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Refined Kolmogorov inequalities for the binomial distribution

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Abstract – This paper presents refined versions of the well-known Kolmogorov maximal inequality for the binomial distribution.

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1. Introduction

Let X_n, $n \in \mathbb{N} = \{1, 2, \dots, \}$, be a sequence of independent identically distributed Bernoulli random variables with mean $0 < p < 1$. For the partial sums $S_n = X_1 + \dots + X_n$, $n \in \mathbb{N}$, Kolmogorov proved in 1963 [5] (see also [6]) the maximal inequalities

$$\mathbb{P}\Big\{\sup_{k\geq n}\Big(\frac{S_k}{k} - p\Big) \geq \varepsilon\Big\} \leq e^{-2n\varepsilon^2(1-\varepsilon)}, \quad \mathbb{P}\Big\{\sup_{k\geq n}\Big(\frac{S_k}{k} - \frac{1}{2}\Big) \geq \varepsilon\Big\} \leq e^{-2n\varepsilon^2}$$

for any $0 \leq \varepsilon \leq 1$. The second inequality corresponds to the parameter $p = 1/2$. Later, the second inequality was reproved by Young et al. [14]. In fact, the second

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Kolmogorov inequality can be easily deduced from the Okamoto [12] result. One can easily deduce two-sided inequalities

$$\mathbb{P}\Big\{\sup_{k\geq n}\Big|\frac{S_k}{k}-p\Big|\geq\varepsilon\Big\}\leq 2e^{-2n\varepsilon^2(1-\varepsilon)},\quad \mathbb{P}\Big\{\sup_{k\geq n}\Big|\frac{S_k}{k}-\frac{1}{2}\Big|\geq\varepsilon\Big\}\leq 2e^{-2n\varepsilon^2}.$$

The Kolmogorov inequalities were improved by Banjević [3], Young et al. [13, 15], Kruglov [8], Antonov and Kruglov [1]. Now we recall some of the sharpest inequalities. Young et al. [13] proved that for any $0\leq\varepsilon<\min\{p,q=1-p\}$ and $0<p<1, n\in\mathbb{N}$, the following inequality holds:

$$\mathbb{P}\Big\{\sup_{k\geq n}\Big|\frac{S_k}{k}-p\Big|\geq\varepsilon\Big\}\leq 2e^{-2n\varepsilon^2-(4/9)n\varepsilon^4}. \tag{1}$$

The same authors, Young et al. [13], strengthened the one-sided inequality as follows:

$$\mathbb{P}\Big\{\sup_{k\geq n}\Big(\frac{S_k}{k}-p\Big)\geq\varepsilon\Big\}\leq (1+2\varepsilon)^{n(1/2+\varepsilon)}(1-2\varepsilon)^{n(1/2-\varepsilon)}, \tag{2}$$

if $p\geq\frac{1}{2}$ or $p+\varepsilon<\frac{1}{2}$. Also, Young et al. [15] proved rather sharp the one-sided inequality of the form

$$\mathbb{P}\Big\{\sup_{k\geq n}\Big(\frac{S_k}{k}-p\Big)\geq\varepsilon\Big\}\leq e^{-2n\varepsilon^2-(4/3)n\varepsilon^4}, \tag{3}$$

for $0\leq\varepsilon<\frac{1}{4}$, $0<p\leq\frac{1}{4}$ or $\frac{1}{2}\leq p\leq\frac{3}{4}$.

Proofs of all the mentioned inequalities are based on previous studies of the function

$$L(p,\varepsilon)=(p+\varepsilon)\ln\Big(1+\frac{\varepsilon}{p}\Big)+(q-\varepsilon)\ln\Big(1-\frac{\varepsilon}{q}\Big)$$

where $0<p<1, q=1-p$ and $0\leq\varepsilon<q$. Progress in the lower bound estimation of this function leads to further progress in the upper bound estimation of the inequalities (1)–(3). For example, inequalities (1) and (2) are based on the following inequalities due to Kraft [7]:

$$L(p,\varepsilon)\geq\begin{cases}2\varepsilon^2+\dfrac{4}{9}\varepsilon^4,\\ \ln\big((1+2\varepsilon)^{1/2+\varepsilon}(1-2\varepsilon)^{1/2-\varepsilon}\big) & \text{if } p\geq\dfrac{1}{2}\text{ or } p+\varepsilon<\dfrac{1}{2}.\end{cases} \tag{4}$$

Inequality (3) is a consequence of the bound

$$L(p,\varepsilon)\geq 2\varepsilon^2+\frac{4}{3}\varepsilon^4\quad\text{for } 0\leq\varepsilon\leq\frac{1}{4},\ 0<p\leq\frac{1}{4}\text{ or }\frac{1}{2}\leq p\leq\frac{3}{4} \tag{5}$$

which was proved by Young et al. [15].

One can easily verify that the first inequality in (4) and inequality (5) hold for $\varepsilon = q$. It follows that

$$-\ln(1-\varepsilon) \geq \begin{cases} 2\varepsilon^2 + \frac{4}{9}\varepsilon^4 & \text{for } 0 < \varepsilon < 1, \\ 2\varepsilon^2 + \frac{4}{3}\varepsilon^4 & \text{for } 0 < \varepsilon \leq \frac{1}{4}. \end{cases} \tag{6}$$

An important feature of inequalities (4) and (5) is that the right part of each inequality is the function of the only argument ε and not of p. It is possible to prove analogues of the mentioned inequalities with right sides depending on both arguments p and ε. Massart [11] proved the following inequality:

$$L(p,\varepsilon) \geq \frac{\varepsilon^2}{2(p+\varepsilon/3)(q-\varepsilon/3)}, \quad 0 \leq \varepsilon \leq q = 1-p. \tag{7}$$

Inequality (7) is an essential tool in the investigations of Kolmogorov and Smirnov statistics. Let ξ_n, $n \in \mathbb{N}$, be independent random variables with the same continuous distribution function F. Let F_n stand for the empirical distribution function constructed with the help of random variables $\xi_1, \ldots, \xi_n$. With the help of (7) Massart [11] proved the following inequality for the Smirnov statistic:

$$\mathbb{P}\Big\{\sqrt{n} \sup_{-\infty<x<\infty} (F_n(x) - F(x)) > \lambda\Big\} \leq 2e^{-2\lambda^2}, \quad \lambda \geq 0. \tag{8}$$

Massart [11] explained the importance of this inequality for some problems from mathematical statistics. As an application of the theory of large deviations Bahadur [2, §3] showed that the right part of (8) can be replaced by $2\exp\{-2\lambda^2 + O(\lambda^3)\}$.

Recall that the sequence of random variables η_n, $n \in \mathbb{N}$, satisfies the large deviation principle with rate function $J(x)$, $x \in \mathbb{R}$, if

$$\limsup_{n\to\infty} \frac{1}{n} \ln \mathbb{P}\{\eta_n \in F\} \leq -\inf_{x\in F} J(x) \quad \text{for each closed set } F \subseteq \mathbb{R},$$
$$\liminf_{n\to\infty} \frac{1}{n} \ln \mathbb{P}\{\eta_n \in G\} \geq -\inf_{x\in G} J(x) \quad \text{for each open set } G \subseteq \mathbb{R}.$$

By the well-known Cramér theorem the sequence S_n/n, $n \in \mathbb{N}$, satisfies the large deviation principle with rate function $J(x) = \infty$ for $x < 0$ and $x > 1$ and

$$J(x) = x\ln\Big(\frac{x}{p}\Big) + (1-x)\ln\Big(\frac{1-x}{q}\Big), \quad 0 \leq x \leq 1,\ q = 1-p.$$

Inequalities (1)–(7) and our refined inequalities add new information about the rate function $J(x)$. Our refined inequalities also contain all one needs to generalize the Cramér theorem about large deviations for the sequence $\sup_{k\geq n}(\frac{S_k}{k} - p)$, $n \in \mathbb{N}$.

