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Height pairings of 1-motives

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Abstract – The purpose of this work is to generalize, in the context of 1-motives, the p-adic height pairings constructed by B. Mazur and J. Tate on abelian varieties. Following their approach, we define a global pairing between the rational points of a 1-motive and its dual. We also provide a local pairing between disjoint zero-cycles of degree zero on a curve, which is done by considering the Picard and Albanese 1-motives associated to the curve.

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1. Introduction

In [12], Mazur and Tate gave a construction of a global pairing on the rational points of paired abelian varieties over a global field, as well as Néron-type local pairings between disjoint zero-cycles and divisors on an abelian variety over a local field. Their approach involved the concept of ρ-splittings of biextensions of abelian groups, which they mainly studied in the case of K-rational sections of a $\mathbb{G}_m$-biextension of abelian varieties over a local field. They proved that, when certain conditions on the base field, the morphism ρ, and the abelian varieties are met, there exist canonical ρ-splittings for this type of biextensions. They went on to construct canonical local pairings between disjoint zero-cycles and divisors on an abelian variety using said ρ-splittings. By considering a global field endowed with a set of places and its respective completions, they were also able to construct a global pairing on the rational points of paired abelian varieties.

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The Poincaré biextension of an abelian variety and its dual defined over a non-archimedean local field of characteristic 0 will be of particular interest to us. When considering this biextension, there is an alternate method of obtaining ρ-splittings, due to Zarhin [16], starting from splittings of the Hodge filtration of the first de Rham cohomology group of the abelian variety. His construction coincides with Mazur and Tate's in the case that ρ is unramified, or when ρ is ramified and the splitting of the Hodge filtration is the one induced by the unit root subspace. In the latter case, the equality of both constructions is a result of Coleman [6] in the case of ordinary reduction, and of Iovita and Werner [10] in the case of semistable ordinary reduction.

For our generalization to 1-motives, we will focus on the ramified case. Following Zarhin's approach, we will construct ρ-splittings of the Poincaré biextension of a 1-motive and its dual starting from a pair of splittings of the Hodge filtrations of their de Rham realizations; this is the content of Section 4. In order to construct pairings from these ρ-splittings, we will require them to be compatible with the canonical linearization associated to the biextension; the conditions under which this happens are studied in Section 3.

In Section 5 we consider a semi-normal irreducible curve C over a finite extension of $\mathbb{Q}_p$ and construct a local pairing between disjoint zero-cycles of degree zero on C and on its regular locus C_{reg}. We do this by considering the Poincaré biextension of the Picard and Albanese 1-motives of C. This construction generalizes the local pairing of Mazur and Tate [12, p. 212] in the case of elliptic curves.

Finally, in Section 6 we consider a 1-motive M over a number field F and a set of places $\mathcal{V}$ of F. For each $v \in \mathcal{V}$ we consider a homomorphism $\rho_v : F_v^* \to \mathbb{Q}_p$, as well as a ρ_v-splitting $\psi_v : P(F_v) \to \mathbb{Q}_p$ on the F_v-rational sections of the Poincaré biextension P of M and its dual $M^\vee$, satisfying certain properties. With this data we construct a global pairing between the F-rational points of M and $M^\vee$ under the following condition on the family $\{\psi_v\}_v$: either ψ_v is compatible with the canonical $L_{F_v} \times_{F_v} L^\vee_{F_v}$-linearization of P_{F_v}, or M_{F_v} has good reduction and ψ_v is zero on the set $P(\mathcal{O}_{F_v})$ of sections of P over the ring of F_v-integers. The pairing is defined similarly to the case of abelian varieties, hence generalizing the global pairing of Mazur and Tate [12, Lem. 3.1] in the case of an abelian variety and its dual.

2. Preliminaries on abelian varieties and 1-motives

2.1 – *ρ-splittings on abelian varieties*

For the definition of biextension of abelian groups and group schemes we refer to [13].

Definition 2.1 ([12, p. 199]). Let A, B, H, Y be abelian groups and P a biextension of (A, B) by H. Let $\rho : H \to Y$ be a homomorphism. A *ρ-splitting* of P is a map $\psi : P \to Y$ such that

(i) $\psi(h + x) = \rho(h) + \psi(x)$, for all $h \in H$ and $x \in P$, and

(ii) for each $a \in A$ (resp. $b \in B$) the restriction of ψ to $P_{a,B}$ (resp. $P_{A,b}$) is a group homomorphism,

where $P_{a,B}$ (resp. $P_{A,b}$) denotes the preimage of P over $\{a\} \times B$ (resp. $A \times \{b\}$).

