbf{E}j_{N_\infty*}L,$$
$$\mathbf{E}f_\infty^{!}L \simeq \mathbf{E}j_{M_\infty}^{-1}\mathbf{E}\check{f}_\infty^{!}\mathbf{E}j_{N_\infty!!}L \simeq \mathbf{E}j_{M_\infty}^{-1}\mathbf{E}\check{f}_\infty^{!}\mathbf{E}j_{N_\infty*}L.$$

Remark that $\check{M}$ and $\check{N}$ are not necessary compact. Hence we obtain:

Lemma 2.4. *In the situation of Sublemma 2.3 ($\check{M}$ and $\check{N}$ are not necessarily compact), we assume that the restriction $\check{f}|_M$ of $\check{f}$ to M induces an isomorphism $M \xrightarrow{\sim} N$. Then there exists an equivalence of triangulated categories:*

$$\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty}) \underset{\mathbf{E}f_\infty^{-1}}{\overset{\mathbf{E}f_{\infty!!}}{\rightleftarrows}} \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{N_\infty}).$$

Proof. This follows from Sublemma 2.3 and the fact that the functor $\mathbf{E}\check{f}_{!!}$ (resp. $\mathbf{E}\check{f}^{-1}$) is an isomorphism over M (resp. N) by the assumption $M \xrightarrow{\sim} N$. ■

At the end of this subsection, we shall apply the above results to our situation. Let X be a smooth algebraic variety over $\mathbb{C}$ and denote by X^{an} the underlying complex manifold of X. Then we can obtain a smooth complete algebraic variety $\widetilde{X}$ such that $X \subset \widetilde{X}$ and $D := \widetilde{X} \setminus X$ is a normal crossing divisor of $\widetilde{X}$ by Hironaka's desingularization theorem [7] (see also [26, Thm. 4.3]). Hence we obtain a bordered space $(X^{\mathrm{an}}, \widetilde{X}^{\mathrm{an}})$ and an equivalence of triangulated categories:

$$\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{(X^{\mathrm{an}},\widetilde{X}^{\mathrm{an}})}) \underset{\mathbf{E}j^{-1}}{\overset{\mathbf{E}j_{!!}}{\rightleftarrows}} \left\{K \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\widetilde{X}^{\mathrm{an}}}) \mid \pi^{-1}\mathbb{C}_{X^{\mathrm{an}}} \otimes K \xrightarrow{\sim} K\right\}.$$

Let $\widetilde{X}_i$ $(i = 1, 2)$ be smooth complete algebraic varieties over $\mathbb{C}$ such that $X \subset \widetilde{X}_i$ $(i = 1, 2)$, then the identity map $\mathrm{id}_{X^{\mathrm{an}}}$ of X^{an} induces an isomorphism of bordered spaces $(X^{\mathrm{an}}, \widetilde{X}_1^{\mathrm{an}}) \simeq (X^{\mathrm{an}}, \widetilde{X}_2^{\mathrm{an}})$ by Lemma 2.2. Hence we have an equivalence of triangulated categories

$$\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{(X^{\mathrm{an}},\widetilde{X}_1^{\mathrm{an}})}) \simeq \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{(X^{\mathrm{an}},\widetilde{X}_2^{\mathrm{an}})}).$$

Remark that this equivalence can be proved by Lemma 2.4. Therefore the bordered space $(X^{\mathrm{an}}, \widetilde{X}^{\mathrm{an}})$ and the triangulated category $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{(X^{\mathrm{an}},\widetilde{X}^{\mathrm{an}})})$ is independent of the choice of a smooth complete variety $\widetilde{X}$. Hence we can write

$$X^{\mathrm{an}}_{\infty} := (X^{\mathrm{an}}, \widetilde{X}^{\mathrm{an}})$$

and

$$\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_{\infty}}) := \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{(X^{\mathrm{an}},\widetilde{X}^{\mathrm{an}})})$$

for a smooth algebraic variety X.

For a morphism $f\colon X \to Y$ of smooth algebraic varieties, we obtain a semi-proper morphism of bordered spaces from X^{an}_{∞} to Y^{an}_{∞} by Sublemma 2.1. We denote it by $f^{\mathrm{an}}_{\infty}\colon X^{\mathrm{an}}_{\infty} \to Y^{\mathrm{an}}_{\infty}$. On the other hand, since there exists a morphism of complete algebraic varieties $\tilde{f}\colon \widetilde{X} \to \widetilde{Y}$ such that $\tilde{f}|_X = f$, we obtain a morphism of bordered spaces from X^{an}_{∞} to Y^{an}_{∞}, see Section 2.1 or [5, §3.2] for the details. It is clear that this morphism is equal to f^{an}_{∞} and hence we can apply Sublemma 2.3 to f^{an}_{∞}. This fact will be used in the proof of Proposition 3.12.

2.4 – *Analytic $\mathbb{C}$-constructible enhanced ind-sheaves*

In this subsection, we shall recall some notions in [9].

Let $\mathbf{X}$ be a complex manifold and $D \subset \mathbf{X}$ a normal crossing divisor in it. Let us take local coordinates $(u_1, \dots, u_l, v_1, \dots, v_{d_X - l})$ of $\mathbf{X}$ such that $D = \{u_1 u_2 \cdots u_l = 0\}$ and set $Y = \{u_1 = u_2 = \cdots = u_l = 0\}$. Then for a meromorphic function $\varphi \in \mathcal{O}_{\mathbf{X}}(*D)$ on $\mathbf{X}$ along D which has the Laurent expansion $\varphi = \sum_{a \in \mathbb{Z}^l} c_a(\varphi)(v) \cdot u^a \in \mathcal{O}_{\mathbf{X}}(*D)$ with respect to $u_1, \dots, u_l$, where $c_a(\varphi)$ are holomorphic functions on Y, we define its order $\mathrm{ord}(\varphi) \in \mathbb{Z}^l$ by $\min(\{a \in \mathbb{Z}^l \mid c_a(\varphi) \neq 0\} \cup \{0\})$ with respect to the partial order on $\mathbb{Z}^l$ if it exists. For any $f \in \mathcal{O}_{\mathbf{X}}(*D)/\mathcal{O}_{\mathbf{X}}$, we take any lift $\tilde{f}$ to $\mathcal{O}_{\mathbf{X}}(*D)$, and we set $\mathrm{ord}(f) := \mathrm{ord}(\tilde{f})$, if the right-hand side exists. Note that it is independent of the choice of a lift $\tilde{f}$. If $\mathrm{ord}(f) \neq 0$, then $c_{\mathrm{ord}(f)}(\tilde{f})$ is independent of the choice of a lift $\tilde{f}$, which is denoted by $c_{\mathrm{ord}(f)}(f)$.

Definition 2.5 ([24, Def. 2.1.2]). In the situation as above, a finite subset $\mathcal{I} \subset \mathcal{O}_{\mathbf{X}}(*D)/\mathcal{O}_{\mathbf{X}}$ is called a good set of irregular values on $(\mathbf{X}, D)$, if the following conditions are satisfied:

- For each element $f \in \mathcal{I}$, $\mathrm{ord}(f)$ exists. If $f \neq 0$ in $\mathcal{O}_{\mathbf{X}}(*D)/\mathcal{O}_{\mathbf{X}}$, then $c_{\mathrm{ord}(f)}(f)$ is invertible on Y.
- For two distinct $f, g \in \mathcal{I}$, $\mathrm{ord}(f - g)$ exists and $c_{\mathrm{ord}(f-g)}(f - g)$ is invertible on Y.

- The set $\{\mathrm{ord}(f-g) \mid f, g \in \mathcal{I}\}$ is totally ordered with respect to the above partial order $\leq$ on $\mathbb{Z}^l$.

DEFINITION 2.6 ([9, Def. 3.6]). We say that an enhanced ind-sheaf[1] $K \in \mathbf{E}^0(\mathrm{I}\mathbb{C}_{\mathbf{X}})$ has a normal form along D if

(i) $\pi^{-1}\mathbb{C}_{\mathbf{X}\setminus D} \otimes K \xrightarrow{\sim} K$,

(ii) for any $x \in \mathbf{X} \setminus D$ there exist an open neighborhood $U_x \subset \mathbf{X} \setminus D$ of x and a non-negative integer k such that $K|_{U_x} \simeq (\mathbb{C}^{\mathrm{E}}_{U_x})^{\oplus k}$,

(iii) for any $x \in D$ there exist an open neighborhood $U_x \subset \mathbf{X}$ of x, a good set of irregular values $\{\varphi_i\}_i$ on $(U_x, D \cap U_x)$ and a finite sectorial open covering $\{U_{x,j}\}_j$ of $U_x \setminus D$ such that
$$\pi^{-1}\mathbb{C}_{U_{x,j}} \otimes K|_{U_x} \simeq \bigoplus_i \mathbb{E}^{\mathrm{Re}\,\varphi_i}_{U_{x,j}|U_x} \quad \text{for any } j,$$
see the end of Section 2.2 for the definition of $\mathbb{E}^{\mathrm{Re}\,\varphi_i}_{U_{x,j}|U_x}$.

Note that any enhanced ind-sheaf which has a normal form along D is an $\mathbb{R}$-constructible enhanced ind-sheaf on $\mathbf{X}$.

A ramification of $\mathbf{X}$ along D on a neighborhood U of $x \in D$ is a finite map $r\colon U^{\mathrm{rm}} \to U$ of complex manifolds of the form
$$z' \mapsto z = (z_1, z_2, \ldots, z_n) = r(z') = (z_1'^{m_1}, \ldots, z_r'^{m_r}, z'_{r+1}, \ldots, z'_n)$$
for some $(m_1, \ldots, m_r) \in (\mathbb{Z}_{>0})^r$, where $(z'_1, \ldots, z'_n)$ is a local coordinate system of U^{rm} and $(z_1, \ldots, z_n)$ is the one of U such that $D \cap U = \{z_1 \cdots z_r = 0\}$.

DEFINITION 2.7 ([9, Def. 3.11]). We say that an enhanced ind-sheaf $K \in \mathbf{E}^0(\mathrm{I}\mathbb{C}_{\mathbf{X}})$ has a quasi-normal form along D if it satisfies (i) and (ii) in Definition 2.6 and, moreover, if for any $x \in D$ there exist an open neighborhood $U_x \subset \mathbf{X}$ of x and a ramification $r_x\colon U_x^{\mathrm{rm}} \to U_x$ of U_x along $D_x := U_x \cap D$ such that $\mathbf{E}r_x^{-1}(K|_{U_x})$ has a normal form along $D_x^{\mathrm{rm}} := r_x^{-1}(D_x)$.

A modification of $\mathbf{X}$ with respect to an analytic hypersurface H is a projective map $m\colon \mathbf{X}^{\mathrm{md}} \to \mathbf{X}$ from a complex manifold $\mathbf{X}^{\mathrm{md}}$ to $\mathbf{X}$ such that $D^{\mathrm{md}} := m^{-1}(H)$ is a normal crossing divisor of $\mathbf{X}^{\mathrm{md}}$ and m induces an isomorphism $\mathbf{X}^{\mathrm{md}} \setminus D^{\mathrm{md}} \xrightarrow{\sim} \mathbf{X} \setminus H$.

(1) In [9], the author defined enhanced ind-sheaves that have a normal form along D under the assumption that they are $\mathbb{R}$-constructible. However, this assumption is not needed.

Definition 2.8 ([9, Def. 3.14]). We say that an enhanced ind-sheaf $K \in \mathbf{E}^0(\mathrm{I}\mathbb{C}_{\mathbf{X}})$ has a modified quasi-normal form along H if it satisfies (i) and (ii) in Definition 2.6 and, moreover, if for any $x \in H$ there exist an open neighborhood $U_x \subset \mathbf{X}$ of x and a modification $m_x \colon U_x^{\mathrm{md}} \to U_x$ of U_x along $H_x := U_x \cap H$ such that $\mathbf{E}m_x^{-1}(K|_{U_x})$ has a quasi-normal form along $D_x^{\mathrm{md}} := m_x^{-1}(H_x)$.

Let us denote by $\mathbf{E}^0_{\mathrm{mero}}(\mathrm{I}\mathbb{C}_{\mathbf{X}(H)})$ the abelian category of enhanced ind-sheaves which have a modified quasi-normal forms along H and set

$$\mathbf{E}^{\mathrm{b}}_{\mathrm{mero}}(\mathrm{I}\mathbb{C}_{\mathbf{X}(H)}) := \left\{K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_{\mathbf{X}}) \mid \mathcal{H}^i(K) \in \mathbf{E}^0_{\mathrm{mero}}(\mathrm{I}\mathbb{C}_{\mathbf{X}(H)}) \text{ for any } i \in \mathbb{Z}\right\}.$$

A complex analytic stratification of $\mathbf{X}$ is a locally finite partition $\{\mathbf{X}_\alpha\}_{\alpha \in A}$ of $\mathbf{X}$ by locally closed analytic subsets $\mathbf{X}_\alpha$ such that for any $\alpha \in A$, $\mathbf{X}_\alpha$ is smooth, $\overline{\mathbf{X}}_\alpha$ and $\partial \mathbf{X}_\alpha := \overline{\mathbf{X}}_\alpha \setminus \mathbf{X}_\alpha$ are complex analytic subsets and $\overline{\mathbf{X}}_\alpha = \bigsqcup_{\beta \in B} X_\beta$ for a subset $B \subset A$.

Definition 2.9 ([9, Def. 3.19]). We say that an enhanced ind-sheaf $K \in \mathbf{E}^0(\mathrm{I}\mathbb{C}_{\mathbf{X}})$ is $\mathbb{C}$-constructible if there exists a complex analytic stratification $\{\mathbf{X}_\alpha\}_\alpha$ of $\mathbf{X}$ such that

$$\pi^{-1}\mathbb{C}_{\overline{\mathbf{X}}_\alpha^{\mathrm{bl}} \setminus D_\alpha} \otimes \mathbf{E}b_\alpha^{-1} K$$

has a modified quasi-normal form along D_α for any α, where $b_\alpha \colon \overline{\mathbf{X}}_\alpha^{\mathrm{bl}} \to \mathbf{X}$ is a sequence of complex blow-ups of $\overline{\mathbf{X}_\alpha}$ along $\partial \mathbf{X}_\alpha = \overline{\mathbf{X}_\alpha} \setminus \mathbf{X}_\alpha$ and $D_\alpha := b_\alpha^{-1}(\partial \mathbf{X}_\alpha)$.

We denote by $\mathbf{E}^0_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{\mathbf{X}})$ the full subcategory of $\mathbf{E}^0(\mathrm{I}\mathbb{C}_{\mathbf{X}})$ whose objects are $\mathbb{C}$-constructible and set

$$\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{\mathbf{X}}) := \left\{K \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathbf{X}}) \mid \mathcal{H}^i(K) \in \mathbf{E}^0_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{\mathbf{X}}) \text{ for any } i \in \mathbb{Z}\right\} \subset \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathbf{X}}).$$

2.5 – *Algebraic $\mathcal{D}$-modules*

In this subsection, we recall some basic notions and results on $\mathcal{D}$-modules. References are made to [2,4,8]. We also refer to [3], [5, §§8, 9], [12, §7], [15, §§3, 4, 7] for analytic $\mathcal{D}$-modules.