In this paper we refine all inequalities (1)–(7). All these refined inequalities are strictly sharper than (1)–(7). With the help of inequalities (12) and (13) instead of (7) and some new technique it is possible to replace the right-hand side of (8) with $2\exp\{-2\lambda^2 - \lambda^4/(36n)\}$. This new inequality is much sharper than (8) especially for big λ.

From the previous remarks it follows that the problem of refinement of the maximal Kolmogorov inequality for the binomial distribution has a long history. Based on the investigations of A. N. Kolmogorov himself and his numerous followers, the required maximal inequality can be written in the form

$$\mathbb{P}\Big\{\sup_{k\geq n}\Big(\frac{S_k}{k} - p\Big) \geq \varepsilon\Big\} \leq \exp\big\{-nL(p,\varepsilon)\big\}.$$

Function $L(p,\varepsilon)$ can be expanded in a series with terms that are polynomials of variables ε and p. It appears natural to estimate the function $L(p,\varepsilon)$ with the simplest polynomials; the inequalities presented in this paper are obtained largely in this way.

Calculations and graphs show that the suggested approximation of the function $L(p,\varepsilon)$ has a high accuracy which implies high accuracy of the inequalities proved in this paper. In particular, for $\varepsilon \to 0$ all inequalities in Theorems 2.1–2.3 and Corollaries 2.4–2.6 turn to equalities.

An investigation of the optimality of the obtained inequalities has not been carried out. Obviously, it is possible to prove more precise inequalities by taking the more complex approximations of the function $L(p,\varepsilon)$. At the same time, these inequalities may be very complicated and, hence, have low applicability.

Our approach to inequalities (4)–(6) is new. It relies upon applications of the Budan–Fourier theorem and the Sturm theorem on the number of real roots of any polynomial of degree $n \in \mathbb{N}$ with real coefficients:

$$\text{(9)} \quad P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0,\ x \in \mathbb{R} = (-\infty,\infty).$$

Let $c_1,\dots,c_n$ be real numbers which are not all zero. Let $c_{k_1},\dots,c_{k_m}$ be the ones which are non-zero. Denote by $W(c_1,\dots,c_n)$ the number of sign changes in the sequence $c_{k_1},\dots,c_{k_m}$ if $m > 1$, and let $W(c_1,\dots,c_n) = 0$ if $m = 1$.

Budan–Fourier theorem. *Let P and $P^{(k)}$, $k = 1,\dots,n$, be polynomial (9) and its derivatives, respectively. Let $W(x)$ be the number of sign changes in the sequence $P(x), P^{(1)}(x),\dots,P^{(n)}(x)$, $x \in \mathbb{R}$. Then for any $a,b \in \mathbb{R}$, $a < b$, the number of roots of P in $(a,b]$, counted with their orders of multiplicity, is equal to $W(a) - W(b) - 2m$ for some non-negative integer m.*

Proof. See, for example, Leung et al. [10]. ∎

Let us denote by $P_0(\lambda) = P(\lambda)$ polynomial (9) and calculate its first derivative $P_1(\lambda) = P'(\lambda)$. Then we proceed as in the Euclidean algorithm to find

$$P_{k-1}(x) = P_k(x)Q_k(x) - P_{k+1}(x), \quad k = 1, \dots, s-1,\ s \le n, \tag{10}$$

where $P_s(x)$ is a constant. The sequence of polynomials $P_0, P_1, \dots, P_s$ is called the *Sturm sequence* for the polynomial P.

STURM'S THEOREM. *Let P be polynomial (9) which may have only simple roots in a segment $[a,b]$. Let $W(x)$ be the number of sign changes in the sequence $P_0(x)$, $P_1(x)$, $\dots$, $P_s(x)$, $x \in \mathbb{R}$. If $P(a)P(b) \neq 0$, then the number of roots of P in (a,b) is equal to $W(a) - W(b)$.*

PROOF. See, for example, Leung et al. [10]. ■

2. Refined inequalities

This section contains the main results of the paper. The principle step in the proofs of (1)–(3) is an application of the maximal inequality due to Banjević [3]. Instead, we use a maximal inequality for a reversed martingale, as it was done for the first time by Kruglov [8]. Theorem 2.1 contains a substantial improvement of inequalities (6). At the same time, it is a special stronger case of Theorem 2.2 with $\varepsilon = q = 1 - p$. It also simplifies the proof of Theorem 2.2, since now we may suppose that $\varepsilon < q$.

THEOREM 2.1. *The following inequalities hold:*

$$-\ln(1-\varepsilon) \ge \begin{cases} 2\varepsilon^2 + \dfrac{5}{8}\varepsilon^4 & \text{for } 0 \le \varepsilon < 1, \\ 2\varepsilon^2 + 3\varepsilon^4 & \text{for } 0 \le \varepsilon \le 0.5, \\ 2\varepsilon^2 + 6\varepsilon^4 & \text{for } 0 \le \varepsilon \le 0.4. \end{cases} \tag{11}$$

THEOREM 2.2. *Let $0 < p < 1$, $0 \le \varepsilon \le q = 1 - p$. The following inequalities hold:*

$$L(p,\varepsilon) \ge \begin{cases} 2\varepsilon^2 + \dfrac{4}{9}\varepsilon^4 + \dfrac{1}{30}\varepsilon^6, & \\ 2\varepsilon^2 + \dfrac{25}{18}\varepsilon^4 & \text{if } p \ge \dfrac{1}{2} \text{ or } p + \varepsilon \le \dfrac{1}{2}, \\ -\dfrac{1}{2}\ln(1-4\varepsilon^2) & \text{if } 0 \le \varepsilon < \dfrac{1}{2},\ p + \varepsilon \le \dfrac{1}{2}. \end{cases} \tag{12}$$

THEOREM 2.3. *Let $0 < p < 1$, $0 \le \varepsilon \le q = 1 - p$. The following inequality holds:*

$$L(p,\varepsilon) \ge \frac{\varepsilon^2}{2(p+\varepsilon/3)(q-\varepsilon/3)} + \frac{\varepsilon^4}{36(p+\varepsilon)^3}. \tag{13}$$

Corollary 2.4. *Let* $0 < p < 1$, $0 \le \varepsilon \le q = 1 - p$. *The following inequalities hold:*

$$\mathbb{P}\Big\{\sup_{k\ge n}\Big(\frac{S_k}{k} - p\Big) \ge \varepsilon\Big\} \le \begin{cases} \exp\Big\{-n\Big(2\varepsilon^2 + \frac{4}{9}\varepsilon^4 + \frac{1}{30}\varepsilon^6\Big)\Big\}, \\ \exp\Big\{-n\Big(2\varepsilon^2 + \frac{25}{18}\varepsilon^4\Big)\Big\} & \text{if } p \ge \frac{1}{2} \text{ or } p + \varepsilon \le \frac{1}{2}, \\ (1 - 4\varepsilon^2)^{n/2} & \text{if } 0 \le \varepsilon < \frac{1}{2},\ p + \varepsilon \le \frac{1}{2}. \end{cases}$$

Corollary 2.5. *Let* $0 < p < 1$, $0 \le \varepsilon \le \min\{p, q\}$, $q = 1 - p$. *The following inequality holds:*

$$\mathbb{P}\Big\{\sup_{k\ge n}\Big|\frac{S_k}{k} - p\Big| \ge \varepsilon\Big\} \le 2\exp\Big\{-n\Big(2\varepsilon^2 + \frac{4}{9}\varepsilon^4 + \frac{1}{30}\varepsilon^6\Big)\Big\}.$$

Corollary 2.6. *Let* $0 < p < 1$, $0 \le \varepsilon \le q = 1 - p$. *The following inequality holds:*

$$\mathbb{P}\Big\{\sup_{k\ge n}\Big(\frac{S_k}{k} - p\Big) \ge \varepsilon\Big\} \le \exp\Big\{-n\Big(\frac{\varepsilon^2}{2(p+\varepsilon/3)(q-\varepsilon/3)} + \frac{\varepsilon^4}{36(p+\varepsilon)^3}\Big)\Big\}.$$

3. Proofs

We postpone the proof of Theorems 2.1–2.3. At first we deduce Corollaries 2.4–2.6 from Theorems 2.1–2.3.