Thus, a ρ-splitting can be seen as a bi-homomorphic map which is compatible with the natural action of H on P. Moreover, ψ induces a trivialization (as biextension) of the pushout of P along ρ, hence its name.

The context in which these maps were classically studied is the following. Consider a field K which is complete with respect to a place v, either archimedean or discrete, A and B abelian varieties over K, P a biextension of (A, B) by $\mathbb{G}_m$, and $\rho : K^* \to Y$ a homomorphism from the group of units of K to an abelian group Y. A key result by Mazur and Tate [12, p. 199] states the existence of canonical ρ-splittings of the set $P(K)$ of K-rational points of P in the following cases:

(i) v is archimedean and $\rho(c) = 0$ for all c such that $|c|_v = 1$,

(ii) v is discrete, ρ is unramified (i.e. $\rho(R^*) = 0$, where R is the valuation ring of K) and Y is uniquely divisible by N, and

(iii) v is discrete, the residue field of K is finite, A has semistable ordinary reduction and Y is uniquely divisible by M,

where N is an integer depending on A and M is an integer depending on A and B. We will mainly focus on case (iii). In this case, the ρ-splitting of $P(K)$ is obtained by extending a local formal splitting of P, which exists and is unique because of the semistable ordinary reduction of A.

In the case of a p-adic base field, when considering $B = A^\vee$ the dual abelian variety of A and $P = P_A$ the Poincaré biextension, there is an alternate method of obtaining ρ-splittings of $P(K)$ starting with a splitting of the Hodge filtration of the first de Rham cohomology of A. This construction is due to Zarhin [16] and is done as follows. Let K be a field which is the completion of a number field with respect to a discrete place v over a prime p and consider a continuous homomorphism $\rho : K^* \to \mathbb{Q}_p$. Recall that, associated to the first de Rham cohomology K-vector space of A, there is a canonical extension

$$0 \to \mathrm{H}^0(A, \Omega^1_{A/K}) \to \mathrm{H}^1_{\mathrm{dR}}(A) \to \mathrm{H}^1(A, \mathcal{O}_A) \to 0 \tag{2.1}$$

coming from the Hodge filtration of $\mathrm{H}^1_{\mathrm{dR}}(A)$. It is known that (2.1) can be naturally identified with the exact sequence of Lie algebras induced by the universal vectorial extension $A^{\vee\#}$ of $A^\vee$:

$$0 \to \omega_A \to A^{\vee\#} \to A^\vee \to 0,$$

where ω_A is the K-vector group scheme representing the sheaf of invariant differentials on A (see [11, Prop. 4.1.7]). Therefore, it is possible to obtain a (uniquely determined) splitting $\eta : A^\vee(K) \to A^{\vee\#}(K)$ at the level of groups from any splitting $r : \mathrm{H}^1(A, \mathcal{O}_A) \to \mathrm{H}^1_{\mathrm{dR}}(A)$ of (2.1) (see [16, Ex. 3.1.5] or [6, Lem. 3.1.1]). Since $A^\vee$ represents the functor $\underline{\mathrm{Ext}}_K(A, \mathbb{G}_m)$, while $A^{\vee\#}$ represents the functor $\underline{\mathrm{Extrig}}_K(A, \mathbb{G}_m)$ of rigidified extensions of A by $\mathbb{G}_m$, the morphism η gives a multiplicative way of associating a rigidification to every extension of A by $\mathbb{G}_m$. Indeed, take a point $a^\vee \in A^\vee(K)$ and let $P_{A,a^\vee}$ be the fiber of the Poincaré bundle P_A over $A \times_K \{a^\vee\}$. Then $\eta(a^\vee)$ corresponds to the extension $P_{A,a^\vee}$ of A by $\mathbb{G}_m$ endowed with a rigidification or, equivalently, a splitting

$$s_{a^\vee} : \mathrm{Lie}\, P_{A,a^\vee}(K) \to \mathrm{Lie}\, \mathbb{G}_m(K)$$

of the exact sequence of Lie algebras induced by the extension $P_{A,a^\vee}$. The composition $\mathrm{Lie}\,\rho \circ t_{a^\vee}$ can then be extended to a group homomorphism $P_{A,a^\vee}(K) \to \mathbb{Q}_p$ (see [16, Thm. 3.1.7]), for every $a^\vee \in A^\vee$, hence obtaining a ρ-splitting

$$\psi_\rho : P_A(K) \to \mathbb{Q}_p.$$

When ρ is unramified, ψ_ρ does not depend on the choice of splitting of (2.1), recovering Mazur and Tate's result for case (ii) (see [16, Thm. 4.1]). On the other hand, when ρ is ramified, ψ_ρ does depend on the chosen splitting of (2.1) (see [16, Thm. 4.3]). Coleman [6] demonstrated that, when A has good ordinary reduction, the canonical ρ-splitting of $P_A(K)$ constructed by Mazur and Tate comes from the splitting of (2.1) induced by the unit root subspace, which is the subspace of $\mathrm{H}^1_{\mathrm{dR}}(A)$ on which the Frobenius map acts with slope 0. Later, Iovita and Werner [10] were able to generalize this result to abelian varieties with semistable ordinary reduction by considering their Raynaud extension, which can be seen as a 1-motive whose abelian part has good ordinary reduction (see also [15]).