Let X be a smooth algebraic variety over $\mathbb{C}$ and denote by d_X its complex dimension. We shall denote by $\mathcal{O}_X$ and $\mathcal{D}_X$ the sheaves of regular functions and algebraic differential operators on X, respectively. Let $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$ be the bounded derived category of left $\mathcal{D}_X$-modules. Moreover, we denote by $\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ and $\mathbf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)$ the full triangulated subcategories of $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$ consisting of objects with algebraic holonomic and algebraic regular holonomic cohomologies, respectively. For a morphism $f \colon X \to Y$ of smooth algebraic varieties, we denote by $\otimes^D$ the tensor product functor, by $\boxtimes^D$ the external tensor product functor, by $\mathbf{D}f_*$ the direct image functor, by $\mathbf{D}f^*$ the inverse image functor, and by $\mathbb{D}_X$ the duality functor for $\mathcal{D}$-modules. See, e.g., [8, §3] for the details. In this paper, for convenience, we set

$$\mathbf{D}f_! := \mathbb{D}_Y \circ \mathbf{D}f_* \circ \mathbb{D}_X, \quad \mathbf{D}f^\bigstar := \mathbb{D}_X \circ \mathbf{D}f^* \circ \mathbb{D}_Y.$$

Remark that in [8, Def. 3.2.13] the functor $\mathbb{D}_X \circ \mathbf{D}f^*(\cdot)[d_X - d_Y] \circ \mathbb{D}_Y$ is denoted by $\mathbf{D}f^\bigstar$. Note that these functors preserve the holonomicity. See [8, Props. 3.2.1, 3.2.2, Thm. 3.2.3 and Cor. 3.2.4] for the details.

We denote by X^{an} the underlying complex manifold of X and by $\tilde{\imath}\colon (X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \to (X, \mathcal{O}_X)$ the morphism of ringed spaces. Since there exists a morphism $\tilde{\imath}^{-1}\mathcal{O}_X \to \mathcal{O}_{X^{\mathrm{an}}}$ of sheaves on X^{an}, we have a canonical morphism $\tilde{\imath}^{-1}\mathcal{D}_X \to \mathcal{D}_{X^{\mathrm{an}}}$. Then we obtain a functor

$$(\cdot)^{\mathrm{an}}\colon \mathrm{Mod}(\mathcal{D}_X) \to \mathrm{Mod}(\mathcal{D}_{X^{\mathrm{an}}}), \quad \mathcal{M} \mapsto \mathcal{M}^{\mathrm{an}} := \mathcal{D}_{X^{\mathrm{an}}} \otimes_{\tilde{\imath}^{-1}\mathcal{D}_X} \tilde{\imath}^{-1}\mathcal{M}.$$

It is called the analytification functor on X. Since the sheaf $\mathcal{D}_{X^{\mathrm{an}}}$ is faithfully flat over $\tilde{\imath}^{-1}\mathcal{D}_X$, the analytification functor is faithful and exact, and hence we obtain $(\cdot)^{\mathrm{an}}\colon \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X) \to \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X^{\mathrm{an}}})$. Note that the analytification functor preserves the holonomicity. Moreover, we have some functorial properties of the analytification functor. See [8, Props. 4.7.1, 4.7.2] for the details.

The classical solution functor on X is defined by

$$\mathrm{Sol}_X\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} \to \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{X^{\mathrm{an}}}), \quad \mathcal{M} \mapsto \mathrm{Sol}_X(\mathcal{M})\colon= \mathrm{Sol}_{X^{\mathrm{an}}}(\mathcal{M}^{\mathrm{an}}).$$

Theorem 2.10 ([2,4], see also [8, Thms. 4.7.7, 7.2.2]). *There exists an equivalence of triangulated categories:*

$$\mathrm{Sol}_X\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)^{\mathrm{op}} \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X).$$

This is an algebraic version of the regular Riemann–Hilbert correspondence. The following result means that the classical solution functor is t-exact with respect to the standard t-structure on $\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ and the perverse t-structure on $\mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$. See, e.g., the proof of [8, Thm. 7.2.5] for the details.

Theorem 2.11. *For any $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$, we have*

(1) $\mathcal{M} \in \mathbf{D}^{\leq 0}_{\mathrm{hol}}(\mathcal{D}_X) \iff \mathrm{Sol}_X(\mathcal{M})[d_X] \in {}^p\mathbf{D}^{\geq 0}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$,

(2) $\mathcal{M} \in \mathbf{D}^{\geq 0}_{\mathrm{hol}}(\mathcal{D}_X) \iff \mathrm{Sol}_X(\mathcal{M})[d_X] \in {}^p\mathbf{D}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$.

Moreover, the above equivalence induces an equivalence of abelian categories:

$$\mathrm{Sol}_X(\cdot)[d_X]\colon \mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Perv}(\mathbb{C}_X).$$

By Theorems 2.10 and 2.11, we have a functor $(\cdot)_{\mathrm{reg}}\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) \to \mathbf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)$ which is defined by $\mathcal{M}_{\mathrm{reg}} := \mathrm{RH}_X(\mathrm{Sol}_X(\mathcal{M}))$. Here $\mathrm{RH}_X\colon \mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)^{\mathrm{op}} \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)$ is the inverse functor of $\mathrm{Sol}_X\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)^{\mathrm{op}} \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$. We call it the regularization functor for algebraic holonomic $\mathcal{D}$-modules. By Theorem 2.11, we also have the functor $(\cdot)_{\mathrm{reg}}\colon \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X) \to \mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X)$ between abelian categories.

At the end of this subsection, we shall recall algebraic meromorphic connections. Let D be a divisor of X, and $j\colon X \setminus D \hookrightarrow X$ the natural embedding. Then we set $\mathcal{O}_X(*D) := j_*\mathcal{O}_X$ and also set $\mathcal{M}(*D) := \mathcal{M} \otimes^{D} \mathcal{O}_X(*D)$ for $\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_X)$. Note that we have $\mathcal{M}(*D) \simeq \mathbf{D}j_*\mathbf{D}j^*\mathcal{M}$. We say that a $\mathcal{D}_X$-module is an algebraic meromorphic connection along D if it is isomorphic as an $\mathcal{O}_X$-module to a coherent $\mathcal{O}_X(*D)$-module. We denote by $\mathrm{Conn}(X; D)$ the category of algebraic meromorphic connections along D. Note that it is full abelian subcategory of $\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$. Moreover, we set

$$\mathbf{D}^{\mathrm{b}}_{\mathrm{mero}}(\mathcal{D}_{X(D)}) := \left\{\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) \mid \mathcal{H}^i(\mathcal{M}) \in \mathrm{Conn}(X; D) \text{ for any } i \in \mathbb{Z}\right\}.$$

We say that a Zariski locally finite partition $\{X_\alpha\}_{\alpha\in A}$ of X by locally closed subvarieties X_α is an algebraic stratification of X if for any $\alpha \in A$, X_α is smooth and there exists a subset $B \subset A$ such that $\overline{X_\alpha} = \bigsqcup_{\beta\in B} X_\beta$.

Lemma 2.12. *For any $\mathcal{M} \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$, there exists an algebraic stratification $\{X_\alpha\}_{\alpha\in A}$ such that any cohomology of $\mathbf{D}i^*_{X_\alpha}(\mathcal{M})$ is an integrable connection on X_α for each $\alpha \in A$.*

This result is known. See, e.g., [8, Thm. 3.3.1] for the details.

Lemma 2.13. *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Then there exists an algebraic stratification $\{X_\alpha\}_{\alpha\in A}$ of X such that for any $\alpha \in A$ and any sequence of complex blow-ups $b_\alpha\colon \overline{X_\alpha}^{\mathrm{bl}} \to X$ of $\overline{X_\alpha}$ along $\overline{X_\alpha} \setminus X_\alpha$ we have $(\mathbf{D}b_\alpha^*\mathcal{M})(*D_\alpha) \in \mathbf{D}^{\mathrm{b}}_{\mathrm{mero}}(\mathcal{D}_{\overline{X_\alpha}^{\mathrm{bl}}(D_\alpha)})$, where $D_\alpha := b_\alpha^{-1}(\overline{X_\alpha} \setminus X_\alpha)$.*

This lemma follows from Lemma 2.12 and will be used in the proof of Proposition 3.4.

The analytification functor $(\cdot)^{\mathrm{an}} : \mathrm{Mod}(\mathcal{D}_X) \to \mathrm{Mod}(\mathcal{D}_{X^{\mathrm{an}}})$ induces

$$(\cdot)^{\mathrm{an}}\colon \mathrm{Conn}(X; D) \to \mathrm{Conn}(X^{\mathrm{an}}; D^{\mathrm{an}}),$$
$$(\cdot)^{\mathrm{an}}\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{mero}}(\mathcal{D}_{X(D)}) \to \mathbf{D}^{\mathrm{b}}_{\mathrm{mero}}(\mathcal{D}_{X^{\mathrm{an}}(D^{\mathrm{an}})})$$

where we set $D^{\mathrm{an}} := X^{\mathrm{an}} \setminus (X \setminus D)^{\mathrm{an}}$ and $\mathrm{Conn}(X^{\mathrm{an}}; D^{\mathrm{an}})$ is an abelian category of meromorphic connections on X^{an} along D^{an}, $\mathbf{D}^{\mathrm{b}}_{\mathrm{mero}}(\mathcal{D}_{X^{\mathrm{an}}(D^{\mathrm{an}})})$ is a full triangulated subcategory of $\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X^{\mathrm{an}}})$ consisting of objects whose cohomologies are in $\mathrm{Conn}(X^{\mathrm{an}}; D^{\mathrm{an}})$.

We note that if X is complete, there exists an equivalence of categories between the abelian category $\mathrm{Conn}(X; D)$ and the one of effective meromorphic connections on X^{an} along D^{an} by [8, §5.3]. However as a consequence of [22, Thm. 4.2] any analytic meromorphic connection is effective. Hence we have:

Lemma 2.14 ([8, (5.3.2)], [22]). *If X is complete, there exists an equivalence of abelian categories: $(\cdot)^{\mathrm{an}}\colon \mathrm{Conn}(X; D) \xrightarrow{\sim} \mathrm{Conn}(X^{\mathrm{an}}; D^{\mathrm{an}})$. Moreover, this induces an equivalence of triangulated categories: $(\cdot)^{\mathrm{an}}\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{mero}}(\mathcal{D}_{X(D)}) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}_{\mathrm{mero}}(\mathcal{D}_{X^{\mathrm{an}}(D^{\mathrm{an}})})$.*

3. Main results

In this section, we define algebraic $\mathbb{C}$-constructible enhanced ind-sheaves and prove that the triangulated category of them is equivalent to the one of algebraic holonomic $\mathcal{D}$-modules (Theorem 3.7).

3.1 – *The condition* (AC)

A result for the analytic case similar to the result in this section is proved in [9, §3.5].

Let X be a smooth algebraic variety over $\mathbb{C}$ and denote by X^{an} the underlying complex analytic manifold of X.

Definition 3.1. We say that an enhanced ind-sheaf $K \in \mathbf{E}^0(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}})$ satisfies the condition (AC) if there exists an algebraic stratification $\{X_\alpha\}_\alpha$ of X such that

$$\pi^{-1}\mathbb{C}_{(\overline{X}^{\mathrm{bl}}_\alpha)^{\mathrm{an}}\setminus D^{\mathrm{an}}_\alpha} \otimes \mathbf{E}(b^{\mathrm{an}}_\alpha)^{-1}K$$

has a modified quasi-normal form along D^{an}_α for any α, where $b_\alpha\colon \overline{X}^{\mathrm{bl}}_\alpha \to X$ is a sequence of blow-ups of $\overline{X_\alpha}$ along

$$\partial X_\alpha := \overline{X_\alpha} \setminus X_\alpha, \quad D_\alpha := b_\alpha^{-1}(\partial X_\alpha) \quad \text{and} \quad D^{\mathrm{an}}_\alpha := (\overline{X}^{\mathrm{bl}}_\alpha)^{\mathrm{an}} \setminus (\overline{X}^{\mathrm{bl}}_\alpha \setminus D_\alpha)^{\mathrm{an}}.$$

We call such a family $\{X_\alpha\}_{\alpha\in A}$ an algebraic stratification adapted to K.

We denote by $\mathbf{E}^0_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_X)$ the full subcategory of $\mathbf{E}^0(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}})$ whose objects satisfy the condition (AC). Note that $\mathbf{E}^0_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_X)$ is the full subcategory of the abelian category $\mathbf{E}^0_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}})$ of $\mathbb{C}$-constructible enhanced ind-sheaves on X^{an}. Moreover, we set

$$\begin{aligned}\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_X) &:= \left\{K \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}}) \mid \mathcal{H}^i(K) \in \mathbf{E}^0_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_X) \text{ for any } i \in \mathbb{Z}\right\}\\ &\subset \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}}).\end{aligned}$$

Remark 3.2. Definition 3.1 does not depend on a choice of a sequence of blow-ups b_α by [9, Sublem. 3.22]. Hence we obtain the following properties:

(1) By Hironaka's desingularization theorem [7] (see also [26, Thm. 4.3]), there exists a smooth complete algebraic variety $\overline{X}_\alpha^{\mathrm{bl}}$ in Definition 3.1.

(2) Let $\{X_\alpha\}_{\alpha\in A}$ be an algebraic stratification of X adapted to $K \in \mathbf{E}^0_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_X)$. Then any algebraic stratification of X which is finer than $\{X_\alpha\}_{\alpha\in A}$ is also adapted to K, see [9, Sublem. 3.22] for the analytic case.

Proposition 3.3. *The category* $\mathbf{E}^0_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_X)$ *is a full abelian subcategory of* $\mathbf{E}^0_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}})$.

Hence the category $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_X)$ *is a full triangulated subcategory of* $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}})$.

This result can be proved by the same arguments as in the proof of [9, Prop. 3.22]. For $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$, we set

$$\mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M}) := \mathrm{Sol}^{\mathrm{E}}_{X^{\mathrm{an}}}(\mathcal{M}^{\mathrm{an}}) \quad (\in \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}})).$$

Then we obtain the following assertion:

Proposition 3.4. *For* $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$*, the enhanced solution complex* $\mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M})$ *of* $\mathcal{M}$ *is an object of* $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_X)$.

Proof. It is enough to show the assertion in the case of $\mathcal{M} \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$.

Let $\mathcal{M} \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$ and we put $K := \mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M}) \in \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}})$. By Lemma 2.13, Lemma 2.14 and [9, Prop. 3.18], there exist an algebraic stratification $\{X_\alpha\}_{\alpha\in A}$ of X and a sequence of blow-ups $b_\alpha\colon \overline{X}_\alpha^{\mathrm{bl}} \to X$ of $\overline{X_\alpha}$ along ∂X_α for each $\alpha \in A$ such that

$$\pi^{-1}\mathbb{C}_{(\overline{X}_\alpha^{\mathrm{bl}})^{\mathrm{an}}\setminus D_\alpha^{\mathrm{an}}} \otimes \mathbf{E}(b_\alpha^{\mathrm{an}})^{-1}K \in \mathbf{E}^{\mathrm{b}}_{\mathrm{mero}}(\mathrm{I}\mathbb{C}_{(\overline{X}_\alpha^{\mathrm{bl}})^{\mathrm{an}}(D_\alpha^{\mathrm{an}})})$$

for any $\alpha \in A$, where $D_\alpha := b_\alpha^{-1}(\partial X_\alpha)$ is a normal crossing divisor. Since the functor

$$\pi^{-1}\mathbb{C}_{(\overline{X}_\alpha^{\mathrm{bl}})^{\mathrm{an}}\setminus D_\alpha^{\mathrm{an}}} \otimes \mathbf{E}(b_\alpha^{\mathrm{an}})^{-1}(\cdot)$$

is t-exact with respect to the standard t-structure (see [6, Prop. 2.7.3 (iv) and Lem. 2.7.5 (i)]), for any $i \in \mathbb{Z}$ there exists an isomorphism in $\mathbf{E}^0_{\mathrm{mero}}(\mathrm{I}\mathbb{C}_{(\overline{X}_\alpha^{\mathrm{bl}})^{\mathrm{an}}(D_\alpha^{\mathrm{an}})})$:

$$\pi^{-1}\mathbb{C}_{(\overline{X}_\alpha^{\mathrm{bl}})^{\mathrm{an}}\setminus D_\alpha^{\mathrm{an}}} \otimes \mathbf{E}(b_\alpha^{\mathrm{an}})^{-1}(\mathcal{H}^i K) \simeq \mathcal{H}^i(\pi^{-1}\mathbb{C}_{(\overline{X}_\alpha^{\mathrm{bl}})^{\mathrm{an}}\setminus D_\alpha^{\mathrm{an}}} \otimes \mathbf{E}(b_\alpha^{\mathrm{an}})^{-1}K). \quad \blacksquare$$

On the other hand, the following proposition can be proved by Lemma 3.6 below and the same arguments as in the proof of [9, Thm. 3.26].