Proof of Corollaries 2.4–2.6. Let $\mathcal{F}_n$ be the σ-algebra generated by the random variables S_k, $k \in \mathbb{N}$, $k \ge n$. Note that the sequence $\{\mathcal{F}_n\}_{n\ge 1}$ decreases, that is, $\mathcal{F}_{n+1} \subseteq \mathcal{F}_n$. On page 369 of the textbook [9] by Laha and Rohatgi one can find a proof of the equality

$$\mathbb{E}\Big(\frac{1}{n}S_n | \mathcal{F}_{n+1}\Big) = \frac{1}{n+1}S_{n+1} \text{ a.s.}$$

for the conditional mathematical expectations. For any real number λ by the conditional Jensen inequality we get

$$\exp\Big\{\frac{\lambda}{n+1}S_{n+1}\Big\} \le \mathbb{E}\Big(\exp\Big\{\frac{\lambda}{n}S_n\Big\} | \mathcal{F}_{n+1}\Big) \text{ a.s.}$$

The sequence $\{\exp\{\lambda n^{-1}S_n\}\}_{n\ge 1}$, according to the previous inequality, is a reversed sub-martingale with respect to the decreasing sequence $\{\mathcal{F}_n\}_{n\ge 1}$ of σ-algebras. For any $\lambda > 0$ by the maximal inequality for sub-martingales (see Laha and Rohatgi [9, p. 433]) we get

$$\mathbb{P}\Big\{\sup_{k\ge n}\Big(\frac{S_k}{k}-p\Big)\ge\varepsilon\Big\}=\mathbb{P}\big\{\sup_{k\ge n}e^{\lambda S_k/k}\ge e^{\lambda(p+\varepsilon)}\big\}\le e^{-\lambda(p+\varepsilon)}\mathbb{E}e^{\lambda S_n/n}.$$

For $\lambda=n\ln((p+\varepsilon)q/(q-\varepsilon)p)$ we get

$$\begin{aligned}\mathbb{P}\Big\{\sup_{k\ge n}\Big(\frac{S_k}{k}-p\Big)\ge\varepsilon\Big\}&\le e^{-\lambda(p+\varepsilon)}\mathbb{E}e^{\lambda S_n/n}\\&=e^{-\lambda(p+\varepsilon)}(q+pe^{\lambda/n})^n\le e^{-nL(p,\varepsilon)}.\end{aligned}\tag{14}$$

Corollaries 2.4 and 2.6 follow from inequality (14) and Theorems 2.2 and 2.3.

Recall that X_n, $n\in\mathbb{N}$, are independent identically distributed Bernoulli random variables with mean $0<p<1$. It follows that $Y_n=1-X_n$, $n\in\mathbb{N}$, are independent identically distributed Bernoulli random variables with mean $q=1-p$. Inequality (14) may be applied to random variables $S'_n=Y_1+\cdots+Y_n$, $n\in\mathbb{N}$, with p replaced by $q=1-p$. Since $S_k/k-p=q-S'_k/k$, we have

$$\Big\{\Big|\frac{S_k}{k}-p\Big|\ge\varepsilon\Big\}=\Big\{\frac{S_k}{k}-p\ge\varepsilon\Big\}\cup\Big\{\frac{S'_k}{k}-q\ge\varepsilon\Big\}.$$

It follows that

$$\mathbb{P}\Big\{\sup_{k\ge n}\Big|\frac{S_k}{k}-p\Big|\ge\varepsilon\Big\}\le\mathbb{P}\Big\{\sup_{k\ge n}\Big(\frac{S_k}{k}-p\Big)\ge\varepsilon\Big\}+\mathbb{P}\Big\{\sup_{k\ge n}\Big(\frac{S'_k}{k}-q\Big)\ge\varepsilon\Big\}.$$

Now we may apply inequality (14) to S_k and S'_k, $k\in\mathbb{N}$, and Theorem 2.2. Corollaries 2.4–2.6 are proved. ■

Proof of Theorem 2.1. The first inequality in (11) means that the function $Q(\varepsilon)=\ln(1-\varepsilon)+2\varepsilon^2+(5/8)\varepsilon^4$ of argument $\varepsilon\in[0,1)$ takes only negative (non-positive) values. We need the following derivatives:

$$\begin{aligned}&Q^{(1)}(\varepsilon)=\frac{P(\varepsilon)}{1-\varepsilon},\quad P(\varepsilon)=-\frac{5}{2}\varepsilon^4+\frac{5}{2}\varepsilon^3-4\varepsilon^2+4\varepsilon-1,\\&P^{(1)}(\varepsilon)=-10\varepsilon^3+\frac{15}{2}\varepsilon^2-8\varepsilon+4,\\&P^{(2)}(\varepsilon)=-30\varepsilon^2+15\varepsilon-8.\end{aligned}$$

The polynomial $P^{(2)}(\varepsilon)$ of second order is negative, since $P^{(2)}(0)=-8<0$ and its discriminant is negative. This means that the function $P(\varepsilon)$ is concave on $[0,1]$. Since $P(0)=P(1)=-1$ and $P(1/2)=5/32>0$, there are ε_1 and ε_2 such that $0<\varepsilon_1<\varepsilon_2<1$ and $P(\varepsilon_1)=P(\varepsilon_2)=0$. Direct calculation shows that $P(15/20)=7/512>0$ and $P(16/20)=-13/125<0$, and hence $15/20<\varepsilon_2<16/20$. Since

$P(\varepsilon) \le 0$ for $\varepsilon \in [0, \varepsilon_1] \cup [\varepsilon_2, 1]$ and $P(\varepsilon) \ge 0$ for $\varepsilon \in [\varepsilon_1, \varepsilon_2]$, we have that $Q(\varepsilon_2)$ is the maximal value of $Q(\varepsilon)$ on $(0, 1)$. Recall that the first derivative $P^{(1)}(\varepsilon)$ decreases. Since $P^{(1)}(15/20) = -2 < 0$, we have $P^{(1)}(\varepsilon) \le 0$ for all $\varepsilon \in [15/20, 16/20]$, and hence $P(\varepsilon)$ decreases on the segment $[15/20, 16/20]$. By the Taylor formula there is a number $\varepsilon' \in [15/20, \varepsilon_2]$ such that

$$\begin{aligned}
Q(\varepsilon_2) &= Q\Big(\frac{15}{20}\Big) + Q'(\varepsilon')\Big(\varepsilon_2 - \frac{15}{20}\Big) \\
&= Q\Big(\frac{15}{20}\Big) + \frac{P(\varepsilon')}{1-\varepsilon'}\Big(\varepsilon_2 - \frac{15}{20}\Big) \\
&\le Q\Big(\frac{15}{20}\Big) + \frac{P(15/20)}{1-16/20}\frac{1}{20} \\
&= Q\Big(\frac{15}{20}\Big) + \frac{P(15/20)}{4} \\
&= -\ln 4 + \frac{356671}{268288} < -0.05686 < 0.
\end{aligned}$$

The first inequality in (11) is proved.