2.2 – 1-*motives*

According to Deligne [8, p. 59], a 1-*motive* M over a field K consists of

(i) a *lattice* L over K, i.e. a group scheme which, locally for the étale topology on K, is isomorphic to a finitely generated free abelian constant group;

(ii) a *semi-abelian variety* G over K, i.e. an extension of an abelian variety A by a torus T; and

(iii) a morphism of K-group schemes $u : L \to G$.

A 1-motive can be considered as a complex of K-group schemes

$$M = [L \xrightarrow{u} G]$$

with the lattice in degree -1 and the semi-abelian variety in degree 0. A *morphism of* 1*-motives* can then be defined as a morphism of the corresponding complexes.

2.2.1. Cartier duality. Associated to a 1-motive M, there is a *Cartier dual* 1*-motive*

$$M^\vee = [L^\vee \xrightarrow{u^\vee} G^\vee]$$

defined as follows (see [8, p. 67]). The lattice $L^\vee := \underline{\mathrm{Hom}}_K(T, \mathbb{G}_m)$ is the Cartier dual of T, the torus $T^\vee := \underline{\mathrm{Hom}}_K(L, \mathbb{G}_m)$ is the Cartier dual of L, the abelian variety $A^\vee$ is the dual abelian variety of A, and the semi-abelian variety $G^\vee$ is the image of the composition $v : L \xrightarrow{u} G \to A$ under the natural isomorphism

$$\mathrm{Hom}_K(L, A) \xrightarrow{\cong} \mathrm{Ext}^1_K(A^\vee, T^\vee).$$

There is a canonical biextension P of $(M, M^\vee)$ by $\mathbb{G}_m$, called the *Poincaré biextension*, expressing the duality between M and $M^\vee$. It is defined as the pullback to $G \times_K G^\vee$ of the Poincaré biextension P_A of $(A, A^\vee)$. The biextension P is naturally endowed with trivializations

$$\tau : L \times_K G^\vee \to P, \quad \tau^\vee : G \times_K L^\vee \to P$$

that coincide over $L \times_K L^\vee$, which complete its structure of biextension of $(M, M^\vee)$ by $\mathbb{G}_m$ (see [8, p. 60]). Using the fact that the group scheme $G^\vee$ represents the fppf-sheaf $\underline{\mathrm{Ext}}_K([L \xrightarrow{v} A], \mathbb{G}_m)$, it is possible to define the map $u^\vee : L^\vee \to G^\vee$ as

$$\begin{aligned} u^\vee : \underline{\mathrm{Hom}}_K(T, \mathbb{G}_m) &\to \underline{\mathrm{Ext}}_K([L \xrightarrow{v} A], \mathbb{G}_m) \\ \chi &\mapsto [L \xrightarrow{\xi} P_{A, v^\vee(x^\vee)}], \end{aligned}$$

where $x^\vee \in L^\vee$ is the element corresponding to $\chi \in \underline{\mathrm{Hom}}_K(T, \mathbb{G}_m)$ and ξ is obtained from the trivialization of P over $L \times_K L^\vee$.

2.2.2. De Rham realization. A 1-motive is endowed with a de Rham realization defined via its universal vectorial extension (see [8, p. 58]). The *universal vectorial extension* of a 1-motive $M = [L \xrightarrow{u} G]$ over K is a two-term complex of K-group schemes

$$M^\natural = [L \xrightarrow{u^\natural} G^\natural]$$

which is an extension of M by the K-vector group $\omega_{G^\vee}$ of invariant differentials on $G^\vee$