Proposition 3.5. *For any* $K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_X)$*, there exists* $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ *such that*

$$K \xrightarrow{\sim} \mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M}).$$

Lemma 3.6. *For any* $K \in \mathbf{E}^{0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_X)$*, there exists an algebraic stratification* $\{X_\alpha\}_\alpha$ *of* X *such that*

$$\pi^{-1}\mathbb{C}_{X_\alpha^{\mathrm{an}}} \otimes K \in \mathrm{Sol}^{\mathrm{E}}_X(\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)).$$

Proof. By Lemma 2.14 and [9, Lem. 3.16], there exist an algebraic stratification $\{X_\alpha\}_\alpha$ of X and algebraic meromorphic connections $\mathcal{M}_\alpha$ on $\overline{X}^{\mathrm{bl}}_\alpha$ along D_α such that

$$\pi^{-1}\mathbb{C}_{(\overline{X}^{\mathrm{bl}}_\alpha)^{\mathrm{an}}\setminus D_\alpha^{\mathrm{an}}} \otimes \mathbf{E}(b_\alpha^{\mathrm{an}})^{-1}K \simeq \mathrm{Sol}^{\mathrm{E}}_{\overline{X}^{\mathrm{bl}}_\alpha}(\mathcal{M}_\alpha) \quad \text{(for any } \alpha\text{)},$$

where $b_\alpha\colon \overline{X}^{\mathrm{bl}}_\alpha \to X$ is a sequence of blow-ups of $\overline{X_\alpha}$ along ∂X_α and $D_\alpha := b_\alpha^{-1}(\partial X_\alpha)$. By applying the direct image functor $\mathbf{E}b^{\mathrm{an}}_{\alpha!!}$ we obtain an isomorphism in $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}})$:

$$\pi^{-1}\mathbb{C}_{X_\alpha^{\mathrm{an}}} \otimes K \xrightarrow{\sim} \mathrm{Sol}^{\mathrm{E}}_X(\mathbf{D}b_{\alpha*}\mathcal{N}_\alpha)[d_X - d_{X_\alpha}] \in \mathrm{Sol}^{\mathrm{E}}_X(\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)).$$

Here we used [5, Cor. 9.4.10 (ii)][2] and [8, Prop. 4.7.2 (ii), Thm. 3.2.3 (i)]. ∎

Hence we obtain an essential surjective functor $\mathrm{Sol}^{\mathrm{E}}_X\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} \to \mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_X)$. This is not fully faithful in general. For example, $\mathcal{D}_{\mathbb{C}}/\mathcal{D}_{\mathbb{C}}\cdot(\partial_z - 1)$ is not isomorphic to $\mathcal{D}_{\mathbb{C}}/\mathcal{D}_{\mathbb{C}}\cdot\partial_z$ as algebraic $\mathcal{D}_{\mathbb{C}}$-modules, although

$$\mathrm{Sol}^{\mathrm{E}}_X(\mathcal{D}_{\mathbb{C}}/\mathcal{D}_{\mathbb{C}}\cdot(\partial_z - 1)) \simeq \mathrm{Sol}^{\mathrm{E}}_X(\mathcal{D}_{\mathbb{C}}/\mathcal{D}_{\mathbb{C}}\cdot\partial_z).$$

However if X is complete, then this is fully faithful.

Theorem 3.7. *Let* X *be a smooth complete algebraic variety over* $\mathbb{C}$*. Then there exists an equivalence of triangulated categories:*

$$\mathrm{Sol}^{\mathrm{E}}_X\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} \xrightarrow{\sim} \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_X).$$

Proof. It is enough to show

$$\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)}(\mathcal{M},\mathcal{N}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_X)}(\mathrm{Sol}^{\mathrm{E}}_X(\mathcal{N}), \mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M}))$$

for any $\mathcal{M},\mathcal{N} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$. By [5, Lem. 4.5.14], this follows by taking the 0-th cohomology in Lemma 3.8 below. ∎

(2) In [5], this result was proved under the assumption that $\mathcal{M}$ has a globally good filtration. However, any analytic holonomic $\mathcal{D}$-module has a globally defined good filtration by [19–21] (see also [27, Thm. 4.3.4]).

Lemma 3.8. *Let X be a smooth complete algebraic variety over $\mathbb{C}$. For any $\mathcal{M}, \mathcal{N} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$, there exists an isomorphism in $\mathbf{D}^{\mathrm{b}}(\mathbb{C})$:*

$$\mathbf{R}\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N}) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}^{\mathrm{E}}(\mathrm{Sol}^{\mathrm{E}}_X(\mathcal{N}), \mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M})).$$

Proof. Let us denote by $p_{X^{\mathrm{an}}}\colon X^{\mathrm{an}} \to \{\mathrm{pt}\}$ the map from X^{an} to the set of one point. Recall that the right-hand side is isomorphic to $\mathbf{R}p_{X^{\mathrm{an}}*}\,\mathbf{R}\mathcal{H}om^{\mathrm{E}}(\mathrm{Sol}^{\mathrm{E}}_X(\mathcal{N}), \mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M}))$.

Since $\mathrm{Sol}^{\mathrm{E}}_X(\mathcal{N}) \in \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_X)$ $(\subset \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}}))$ is stable (see [5, §4.9]), we have

$$\mathbf{R}\mathcal{H}om^{\mathrm{E}}(\mathrm{Sol}^{\mathrm{E}}_X(\mathcal{N}), \mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M})) \simeq \mathbf{R}\mathcal{H}om^{\mathrm{E}}(\mathbb{C}^{\mathrm{E}}_{X^{\mathrm{an}}}, \mathbf{R}\mathcal{I}hom^{+}(\mathrm{Sol}^{\mathrm{E}}_X(\mathcal{N}), \mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M}))).$$

Note that there exists an isomorphisms in $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}})$:

$$\mathbf{R}\mathcal{I}hom^{+}(\mathrm{Sol}^{\mathrm{E}}_X(\mathcal{N}), \mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M})) \simeq \mathrm{DR}^{\mathrm{E}}_X(\mathcal{N} \otimes^{D} \mathbb{D}_X\mathcal{M})[-d_X],$$

by [5, Prop. 4.9.13 (2), Thm. 9.4.8 and Cors. 9.4.9 and 9.4.10 (iii)].

By [5, Thm. 9.1.2 (iii)], [8, Prop. 4.7.2 (ii)] and the fact that the map $p_{X^{\mathrm{an}}}$ is proper there exists an isomorphism in $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\{\mathrm{pt}\}})$:

$$\mathbf{E}p_{X^{\mathrm{an}}*}\mathrm{DR}^{\mathrm{E}}_X(\mathcal{N} \otimes^{D} \mathbb{D}_X\mathcal{M}) \simeq \mathrm{DR}^{\mathrm{E}}_{\{\mathrm{pt}\}}(\mathbf{D}p_{X*}(\mathcal{N} \otimes^{D} \mathbb{D}_X\mathcal{M})).$$

Hence we have isomorphisms in $\mathbf{D}^{\mathrm{b}}(\mathbb{C})$:

$$\begin{aligned}
&\mathbf{R}p_{X^{\mathrm{an}}*}\,\mathbf{R}\mathcal{H}om^{\mathrm{E}}(\mathrm{Sol}^{\mathrm{E}}_X(\mathcal{N}), \mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M}))\\
&\simeq \mathbf{R}p_{X^{\mathrm{an}}*}\,\mathbf{R}\mathcal{H}om^{\mathrm{E}}(\mathbb{C}^{\mathrm{E}}_{X^{\mathrm{an}}}, \mathbf{R}\mathcal{I}hom^{+}(\mathrm{Sol}^{\mathrm{E}}_X(\mathcal{N}), \mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M})))\\
&\simeq \mathbf{R}p_{X^{\mathrm{an}}*}\,\mathbf{R}\mathcal{H}om^{\mathrm{E}}(\mathbb{C}^{\mathrm{E}}_{X^{\mathrm{an}}}, \mathrm{DR}^{\mathrm{E}}_X(\mathcal{N} \otimes^{D} \mathbb{D}_X\mathcal{M})[-d_X])\\
&\simeq \mathbf{R}\mathcal{H}om^{\mathrm{E}}(\mathbb{C}^{\mathrm{E}}_{\{\mathrm{pt}\}}, \mathbf{E}p_{X^{\mathrm{an}}*}\mathrm{DR}^{\mathrm{E}}_X(\mathcal{N} \otimes^{D} \mathbb{D}_X\mathcal{M})[-d_X])\\
&\simeq \mathbf{R}\mathcal{H}om^{\mathrm{E}}(\mathbb{C}^{\mathrm{E}}_{\{\mathrm{pt}\}}, \mathrm{DR}^{\mathrm{E}}_{\{\mathrm{pt}\}}(\mathbf{D}p_{X*}(\mathcal{N} \otimes^{D} \mathbb{D}_X\mathcal{M}))[-d_X])\\
&\simeq \mathbf{D}p_{X*}(\mathcal{N} \otimes^{D} \mathbb{D}_X\mathcal{M})[-d_X],
\end{aligned}$$

where in the third isomorphism we used [5, Lem. 4.5.17] and in the last isomorphism we used the fact that

$$\mathbf{R}\mathcal{H}om^{\mathrm{E}}(\mathbb{C}^{\mathrm{E}}_{\{\mathrm{pt}\}}, \mathrm{DR}^{\mathrm{E}}_{\{\mathrm{pt}\}}(\cdot)) \simeq \mathbf{R}\mathcal{H}om^{\mathrm{E}}(\mathbb{C}^{\mathrm{E}}_{\{\mathrm{pt}\}}, e_{\{\mathrm{pt}\}}(\cdot)) \simeq \mathrm{id}$$

(see, e.g., [6, Ex. 3.5.9] for $\mathrm{DR}^{\mathrm{E}}_{\{\mathrm{pt}\}} = e_{\{\mathrm{pt}\}}$ and see the proof of [5, Prop. 4.7.15] for the second isomorphism). On the other hand, we have an isomorphism

$$\mathbf{D}p_{X*}(\mathcal{N} \otimes^{D} \mathbb{D}_X\mathcal{M})[-d_X] \simeq \mathbf{R}\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N})$$

in $\mathbf{D}^{\mathrm{b}}(\mathbb{C})$ by [8, Cor. 2.6.15]. Hence the proof is complete. ■

We obtain the following corollary of Theorem 3.7, which is known to experts.

COROLLARY 3.9. *Let X be a smooth complete algebraic variety over $\mathbb{C}$. Then the analytification functor* $(\cdot)^{\mathrm{an}}\colon \mathrm{Mod}(\mathcal{D}_X) \to \mathrm{Mod}(\mathcal{D}_{X^{\mathrm{an}}})$ *induces fully faithful embeddings*

$$\begin{aligned}(\cdot)^{\mathrm{an}}&\colon \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X) \hookrightarrow \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{X^{\mathrm{an}}}),\\ (\cdot)^{\mathrm{an}}&\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) \hookrightarrow \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X^{\mathrm{an}}}).\end{aligned}$$

3.2 – *The general case*

In this subsection, we consider the general case. Thanks to Hironaka's desingularization theorem [7] (see also [26, Thm. 4.3]), for any smooth algebraic variety X over $\mathbb{C}$ there exists a smooth complete algebraic variety $\widetilde{X}$ such that $X \subset \widetilde{X}$ and $D := \widetilde{X} \setminus X$ is a normal crossing divisor of $\widetilde{X}$.

Let us consider a bordered space $X^{\mathrm{an}}_\infty = (X^{\mathrm{an}}, \widetilde{X}^{\mathrm{an}})$ and the triangulated category $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_\infty})$ of enhanced ind-sheaves on X^{an}_∞. Recall that $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_\infty})$ does not depend on the choice of $\widetilde{X}$ and there exists an equivalence of triangulated categories:

$$\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_\infty}) \underset{\mathbf{E}j^{-1}}{\overset{\mathbf{E}j_{!!}}{\rightleftarrows}} \big\{K \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\widetilde{X}^{\mathrm{an}}}) \mid \pi^{-1}\mathbb{C}_{X^{\mathrm{an}}} \otimes K \xrightarrow{\sim} K\big\},$$

where we denote by $j\colon X^{\mathrm{an}}_\infty \to \widetilde{X}^{\mathrm{an}}$ the morphism of bordered spaces given by the open embedding $X \hookrightarrow \widetilde{X}$ for simplicity, see Section 2.3 for the details.

We shall denote the open embedding $X \hookrightarrow \widetilde{X}$ by the same symbol j and set

$$\mathrm{Sol}^{\mathrm{E}}_{X_\infty}(\mathcal{M}) := \mathbf{E}j^{-1}\mathrm{Sol}^{\mathrm{E}}_{\widetilde{X}}(\mathbf{D}j_*\mathcal{M}) \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_\infty})$$

for any $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$. Note that for any $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ we have

$$\mathrm{Sol}^{\mathrm{E}}_{\widetilde{X}}(\mathbf{D}j_*\mathcal{M}) \in \big\{K \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\widetilde{X}^{\mathrm{an}}}) \mid \pi^{-1}\mathbb{C}_{X^{\mathrm{an}}} \otimes K \xrightarrow{\sim} K\big\}.$$

Furthermore, since $\mathbf{D}j_*\mathcal{M} \simeq (\mathbf{D}j_!\mathcal{M})(*D)$, for any $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ we have

$$\mathrm{Sol}^{\mathrm{E}}_{X_\infty}(\mathcal{M}) \simeq \mathbf{E}j^{-1}\mathrm{Sol}^{\mathrm{E}}_{\widetilde{X}}(\mathbf{D}j_!\mathcal{M})$$

in $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_\infty})$.

Moreover, we obtain some functorial properties of the enhanced solution functor $\mathrm{Sol}^{\mathrm{E}}_{X_\infty}$ on X_∞, see Proposition 3.12 below.

DEFINITION 3.10. We say that an enhanced ind-sheaf $K \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_\infty})$ is algebraic $\mathbb{C}$-constructible on X^{an}_∞ if $\mathbf{E}j_{!!}K \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\widetilde{X}^{\mathrm{an}}})$ is an object of $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{\widetilde{X}})$.

We denote by $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty})$ the full triangulated subcategory of $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_\infty})$ consisting of algebraic $\mathbb{C}$-constructible enhanced ind-sheaves on X^{an}_∞.

By [6, Lem. 3.3.2] and the fact that the triangulated category $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}})$ is a full triangulated subcategory of $\mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}})$, the triangulated category $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_\infty})$ is also a full triangulated subcategory of $\mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_\infty})$.

Then we obtain the first main theorem of this paper.