The second inequality in (11) means that the function $Q(\varepsilon) = \ln(1-\varepsilon) + 2\varepsilon^2 + 3\varepsilon^4$ of argument $\varepsilon \in [0, 1/2]$ takes only negative (non-positive) values. Again we need the following derivatives:

$$\begin{aligned}
&Q^{(1)}(\varepsilon) = \frac{P(\varepsilon)}{1-\varepsilon}, \quad P(\varepsilon) = -12\varepsilon^4 + 12\varepsilon^3 - 4\varepsilon^2 + 4\varepsilon - 1, \\
&P^{(1)}(\varepsilon) = -48\varepsilon^3 + 36\varepsilon^2 - 8\varepsilon + 4, \\
&P^{(2)}(\varepsilon) = -144\varepsilon^2 + 72\varepsilon - 8, \\
&P^{(3)}(\varepsilon) = -288\varepsilon + 72.
\end{aligned}$$

Note that we use the same symbols for the polynomial and derivatives as above. This cannot lead to misunderstanding since they are used only for local proofs.

We intend to apply the Budan–Fourier theorem and the Sturm theorem to the polynomial $P^{(1)}(\varepsilon)$ to prove the inequality $P^{(1)}(\varepsilon) \ge 0$ for all $\varepsilon \in [0, 1/2]$. Direct calculation shows that

$$\begin{aligned}
&P^{(1)}(0) = 4 > 0, && P^{(1)}(1/2) = 3 > 0, \\
&P^{(2)}(0) = -8 < 0, && P^{(2)}(1/2) = -8 < 0, \\
&P^{(3)}(0) = 72 > 0, && P^{(3)}(1/2) = -72 < 0, \\
&P^{(4)}(0) = -288 < 0, && P^{(4)}(1/2) = -288 < 0.
\end{aligned}$$

The numbers of sign changes of the sequence $P^{(1)}(\varepsilon), \ldots, P^{(4)}(\varepsilon)$ at points $\varepsilon = 0$ and $\varepsilon = 0.4$ are not equal: $W(0) = 3$ and $W(0.4) = 1$. By the Budan–Fourier theorem, the polynomial $P^{(1)}(\varepsilon)$ may have two roots or it has no roots in the segment $[0, 1/2]$. Suppose that it has two roots, say, ε_1 and ε_2. If $\varepsilon_1 = \varepsilon_2$, then $P^{(1)}(\varepsilon) = (\varepsilon - \varepsilon_1)^2 Z(\varepsilon)$ where the polynomial $Z(\varepsilon)$ has no roots in $[0, 1/2]$. Since $P^{(1)}(0) = 4 > 0$, the polynomial $P^{(1)}(\varepsilon)$ is positive for all $\varepsilon \in [0, 1/2]$. If $\varepsilon_1 \neq \varepsilon_2$, then both roots are simple. In this case we may apply the Sturm theorem to the polynomial $P_0(\varepsilon) = P^{(1)}(\varepsilon)$. Let us calculate the Sturm sequence (10) for $P_0(\varepsilon)$:

$$\begin{aligned}
P_0(\varepsilon) &= -48\varepsilon^3 + 36\varepsilon^2 - 8\varepsilon + 4,\\
P_1(\varepsilon) &= -144\varepsilon^2 + 72\varepsilon - 8,\\
P_2(\varepsilon) &= -\frac{2}{3}\varepsilon - \frac{10}{3},\\
P_3(\varepsilon) &= 3968.
\end{aligned}$$

Direct calculation shows that

$$\begin{aligned}
P_0(0) &= 4 > 0, & P_0(1/2) &= 3 > 0,\\
P_1(0) &= -8 < 0, & P_1(1/2) &= -8 < 0,\\
P_2(0) &= -10/3 < 0, & P_2(1/2) &= -11/3 < 0,\\
P_3(p) &= 3968 > 0, & P_3(1/2) &= 3968 > 0.
\end{aligned}$$

The numbers of sign changes $W(0) = 2$ and $W(1/2) = 2$ of the sequence $P_0(\varepsilon), \ldots, P_3(\varepsilon)$ at points $\varepsilon = 0$ and $\varepsilon = 1/2$ are equal, and hence the polynomial $P_0(\varepsilon) = P^{(1)}(\varepsilon)$ has no roots in the segment $[0, 1/2]$. Since $P^{(1)}(0) = 4 > 0$, the first derivative $P^{(1)}(\varepsilon)$ is positive for all $\varepsilon \in [0, 1/2]$. This means that the function $P(\varepsilon)$ increases on the segment $[0, 1/2]$. It has the only root $\varepsilon_0 \in (0, 1/2)$ since $P(0) = -1$ and $P(1/2) = 3/4$. It follows that the function $Q(\varepsilon)$ decreases on $[0, \varepsilon_0]$ and increases on $[\varepsilon_0, 1/2]$. Since $Q(0) = 0$ and $Q(1/2) < -0.00564 < 0$, we have $Q(\varepsilon) \leq 0$ for $\varepsilon \in [0, 1/2]$.

The third inequality in (11) means that the function

$$Q(\varepsilon) = \ln(1 - \varepsilon) + 2\varepsilon^2 + 6\varepsilon^4$$

takes only negative (non-positive) values for all $\varepsilon \in [0, 0.4]$. Again we need the following derivatives:

$$\begin{aligned}
Q^{(1)}(\varepsilon) &= \frac{P(\varepsilon)}{1-\varepsilon}, \quad P(\varepsilon) = -24\varepsilon^4 + 24\varepsilon^3 - 4\varepsilon^2 + 4\varepsilon - 1,\\
P^{(1)}(\varepsilon) &= -96\varepsilon^3 + 72\varepsilon^2 - 8\varepsilon + 4,\\
P^{(2)}(\varepsilon) &= -288\varepsilon^2 + 144\varepsilon - 8,\\
P^{(3)}(\varepsilon) &= -576\varepsilon + 144.
\end{aligned}$$

Prove that the polynomial $P^{(1)}(\varepsilon)$ is positive for all $\varepsilon \in [0, 0.4]$. Direct calculation shows that

$$
\begin{aligned}
&P^{(1)}(0) = 4 > 0, && P^{(1)}(0.4) = 772/125 > 0,\\
&P^{(2)}(0) = -8 < 0, && P^{(2)}(0.4) = 88/25 > 0,\\
&P^{(3)}(0) = 144 > 0, && P^{(3)}(0.4) = -432/5 < 0,\\
&P^{(4)}(0) = -576 < 0, && P^{(4)}(0.4) = -576 < 0.
\end{aligned}
$$

By the Budan–Fourier theorem the polynomial $P^{(1)}(\varepsilon)$ may have two roots or it has no roots in the segment $[0, 0.4]$. Suppose that it has two roots, say, ε_1 and ε_2. If $\varepsilon_1 = \varepsilon_2$, then one can prove as above that the polynomial $P^{(1)}(\varepsilon)$ is positive for all $\varepsilon \in [0, 1/2]$. If $\varepsilon_1 \neq \varepsilon_2$, then both roots are simple. In this case we may apply the Sturm theorem to the polynomial $P_0(\varepsilon) = P^{(1)}(\varepsilon)$. Let us calculate the Sturm sequence (10) for $P_0(\varepsilon) = P^{(1)}(\varepsilon)$:

$$
\begin{aligned}
P_0(\varepsilon) &= -96\varepsilon^3 + 72\varepsilon^2 - 8\varepsilon + 4,\\
P_1(\varepsilon) &= -288\varepsilon^2 + 144\varepsilon - 8,\\
P_2(\varepsilon) &= -\frac{20}{3}\varepsilon - \frac{10}{3},\\
P_3(\varepsilon) &= 152.
\end{aligned}
$$