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & L \arrow[r, equal] \arrow[d, "u^\natural"] & L \arrow[r] \arrow[d, "u"] & 0 \\
0 \arrow[r] & \omega_{G^\vee} \arrow[r] & G^\natural \arrow[r] & G \arrow[r] & 0
\end{tikzcd}$$

and satisfies the following universal property: for all K-vector groups V, the map

$$\mathrm{Hom}_{\mathcal{O}_K}(\omega_{G^\vee}, V) \to \mathrm{Ext}^1_K(M, V),$$

which sends a morphism $\omega_{G^\vee} \to V$ of vector groups to the extension of M by V induced by pushout, is an isomorphism. It is well known that the universal vectorial extension of a 1-motive always exists. The *de Rham realization* of M is then defined as

$$\mathrm{T}_{\mathrm{dR}}(M) := \mathrm{Lie}\, G^\natural.$$

This is endowed with a *Hodge filtration*, defined as follows:

$$F^i\,\mathrm{T}_{\mathrm{dR}}(M) = \begin{cases} \mathrm{T}_{\mathrm{dR}}(M) & \text{if } i \leq -1, \\ \omega_{G^\vee} & \text{if } i = 0, \\ 0 & \text{if } i \geq 1. \end{cases}$$

We mention some properties concerning schemes involved in universal vectorial extensions.

LEMMA 2.2. (i) *The group scheme $G^\natural$ represents the fppf-sheaf*

$$S \mapsto \big\{(g, \nabla) \mid g \in G(S) \text{ and } \nabla \text{ is a } \natural\text{-structure on the extension } [L_S^\vee \to P_{g,G^\vee}] \text{ of } M_S^\vee \text{ by } \mathbb{G}_{m,S} \text{ associated to } g\big\}.$$

(ii) *If we regard the semi-abelian variety G as the* 1*-motive* $G[0] = [0 \to G]$*, then its universal vectorial extension is a group scheme $G^\#$ which is an extension of G by the vector group $\omega_{A^\vee}$. Moreover, $G^\#$ represents the fppf-sheaf*

$$S \mapsto \big\{(g, \nabla) \mid g \in G(S) \text{ and } \nabla \text{ is a } \natural\text{-structure on the extension of } [L_S^\vee \xrightarrow{v^\vee} A_S^\vee] \text{ by } \mathbb{G}_{m,S} \text{ associated to } g\big\}.$$

(iii) *If we regard the abelian variety A as the* 1*-motive* $A[0] = [0 \to A]$*, then its universal vectorial extension is a group scheme $A^\#$ which is an extension of A by the vector group $\omega_{A^\vee}$. Moreover, $A^\#$ represents the fppf-sheaf*

$$S \mapsto \big\{(a, \nabla) \mid a \in A(S) \text{ and } \nabla \text{ is a } \natural\text{-structure on the extension } P_{a,A^\vee} \text{ of } A_S^\vee \text{ by } \mathbb{G}_{m,S}\big\}.$$

(iv) *If we regard the lattice L as the 1-motive $L[1] = [L \to 0]$, then its universal vectorial extension is the complex $[L \to \omega_{T^\vee}]$. Via the identifications $L = \underline{\mathrm{Hom}}_K(T^\vee, \mathbb{G}_m)$ and $\omega_{T^\vee} = \underline{\mathrm{Hom}}_{\mathcal{O}_K}(\mathrm{Lie}\, T^\vee, \mathcal{O}_K)$, this map is described as*

$$\begin{aligned}\underline{\mathrm{Hom}}_K(T^\vee, \mathbb{G}_m) &\to \underline{\mathrm{Hom}}_{\mathcal{O}_K}(\mathrm{Lie}\, T^\vee, \mathcal{O}_K)\\ \chi &\mapsto \mathrm{Lie}\, \chi.\end{aligned}$$

Proof. Parts (i) and (ii) follow from [4, Prop. 3.8] and [4, Lem. 5.2], respectively. Part (iii) follows from [11, Props. 2.6.7 and 3.2.3 (a)] (see also [6, Thm. 0.3.1]). And, finally, (iv) follows from [1, Lem. 2.2.2], once we notice that there is a natural isomorphism $L \otimes_{\mathbb{Z}} \mathbb{G}_a \cong \omega_{T^\vee}$ mapping $x \otimes 1 \mapsto \mathrm{Lie}\, \chi$. ∎

Let $P^\natural$ be the biextension of $(M^\natural, M^{\vee\natural})$ by $\mathbb{G}_m$ obtained from P by pullback. There is a canonical connection ∇ on $P^\natural$ which endows it with a $\natural$-structure (see [8, Prop. 10.2.7.4]). Its curvature is an invariant 2-form on $G^\natural \times_K G^{\vee\natural}$ and therefore it determines an alternating pairing R on $\mathrm{Lie}\, G^\natural \times_K \mathrm{Lie}\, G^{\vee\natural}$ with values in $\mathrm{Lie}\, \mathbb{G}_m$. Since the restriction of R to $\mathrm{Lie}\, G^\natural$ and $\mathrm{Lie}\, G^{\vee\natural}$ is zero, this map induces a pairing

$$\Phi : \mathrm{Lie}\, G^\natural \times_K \mathrm{Lie}\, G^{\vee\natural} \to \mathrm{Lie}\, \mathbb{G}_m.$$