THEOREM 3.11. *Let X be a smooth algebraic variety over $\mathbb{C}$. There exists an equivalence of triangulated categories:*

$$\mathrm{Sol}^{\mathrm{E}}_{X_\infty}\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} \xrightarrow{\sim} \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty}).$$

PROOF. For any $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$, we have $\mathrm{Sol}^{\mathrm{E}}_{\widetilde{X}}(\mathbf{D}j_*\mathcal{M}) \in \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{\widetilde{X}})$ by Proposition 3.4. Then there exist isomorphisms:

$$\begin{aligned}\mathbf{E}j_{!!}\mathrm{Sol}^{\mathrm{E}}_{X_\infty}(\mathcal{M}) &\simeq \mathbf{E}j_{!!}\mathbf{E}j^{-1}\mathrm{Sol}^{\mathrm{E}}_{\widetilde{X}}(\mathbf{D}j_*\mathcal{M})\\ &\simeq \pi^{-1}\mathbb{C}_{X^{\mathrm{an}}} \otimes \mathrm{Sol}^{\mathrm{E}}_{\widetilde{X}}(\mathbf{D}j_*\mathcal{M})\\ &\simeq \mathrm{Sol}^{\mathrm{E}}_{\widetilde{X}}(\mathbf{D}j_*\mathcal{M}).\end{aligned}$$

Hence, by Theorem 3.7, we obtain $\mathrm{Sol}^{\mathrm{E}}_{X_\infty}(\mathcal{M}) \in \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty})$ for any $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$. Moreover, by Theorem 3.7 and the fact that the direct image functor $\mathbf{D}j_*$ of the open embedding $j\colon X \hookrightarrow \widetilde{X}$ is fully faithful, we obtain a fully faithful embedding

$$\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) \hookrightarrow \left\{K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{\widetilde{X}^{\mathrm{an}}}) \mid \pi^{-1}\mathbb{C}_{X^{\mathrm{an}}} \otimes K \xrightarrow{\sim} K\right\}, \quad \mathcal{M} \mapsto \mathrm{Sol}^{\mathrm{E}}_{\widetilde{X}}(\mathbf{D}j_*\mathcal{M}).$$

Thus the enhanced solution functor $\mathrm{Sol}^{\mathrm{E}}_{X_\infty}$ induces a fully faithful embedding

$$\mathrm{Sol}^{\mathrm{E}}_{X_\infty}\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} \hookrightarrow \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty})$$

by the definition of $\mathrm{Sol}^{\mathrm{E}}_{X_\infty} := \mathbf{E}j^{-1}\mathrm{Sol}^{\mathrm{E}}_{\widetilde{X}}(\mathbf{D}j_*(\cdot))$.

On the other hand, let $K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty})$. Then

$$\mathbf{E}j_{!!}K \in \left\{K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{\widetilde{X}^{\mathrm{an}}}) \mid \pi^{-1}\mathbb{C}_{X^{\mathrm{an}}} \otimes K \xrightarrow{\sim} K\right\}$$

by the definition of the algebraic $\mathbb{C}$-constructability for enhanced ind-sheaves on X^{an}_∞. Hence there exists an object $\mathcal{N}$ of $\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{\widetilde{X}})$ such that

$$\mathbf{E}j_{!!}K \simeq \mathrm{Sol}^{\mathrm{E}}_{\widetilde{X}}(\mathcal{N})$$

by Theorem 3.7. Moreover, since $\pi^{-1}\mathbb{C}_{X^{\mathrm{an}}} \otimes \mathbf{E}j_{!!}K \xrightarrow{\sim} \mathbf{E}j_{!!}K$, we have

$$\mathrm{Sol}^{\mathrm{E}}_{\widetilde{X}}(\mathcal{N}) \simeq \pi^{-1}\mathbb{C}_{X^{\mathrm{an}}} \otimes \mathrm{Sol}^{\mathrm{E}}_{\widetilde{X}}(\mathcal{N}) \simeq \mathrm{Sol}^{\mathrm{E}}_{\widetilde{X}}(\mathcal{N}(*D)) \simeq \mathrm{Sol}^{\mathrm{E}}_{\widetilde{X}}(\mathbf{D}j_*\mathbf{D}j^*\mathcal{N}).$$

We set $\mathcal{M} := \mathbf{D}j^*\mathcal{N} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$. Then there exists an isomorphisms in $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_\infty})$:

$$\mathrm{Sol}^{\mathrm{E}}_{X_\infty}(\mathcal{M}) \simeq \mathbf{E}j^{-1}\mathrm{Sol}^{\mathrm{E}}_{\widetilde{X}}(\mathcal{N}) \simeq \mathbf{E}j^{-1}\mathbf{E}j_{!!}K \simeq K.$$

This completes the proof. ∎

At the end of this subsection, we shall prove that the algebraic $\mathbb{C}$-constructability is closed under many operations. This follows from some functorial properties of the enhanced solution functor $\mathrm{Sol}^{\mathrm{E}}_{X_\infty}$ on X_∞. See Section 2.5 for the notations of operations of $\mathcal{D}_X$-modules.

Proposition 3.12. (1) *For $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$, there is an isomorphism in $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_\infty})$:*

$$\mathrm{D}^{\mathrm{E}}_{X^{\mathrm{an}}_\infty}\mathrm{Sol}^{\mathrm{E}}_{X_\infty}(\mathcal{M}) \simeq \mathrm{Sol}^{\mathrm{E}}_{X_\infty}(\mathbb{D}_X\mathcal{M})[2d_X].$$

(2) *Let $f\colon X \to Y$ be a morphism of smooth algebraic varieties. Then for $\mathcal{N} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_Y)$ there are isomorphisms in $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_\infty})$:*

$$\mathbf{E}(f^{\mathrm{an}}_\infty)^{-1}\mathrm{Sol}^{\mathrm{E}}_{Y_\infty}(\mathcal{N}) \simeq \mathrm{Sol}^{\mathrm{E}}_{X_\infty}(\mathbf{D}f^{*}\mathcal{N}),$$
$$\mathbf{E}(f^{\mathrm{an}}_\infty)^{!}\mathrm{Sol}^{\mathrm{E}}_{Y_\infty}(\mathcal{N}) \simeq \mathrm{Sol}^{\mathrm{E}}_{X_\infty}(\mathbf{D}f^{\bigstar}\mathcal{N}).$$

(3) *Let $f\colon X \to Y$ be a morphism of smooth algebraic varieties. For $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$, there are isomorphisms in $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{Y^{\mathrm{an}}_\infty})$:*

$$\mathbf{E}f^{\mathrm{an}}_{\infty *}\mathrm{Sol}^{\mathrm{E}}_{X_\infty}(\mathcal{M})[d_X] \simeq \mathrm{Sol}^{\mathrm{E}}_{Y_\infty}(\mathbf{D}f_{!}\mathcal{M})[d_Y],$$
$$\mathbf{E}f^{\mathrm{an}}_{\infty !!}\mathrm{Sol}^{\mathrm{E}}_{X_\infty}(\mathcal{M})[d_X] \simeq \mathrm{Sol}^{\mathrm{E}}_{Y_\infty}(\mathbf{D}f_{*}\mathcal{M})[d_Y].$$

(4) *For $\mathcal{M}_1, \mathcal{M}_2 \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$, there exists an isomorphism in $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_\infty})$:*

$$\mathrm{Sol}^{\mathrm{E}}_{X_\infty}(\mathcal{M}_1) \otimes^{+} \mathrm{Sol}^{\mathrm{E}}_{X_\infty}(\mathcal{M}_2) \simeq \mathrm{Sol}^{\mathrm{E}}_{X_\infty}(\mathcal{M}_1 \otimes^{D} \mathcal{M}_2).$$

Moreover, for any $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ and any $\mathcal{N} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_Y)$, there exists an isomorphism in $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_\infty \times Y^{\mathrm{an}}_\infty})$:

$$\mathrm{Sol}^{\mathrm{E}}_{X_\infty}(\mathcal{M}) \boxtimes^{+} \mathrm{Sol}^{\mathrm{E}}_{Y_\infty}(\mathcal{N}) \simeq \mathrm{Sol}^{\mathrm{E}}_{X_\infty \times Y_\infty}(\mathcal{M} \boxtimes^{D} \mathcal{N}).$$

Proof. (1) Let us recall that there exists an isomorphism $\mathbb{D}_{\widetilde{X}}\mathbf{D}j_*\mathcal{M} \simeq \mathbf{D}j_!\mathbb{D}_X\mathcal{M}$ in $\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{\widetilde{X}})$. Hence we have isomorphisms

$$\begin{aligned}\mathrm{D}^{\mathrm{E}}_{X^{\mathrm{an}}_\infty}\mathrm{Sol}^{\mathrm{E}}_{X_\infty}(\mathcal{M}) &\simeq \mathrm{D}^{\mathrm{E}}_{X^{\mathrm{an}}_\infty}\mathbf{E}j^{-1}\mathrm{Sol}^{\mathrm{E}}_{\widetilde{X}}(\mathbf{D}j_*\mathcal{M})\\ &\simeq \mathbf{E}j^{-1}\mathrm{Sol}^{\mathrm{E}}_{\widetilde{X}}(\mathbb{D}_{\widetilde{X}}\mathbf{D}j_*\mathcal{M})[2d_X]\\ &\simeq \mathbf{E}j^{-1}\mathrm{Sol}^{\mathrm{E}}_{\widetilde{X}}(\mathbf{D}j_!\mathbb{D}_X\mathcal{M})[2d_X]\\ &\simeq \mathrm{Sol}^{\mathrm{E}}_{X_\infty}(\mathbb{D}_X\mathcal{M})[2d_X],\end{aligned}$$

where in the second isomorphism we used [5, Thm. 9.4.8] and [8, Prop. 4.7.1].

(2) Since the proofs of (2) and (3) are similar, we shall skip the proof of (2).

(3) By [6, Prop. 3.3.3 (iv)] and the fact that the morphism $f_\infty^{\rm an}\colon X_\infty^{\rm an}\to Y_\infty^{\rm an}$ of bordered spaces is semi-proper, for $K\in\mathbf{E}^{\rm b}_{\mathbb{R}\text{-c}}({\rm I}\mathbb{C}_{X_\infty^{\rm an}})$ we have an isomorphism in $\mathbf{E}^{\rm b}({\rm I}\mathbb{C}_{Y_\infty^{\rm an}})$:

$$\mathbf{E}f_{\infty *}^{\rm an}K\simeq {\rm D}^{\rm E}_{Y_\infty^{\rm an}}\mathbf{E}f_{\infty !!}^{\rm an}{\rm D}^{\rm E}_{X_\infty^{\rm an}}(K).$$

Hence it is enough to prove the second part of (3). In fact, we have isomorphisms

$$\begin{aligned}\mathbf{E}f_{\infty !!}^{\rm an}{\rm Sol}^{\rm E}_{X_\infty}(\mathcal{M})&\simeq \mathbf{E}j_Y^{-1}\mathbf{E}(\tilde{f}^{\rm an})_{!!}\mathbf{E}j_{X!!}{\rm Sol}^{\rm E}_{X_\infty}(\mathcal{M})\\&\simeq \mathbf{E}j_Y^{-1}\mathbf{E}(\tilde{f}^{\rm an})_{!!}{\rm Sol}^{\rm E}_{\tilde{X}}(\mathbf{D}j_{X*}\mathcal{M})\\&\simeq \mathbf{E}j_Y^{-1}{\rm Sol}^{\rm E}_{\tilde{Y}}(\mathbf{D}\tilde{f}_*\mathbf{D}j_{X*}\mathcal{M})[d_Y-d_X]\\&\simeq \mathbf{E}j_Y^{-1}{\rm Sol}^{\rm E}_{\tilde{Y}}(\mathbf{D}j_{Y*}\mathbf{D}f_*\mathcal{M})[d_Y-d_X]\\&\simeq {\rm Sol}^{\rm E}_{Y_\infty}(\mathbf{D}f_*\mathcal{M})[d_Y-d_X],\end{aligned}$$

where in the second (resp. third) isomorphism we used Sublemma 2.3 (1) (resp. [5, Cor. 9.4.10 (ii)] and [8, Prop. 4.7.2 (ii)]).

(4) By (2) it is enough to show the first part of (4). Recall that there exists an isomorphism $\mathbf{D}j_*\mathcal{M}_1\otimes^D\mathbf{D}j_*\mathcal{M}_2\simeq\mathbf{D}j_*(\mathcal{M}_1\otimes^D\mathcal{M}_2)$ in $\mathbf{D}^{\rm b}_{\rm hol}(\mathcal{D}_{\tilde{X}})$ by [8, Cor. 1.7.5]. Hence we have isomorphisms in $\mathbf{E}^{\rm b}({\rm I}\mathbb{C}_{X_\infty^{\rm an}})$:

$$\begin{aligned}{\rm Sol}^{\rm E}_{X_\infty}(\mathcal{M}_1\otimes^D\mathcal{M}_2)&\simeq \mathbf{E}j^{-1}{\rm Sol}^{\rm E}_{\tilde{X}}(\mathbf{D}j_*(\mathcal{M}_1\otimes^D\mathcal{M}_2))\\&\simeq \mathbf{E}j^{-1}{\rm Sol}^{\rm E}_{\tilde{X}}(\mathbf{D}j_*\mathcal{M}_1\otimes^D\mathbf{D}j_*\mathcal{M}_2)\\&\simeq \mathbf{E}j^{-1}{\rm Sol}^{\rm E}_{\tilde{X}}(\mathbf{D}j_*\mathcal{M}_1)\otimes^+\mathbf{E}j^{-1}{\rm Sol}^{\rm E}_{\tilde{X}}(\mathbf{D}j_*\mathcal{M}_2)\\&\simeq {\rm Sol}^{\rm E}_{X_\infty}(\mathcal{M}_1)\otimes^+{\rm Sol}^{\rm E}_{X_\infty}(\mathcal{M}_2),\end{aligned}$$

where in the third isomorphism we used [5, Cor. 9.4.10 (ii), (iv)]. ∎

Corollary 3.13. *Let $f\colon X\to Y$ be a morphism of smooth algebraic varieties and $K, K_1, K_2\in\mathbf{E}^{\rm b}_{\mathbb{C}\text{-c}}({\rm I}\mathbb{C}_{X_\infty})$, $L\in\mathbf{E}^{\rm b}_{\mathbb{C}\text{-c}}({\rm I}\mathbb{C}_{Y_\infty})$. Then we have*

(1) ${\rm D}^{\rm E}_{X_\infty^{\rm an}}(K)\in\mathbf{E}^{\rm b}_{\mathbb{C}\text{-c}}({\rm I}\mathbb{C}_{X_\infty})$ *and* $K\xrightarrow{\sim}{\rm D}^{\rm E}_{X_\infty^{\rm an}}{\rm D}^{\rm E}_{X_\infty^{\rm an}}K$,

(2) $K_1\otimes^+K_2$, $\mathbf{R}\mathcal{I}hom^+(K_1,K_2)$ *and* $K\boxtimes^+L$ *are algebraic* $\mathbb{C}$*-constructible,*

(3) $\mathbf{E}(f_\infty^{\rm an})^{-1}L$ *and* $\mathbf{E}(f_\infty^{\rm an})^!L$ *are algebraic* $\mathbb{C}$*-constructible,*

(4) $\mathbf{E}f_{\infty !!}^{\rm an}K$ *and* $\mathbf{E}f_{\infty *}^{\rm an}K$ *are algebraic* $\mathbb{C}$*-constructible.*

Proof. Since the proofs of these assertions in the corollary are similar, we only prove (1). By Theorem 3.11, there exists $\mathcal{M}\in\mathbf{D}^{\rm b}_{\rm hol}(\mathcal{D}_X)$ such that $K\simeq{\rm Sol}^{\rm E}_{X_\infty}(\mathcal{M})$. Then by Proposition 3.12 (1) we obtain

$${\rm D}^{\rm E}_{X_\infty^{\rm an}}(K)\simeq{\rm D}^{\rm E}_{X_\infty^{\rm an}}{\rm Sol}^{\rm E}_{X_\infty}(\mathcal{M})\simeq{\rm Sol}^{\rm E}_{X_\infty}(\mathbb{D}_X\mathcal{M})[2d_X]\simeq{\rm Sol}^{\rm E}_{X_\infty}(\mathbb{D}_X(\mathcal{M}[2d_X])).$$

By Theorem 3.11 and the fact that $\mathbb{D}_X(\mathcal{M}[2d_X]) \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ we have $\mathrm{D}^{\mathrm{E}}_{X^{\mathrm{an}}_\infty}(K) \in \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty})$. Moreover, the second part of (1) follows from [6, Prop. 3.3.3 (ii)] and the fact that any $\mathbb{C}$-constructible enhanced ind-sheaf is $\mathbb{R}$-constructible. ■

Let us recall that for any $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{X^{\mathrm{an}}})$ we have

$$e_{X^{\mathrm{an}}_\infty}(\mathcal{F}) \simeq \mathbf{E}j^{-1}(e_{\widetilde{X}^{\mathrm{an}}}(\mathbf{R}j^{\mathrm{an}}_!(\mathcal{F}))),$$

see Section 2.2 for the details.