Direct calculation shows that

$$
\begin{aligned}
&P_0(0) = 4 > 0, && P_0(0.4) = 772/125 > 0,\\
&P_1(0) = -8 < 0, && P_1(0.4) = 88/25 > 0,\\
&P_2(0) = -10/3 < 0, && P_2(0.4) = -6 < 0,\\
&P_3(0) = 152 > 0, && P_3(0.4) = 152 > 0.
\end{aligned}
$$

The numbers of sign changes of the sequence $P_0(\varepsilon), \dots, P_3(\varepsilon)$ at points $\varepsilon = 0$ and $\varepsilon = 0.4$ are equal: $W(0) = 2$ and $W(0.4) = 2$. Hence the polynomial $P_0(\varepsilon) = P^{(1)}(\varepsilon)$ has no roots in the segment $[0, 0.4]$. Since $P_0(0) = P^{(1)}(0) = 4 > 0$, the first derivative $P^{(1)}(\varepsilon)$ is positive for all $\varepsilon \in [0, 0.4]$. This means that the function $P(\varepsilon)$ increases on the segment $[0, 0.4]$. It has the only root $\varepsilon' \in (0, 0.4)$, since $P(0) = -1$ and $P(0.4) = 551/625$. It follows that the function $Q(\varepsilon)$ decreases on $[0, \varepsilon']$ and increases on $[\varepsilon', 0.4]$. Since $Q(0) = 0$ and $Q(0.4) = -0.0372256 < 0$, all values of the function $Q(\varepsilon)$ are negative. Theorem 2.1 is proved. ■

Proof of Theorem 2.2. All inequalities (12) hold for $\varepsilon = 0$. They hold for $\varepsilon = q$ by Theorem 2.1. From now on, we suppose that $0 < \varepsilon < q$.

Proof of the second inequality in (12) *for* $p \geq 1/2$. Let us rewrite it

$$L(p,\varepsilon) \geq 2\varepsilon^2 + \frac{25}{18}\varepsilon^4. \tag{15}$$

If inequality (15) holds for all $1/2 < p < 1$, it holds for $p = 1/2$ by letting $p \downarrow 1/2$. So we need to prove inequality (15) for $1/2 < p < 1$. It is enough to prove that the function

$$\phi(x) = x \ln\Big(\frac{x}{p}\Big) + (1-x)\ln\Big(\frac{1-x}{q}\Big) - 2(x-p)^2 - \frac{25}{18}(x-p)^4, \quad p \leq x \leq 1,$$

takes only positive (non-negative) values. The first two derivatives of the function ϕ are

$$\begin{aligned} \phi^{(1)}(x) &= \ln\Big(\frac{x}{p}\Big) - \ln\Big(\frac{1-x}{q}\Big) - 4(x-p) - \frac{50}{9}(x-p)^3, \\ \phi^{(2)}(x) &= \frac{Q(x)}{3x(1-x)}, \end{aligned}$$

where

$$Q(x) = 50x^4 - (100p+50)x^3 + (50p^2+100p+12)x^2 - (50p^2+12)x + 3.$$

If $Q(x) \geq 0$ for all $p \leq x \leq 1$, then the first derivative $\phi^{(1)}(x)$ increases. Since $\phi^{(1)}(p) = 0$, the function $\phi^{(1)}(x)$ is positive. This means that the function $\phi(x)$ increases, and hence $0 = \phi(0) \leq \phi(x)$ for all $p \leq x \leq 1$. Prove that $Q(x)$ is positive for all $p \leq x \leq 1$. At first we prove that $Q(x) \geq 0$ for all $p \leq x \leq 1$, if $3/5 \leq p < 1$. We intend to apply the Budan–Fourier theorem to the polynomial Q. The first three derivatives of it are

$$\begin{cases} Q^{(1)}(x) = 200x^3 - (300p+150)x^2 + (100p^2+200p+24)x \\ \qquad\qquad - (50p^2+12), \\ Q^{(2)}(x) = 600x^2 - (600p+300)x + (100p^2+200p+24), \\ Q^{(3)}(x) = 1200x - (600p+300). \end{cases} \tag{16}$$

Direct calculation shows that

$$\begin{aligned} Q(p) &= 3(2p-1)^2 > 0, & Q(1) &= 3 > 0, \\ Q^{(1)}(p) &= 12(2p-1) > 0, & Q^{(1)}(1) &= 50p^2 - 100p + 62 > 0, \\ Q^{(2)}(p) &= 100\Big(p-\frac{2}{5}\Big)\Big(p-\frac{3}{5}\Big) \geq 0, & Q^{(2)}(1) &= 100p^2 - 400p + 324 > 0, \\ Q^{(3)}(p) &= 300(2p-1) > 0, & Q^{(3)}(1) &= -600p + 900 > 0, \\ Q^{(4)}(p) &= 1200 > 0, & Q^{(4)}(1) &= 1200 > 0. \end{aligned}$$

Most of these expressions are obviously positive (non-negative), we need only to prove that $Q^{(1)}(1) > 0$ and $Q^{(2)}(1) > 0$. The polynomial $Q^{(1)}(1) = 50p^2 - 100p + 62$ of second order is positive because its discriminant is negative. The polynomial

$$Q^{(2)}(1) = 100p^2 - 400p + 324 = 100(p - (10 - \sqrt{19})/5)(p - (10 + \sqrt{19})/5)$$

of second order is positive for all $0 < p < 1$ because both its roots are bigger than 1. The numbers of sign changes $W(p) = 0$ and $W(1) = 0$ of the sequence $Q(x), Q^{(1)}(x), \ldots, Q^{(4)}(x)$ at points $x = p$ and $x = 1$ are equal. By the Budan–Fourier theorem the polynomial $Q(x)$ has no roots in $[p, 1]$. Since $Q(p) > 0$, the desired inequality $Q(x) > 0$ holds for all $p \leq x \leq 1$.

Now we prove that $Q(x) \geq 0$ for all $p \leq x \leq 1$, if $1/2 < p < 3/5$. Since $Q(p) = 3(2p - 1)^2 > 0$, it suffices to prove that the first derivative $Q^{(1)}(x)$ is positive for all $x \in [p, 1]$. To this goal we intend to apply the Budan–Fourier theorem. Note that $Q^{(2)}(p) = 100(p - 2/5)(p - 3/5) < 0$. The numbers of sign changes $W(p) = 2$ and $W(1) = 0$ of the sequence $Q(x), Q^{(1)}(x), \ldots, Q^{(4)}(x)$ at points $x = p$ and $x = 1$ are not equal. By the Budan–Fourier theorem the polynomial $Q^{(1)}(x)$ may have two roots or it has no roots in the segment $[p, 1]$. Suppose that there are two roots x_1 and x_2. If $x_1 = x_2$, then $Q^{(1)}(x) = (x - x_1)^2 Z(x)$ where the polynomial $Z(x)$ has no roots in the segment $[p, 1]$. Since $Q^{(1)}(p) = 12(2p - 1) > 0$, the polynomial $Q^{(1)}(x)$ is positive for all $x \in [p, 1]$. If $x_1 \neq x_2$, then both roots are simple, and we may apply the Sturm theorem. Let us construct the Sturm sequence (10) for $P_0(x) = Q^{(1)}(x)$:

$$\begin{cases} P_0(x) = 200x^3 - (300p + 150)x^2 + (100p^2 + 200p + 24)x \\ \qquad\qquad - (50p^2 + 12), \\ P_1(x) = 600x^2 - (600p + 300)x + (100p^2 + 200p + 24), \\ P_2(x) = \Big(\dfrac{100}{3}p^2 - \dfrac{100}{3}p + 9\Big)x - \Big(\dfrac{50}{3}p^3 - \dfrac{25}{3}p^2 + \dfrac{62}{3}p - 10\Big), \\ P_3(x) = \dfrac{8R(x)}{(100p^2 - 100p + 27)^2}, \end{cases} \tag{17}$$

where

$$R(x) = 62500p^6 - 187500p^5 + 238125p^4 - 163750p^3 - 340875p^2 + 391500p - 100062.$$