Deligne's pairing is then defined as

$$(\cdot, \cdot)^{\mathrm{Del}}_M := -\Phi : \mathrm{T}_{\mathrm{dR}}(M) \times_K \mathrm{T}_{\mathrm{dR}}(M^\vee) \to \mathrm{Lie}\, \mathbb{G}_m.$$

2.2.3. Albanese and Picard 1-motives. Let C_0 be a curve over a field K of characteristic 0, i.e. a purely 1-dimensional variety. Note that originally Deligne considered only algebraically closed fields, but these constructions can also be done over an arbitrary field of characteristic 0 (see [3, pp. 87–90]). Consider the commutative diagram

$$\begin{tikzcd} C' \arrow[r, hook, "j'"] \arrow[d, two heads, "\pi"'] \arrow[dd, bend right=60, "q"'] & \overline{C}' \arrow[d, two heads, "\overline{\pi}"] \\ C \arrow[r, hook, "j"] \arrow[d, two heads, "\pi_0"'] & \overline{C} \\ C_0 & \end{tikzcd}$$

where C' is the normalization of C_0, $\overline{C}'$ is a smooth compactification of C', and $\overline{C}$ (resp. C) is the curve obtained from $\overline{C}'$ (resp. C') by contracting each of the finite sets $q^{-1}(x)$, for $x \in C_0$. Notice that $\overline{C}$ is projective and C is semi-normal. Let S be the set of singular points of C, $S' := \pi^{-1}(S)$, and $F := \overline{C}' - C' = \overline{C} - C$.

The *cohomological Albanese* 1*-motive of* C_0 is defined as

$$\mathrm{Alb}^+(C_0) = [u_{\mathrm{Alb}} : \mathrm{Div}^0_F(\bar{C}') \to \mathrm{Pic}^0(\bar{C})],$$

where:

(i) $\mathrm{Pic}^0(\bar{C})$ denotes the group of isomorphism classes of invertible sheaves on $\bar{C}$ which are algebraically equivalent to 0. This is a semi-abelian variety: the map $\bar{\pi}^* : \mathrm{Pic}^0(\bar{C}) \to \mathrm{Pic}^0(\bar{C}')$ is surjective and its kernel is a torus.

(ii) $\mathrm{Div}^0_F(\bar{C}')$ denotes the group of Weil divisors D on $\bar{C}'$ such that $\operatorname{supp} D \subset F$ and $\mathcal{O}(D) \in \mathrm{Pic}^0(\bar{C}')$.

(iii) u_{Alb} is the map $D \mapsto \mathcal{O}(D)$ attaching to a divisor D the corresponding invertible sheaf $\mathcal{O}(D)$.

The *homological Picard* 1*-motive of* C_0 is defined as

$$\mathrm{Pic}^-(C_0) = \left[u_{\mathrm{Pic}} : \mathrm{Div}^0_{S'/S}(\bar{C}', F) \to \mathrm{Pic}^0(\bar{C}', F)\right],$$

where:

(i) $\mathrm{Pic}^0(\bar{C}', F)$ denotes the group of isomorphism classes of pairs $(\mathcal{L}, \phi)$, where $\mathcal{L}$ is an invertible sheaf on $\bar{C}'$ algebraically equivalent to 0 and $\phi : \mathcal{L}|_F \to \mathcal{O}_F$ is a trivialization of $\mathcal{L}$ over F. This is a semi-abelian variety: the natural map $\mathrm{Pic}^0(\bar{C}', F) \to \mathrm{Pic}^0(\bar{C}')$ is surjective and its kernel is a torus.

(ii) $\mathrm{Div}^0_{S'/S}(\bar{C}', F)$ denotes the group of Weil divisors D on $\bar{C}'$ which belong to the kernel of $\bar{\pi}_* : \mathrm{Div}^0_{S'}(\bar{C}') \to \mathrm{Div}^0_S(\bar{C})$ and satisfy $\operatorname{supp} D \cap F = \emptyset$.

(iii) u_{Pic} is the map $D \mapsto \mathcal{O}(D)$ attaching to a divisor D the corresponding invertible sheaf $\mathcal{O}(D)$.

An important fact is that the dual of $\mathrm{Pic}^-(C_0)$ is $\mathrm{Alb}^+(C_0)$, and viceversa.

3. Linearizations of biextensions

For the entirety of this section, we fix a field K. The following is inspired by [14, Def. 1.6].