Proposition 3.14. *The functor* $e_{X^{\mathrm{an}}_\infty}\colon \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{X^{\mathrm{an}}}) \hookrightarrow \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_\infty})$ *induces an embedding*

$$\mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X) \hookrightarrow \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty})$$

and we have a commutative diagram

$$\begin{tikzcd}
\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} \arrow[r, "\mathrm{Sol}^{\mathrm{E}}_{X_\infty}", "\sim"'] & \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty}) \\
\mathbf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)^{\mathrm{op}} \arrow[u, phantom, "\cup"] \arrow[r, "\sim", "\mathrm{Sol}_X"'] \arrow[ur, phantom, "\circlearrowleft"] & \mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X). \arrow[u, hook, "e_{X^{\mathrm{an}}_\infty}"']
\end{tikzcd}$$

Proof. It is enough to show the last part. Let $\mathcal{M}$ be an object of $\mathbf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)$. Then $(\mathbf{D}j_*\mathcal{M})^{\mathrm{an}} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_{\widetilde{X}^{\mathrm{an}}})$ by the definition of algebraic regular holonomic. Hence we have isomorphisms in $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\widetilde{X}^{\mathrm{an}}})$:

$$\mathrm{Sol}^{\mathrm{E}}_{\widetilde{X}}(\mathbf{D}j_*\mathcal{M}) \simeq \mathrm{Sol}^{\mathrm{E}}_{\widetilde{X}^{\mathrm{an}}}((\mathbf{D}j_*\mathcal{M})^{\mathrm{an}}) \simeq e_{\widetilde{X}^{\mathrm{an}}}(\mathrm{Sol}_{\widetilde{X}^{\mathrm{an}}}((\mathbf{D}j_*\mathcal{M})^{\mathrm{an}}))$$

by [5, Prop. 9.1.3]. On the other hand, we have an isomorphism in $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_{\widetilde{X}^{\mathrm{an}}})$:

$$\mathrm{Sol}_{\widetilde{X}^{\mathrm{an}}}((\mathbf{D}j_*\mathcal{M})^{\mathrm{an}}) \simeq \mathbf{R}j^{\mathrm{an}}_!(\mathrm{Sol}_{X^{\mathrm{an}}}(\mathcal{M}^{\mathrm{an}})) \simeq \mathbf{R}j^{\mathrm{an}}_!(\mathrm{Sol}_X(\mathcal{M})),$$

see, e.g., [8, Thm. 7.1.1]. Hence there exist isomorphisms in $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_\infty})$:

$$\begin{aligned}
\mathrm{Sol}^{\mathrm{E}}_{X_\infty}(\mathcal{M}) &\simeq \mathbf{E}j^{-1}\mathrm{Sol}^{\mathrm{E}}_{\widetilde{X}}(\mathbf{D}j_*\mathcal{M}) \\
&\simeq \mathbf{E}j^{-1}e_{\widetilde{X}^{\mathrm{an}}}(\mathrm{Sol}_{\widetilde{X}^{\mathrm{an}}}((\mathbf{D}j_*\mathcal{M})^{\mathrm{an}}) \\
&\simeq \mathbf{E}j^{-1}e_{\widetilde{X}^{\mathrm{an}}}\mathbf{R}j^{\mathrm{an}}_!(\mathrm{Sol}_X(\mathcal{M})) \\
&\simeq e_{X^{\mathrm{an}}_\infty}(\mathrm{Sol}_X(\mathcal{M})).
\end{aligned}$$ ■

Let us recall that for any $K \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_\infty})$ we have

$$\mathrm{sh}_{X^{\mathrm{an}}_\infty}(K) \simeq (j^{\mathrm{an}})^{-1}(\mathrm{sh}_{\widetilde{X}^{\mathrm{an}}}(\mathbf{E}j_{!!}(K))),$$

see Section 2.2 for the details.

Lemma 3.15. *For any $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$, there exists an isomorphism in $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_{X^{\mathrm{an}}})$:*

$$\mathrm{sh}_{X^{\mathrm{an}}_\infty}(\mathrm{Sol}^{\mathrm{E}}_{X_\infty}(\mathcal{M})) \simeq \mathrm{Sol}_X(\mathcal{M}).$$

Proof. By the definition of $\mathrm{Sol}^{\mathrm{E}}_{X_\infty}$ and $\mathrm{sh}_{X^{\mathrm{an}}_\infty}$ we have isomorphisms in $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_{X^{\mathrm{an}}})$:

$$\begin{aligned}
\mathrm{sh}_{X^{\mathrm{an}}_\infty}(\mathrm{Sol}^{\mathrm{E}}_{X_\infty}(\mathcal{M})) &\simeq (j^{\mathrm{an}})^{-1}\mathrm{sh}_{\widetilde{X}^{\mathrm{an}}}\mathbf{E}j_{!!}\mathbf{E}j^{-1}\mathrm{Sol}^{\mathrm{E}}_{\widetilde{X}}(\mathbf{D}j_*\mathcal{M})\\
&\simeq (j^{\mathrm{an}})^{-1}\mathrm{sh}_{\widetilde{X}^{\mathrm{an}}}\mathrm{Sol}^{\mathrm{E}}_{\widetilde{X}}(\mathbf{D}j_*\mathcal{M})\\
&\simeq (j^{\mathrm{an}})^{-1}\mathrm{Sol}_{\widetilde{X}}(\mathbf{D}j_*\mathcal{M})\\
&\simeq \mathrm{Sol}_X(\mathbf{D}j^*\mathbf{D}j_*\mathcal{M})\\
&\simeq \mathrm{Sol}_X(\mathcal{M}),
\end{aligned}$$

where in the third isomorphism we used [5, Lem. 9.5.5]. ∎

Proposition 3.16. *The functor $\mathrm{sh}_{X^{\mathrm{an}}_\infty}\colon \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_\infty}) \to \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{X^{\mathrm{an}}})$ induces*

$$\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty}) \to \mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$$

and hence we have a commutative diagram

$$\begin{tikzcd}
\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} \arrow[r, "\mathrm{Sol}^{\mathrm{E}}_{X_\infty}", "\sim"'] \arrow[d, "(\cdot)_{\mathrm{reg}}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty}) \arrow[d, "\mathrm{sh}_{X^{\mathrm{an}}_\infty}"] \\
\mathbf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)^{\mathrm{op}} \arrow[r, "\sim", "\mathrm{Sol}_X"'] & \mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X).
\end{tikzcd}$$

Proof. The first part follows from Theorem 3.11 and Lemma 3.15. Moreover, since there exists an isomorphism $\mathrm{Sol}_X(\mathcal{M}) \simeq \mathrm{Sol}_X(\mathcal{M}_{\mathrm{reg}})$ in $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_{X^{\mathrm{an}}})$ for any $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ (see Section 2.5 for the details), we obtain the commutative diagram. ∎

3.3 – *Perverse t-structure*

In this subsection, we define a t-structure on the triangulated category $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty})$ of algebraic $\mathbb{C}$-constructible enhanced ind-sheaves on X^{an}_∞ and prove that its heart is equivalent to the abelian category $\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$ of algebraic holonomic $\mathcal{D}_X$-modules. A similar results for the analytic case is proved in [9, §4].

We denote by $\mathrm{D}_{X^{\mathrm{an}}}\colon \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{X^{\mathrm{an}}})^{\mathrm{op}} \to \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{X^{\mathrm{an}}})$ the Verdier dual functor for sheaves, see [11, §3] for the definition. By the same arguments as in the proof of [9, Lem. 4.1], the sheafification functor $\mathrm{sh}_{X^{\mathrm{an}}_\infty}\colon \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty}) \to \mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$ commutes with the duality functor as follows.

Lemma 3.17. *For any $K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty})$, there exists an isomorphism in $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_{X^{\mathrm{an}}})$:*

$$\mathrm{sh}_{X^{\mathrm{an}}_\infty}(\mathrm{D}^{\mathrm{E}}_{X^{\mathrm{an}}_\infty}(K)) \simeq \mathrm{D}_{X^{\mathrm{an}}}(\mathrm{sh}_{X^{\mathrm{an}}_\infty}(K)).$$

Let us recall that the triangulated category $\mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$ has the t-structure

$$({}^p\mathbf{D}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathbb{C}_X), {}^p\mathbf{D}^{\geq 0}_{\mathbb{C}\text{-c}}(\mathbb{C}_X))$$

which is called perverse t-structure; denote by $\mathrm{Perv}(\mathbb{C}_X)$ the heart of its t-structure, see [1] (also [8, Thm. 8.1.27]) for the details.

Definition 3.18. We define full subcategories of $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty})$ by

$$\begin{aligned}
{}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty}) &:= \left\{K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty}) \mid \mathrm{sh}_{X^{\mathrm{an}}_\infty}(K) \in {}^p\mathbf{D}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)\right\},\\
{}^p\mathbf{E}^{\geq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty}) &:= \left\{K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty}) \mid \mathrm{D}^{\mathrm{E}}_{X^{\mathrm{an}}_\infty}(K) \in {}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty})\right\}\\
&= \left\{K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty}) \mid \mathrm{sh}_{X^{\mathrm{an}}_\infty}(K) \in {}^p\mathbf{D}^{\geq 0}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)\right\}.
\end{aligned}$$

(The last equality follows from Lemma 3.17.)

The next theorem 3.19 follows from Theorem 2.11 and Lemma 3.15.

Theorem 3.19. *Let X be a smooth algebraic variety over $\mathbb{C}$ and $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$. Then we have*

(1) $\mathcal{M} \in \mathbf{D}^{\leq 0}_{\mathrm{hol}}(\mathcal{D}_X) \iff \mathrm{Sol}^{\mathrm{E}}_{X_\infty}(\mathcal{M})[d_X] \in {}^p\mathbf{E}^{\geq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty})$,

(2) $\mathcal{M} \in \mathbf{D}^{\geq 0}_{\mathrm{hol}}(\mathcal{D}_X) \iff \mathrm{Sol}^{\mathrm{E}}_{X_\infty}(\mathcal{M})[d_X] \in {}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty})$.

Therefore, the pair $({}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty}), {}^p\mathbf{E}^{\geq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty}))$ is a t-structure on $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty})$ and its heart is equivalent to the abelian category $\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$ *of holonomic $\mathcal{D}_X$-modules.*

Definition 3.20. We say that $K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty})$ is an algebraic enhanced perverse ind-sheaf on X^{an}_∞ if

$$K \in \mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty}) := {}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty}) \cap {}^p\mathbf{E}^{\geq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty}).$$

By the definition of the t-structure $({}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty}), {}^p\mathbf{E}^{\geq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty}))$, the duality functor $\mathrm{D}^{\mathrm{E}}_{X^{\mathrm{an}}_\infty}$ induces an equivalence of abelian categories:

$$\mathrm{D}^{\mathrm{E}}_{X^{\mathrm{an}}_\infty} \colon \mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty}).$$

By Proposition 3.14 and the fact that there exists an isomorphism $\mathrm{id} \xrightarrow{\sim} \mathrm{sh}_{X^{\mathrm{an}}_\infty} \circ e_{X^{\mathrm{an}}_\infty}$ of functors, we obtain:

Proposition 3.21. *The embedding functor* $e_{X_\infty^{\mathrm{an}}}\colon \mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X) \hookrightarrow \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty})$ *is t-exact with respect to the perverse t-structures and hence it induces an embedding:*

$$\mathrm{Perv}(\mathbb{C}_X) \hookrightarrow \mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty}).$$

Moreover, we obtain a commutative diagram:

$$\begin{tikzcd}
\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} \arrow[r, "\mathrm{Sol}^{\mathrm{E}}_{X_\infty}(\cdot)[d_X]", "\sim"'] & \mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty}) \\
\mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X)^{\mathrm{op}} \arrow[u, phantom, "\cup"] \arrow[r, "\sim", "\mathrm{Sol}_X(\cdot)[d_X]"'] \arrow[ur, phantom, "\circlearrowleft"] & \mathrm{Perv}(\mathbb{C}_X). \arrow[u, hook, "e_{X_\infty^{\mathrm{an}}}"']
\end{tikzcd}$$

By the definition of the t-structure $({}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty}), {}^p\mathbf{E}^{\geq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty}))$, the sheafification functor $\mathrm{sh}_{X_\infty^{\mathrm{an}}}$ induces a functor

$$\mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty}) \to \mathrm{Perv}(\mathbb{C}_X).$$

Moreover, by Proposition 3.16 we obtain a commutative diagram:

$$\begin{tikzcd}
\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} \arrow[r, "\mathrm{Sol}^{\mathrm{E}}_{X_\infty}(\cdot)[d_X]", "\sim"'] \arrow[d, "(\cdot)_{\mathrm{reg}}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty}) \arrow[d, "\mathrm{sh}_{X_\infty^{\mathrm{an}}}"] \\
\mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X)^{\mathrm{op}} \arrow[r, "\sim", "\mathrm{Sol}_X(\cdot)[d_X]"'] & \mathrm{Perv}(\mathbb{C}_X).
\end{tikzcd}$$

Recall that there exists a generalized t-structure $({}^{\frac{1}{2}}\mathbf{E}^{\leq c}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty^{\mathrm{an}}}), {}^{\frac{1}{2}}\mathbf{E}^{\geq c}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty^{\mathrm{an}}}))_{c\in\mathbb{R}}$ on $\mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty^{\mathrm{an}}})$ by [6, Thm. 3.5.2 (i)]. Then the pair $({}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty}), {}^p\mathbf{E}^{\geq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty}))$ is related to this as follows:

Proposition 3.22. *We have*

$$\begin{aligned}
{}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty}) &= {}^{\frac{1}{2}}\mathbf{E}^{\leq 0}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty^{\mathrm{an}}}) \cap \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty}),\\
{}^p\mathbf{E}^{\geq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty}) &= {}^{\frac{1}{2}}\mathbf{E}^{\geq 0}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty^{\mathrm{an}}}) \cap \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty}).
\end{aligned}$$

Proof. By Corollary 3.13 (1) and the facts

$$\begin{aligned}
{}^p\mathbf{E}^{\geq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty}) &= \left\{K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty}) \mid \mathrm{D}^{\mathrm{E}}_{X_\infty^{\mathrm{an}}}(K) \in {}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty})\right\},\\
{}^{\frac{1}{2}}\mathbf{E}^{\geq 0}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty^{\mathrm{an}}}) &= \left\{K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty^{\mathrm{an}}}) \mid \mathrm{D}^{\mathrm{E}}_{X_\infty^{\mathrm{an}}}(K) \in {}^{\frac{1}{2}}\mathbf{E}^{\leq 0}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty^{\mathrm{an}}})\right\},
\end{aligned}$$

it is enough to show the first part. Let us recall that for any $K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_\infty})$, we have

$$K \in {}^{\frac{1}{2}}\mathbf{E}^{\le 0}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_\infty}) \Longleftrightarrow \mathbf{E}j_{X^{\mathrm{an}}_\infty !!}K \in {}^{\frac{1}{2}}\mathbf{E}^{\le 0}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_{\widetilde{X}^{\mathrm{an}}})$$

by [6, Lem. 3.3.2, Prop. 3.5.6 (i), (iv)]. Hence the first part follows from Lemma 3.23 (1) and Sublemma 3.24 (1) below. ∎

Lemma 3.23. *For any $K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty})$, we have*

(1) $K \in {}^{p}\mathbf{E}^{\le 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty}) \Longleftrightarrow \mathbf{E}j_{X^{\mathrm{an}}_\infty !!}K \in {}^{p}\mathbf{E}^{\le 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{\widetilde{X}})$,

(2) $K \in {}^{p}\mathbf{E}^{\ge 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty}) \Longleftrightarrow \mathbf{E}j_{X^{\mathrm{an}}_\infty !!}K \in {}^{p}\mathbf{E}^{\ge 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{\widetilde{X}})$.