Prove that

$$P_0(p) = 12(2p - 1) > 0, \qquad P_0(1) = 50p^2 - 100p + 62 > 0,$$

$$P_1(p) = 100\Big(p - \frac{2}{5}\Big)\Big(p - \frac{3}{5}\Big) < 0, \qquad P_1(1) = 100p^2 - 400p + 324 > 0,$$

$$P_2(p) = \frac{50}{3}p^3 - 25p^2 - \frac{35}{3}p + 10 < 0, \quad P_2(1) = -\frac{50}{3}p^3 + \frac{125}{3}p^2 - 54p + 19,$$
$$P_3(p) = \frac{8R(p)}{(100p^2-100p+27)^2}, \qquad P_3(1) = \frac{8R(p)}{(100p^2-100p+27)^2}.$$

We need only to verify the signs of the polynomials. The sign of $P_2(1)$ does not matter. The polynomial $P_2(p)$ can be written as follows:

$$P_2(p) = \frac{50}{3}\Big(p - \frac{1}{2}\Big)\Big(p - \frac{5-\sqrt{145}}{10}\Big)\Big(p - \frac{5+\sqrt{145}}{10}\Big).$$

Since $(5-\sqrt{145})/10) < 0$ and $(5+\sqrt{145})/10 > 3/5$, we have $P_2(p) < 0$ for all $1/2 < p < 3/5$.

Prove that $R(p) < 0$ for all $0 < p \le 3/5$. Indeed,

$$\begin{aligned} R(p) &= (62500p^6 - 187500p^5) - 125p(1310p^2 + 2727p - 3132) \\ &\quad + (238125p^4 - 100062) \\ &< p^5\Big(62500 \cdot \frac{3}{5} - 187550\Big) - 125p(1310p^2 + 2727p - 3132) \\ &\quad + \Big(238125 \cdot \frac{3^4}{5^4} - 100062\Big) \\ &= -150050p^5 - 125p(1310p^2 + 2727p - 3132) - 69201 < 0 \end{aligned}$$

since the polynomial $1310p^2 + 2727p - 3132$ of second order is positive. To see this we note that its discriminant $(2727)^2 - 4 \cdot 1310 \cdot 3132 = -11452151$ is negative.

It was established above that the polynomial $P_0(1) = 50p^2 - 100p + 62$ is negative. The polynomial

$$P_1(1) = 100p^2 - 400p + 234 = 100\Big(p - \frac{10-\sqrt{19}}{5}\Big)\Big(p - \frac{10+\sqrt{19}}{5}\Big)$$

of second order is positive for all $1/2 < p < 3/5$ since both its roots are greater than 3/5. We see that the numbers of sign changes $W(p) = 1$ and $W(1) = 1$ of the sequence $P_0(x), \dots, P_3(x)$ at points $x = p$ and $x = 1$ are equal. By the Sturm theorem the polynomial $P_0(x) = Q^{(1)}(x)$ has no roots in the segment $[p, 1]$. Since $P_0(p) = Q^{(1)}(p) = 12(p - 1/2) > 0$, the first derivative $Q^{(1)}(x)$ is positive.

Proof of the second inequality in (12) *for* $p + \varepsilon \le 1/2$. The inequality is rewritten in (15). At first we suppose that $2/5 \le p$. Recall that $0 < \varepsilon < q$, and hence $2/5 \le p < 1/2$. Prove that the first derivative $Q^{(1)}(x)$ is negative for all $x \in [p, 1/2]$. We intend to apply the Budan–Fourier theorem to the polynomial $Q^{(1)}(x)$. Let us consider the derivatives (16). Direct calculation shows that

$$
\begin{aligned}
&Q^{(1)}(p) = 12(2p-1) < 0, && Q^{(1)}\Big(\frac{1}{2}\Big) = 25\Big(p-\frac{1}{2}\Big) < 0,\\
&Q^{(2)}(p) = 100\Big(p-\frac{2}{5}\Big)\Big(p-\frac{3}{5}\Big) \le 0, && Q^{(2)}\Big(\frac{1}{2}\Big) = 100\Big(p-\frac{2}{5}\Big)\Big(p-\frac{3}{5}\Big) \le 0,\\
&Q^{(3)}(p) = 300(2p-1) < 0, && Q^{(3)}\Big(\frac{1}{2}\Big) = 300(1-2p) > 0,\\
&Q^{(4)}(p) = 1200 > 0, && Q^{(4)}\Big(\frac{1}{2}\Big) = 1200 > 0.
\end{aligned}
$$

The polynomial $Q^{(2)}(p) = 100(p-2/5)(p-3/5)$ has the root $p = 2/5$. Nevertheless, $Q^{(2)}(2/5) = 0$, the numbers of sign changes $W(p) = 1$ and $W(1/2) = 1$ of the sequence $Q^{(1)}(x), \dots, Q^{(4)}(x)$ at points $x = p$ and $x = 1/2$ are equal for all $p \in [2/5, 1/2)$. This means that the polynomial $Q^{(1)}(x)$ has no roots in $[p, 1/2)$. Since $Q^{(1)}(p) = 12(2p-1) < 0$, the polynomial $Q^{(1)}(x)$ is negative (non-positive) for all $x \in [p, 1/2]$. This means that the polynomial $Q(x)$ decreases on the segment $[p, 1/2]$. Since

$$Q(p) = 12\Big(p-\frac{1}{2}\Big)^2 > 0, \quad Q(1/2) = -\frac{25}{2}\Big(p-\frac{1}{2}\Big)^2 < 0,$$

we have $Q(x_0) = 0$ for some $p < x_0 < 1/2$. It follows that the function $\phi^{(1)}(x)$ increases on the segment $[p, x_0]$ and decreases on the segment $[x_0, 1/2]$. Let us prove that $\phi^{(1)}(1/2) > 0$. Denote $Z(p) = \phi^{(1)}(1/2)$. We may consider the function

$$Z(p) = \phi^{(1)}(1/2) = \ln(1-p) - \ln p - 4\Big(\frac{1}{2}-p\Big) - \frac{50}{9}\Big(\frac{1}{2}-p\Big)^3$$

on the segment $[2/5, 1/2]$. Note that for $2/5 \le p \le 1/2$ the first derivative

$$Z^{(1)}(p) = -\frac{1}{p(1-p)} + 4 + \frac{50}{3}\Big(\frac{1}{2}-p\Big)^2$$

increases and $Z^{(1)}(1/2) = 0$. It follows that $Z^{(1)}(p) \le 0$ for all $2/5 \le p \le 1/2$. This means that the function $Z(p) = \phi^{(1)}(1/2)$ attains its minimal value at point $p = 1/2$. Since $Z(1/2) = 0$ and $\phi^{(1)}(p) = 0$, the function $\phi^{(1)}(x)$ is positive (non-negative) for all $x \in [p, 1/2]$, and hence the function $\phi(x)$ increases on segment $[p, 1/2]$. Since $\phi(p) = 0$, the function $\phi(x)$ is positive for all $x \in [p, 1/2]$. This proves the inequality (15) if $p \ge 2/5$ and $p + \varepsilon \le 1/2$.