Definition 3.1. Let $C = [A \xrightarrow{u} B]$, $C' = [A' \xrightarrow{u'} B']$ be complexes of commutative group schemes over K. Let

$$\begin{aligned} \sigma : A \times_K B &\to B \\ (a, b) &\mapsto u(a) + b \end{aligned}$$

be the A-action on B induced by u, and define $\sigma' : A' \times_K B'$ analogously. Let P be a biextension of (B, B') by $\mathbb{G}_m$. We define an $A \times_K A'$-*linearization* of P as an $A \times_K A'$-action on P,

$$\Sigma : (A \times_K A') \times_K P \to P,$$

satisfying the following conditions:

(i) $\mathbb{G}_m$-*equivariance*: For $a \in A$, $a' \in A'$, $c \in \mathbb{G}_m$ and $x \in P$,

$$\Sigma(a, a', c + x) = c + \Sigma(a, a', x).$$

(ii) *Compatibility with σ and σ'*: For $a \in A$ and $a' \in A'$, if $x \in P$ lies above $(b, b') \in B \times_K B'$, then $\Sigma(a, a', x)$ lies above $(\sigma(a, b), \sigma'(a', b'))$.

(iii) *Compatibility with the partial group structures of P*: For $a \in A$, $a'_1, a'_2 \in A'$ and $x_1, x_2 \in P$ lying above $b \in B$,

$$\Sigma(a, a'_1 + a'_2, x_1 +_1 x_2) = \Sigma(a, a'_1, x_1) +_1 \Sigma(a, a'_2, x_2),$$

and for $a_1, a_2 \in A$, $a' \in A'$ and $x_1, x_2 \in P$ lying above $b' \in B'$,

$$\Sigma(a_1 + a_2, a', x_1 +_2 x_2) = \Sigma(a_1, a', x_1) +_2 \Sigma(a_2, a', x_2).$$

Remark 3.2. An action $\Sigma : (A \times_K A') \times_K P \to P$ satisfying conditions (i) and (ii) is an $A \times_K A'$-linearization of the line bundle P in the sense of [14, Def. 1.6]; this can be summed up as saying that Σ is a "bundle action" lifting the actions σ and σ'. Notice that σ and σ' are homomorphisms, and so condition (iii) may then be interpreted as a lifting to P of the compatibility of σ and σ' with the group structures of B and B'. In the rest of the article, we will only use the term *linearization* in the sense of Definition 3.1 above.

Remark 3.3. By considering constant group schemes, we will also be able to talk about linearizations of biextensions of abelian groups.

Let $C = [A \xrightarrow{u} B]$ and $C' = [A' \xrightarrow{u'} B']$ be as in Definition 3.1 and consider a biextension P of (B, B') by $\mathbb{G}_m$. Whenever P has the structure of biextension of (C, C') by $\mathbb{G}_m$ with trivializations

$$\tau : A \times_K B' \to P, \quad \tau' : B \times_K A' \to P,$$

we can define an $A \times_K A'$-linearization of P as

$$\begin{aligned} \Sigma : (A \times_K A') \times_K P &\to P \\ (a, a', x) &\mapsto [\tau'(u(a), a') +_2 \tau'(b, a')] +_1 [\tau(a, b') +_2 x], \end{aligned}$$

where $x \in P$ lies above $(b, b') \in B \times_K B'$. This construction is due to [5, Thm. 6.8] (see also [15, p. 306]). Conversely, given an $A \times_K A'$-linearization

$$\Sigma : (A \times_K A') \times_K P \to P$$

of P, we can define a biextension structure of (C, C') by $\mathbb{G}_m$ on P as the one determined by the trivializations

$$\begin{aligned} \tau : A \times_K B' &\to P & \qquad \tau' : B \times_K A' &\to P \\ (a, b') &\mapsto \Sigma(a, 0, 0_{b'}), & (b, a') &\mapsto \Sigma(0, a', 0_b), \end{aligned}$$

where $0_b, 0_{b'}$ are the zero elements in the groups $(P_{b,B'}, +_1), (P_{B,b'}, +_2)$, respectively. These constructions are inverses of each other.

Proposition 3.4. *Let C, C' and P be as in Definition 3.1 and suppose that $u(K)$ and $u'(K)$ are injective. Then an $A \times_K A'$-linearization Σ of P induces a biextension $Q(K)$ of $(B(K)/A(K), B'(K)/A'(K))$ by K^*.*