Proof. Since the proof of (2) is similar, we only prove (1).

First, we assume $K \in {}^{p}\mathbf{E}^{\le 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty})$. Then there exists an object $\mathcal{M} \in \mathbf{D}^{\ge 0}_{\mathrm{hol}}(\mathcal{D}_X)$ such that

$$K \simeq \mathrm{Sol}^{\mathrm{E}}_{X_\infty}(\mathcal{M})[d_X]$$

by Theorems 3.11 and 3.19 (2). Furthermore, since the canonical embedding $j\colon X \hookrightarrow \widetilde{X}$ is affine, we have $\mathbf{D}j_*\mathcal{M} \in \mathbf{D}^{\ge 0}_{\mathrm{hol}}(\mathcal{D}_{\widetilde{X}})$, and hence we have

$$\mathbf{E}j_{X^{\mathrm{an}}_\infty !!}K \simeq \mathrm{Sol}^{\mathrm{E}}_{\widetilde{X}}(\mathbf{D}j_*\mathcal{M})[d_X] \in {}^{p}\mathbf{E}^{\le 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{\widetilde{X}})$$

by Theorem 3.19 (2).

We assume $\mathbf{E}j_{X^{\mathrm{an}}_\infty !!}K \in {}^{p}\mathbf{E}^{\le 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{\widetilde{X}})$. Then we obtain

$$\mathrm{sh}_{\widetilde{X}^{\mathrm{an}}}(\mathbf{E}j_{X^{\mathrm{an}}_\infty !!}K) \in {}^{p}\mathbf{D}^{\le 0}_{\mathbb{C}\text{-c}}(\mathbb{C}_{\widetilde{X}}).$$

Since the functor

$$(j^{\mathrm{an}})^{-1}\colon \mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_{\widetilde{X}}) \to \mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$$

is t-exact with respect to the perverse t-structures, we have

$$\mathrm{sh}_{X^{\mathrm{an}}_\infty}(K) \simeq (j^{\mathrm{an}})^{-1}(\mathrm{sh}_{\widetilde{X}^{\mathrm{an}}}(\mathbf{E}j_{X^{\mathrm{an}}_\infty !!}K)) \in {}^{p}\mathbf{D}^{\le 0}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$$

and hence $K \in {}^{p}\mathbf{E}^{\le 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty})$. ∎

Let us recall that the triangulated category $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_X)$ is a full triangulated subcategory of $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}})$ and $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}})$ has the t-structure $({}^{p}\mathbf{E}^{\le 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}}), {}^{p}\mathbf{E}^{\ge 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}}))$, see Sections 2.4, 3.1 and [9, Def. 4.2, Thm. 4.4] for the details.

Sublemma 3.24. *Let X be a smooth complete algebraic variety over $\mathbb{C}$. Then we have*

(1) ${}^{p}\mathbf{E}^{\le 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_X) = {}^{p}\mathbf{E}^{\le 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}}) \cap \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_X) = {}^{\frac{1}{2}}\mathbf{E}^{\le 0}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}}) \cap \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_X)$,

(2) ${}^{p}\mathbf{E}^{\ge 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_X) = {}^{p}\mathbf{E}^{\ge 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}}) \cap \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_X) = {}^{\frac{1}{2}}\mathbf{E}^{\ge 0}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}}) \cap \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_X)$.

Proof. Since the proof of (2) is similar, we only prove (1). By [9, Cor. 4.5], it is enough to prove

$$ {}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_X) = {}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}}) \cap \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_X). $$

Let K be an object of ${}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_X)$. By Proposition 3.5 and Theorem 3.19 (2), there exists an object $\mathcal{M}$ of $\mathbf{D}^{\geq 0}_{\mathrm{hol}}(\mathcal{D}_X)$ such that $K \simeq \mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M})[d_X]$ $(:= \mathrm{Sol}^{\mathrm{E}}_{X^{\mathrm{an}}}(\mathcal{M}^{\mathrm{an}})[d_X])$. Since the analytification functor $(\cdot)^{\mathrm{an}}\colon \mathrm{Mod}(\mathcal{D}_X) \to \mathrm{Mod}(\mathcal{D}_{X^{\mathrm{an}}})$ is exact, we have $\mathcal{M}^{\mathrm{an}} \in \mathbf{D}^{\geq 0}_{\mathrm{hol}}(\mathcal{D}_{X^{\mathrm{an}}})$, and hence we have

$$ \mathrm{Sol}^{\mathrm{E}}_{X^{\mathrm{an}}}(\mathcal{M}^{\mathrm{an}})[d_X] \in {}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}}) $$

by [9, Thm. 4.4 (2)]. Therefore we obtain

$$ K \in {}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}}) \cap \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_X). $$

Let K be an object of ${}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}}) \cap \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_X)$. Since $K \in \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_X)$, there exists an object $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ such that $K \simeq \mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M})[d_X]$ by Proposition 3.5. Since $K \in {}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}})$, we have $\mathcal{M}^{\mathrm{an}} \in \mathbf{D}^{\geq 0}_{\mathrm{hol}}(\mathcal{D}_{X^{\mathrm{an}}})$ by [9, Thm. 4.4 (2)], and hence we obtain $\mathcal{M} \in \mathbf{D}^{\geq 0}_{\mathrm{hol}}(\mathcal{D}_X)$ because the analytification functor $(\cdot)^{\mathrm{an}}\colon \mathrm{Mod}(\mathcal{D}_X) \to \mathrm{Mod}(\mathcal{D}_{X^{\mathrm{an}}})$ is exact and faithful. Therefore, by Theorem 3.19 (2), we have

$$ K \simeq \mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M})[d_X] \in {}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_X). \qquad \blacksquare $$

Thanks to [6, Prop. 3.5.6], the next proposition follows from Corollary 3.13 (3), (4) and Proposition 3.22. We skip its proof.

Proposition 3.25. *Let $f\colon X \to Y$ be a morphism of smooth algebraic varieties. We assume that there exists a non-negative integer d such that $\dim f^{-1}(y) \leq d$ for any $y \in Y$.*

(1) *For any $K \in {}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty})$, we have $\mathbf{E}f^{\mathrm{an}}_{\infty !!}K \in {}^p\mathbf{E}^{\leq d}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{Y_\infty})$.*

(2) *For any $K \in {}^p\mathbf{E}^{\geq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty})$, we have $\mathbf{E}f^{\mathrm{an}}_{\infty *}K \in {}^p\mathbf{E}^{\geq -d}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{Y_\infty})$.*

(3) *For any $L \in {}^p\mathbf{E}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{Y_\infty})$, we have $\mathbf{E}(f^{\mathrm{an}}_\infty)^{-1}L \in {}^p\mathbf{E}^{\leq d}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty})$.*

(4) *For any $L \in {}^p\mathbf{E}^{\geq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{Y_\infty})$, we have $\mathbf{E}(f^{\mathrm{an}}_\infty)^{!}L \in {}^p\mathbf{E}^{\geq -d}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty})$.*

Corollary 3.26. *Let X be a smooth algebraic variety over $\mathbb{C}$ and Z a locally closed smooth subvariety of X. We denote by $i_{Z^{\mathrm{an}}_\infty}\colon Z^{\mathrm{an}}_\infty \to X^{\mathrm{an}}_\infty$ the morphism of bordered spaces induced by the natural embedding $Z \hookrightarrow X$.*

(1) *$\mathbf{E}i_{Z^{\mathrm{an}}_\infty *}$ and $\mathbf{E}i^{!}_{Z^{\mathrm{an}}_\infty}$ are left t-exact with respect to the perverse t-structures.*

(2) *$\mathbf{E}i_{Z^{\mathrm{an}}_\infty !!}$ and $\mathbf{E}i^{-1}_{Z^{\mathrm{an}}_\infty}$ are right t-exact with respect to the perverse t-structures.*

In particular, if Z is open (resp. closed), then the functor

$$\mathbf{E}i^{-1}_{Z^{\mathrm{an}}_\infty} \simeq \mathbf{E}i^{!}_{Z^{\mathrm{an}}_\infty} \quad (\textit{resp. } \mathbf{E}i_{Z^{\mathrm{an}}_\infty !!} \simeq \mathbf{E}i_{Z^{\mathrm{an}}_\infty *})$$

is t-exact with respect to the perverse t-structures.

Remark 3.27. Let X be a smooth algebraic variety over $\mathbb{C}$ and Z a locally closed smooth subvariety of X. We assume that the natural embedding $i_Z\colon Z \hookrightarrow X$ is affine.

Then we have exact functors

$$\mathbf{D}i_{Z*}, \mathbf{D}i_{Z!}\colon \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_Z) \to \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X).$$

Hence by Proposition 3.12 (3) and Theorem 3.19, we obtain exact functors

$$\mathbf{E}i_{Z^{\mathrm{an}}_\infty *}, \mathbf{E}i_{Z^{\mathrm{an}}_\infty !!}\colon \mathrm{Perv}(\mathrm{I}\mathbb{C}_{Z_\infty}) \to \mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty}).$$

Note that there exists a canonical morphism

$$\mathbf{E}i_{Z^{\mathrm{an}}_\infty !!} \to \mathbf{E}i_{Z^{\mathrm{an}}_\infty *}$$

of functors $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{Z_\infty}) \to \mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$ and it is an isomorphism if Z is closed.

Notation 3.28. For a functor $\mathcal{F}\colon \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_\infty}) \to \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{Y_\infty})$, we set

$${}^{p}\mathcal{F} := {}^{p}\mathcal{H}^0 \circ \mathcal{F}\colon \mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty}) \to \mathrm{Perv}(\mathrm{I}\mathbb{C}_{Y_\infty}),$$

where ${}^{p}\mathcal{H}^0$ is the 0-th cohomology functor with respect to the perverse t-structures.

In this paper, for an object $K \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_\infty})$, let us define the support of K by the complement of the union of open subsets U^{an} of X^{an} such that $K|_{U^{\mathrm{an}}_\infty} := \mathbf{E}i^{-1}_{U^{\mathrm{an}}_\infty} K \simeq 0$ and denote it by $\mathrm{supp}(K)$. Namely, we set

$$\mathrm{supp}(K) := \Bigg(\bigcup_{\substack{U^{\mathrm{an}} \underset{\mathrm{open}}{\subset} X^{\mathrm{an}},\ K|_{U^{\mathrm{an}}_\infty} = 0}} U^{\mathrm{an}} \Bigg)^{c} \subset X^{\mathrm{an}}.$$

Note that we have

$$\bigcup_{\substack{U^{\mathrm{an}} \underset{\mathrm{open}}{\subset} X^{\mathrm{an}},\ K|_{U^{\mathrm{an}}_\infty} = 0}} U^{\mathrm{an}} = \bigcup_{\substack{V^{\mathrm{an}} \underset{\mathrm{open}}{\subset} X^{\mathrm{an}},\ K|_{V^{\mathrm{an}}} = 0}} V^{\mathrm{an}}.$$

Moreover, for a closed smooth subvariety Z of X, we set

$$\mathrm{Perv}_Z(\mathrm{I}\mathbb{C}_{X_\infty}) := \big\{K \in \mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty}) \mid \mathrm{supp}(K) \subset Z^{\mathrm{an}}\big\}.$$

Proposition 3.29. *Let X be a smooth algebraic variety over $\mathbb{C}$ and Z a closed smooth subvariety of X. Then we have an equivalence of abelian categories:*

$$\mathrm{Perv}_Z(\mathrm{I}\mathbb{C}_{X_\infty}) \underset{\mathbf{E}i_{Z^{\mathrm{an}}_\infty !!}}{\overset{{}^p\mathbf{E}i^{-1}_{Z^{\mathrm{an}}_\infty}}{\overset{\sim}{\rightleftarrows}}} \mathrm{Perv}(\mathrm{I}\mathbb{C}_{Z_\infty}).$$

Furthermore, for any $K \in \mathrm{Perv}_Z(\mathrm{I}\mathbb{C}_{X_\infty})$ there exists an isomorphism in $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{Z_\infty})$:

$${}^p\mathbf{E}i^{-1}_{Z^{\mathrm{an}}_\infty} K \simeq {}^p\mathbf{E}i^{!}_{Z^{\mathrm{an}}_\infty} K.$$

Proof. Let L be an object of $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{Z_\infty})$. Then we have $\mathbf{E}i_{Z^{\mathrm{an}}_\infty !!}L \in \mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$ by Corollary 3.26. Furthermore, since

$$\mathbf{E}i^{-1}_{X^{\mathrm{an}}_\infty \setminus Z^{\mathrm{an}}_\infty} \mathbf{E}i_{Z^{\mathrm{an}}_\infty !!}L \simeq 0,$$

we obtain $\mathrm{supp}(\mathbf{E}i_{Z^{\mathrm{an}}_\infty !!}L) \subset Z^{\mathrm{an}}$. This implies that

$$\mathbf{E}i_{Z^{\mathrm{an}}_\infty !!}L \in \mathrm{Perv}_Z(\mathrm{I}\mathbb{C}_{X_\infty}).$$

Note that there exists an isomorphism $\mathbf{E}i^{-1}_{Z^{\mathrm{an}}_\infty} \mathbf{E}i_{Z^{\mathrm{an}}_\infty !!}L \xrightarrow{\sim} L$ in $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{Z^{\mathrm{an}}_\infty})$. Thus we have

$${}^p\mathbf{E}i^{-1}_{Z^{\mathrm{an}}_\infty} \mathbf{E}i_{Z^{\mathrm{an}}_\infty !!}L \xrightarrow{\sim} L.$$

Let K be an object of $\mathrm{Perv}_Z(\mathrm{I}\mathbb{C}_{X_\infty})$. Then we have

$$\pi^{-1}\mathbb{C}_{(X\setminus Z)^{\mathrm{an}}} \otimes K \simeq \mathbf{E}i_{(X\setminus Z)^{\mathrm{an}}_\infty !!}\mathbf{E}i^{-1}_{(X\setminus Z)^{\mathrm{an}}_\infty} K \simeq 0,$$

where in the first isomorphism we used [6, Lem. 2.7.6]. Thus we obtain

$$K \xrightarrow{\sim} \pi^{-1}\mathbb{C}_{Z^{\mathrm{an}}} \otimes K \quad (\simeq \mathbf{E}i_{Z^{\mathrm{an}}_\infty !!}\mathbf{E}i^{-1}_{Z^{\mathrm{an}}_\infty} K).$$

Since the functor $\mathbf{E}i_{Z^{\mathrm{an}}_\infty !!}$ is t-exact with respect to the perverse t-structures, we obtain

$$K \xrightarrow{\sim} \mathbf{E}i_{Z^{\mathrm{an}}_\infty !!}{}^p\mathbf{E}i^{-1}_{Z^{\mathrm{an}}_\infty} K.$$

Therefore the proof of the first part is complete.