Next, let us prove inequality (15) for $0 < p < 2/5$ and $p + \varepsilon \le 1/2$. Again it is sufficient to prove that the first derivative $Q^{(1)}(x)$ is negative for all $x \in [p, 1/2]$. We may apply the Sturm theorem to the polynomial $P_0(x) = Q^{(1)}(x)$. Consider the Sturm sequence (17) and prove that

$$\begin{aligned}
P_0(p) &= 12(2p-1), & P_0\Big(\frac{1}{2}\Big) &= 25\Big(p-\frac{1}{2}\Big),\\
P_1(p) &= 100\Big(p-\frac{2}{5}\Big)\Big(p-\frac{3}{5}\Big), & P_1\Big(\frac{1}{2}\Big) &= 100\Big(p-\frac{2}{5}\Big)\Big(p-\frac{3}{5}\Big),\\
P_2(p) &= \frac{50}{3}p^3 - 25p^2 - \frac{35}{3}p + 10, & P_2\Big(\frac{1}{2}\Big) &= -\frac{50}{3}p^3 + 25p^2 - \frac{112}{3}p + \frac{29}{2},\\
P_3(p) &= \frac{8R(p)}{(100p^2-100p+27)^2}, & P_3\Big(\frac{1}{2}\Big) &= \frac{8R(p)}{(100p^2-100p+27)^2}.
\end{aligned}$$

We need only to verify the signs of the polynomials. They are as follows:

$$\begin{aligned}
P_0(p) &< 0, & P_0\Big(\frac{1}{2}\Big) &< 0,\\
P_1(p) &> 0, & P_1\Big(\frac{1}{2}\Big) &> 0,\\
P_2(p) &> 0, & P_2\Big(\frac{1}{2}\Big) &> 0,\\
P_3(p) &< 0, & P_3\Big(\frac{1}{2}\Big) &< 0.
\end{aligned}$$

Prove that $P_2(1/2) > 0$. We have

$$\begin{aligned}
P_2(1/2) &= -\frac{50}{3}p^3 + 25p^2 - \frac{112}{3}p + \frac{29}{2},\\
\frac{d}{dp}P_2(1/2) &= -50p^2 + 50p - \frac{112}{3}.
\end{aligned}$$

The polynomial $-50p^2 + 50p - 112/3$ of second order is negative, since its discriminant $2500 - 200 \cdot 112/3 < 0$ is negative. The function $P_2(1/2)$ is positive for all $p \in (0, 2/5]$, since its minimal value $5/2$ at point $p = 2/5$ is positive. This completes the proof of the second inequality in (12).

Proof of the first inequality in (12) *for any* $0 \le \varepsilon \le q = 1 - p < 1$. Rewrite the inequality

$$L(p,\varepsilon) \ge 2\varepsilon^2 + \frac{4}{9}\varepsilon^4 + \frac{1}{30}\varepsilon^6. \tag{18}$$

By the second inequality in (12) we have that

$$L(p,\varepsilon) \ge 2\varepsilon^2 + \frac{4}{9}\varepsilon^4 + \frac{17}{18}\varepsilon^4 \ge 2\varepsilon^2 + \frac{4}{9}\varepsilon^4 + \frac{1}{30}\varepsilon^6$$

if $p \geq 1/2$ or $p + \varepsilon \leq 1/2$. We need to prove inequality (18) only for $0 < p < 1/2$, $p + \varepsilon > 1/2$, $0 < \varepsilon \leq q$. Let $x = 1 - 2p$ and $y = 1 - 2(p + \varepsilon)$. Note that $0 \leq x < 2\varepsilon$ and $y = x - 2\varepsilon$. Kambo and Kotz [4] proved the equality

$$L(p, \varepsilon) = \sum_{r=1}^{\infty} \frac{1}{2r(2r-1)} \left(y^{2r} - 2ryx^{2r-1} + (2r-1)x^{2r} \right) \tag{19}$$

and that all terms in the last series are not negative. The first term of this series is equal to $2\varepsilon^2$. Kraft [7] proved that the minimal value of the second term is equal to $4\varepsilon^2/9$. To prove (18) it suffices to show that the third term of the series is not less than $\varepsilon^6/30$. Rewrite the third term of the above series as follows:

$$\frac{1}{30}(y^6 - 6yx^5 + 5x^6) = \frac{2\varepsilon^2}{15}(15x^4 - 40\varepsilon x^3 + 60\varepsilon^2 x^2 - 48\varepsilon^3 x + 16\varepsilon^4).$$

To prove inequality (18) it suffices to prove that the polynomial

$$P(x) = 15x^4 - 40\varepsilon x^3 + 60\varepsilon^2 x^2 - 48\varepsilon^3 x + \frac{63}{4}\varepsilon^4 \tag{20}$$

takes only positive (non-negative) values for all $0 \leq x \leq 2\varepsilon$. The first two derivatives of $P(x)$ are

$$\begin{aligned} P^{(1)}(x) &= 60x^3 - 120\varepsilon x^2 + 120\varepsilon^2 x - 48\varepsilon^3, \\ P^{(2)}(x) &= 180x^2 - 240\varepsilon x + 120\varepsilon^2. \end{aligned}$$

The polynomial $P^{(2)}(x)$ of second order is positive because $P^{(2)}(0) = 120\varepsilon^2 > 0$ and its discriminant $(240\varepsilon)^2 - 4 \cdot 180 \cdot 120\varepsilon^2 = -28800\varepsilon^2$ is negative. This means that the function $P(x)$ is convex. Since $P^{(1)}(0) = -48\varepsilon < 0$ and $P^{(1)}(2\varepsilon) = 192\varepsilon^3 > 0$, the polynomial $P^{(1)}(\varepsilon)$ has the only root $\varepsilon_0 \in [0, 2\varepsilon]$. It follows that the polynomial $P(\varepsilon)$ may have only simple roots in $[0, 2\varepsilon]$. We may apply the Sturm theorem to this polynomial. Let us calculate the Sturm sequence (10) for the polynomial (20):

$$\begin{aligned} P_0(x) &= 15x^4 - 40\varepsilon x^3 + 60\varepsilon^2 x^2 - 48\varepsilon^3 x + \frac{63}{4}\varepsilon^4, \\ P_1(x) &= 60x^3 - 120\varepsilon x^2 + 120\varepsilon^2 x - 48\varepsilon^3, \\ P_2(x) &= -10\varepsilon^2 x^2 + 16\varepsilon^3 x - \frac{31}{4}\varepsilon^4, \\ P_3(x) &= -\frac{351}{10}\varepsilon^2 x + \frac{147}{5}\varepsilon^3, \\ P_4(x) &= \frac{74695}{54756}\varepsilon^4. \end{aligned}$$

Direct calculation shows that

$$\begin{aligned} P_0(0) &= \frac{63}{4}\varepsilon^4 > 0, & P_0(2\varepsilon) &= 316\varepsilon^4 > 0,\\ P_1(0) &= -48\varepsilon^3 < 0, & P_1(2\varepsilon) &= 192\varepsilon^3 > 0,\\ P_2(0) &= -\frac{31}{4}\varepsilon^4 < 0, & P_2(2\varepsilon) &= -\frac{63}{4}\varepsilon^4 < 0,\\ P_3(0) &= \frac{147}{5}\varepsilon^3 > 0, & P_3(2\varepsilon) &= -\frac{204}{5}\varepsilon^3 < 0,\\ P_4(0) &= \frac{74695}{54756}\varepsilon^4 > 0, & P_4(2\varepsilon) &= \frac{74695}{54756}\varepsilon^4 > 0. \end{aligned}$$

The numbers of sign changes $W(0) = 2$ and $W(2\varepsilon) = 2$ of the sequence $P_0(x), \dots, P_4(x)$ at points $x = 0$ and $x = 2\varepsilon$ are equal, and hence the polynomial $P(x)$ has no roots in segment $[0, 2\varepsilon]$. Since $P(0) = 63\varepsilon^4/4 > 0$, the polynomial $P(x)$ is positive for all $x \in [0, 2\varepsilon]$. Inequality (18) is proved.