Proof. Notice that $P(K)$ is a biextension of $(B(K), B'(K))$ by K^* and that

$$\Sigma(K) : (A(K) \times A'(K)) \times P(K) \to P(K)$$

is an $A(K) \times A'(K)$-linearization of $P(K)$. We define $Q(K)$ as the set consisting of the orbits

$$[x] := \left\{ \Sigma(a, a', x) \mid a \in A(K),\ a' \in A'(K) \right\}$$

of elements $x \in P(K)$ under Σ. Then $Q(K)$ maps surjectively onto $B(K)/A(K) \times B'(K)/A'(K)$ and is endowed with a K^*-action which is free and transitive on fibers. To see that it is a biextension it is then enough to prove that $+_1$ and $+_2$ induce partial group structures on $Q(K)$. For this, take elements $x_1, x_2 \in P(K)$ lying above $(b_1, b_1'), (b_2, b_2') \in B(K) \times B'(K)$, respectively, such that the orbits of b_1 and b_2 under σ are equal. This is equivalent to having

$$b_1 = \sigma(a, b_2),$$

for some (unique) $a \in A(K)$. Then x_1 and $\Sigma(a, 0, x_2)$ project to $b_1 \in B(K)$ and we are able to define

$$[x_1] +_1 [x_2] := [x_1 +_1 \Sigma(a, 0, x_2)].$$

This is well defined and commutative. We define the partial group structure $+_2$ in the analogous way. ∎

Consider a pair of 1-motives $M = [L \xrightarrow{u} G]$, $M' = [L' \xrightarrow{u'} G']$ over K and a biextension P of (M, M') by $\mathbb{G}_m$. For our purposes, we give the following definition which is inspired by [9, p. 326].

Definition 3.5. We define the *group of K-points of M*, denoted $M(K)$, as

$$M(K) := \mathrm{Ext}^1_K(M^\vee, \mathbb{G}_m).$$

Consider the following short exact sequence of complexes:

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & L^\vee \arrow[r, equal] \arrow[d, "u^\vee"] & L^\vee \arrow[r] \arrow[d, "v^\vee"] & 0 \\ 0 \arrow[r] & T^\vee \arrow[r] & G^\vee \arrow[r] & A^\vee \arrow[r] & 0 \end{tikzcd}$$

and the long exact sequence of abelian groups that it induces:

$$\cdots \to L(K) \xrightarrow{u(K)} G(K) \to M(K) \to \mathrm{Ext}^1_K(T^\vee, \mathbb{G}_m) \to \cdots .$$

It follows that, when $T^\vee$ is split (or, equivalently, when L is constant), the group of K-points of M can be described as

$$M(K) = G(K)/\operatorname{Im}(u(K)).$$

If L, L' are constant and $u(K)$, $u'(K)$ are injective, then $P(K)$ induces a biextension of $(M(K), M'(K))$ by K^*, by Proposition 3.4. When $M' = M^\vee$ and P is the Poincaré biextension, we will denote by $Q_M(K)$ the induced biextension of $(M(K), M^\vee(K))$ by K^*.

We will now introduce the concept of *compatibility* between a linearization and a ρ-splitting of a biextension (see Definition 2.1 for the definition of ρ-splitting of a biextension).

Definition 3.6. Let $C = [A \xrightarrow{u} B]$, $C' = [A' \xrightarrow{u'} B']$ be complexes of commutative group schemes over K and P a biextension of (C, C') by $\mathbb{G}_m$. Let Y be an abelian group and $\rho : K^* \to Y$ a homomorphism. We will say that a ρ-splitting $\psi : P(K) \to Y$ of $P(K)$ is *compatible* with the $A \times_K A'$-linearization Σ of P if any of the following equivalent conditions is satisfied:

(i) $\psi(\Sigma(a, a', x)) = \psi(x)$, for all $a \in A(K)$, $a' \in A'(K)$ and $x \in P(K)$,

(ii) $\psi \circ \tau$ and $\psi \circ \tau'$ vanish on $A(K) \times B'(K)$ and $B(K) \times A'(K)$, respectively.

Remark 3.7. Assuming that $u(K)$ and $u'(K)$ are injective in Definition 3.6, a ρ-splitting ψ is compatible with the $A \times_K A'$-linearization if and only if it induces a ρ-splitting on the biextension $Q(K)$ of Proposition 3.4.

4. ρ-splittings in the ramified case

Let K be a finite extension of $\mathbb{Q}_p$ and consider a 1-motive $M = [L \xrightarrow{u} G]$ over K with dual

$$M^\vee = [L^\vee \xrightarrow{u^\vee} G^\vee].$$

We will assume that L and T are split (or, equivalently, that $L^\vee$ and $T^\vee$ are split). Let

$$M^\natural = [L \xrightarrow{u^\natural} G^\natural] \quad \text{and} \quad M^{\vee\natural} = [L \xrightarrow{u^{\vee\natural}} G^{\vee\natural}]$$

be their corresponding universal vectorial extensions. We will denote Deligne's pairing associated to M and its dual as

$$(\cdot,\cdot)_M^{\mathrm{Del}} : \mathrm{T}_{\mathrm{dR}}(M) \times_K \mathrm{T}_{\mathrm{dR}}(M^\vee) \to \mathrm{Lie}\,\mathbb{G}_m = \mathbb{G}_a.$$