For any $K \in \mathrm{Perv}_Z(\mathrm{I}\mathbb{C}_{X_\infty})$, there exists an object $L \in \mathrm{Perv}(\mathrm{I}\mathbb{C}_{Z_\infty})$ such that

$$K \simeq \mathbf{E}i_{Z^{\mathrm{an}}_\infty !!}L \simeq \mathbf{E}i_{Z^{\mathrm{an}}_\infty *}L.$$

Hence we have isomorphisms in $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{Z_\infty})$:

$${}^p\mathbf{E}i^{-1}_{Z^{\mathrm{an}}_\infty} K \simeq {}^p\mathbf{E}i^{-1}_{Z^{\mathrm{an}}_\infty} \mathbf{E}i_{Z^{\mathrm{an}}_\infty *}L \simeq L \simeq {}^p\mathbf{E}i^{!}_{Z^{\mathrm{an}}_\infty} \mathbf{E}i_{Z^{\mathrm{an}}_\infty !!}L \simeq {}^p\mathbf{E}i^{!}_{Z^{\mathrm{an}}_\infty} K.$$

This completes the proof of the second part. ∎

Remark 3.30. By using Theorem 3.19, Proposition 3.29 also follows from Kashiwara's equivalence, see, e.g., [8, Thm. 1.6.1].

3.4 – *Minimal extensions*

In this subsection, we shall consider simple objects of $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$, and a counterpart of minimal extensions of algebraic holonomic $\mathcal{D}$-modules.

Definition 3.31. Let X be a smooth algebraic variety over $\mathbb{C}$. A non-zero algebraic enhanced perverse ind-sheaf $K \in \mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$ is called simple if it contains no subobjects in $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$ other than K or 0.

Proposition 3.32. *Let X be a smooth algebraic variety over $\mathbb{C}$. For any simple algebraic perverse sheaf $\mathcal{F} \in \mathrm{Perv}(\mathbb{C}_X)$, the natural embedding $e_{X_\infty^{\mathrm{an}}}(\mathcal{F})$ of $\mathcal{F}$ is also simple.*

Proof. Let $\mathcal{F} \in \mathrm{Perv}(\mathbb{C}_X)$ be a simple algebraic perverse sheaf on X and $K \in \mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$ a subobject of $e_{X_\infty^{\mathrm{an}}}(\mathcal{F})$ which is not isomorphic to $e_{X_\infty^{\mathrm{an}}}(\mathcal{F})$. Then there exists $\mathcal{M} \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$ such that $K \simeq \mathcal{S}ol^{\mathrm{E}}_{X_\infty}(\mathcal{M})[d_X]$ by Theorem 3.19. Since the functor $\mathrm{sh}_{X_\infty^{\mathrm{an}}}\colon \mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty}) \to \mathrm{Perv}(\mathbb{C}_X)$ is t-exact with respect to the perverse t-structures, we obtain $\mathrm{sh}_{X_\infty^{\mathrm{an}}}(K) \subset \mathrm{sh}_{X_\infty^{\mathrm{an}}} e_{X_\infty^{\mathrm{an}}}(\mathcal{F})$. Then $\mathcal{S}ol_X(\mathcal{M}_{\mathrm{reg}})[d_X]$ $(\simeq \mathrm{sh}_{X_\infty^{\mathrm{an}}}(K))$ is a subobject of $\mathcal{F}$ which is not isomorphic to $\mathcal{F}$. Since $\mathcal{F}$ is simple, we obtain $\mathcal{S}ol_X(\mathcal{M}_{\mathrm{reg}})[d_X] \simeq 0$, and hence $\mathcal{M}_{\mathrm{reg}} \simeq 0$. This implies that $\mathcal{M} \simeq 0$, thus we have $K \simeq 0$. ∎

In this paper, we shall say that $K \in \mathbf{E}^0(\mathrm{I}\mathbb{C}_{X_\infty^{\mathrm{an}}})$ is an enhanced local system on X_∞ if for any $x \in X$ there exist an open neighborhood $U \subset X$ of x and a non-negative integer k such that $K|_{U_\infty^{\mathrm{an}}} \simeq (\mathbb{C}^{\mathrm{E}}_{U_\infty^{\mathrm{an}}})^{\oplus k}$. Note that for any enhanced local system K on X_∞ there exists an integrable connection $\mathcal{L}$ on X such that

$$K[d_X] \simeq \mathcal{S}ol^{\mathrm{E}}_{X_\infty}(\mathcal{L})[d_X] \in \mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty}).$$

Proposition 3.33. (1) *Let Z be a locally closed smooth connected subvariety of a smooth algebraic variety X and K a simple algebraic enhanced perverse ind-sheaf on X_∞. We assume that the natural embedding $i_Z\colon Z \hookrightarrow X$ is affine. Then the image of the canonical morphism $\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!}K \to \mathbf{E}i_{Z_\infty^{\mathrm{an}}*}K$ is also simple, and it is characterized as the unique simple submodule (resp. unique simple quotient module) of $\mathbf{E}i_{Z_\infty^{\mathrm{an}}*}K$ (resp. $\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!}K$).*

(2) *For any simple algebraic enhanced perverse ind-sheaf K on X_∞, there exist a locally closed smooth connected subvariety Z whose natural embedding is affine and a simple enhanced local system L on Z_∞ such that*

$$K \simeq \mathrm{Im}(\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!}L[d_Z] \to \mathbf{E}i_{Z_\infty^{\mathrm{an}}*}L[d_Z]).$$

(3) *Let* (Z, L) *be as in* (1) *and* (Z', L') *be another such pair. Then we have*

$$\mathrm{Im}(\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!}L[d_Z] \to \mathbf{E}i_{Z_\infty^{\mathrm{an}}*}L[d_Z]) \simeq \mathrm{Im}(\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!}L'[d_{Z'}] \to \mathbf{E}i_{Z_\infty^{\mathrm{an}}*}L'[d_{Z'}])$$

if and only if $\overline{Z} = \overline{Z'}$ *and there exists an open dense subset* U *of* $Z \cap Z'$ *such that* $L|_{U_\infty^{\mathrm{an}}} \simeq L'|_{U_\infty^{\mathrm{an}}}$.

Proof. Let us denote by $\mathcal{L}(Z; \mathcal{M})$ the minimal extension of $\mathcal{M} \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_Z)$ along Z, see, e.g., [8, §3.4] for the definition. Then there exist isomorphisms in $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$:

$$\begin{aligned}\mathrm{Sol}^{\mathrm{E}}_{X_\infty}(\mathcal{L}(Z; \mathcal{M}))[d_X] &\simeq \mathrm{Sol}^{\mathrm{E}}_{X_\infty}(\mathrm{Im}(\mathbf{D}i_{Z!}\mathcal{M} \to \mathbf{D}i_{Z*}\mathcal{M}))[d_X] \\ &\simeq \mathrm{Im}(\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!}\mathrm{Sol}^{\mathrm{E}}_{Z_\infty}(\mathcal{M})[d_Z] \to \mathbf{E}i_{Z_\infty^{\mathrm{an}}*}\mathrm{Sol}^{\mathrm{E}}_{Z_\infty}(\mathcal{M})[d_Z]),\end{aligned}$$

where in the second isomorphism we used Theorem 3.19 and Proposition 3.12 (3). Therefore this theorem follows from Theorem 3.19 and [8, Thm. 3.4.2]. ∎

From now on, we shall consider the image of a canonical morphism

$${}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!}K \to {}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}*}K$$

for a locally closed smooth subvariety Z of X whose natural embedding $i_Z \colon Z \hookrightarrow X$ is not necessarily affine and $K \in \mathrm{Perv}(\mathrm{I}\mathbb{C}_{Z_\infty})$. In this paper, we shall define minimal extensions of algebraic enhanced perverse ind-sheaves as follows.

Definition 3.34. For any $K \in \mathrm{Perv}(\mathrm{I}\mathbb{C}_{Z_\infty})$, we call the image of the canonical morphism ${}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!}K \to {}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}*}K$ the minimal extension of K along Z, and denote it by ${}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!*}K$.

Note that we have a functor ${}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!*} \colon \mathrm{Perv}(\mathrm{I}\mathbb{C}_{Z_\infty}) \to \mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$.

Remark 3.35. (1) If Z is open, then we have $({}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!*}K)|_{Z_\infty^{\mathrm{an}}} \simeq K$.
(2) If Z is closed, then we have

$${}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!*}K \simeq \mathbf{E}i_{Z_\infty^{\mathrm{an}}*}K \simeq \mathbf{E}i_{Z_\infty^{\mathrm{an}}!!}K \in \mathrm{Perv}_Z(\mathrm{I}\mathbb{C}_{X_\infty}).$$

(3) If the natural embedding $i_Z \colon Z \hookrightarrow X$ is affine, then we have

$${}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!*}(\cdot) \simeq \mathrm{Im}(\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!}(\cdot) \to \mathbf{E}i_{Z_\infty^{\mathrm{an}}*}(\cdot)).$$

Moreover, for any $\mathcal{M} \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_Z)$ there exists an isomorphism in $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$:

$${}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!*}\mathrm{Sol}^{\mathrm{E}}_{Z_\infty}(\mathcal{M})[d_Z] \simeq \mathrm{Sol}^{\mathrm{E}}_{X_\infty}(\mathcal{L}(Z; \mathcal{M}))[d_X].$$

The minimal extension functor ${}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!*}$ commutes with the duality functor of algebraic enhanced perverse ind-sheaves.

Proposition 3.36. *In the situation as above, there exists a commutative diagram:*

$$\begin{tikzcd}
\mathrm{Perv}(\mathrm{I}\mathbb{C}_{Z_\infty})^{\mathrm{op}} \arrow[r, "{}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!*}"] \arrow[d, "\mathrm{D}^{\mathrm{E}}_{Z_\infty^{\mathrm{an}}}"', "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})^{\mathrm{op}} \arrow[d, "\mathrm{D}^{\mathrm{E}}_{Z_\infty^{\mathrm{an}}}", "\wr"'] \\
\mathrm{Perv}(\mathrm{I}\mathbb{C}_{Z_\infty}) \arrow[r, "{}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!*}"'] & \mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty}).
\end{tikzcd}$$

Proof. Note that we have isomorphisms in $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$:

$$\mathrm{D}^{\mathrm{E}}_{X_\infty^{\mathrm{an}}}({}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!}K) \simeq {}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}*}\mathrm{D}^{\mathrm{E}}_{Z_\infty^{\mathrm{an}}}K,$$
$$\mathrm{D}^{\mathrm{E}}_{X_\infty^{\mathrm{an}}}({}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}*}K) \simeq {}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!}\mathrm{D}^{\mathrm{E}}_{Z_\infty^{\mathrm{an}}}K.$$

Therefore we obtain isomorphisms in $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$:

$$\begin{aligned}\mathrm{D}^{\mathrm{E}}_{X_\infty^{\mathrm{an}}}({}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!*}K) &\simeq \mathrm{Im}({}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!}\mathrm{D}^{\mathrm{E}}_{Z_\infty^{\mathrm{an}}}K \to {}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}*}\mathrm{D}^{\mathrm{E}}_{Z_\infty^{\mathrm{an}}}K) \\ &\simeq {}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!*}\mathrm{D}^{\mathrm{E}}_{Z_\infty^{\mathrm{an}}}K.\end{aligned}$$

■

We denote by ${}^p\mathbf{R}i_{Z!*}\mathcal{F}$ the minimal extension of a perverse sheaf $\mathcal{F}$ along Z, see [1] for the details. By the same arguments as in the proof of the previous proposition, we have:

Proposition 3.37. *In the situation as above, there exists a commutative diagram:*

$$\begin{tikzcd}
\mathrm{Perv}(\mathrm{I}\mathbb{C}_{Z_\infty}) \arrow[r, "{}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!*}"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty}) \\
\mathrm{Perv}(\mathbb{C}_Z) \arrow[r, "{}^p\mathbf{R}i_{Z^{\mathrm{an}}!*}"'] \arrow[u, hook, "e_{Z_\infty^{\mathrm{an}}}"] & \mathrm{Perv}(\mathbb{C}_X). \arrow[u, hook, "e_{X_\infty^{\mathrm{an}}}"']
\end{tikzcd}$$

The following lemma will be used in the proof of Theorem 3.44.

Lemma 3.38. *Let X be a smooth algebraic variety over $\mathbb{C}$, and let Z and W be locally closed smooth subvarieties of X. We assume $W \subset Z$ and we consider a commutative diagram:*

$$\begin{tikzcd}
Z_\infty^{\mathrm{an}} \arrow[r, "i_{Z_\infty^{\mathrm{an}}}"] & X_\infty^{\mathrm{an}}, \\
W_\infty^{\mathrm{an}} \arrow[u, "k"] \arrow[ur, "i_{W_\infty^{\mathrm{an}}}"'] &
\end{tikzcd}$$

where $i_{Z_\infty^{\mathrm{an}}}, i_{W_\infty^{\mathrm{an}}}$ and k are the morphisms of bordered spaces induced by the natural embeddings, respectively. Then for any $K \in \mathrm{Perv}(\mathrm{I}\mathbb{C}_{W_\infty})$ we have

(1) ${}^p\mathbf{E}i_{W_\infty^{\mathrm{an}}!!}K \simeq {}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!}{}^p\mathbf{E}k_{!!}K$ *and* ${}^p\mathbf{E}i_{W_\infty^{\mathrm{an}}*}K \simeq {}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}*}{}^p\mathbf{E}k_{*}K$,

(2) ${}^p\mathbf{E}i_{W_\infty^{\mathrm{an}}!!*}K \simeq {}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!*}{}^p\mathbf{E}k_{!!*}K$.

Proof. Let K be an object of $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{W_\infty})$.

(1) Since the proof of the first assertion of (1) is similar, we shall only prove the second one. Recall that the functor $\mathbf{E}i_{Z_\infty^{\mathrm{an}}*}$ is left t-exact with respect to the perverse t-structures. Hence for any $K \in \mathrm{Perv}(\mathrm{I}\mathbb{C}_{W_\infty})$, we have an isomorphism in $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$:

$${}^p\mathbf{E}i_{W_\infty^{\mathrm{an}}*}K \simeq {}^p\mathcal{H}^0\mathbf{E}i_{W_\infty^{\mathrm{an}}*}K \simeq {}^p\mathcal{H}^0(\mathbf{E}i_{Z_\infty^{\mathrm{an}}*}{}^p\mathcal{H}^0(\mathbf{E}k_*K)) \simeq {}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}*}{}^p\mathbf{E}k_*K.$$

See, e.g., [8, Prop. 8.1.15 (i)] for the second isomorphism.