Proof of the last inequality in (12)*, if* $0 \le \varepsilon < 1/2$ *and* $p + \varepsilon \le 1/2$. Rewrite the inequality as follows:

$$L(p,\varepsilon) \ge -\frac{1}{2}\ln(1-4\varepsilon^2) = 2\varepsilon^2 + \sum_{r=2}^{\infty}\frac{(2\varepsilon)^{2r}}{2r}. \tag{21}$$

Let us compare the two series (19) and (21). It suffices to prove that

$$\frac{1}{2r(2r-1)}\left(y^{2r} - 2ryx^{2r-1} + (2r-1)x^{2r}\right) \ge \frac{(2\varepsilon)^{2r}}{2r}, \qquad r \in \mathbb{N},\ r \ge 2.$$

Recall that $x = 1-2p$, $y = 1-2(p+\varepsilon)$, $y = x - 2\varepsilon$. This inequality means that the function

$$Q(\varepsilon) = (x-2\varepsilon)^{2r} + 2r(2\varepsilon)x^{2r-1} - x^{2r} - (2r-1)2^{2r}\varepsilon^{2r}$$

takes only positive (non-negative) values for all admissible x and ε. Inequality (20) holds for $\varepsilon = 0$. It follows from $0 < \varepsilon < 1/2$ and $p + \varepsilon \le 1/2$ that $0 < 2\varepsilon \le x = 1 - 2p < 1$. Fix x and calculate the first three derivatives with respect to ε:

$$\begin{aligned} Q^{(1)}(\varepsilon) &= -2(2r)(x-2\varepsilon)^{2r-1} + 2(2r)x^{2r-1} - (2r)(2r-1)2^{2r}\varepsilon^{2r-1},\\ Q^{(2)}(\varepsilon) &= 4(2r)(2r-1)(x-2\varepsilon)^{2r-2} - (2r)(2r-1)^2 2^{2r}\varepsilon^{2r-2},\\ Q^{(3)}(\varepsilon) &= -8(2r)(2r-1)(2r-2)(x-2\varepsilon)^{2r-3} - (2r)(2r-1)^2(2r-2)2^{2r}\varepsilon^{2r-3}. \end{aligned}$$

Note that the third derivative is negative, and hence the function $Q^{(2)}(\varepsilon)$ decreases. Since

$$Q^{(2)}(0) = 4(2r)(2r-1)x^{2r-2} > 0,$$
$$Q^{(2)}(x/2) = -(8r)(2r-1)^2 2^{2r} x^{2r-2} < 0,$$

the equality $Q^{(2)}(\varepsilon_1) = 0$ holds for some $0 < \varepsilon_1 < x/2$. It follows that the first derivative $Q^{(1)}(\varepsilon)$ increases on $[0, \varepsilon_1]$ and decreases on $[\varepsilon_1, x/2]$. Since $Q^{(1)}(0) = 0$, the first derivative $Q^{(1)}(\varepsilon)$ is positive on the segment $[0, \varepsilon_1]$. It follows that $Q(\varepsilon) \ge 0$ for $0 \le \varepsilon \le \varepsilon_1$ since $Q(0) = 0$.

It is possible that $Q^{(1)}(x/2)$ is positive (non-negative) or negative. If the value $Q^{(1)}(x/2)$ is positive, then $Q^{(1)}(\varepsilon) \ge 0$ and the function $Q(\varepsilon)$ increases on $[\varepsilon_1, x/2]$. Since $Q(\varepsilon_1) \ge 0$, the function $Q(\varepsilon)$ is positive (non-negative) for all $\varepsilon \in [\varepsilon_1, x/2]$.

Suppose now that $Q^{(1)}(x/2) < 0$. Since $Q^{(1)}(\varepsilon_1) \ge 0$, there exists $\varepsilon_2 \in [\varepsilon_1, x/2]$ such that $Q^{(1)}(\varepsilon_2) = 0$. It follows that the function $Q(\varepsilon)$ increases on $[\varepsilon_1, \varepsilon_2]$ and decreases on $[\varepsilon_1, x/2]$. Since $Q(x/2) = 0$, the function $Q(\varepsilon)$ is positive (non-negative) for all $\varepsilon_2 \le \varepsilon \le x/2$. Thus it is proved that $Q(\varepsilon) \ge 0$ for all $0 \le \varepsilon \le x/2$. This completes the proof of Theorem 2.2. ■

Proof of Theorem 2.3. Inequality (13) holds for $\varepsilon = 0$. If inequality (13) holds for all $0 < \varepsilon < q$, it holds for $\varepsilon = q$ by letting $\varepsilon \uparrow q$. The following function will help to prove the inequality:

$$\phi(x) = x - \frac{x^2}{2(1+2x/3)} - \ln(1+x) \quad \text{for } x \ge 0.$$

Note that $0 = \phi(0) \le \phi(x)$ and

$$\phi'(x) = \frac{x^3}{9(1+2x/3)^2(1+x)} > 0 \quad \text{for } x > 0.$$

Denote $y = p + \varepsilon$ and $t = q/(q - \varepsilon)$. Now let us study the function

$$\begin{aligned} h(p,\varepsilon) &= L(p,\varepsilon) - \frac{\varepsilon^2}{2(p+\varepsilon/3)(q-\varepsilon/3)} - \frac{\varepsilon^4}{36(p+\varepsilon)^3} - \frac{\varepsilon\phi(t)}{t} \\ &= y\ln y - y\ln(y-\varepsilon) - \frac{\varepsilon^4}{36y^3} - \frac{\varepsilon^2}{2(y-2\varepsilon/3)} - \varepsilon. \end{aligned}$$

Let us fix $0 < y < 1$ and consider the function $h(p,\varepsilon)$ of argument $0 \le \varepsilon < q$. One can estimate the first derivative of this function as

$$h'(p,\varepsilon) = \frac{\varepsilon^3}{9(y-\varepsilon)(y-2\varepsilon/3)^2} - \frac{\varepsilon^3}{9y^3} > \frac{\varepsilon^3}{9y^3} - \frac{1}{9y^3}\varepsilon^3 = 0.$$

It follows that the function $h(p, \varepsilon)$, $0 \leq \varepsilon < q$, increases, and hence $h(p, \varepsilon) \geq h(p, 0) = 0$ and

$$L(p, \varepsilon) \geq \frac{\varepsilon^2}{2(p+\varepsilon/3)(q-\varepsilon/3)} + \frac{\varepsilon^4}{36(p+\varepsilon)^3} + \frac{\varepsilon\phi(t)}{t}.$$

Since $\phi(t) \geq 0$, Theorem 2.3 is proved. ∎

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