(2) Recall that there exist canonical morphisms:

$$\begin{gathered}
{}^p\mathbf{E}i_{W_\infty^{\mathrm{an}}!!}K \twoheadrightarrow {}^p\mathbf{E}i_{W_\infty^{\mathrm{an}}!!*}K \hookrightarrow {}^p\mathbf{E}i_{W_\infty^{\mathrm{an}}*}K,\\
{}^p\mathbf{E}k_{!!}K \twoheadrightarrow {}^p\mathbf{E}k_{!!*}K \hookrightarrow {}^p\mathbf{E}k_*K,\\
{}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!}{}^p\mathbf{E}k_{!!*}K \twoheadrightarrow {}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!*}{}^p\mathbf{E}k_{!!*}K \hookrightarrow {}^p\mathbf{E}i_{z_\infty^{\mathrm{an}}*}{}^p\mathbf{E}k_{!!*}K,
\end{gathered}$$

where $\twoheadrightarrow$ (resp. $\hookrightarrow$) is an epimorphism (resp. a monomorphism) in the abelian category of algebraic enhanced perverse ind-sheaves. Since the functor $\mathbf{E}i_{Z_\infty^{\mathrm{an}}*}$ (resp. $\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!}$) is left (resp. right) t-exact with respect to the perverse t-structures, the canonical morphism

$${}^p\mathbf{E}i_{W_\infty^{\mathrm{an}}!!}K = {}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!}{}^p\mathbf{E}k_{!!}K \to {}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}*}{}^p\mathbf{E}k_*K = {}^p\mathbf{E}i_{W_\infty^{\mathrm{an}}*}K$$

can be decomposed as follows:

$$\begin{aligned}
{}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!}{}^p\mathbf{E}k_{!!}K \twoheadrightarrow {}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!}{}^p\mathbf{E}k_{!!*}K &\twoheadrightarrow {}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!*}{}^p\mathbf{E}k_{!!*}K\\
&\hookrightarrow {}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}*}{}^p\mathbf{E}k_{!!*}K \hookrightarrow {}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}*}{}^p\mathbf{E}k_*K.
\end{aligned}$$

This implies that

$$\mathrm{Im}({}^p\mathbf{E}i_{W_\infty^{\mathrm{an}}!!}K \to {}^p\mathbf{E}i_{W_\infty^{\mathrm{an}}*}K) \simeq {}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!*}{}^p\mathbf{E}k_{!!*}K. \qquad \blacksquare$$

Recall that in the case when Z is a closed smooth subvariety of X, minimal extensions along Z can be characterized by Proposition 3.29, see also Remark 3.35 (2). On the other hand, in the case when Z is open and its complement is a smooth subvariety, the minimal extensions along Z can be characterized as follows. Let U be such an open subset of X and set $W := X \setminus U$.

Proposition 3.39. *In the situation as above, the minimal extension* ${}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}!!*}K$ *of* $K \in \mathrm{Perv}(\mathrm{I}\mathbb{C}_{U_\infty})$ *along* U *is characterized as the unique algebraic enhanced perverse ind-sheaf* L *on* X_∞ *satisfying the conditions*

(1) $\mathbf{E}i_{U_\infty^{\mathrm{an}}}^{-1}L \simeq K$,

(2) $\mathbf{E}i_{W_\infty^{\mathrm{an}}}^{-1}L \in \mathbf{E}^{\leq -1}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{W_\infty})$,

(3) $\mathbf{E}i_{W_\infty^{\mathrm{an}}}^{!}L \in \mathbf{E}^{\geq 1}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{W_\infty})$.

Proof. Let K be an object of $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{U_\infty})$. Set $L := {}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}!!*}K \in \mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$. Then we have $\mathbf{E}i_{U_\infty^{\mathrm{an}}}^{-1}L \simeq K$ by the definition of L. Moreover, we have

$$\mathbf{E}i_{W_\infty^{\mathrm{an}}}^{-1}L \in \mathbf{E}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{W_\infty}) \quad (\text{resp. } \mathbf{E}i_{W_\infty^{\mathrm{an}}}^{!}L \in \mathbf{E}^{\geq 0}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{W_\infty}))$$

by Corollary 3.26 (2) (resp. (1)).

By using [6, Lem. 2.7.7], there exist distinguished triangles in $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X_\infty})$:

$$\mathbf{E}i_{U_\infty^{\mathrm{an}}!!}K \to L \to \mathbf{E}i_{W_\infty^{\mathrm{an}}!!}\mathbf{E}i_{W_\infty^{\mathrm{an}}}^{-1}L \xrightarrow{+1},$$
$$\mathbf{E}i_{W_\infty^{\mathrm{an}}*}\mathbf{E}i_{W_\infty^{\mathrm{an}}}^{!}L \to L \to \mathbf{E}i_{U_\infty^{\mathrm{an}}*}K \xrightarrow{+1}.$$

By Corollary 3.26, we have ${}^p\mathcal{H}^1\mathbf{E}i_{U_\infty^{\mathrm{an}}!!}K \simeq 0$ and ${}^p\mathcal{H}^{-1}\mathbf{E}i_{U_\infty^{\mathrm{an}}*}K \simeq 0$, and hence there exist exact sequences in $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$:

$${}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}!!}K \to L \to {}^p\mathcal{H}^0(\mathbf{E}i_{W_\infty^{\mathrm{an}}!!}\mathbf{E}i_{W_\infty^{\mathrm{an}}}^{-1}L) \to 0,$$
$$0 \to {}^p\mathcal{H}^0(\mathbf{E}i_{W_\infty^{\mathrm{an}}*}{}^p\mathbf{E}i_{W_\infty^{\mathrm{an}}}^{!}L) \to L \to {}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}*}K.$$

Since the morphism ${}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}!!}K \to L$ (resp. $L \to {}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}*}K$) is an epimorphism (resp. a monomorphism), we obtain

$${}^p\mathcal{H}^0(\mathbf{E}i_{W_\infty^{\mathrm{an}}!!}\mathbf{E}i_{W_\infty^{\mathrm{an}}}^{-1}L) \simeq 0 \quad (\text{resp. } {}^p\mathcal{H}^0(\mathbf{E}i_{W_\infty^{\mathrm{an}}*}\mathbf{E}i_{W_\infty^{\mathrm{an}}}^{!}L) \simeq 0).$$

Therefore we have

$${}^p\mathcal{H}^0(\mathbf{E}i_{W_\infty^{\mathrm{an}}}^{-1}L) \simeq 0 \quad \text{and} \quad {}^p\mathcal{H}^0(\mathbf{E}i_{W_\infty^{\mathrm{an}}}^{!}L) \simeq 0,$$

and hence

$$\mathbf{E}i_{W_\infty^{\mathrm{an}}}^{-1}L \in \mathbf{E}^{\leq -1}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{W_\infty}) \quad \text{and} \quad \mathbf{E}i_{W_\infty^{\mathrm{an}}}^{!}L \in \mathbf{E}^{\geq 1}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{W_\infty}).$$

Let L be an object of $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$ which satisfies the conditions (1)–(3) as above. By (1), we obtain morphisms in $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$:

$${}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}!!}K \xrightarrow{\alpha} L \xrightarrow{\beta} {}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}*}K.$$

It is enough to show that the morphism α (resp. β) is an epimorphism (resp. a monomorphism). Since the proofs are similar, we only prove that β is a monomorphism.

Let us consider exact sequences $0 \to \operatorname{Ker}\beta \to L \to {}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}*}K$ in $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$. Since $\mathbf{E}i_{U_\infty^{\mathrm{an}}}^{-1}(\operatorname{Ker}\beta) \simeq 0$, we have $\operatorname{Ker}\beta \in \mathrm{Perv}_W(\mathrm{I}\mathbb{C}_{X_\infty})$, and hence there exist $L_\beta \in \mathrm{Perv}(\mathrm{I}\mathbb{C}_{W_\infty})$ such that $\mathbf{E}i_{W_\infty^{\mathrm{an}}!!}(L_\beta) \simeq \operatorname{Ker}\beta$ by Proposition 3.29. Moreover, we have exact sequences in $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{W_\infty})$:

$$0 \to (L_\beta \simeq)\ {}^p\mathbf{E}i_{W_\infty^{\mathrm{an}}}^! \mathbf{E}i_{W_\infty^{\mathrm{an}}!!}L_\beta \to {}^p\mathbf{E}i_{W_\infty^{\mathrm{an}}}^! L \to {}^p\mathbf{E}i_{W_\infty^{\mathrm{an}}}^! {}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}*}K.$$

Since L satisfies the condition (3), we have ${}^p\mathbf{E}i_{W_\infty^{\mathrm{an}}}^! L \simeq 0$. Thus we obtain $L_\beta \simeq 0$. This implies that the morphism β is a monomorphism. ■

Furthermore, the minimal extensions along U have the following properties.

Proposition 3.40. *In the situation as above, for any* $K \in \mathrm{Perv}(\mathrm{I}\mathbb{C}_{U_\infty})$,

(1) ${}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}*}K \in \mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$ *has no non-trivial subobject in* $\mathrm{Perv}_W(\mathrm{I}\mathbb{C}_{X_\infty})$,

(2) ${}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}!!}K \in \mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$ *has no non-trivial quotient object in* $\mathrm{Perv}_W(\mathrm{I}\mathbb{C}_{X_\infty})$.

Proof. Since the proof of (2) is similar, we only prove (1).

Let $L \in \mathrm{Perv}_W(\mathrm{I}\mathbb{C}_{X_\infty})$ be a subobject of ${}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}*}K \in \mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$. Since we have $\mathbf{E}i_{W_\infty^{\mathrm{an}}!!}{}^p\mathbf{E}i_{W_\infty^{\mathrm{an}}}^! L \xrightarrow{\sim} L$ in $\mathrm{Perv}_W(\mathrm{I}\mathbb{C}_{X_\infty})$, it is enough to prove ${}^p\mathbf{E}i_{W_\infty^{\mathrm{an}}}^! L \simeq 0$. We have a monomorphism ${}^p\mathbf{E}i_{W_\infty^{\mathrm{an}}}^! L \hookrightarrow {}^p\mathbf{E}i_{W_\infty^{\mathrm{an}}}^! {}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}*}K$ in $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$, because the functor ${}^p\mathbf{E}i_{W_\infty^{\mathrm{an}}}^!$ is left t-exact with respect to the perverse t-structures by Corollary 3.26 (1). Then we have

$${}^p\mathbf{E}i_{W_\infty^{\mathrm{an}}}^! {}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}*}K \simeq {}^p\mathcal{H}^0(\mathbf{E}i_{W_\infty^{\mathrm{an}}}^! \mathbf{E}i_{U_\infty^{\mathrm{an}}*}K) \simeq 0,$$

and hence ${}^p\mathbf{E}i_{W_\infty^{\mathrm{an}}}^! L \simeq 0$. ■

Recall that there exist the canonical morphisms ${}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}!!}K \twoheadrightarrow {}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}!!*}K \hookrightarrow {}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}*}K$ in $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$. Hence we have:

Corollary 3.41. *In the situation as above, for any* $K \in \mathrm{Perv}(\mathrm{I}\mathbb{C}_{U_\infty})$, *the minimal extension* ${}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}!!*}K$ *has neither a non-trivial subobject nor a non-trivial quotient object in* $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$ *whose support is contained in* W^{an}.

Corollary 3.42. *In the situation as above, the following holds:*

(1) *For an exact sequence* $0 \to K \to L$ *in* $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{U_\infty})$, *the associated sequence* $0 \to {}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}!!*}K \to {}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}!!*}L$ *in* $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$ *is also exact.*

(2) *For an exact sequence* $K \to L \to 0$ *in* $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{U_\infty})$, *the associated sequence* ${}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}!!*}K \to {}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}!!*}L \to 0$ *in* $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$ *is also exact.*

Proof. Since the proof of (2) is similar, we only prove (1).

Let $\alpha\colon K \hookrightarrow L$ be a monomorphism in $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{U_\infty})$. Since $({}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}!!*}K)|_{U_\infty^{\mathrm{an}}} \simeq K$ and $({}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}!!*}L)|_{U_\infty^{\mathrm{an}}} \simeq L$, we obtain

$$(\mathrm{Ker}({}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}!!*}\alpha))|_{U_\infty^{\mathrm{an}}} \simeq \mathrm{Ker}(({}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}!!*}\alpha)|_{U_\infty^{\mathrm{an}}}) \simeq \mathrm{Ker}\,\alpha \simeq 0.$$

This implies that $\mathrm{Ker}({}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}!!*}\alpha) \in \mathrm{Perv}_W(\mathrm{I}\mathbb{C}_{X_\infty})$. Hence $\mathrm{Ker}({}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}!!*}\alpha) \simeq 0$ by Proposition 3.40 (1). Therefore ${}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}!!*}\alpha$ is a monomorphism in $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$. ∎

Corollary 3.43. *In the situation as above, for any simple object in* $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{U_\infty})$*, its minimal extension along U is also a simple object in* $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$.

Proof. Let K be a simple object in $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{U_\infty})$ and L a subobject of ${}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}!!*}K$. Then we have an exact sequence $0 \to L \to {}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}!!*}K \to L' \to 0$ in $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$.

Since $({}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}!!*}K)|_{U_\infty^{\mathrm{an}}} \simeq K$, there exists an exact sequence

$$0 \to \mathbf{E}i_{U_\infty^{\mathrm{an}}}^{-1}L \to K \to \mathbf{E}i_{U_\infty^{\mathrm{an}}}^{-1}L' \to 0$$

in $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{U_\infty})$. Hence we have $\mathbf{E}i_{U_\infty^{\mathrm{an}}}^{-1}L \simeq 0$ or $\mathbf{E}i_{U_\infty^{\mathrm{an}}}^{-1}L' \simeq 0$ because K is simple. This implies that $\mathrm{supp}(L) \subset W^{\mathrm{an}}$ or $\mathrm{supp}(L') \subset W^{\mathrm{an}}$, and hence $L \simeq 0$ or $L' \simeq 0$ by Corollary 3.41. Therefore ${}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}!!*}K$ is a simple object in $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$. ∎

Therefore by Proposition 3.29 and Lemma 3.38, we obtain the following results.

Theorem 3.44. *Let X be a smooth algebraic variety over $\mathbb{C}$, and Z a locally closed smooth subvariety of X whose natural embedding $i_Z\colon Z \hookrightarrow X$ is not necessarily affine. We assume that $Z = U \cap W$ where $U \subset X$ is an open subset whose complement $X \setminus U$ is smooth and $W \subset X$ is a closed subvariety.*

(1) (i) *For an exact sequence $0 \to K \to L$ in* $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{Z_\infty})$*, the associated sequence $0 \to {}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!*}K \to {}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!*}L$ in* $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$ *is also exact.*

(ii) *For an exact sequence $K \to L \to 0$ in* $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{Z_\infty})$*, the associated sequence ${}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!*}K \to {}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!*}L \to 0$ in* $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{X_\infty})$ *is also exact.*

(2) *For any simple object in* $\mathrm{Perv}(\mathrm{I}\mathbb{C}_{Z_\infty})$*, its minimal extension along Z is also simple.*

Proof. Let us consider a commutative diagram:

$$\begin{tikzcd} U_\infty^{\mathrm{an}} \arrow[r, "i_{U_\infty^{\mathrm{an}}}"] & X_\infty^{\mathrm{an}} . \\ Z_\infty^{\mathrm{an}} \arrow[u, "k"] \arrow[ur, "i_{Z_\infty^{\mathrm{an}}}"'] & \end{tikzcd}$$

Then we have ${}^p\mathbf{E}i_{Z_\infty^{\mathrm{an}}!!*} = {}^p\mathbf{E}i_{U_\infty^{\mathrm{an}}!!*} \circ {}^p\mathbf{E}k_{!!*}$ by Lemma 3.38. Therefore the assertion (1) (resp. (2)) follows from Corollary 3.42 (resp. Corollary 3.43) and Proposition 3.29. ∎

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