Let $P^\natural$ be the canonical biextension of $(M^\natural, M^{\vee\natural})$ by $\mathbb{G}_m$. We will denote by $e_{P^\natural/G^\natural}$ and $e_{P^\natural/G^{\vee\natural}}$ the zero sections of $P^\natural$ over $G^\natural$ and $G^{\vee\natural}$, respectively, and by $\pi^\natural : P^\natural \to G^\natural \times_K G^{\vee\natural}$ the projection:

$$\begin{tikzcd}
 & P^\natural \arrow[d, "\pi^\natural"] & \\
G^\natural \times_K \{0\} \arrow[r, hook] \arrow[ur, bend left, "e_{P^\natural/G^\natural}"] & G^\natural \times_K G^{\vee\natural} & \{0\} \times_K G^{\vee\natural}. \arrow[l, hook'] \arrow[ul, bend right, "e_{P^\natural/G^{\vee\natural}}"']
\end{tikzcd}$$

The canonical connection on $P^\natural$ determines, and is determined by, a normal bi-invariant 1-form $\omega \in \Omega^1_{P^\natural/K}$ (see [6, Prop. 2.1]). In particular, if we denote by ω_1 and ω_2 the images of ω under the canonical maps

$$\Omega^1_{P^\natural/K} \to \Omega^1_{P^\natural/G^{\vee\natural}} \quad \text{and} \quad \Omega^1_{P^\natural/K} \to \Omega^1_{P^\natural/G^\natural},$$

then ω_1 and ω_2 are invariant differentials over $G^{\vee\natural}$ and $G^\natural$, respectively. Let

$$r_1 : \mathrm{Lie}(P^\natural/G^{\vee\natural}) \to \mathbb{G}_{a,G^{\vee\natural}} \quad \text{and} \quad r_2 : \mathrm{Lie}(P^\natural/G^\natural) \to \mathbb{G}_{a,G^\natural}$$

be the homomorphisms corresponding to ω_1 and ω_2, respectively.

We fix a branch $\lambda : K^* \to K$ of the p-adic logarithm for the rest of the section. For a commutative algebraic group H over K we will denote by $\lambda_H : H(K) \to \mathrm{Lie}\,H(K)$ the uniquely determined homomorphism of Lie groups extending λ as constructed in [17, §1]. We have the following result:

Lemma 4.1. *Let $h \in G^\natural(K)$, $h^\vee \in G^{\vee\natural}(K)$ and $y \in P^\natural(K)$ be such that $\pi^\natural(y) = (h, h^\vee)$. Then*

$$(\lambda_{G^\natural}(h), \lambda_{G^{\vee\natural}}(h^\vee))_M^{\mathrm{Del}} = r_{1,h^\vee} \circ \lambda_{P^\natural_{G^\natural,h^\vee}}(y) - r_{2,h} \circ \lambda_{P^\natural_{h,G^{\natural\vee}}}(y).$$

Proof. Let $\mathcal{T}P^\natural$ denote the tangent sheaf of $P^\natural$. Notice that the germ of $\mathcal{T}P^\natural$ at $y \in P^\natural$ can be expressed as the contracted product of $\mathbb{G}_a$-torsors

$$(\mathcal{T}P^\natural)_y = \mathrm{Lie}\, P^\natural_{G^\natural,h^\vee} \wedge^{\mathbb{G}_a} \mathrm{Lie}\, P^\natural_{h,G^{\vee\natural}},$$

where $(h,h^\vee) = \pi^\natural(y)$. Let $F_1, F_2 \in \Gamma(P^\natural, \mathcal{T}P^\natural)$ be the global sections given by

$$F_1(y) = \lambda_{P^\natural_{G^\natural,h^\vee}}(y) \wedge \lambda_{P^\natural_{h,G^{\vee\natural}}}(e_{P^\natural/G^\natural}(h)) \in \mathrm{Lie}\, P^\natural_{G^\natural,h^\vee} \wedge^{\mathbb{G}_a} \mathrm{Lie}\, P^\natural_{h,G^{\vee\natural}},$$
$$F_2(y) = \lambda_{P^\natural_{G^\natural,h^\vee}}(e_{P^\natural/G^{\vee\natural}}(h^\vee)) \wedge \lambda_{P^\natural_{h,G^{\vee\natural}}}(y) \in \mathrm{Lie}\, P^\natural_{G^\natural,h^\vee} \wedge^{\mathbb{G}_a} \mathrm{Lie}\, P^\natural_{h,G^{\vee\natural}}.$$

We have the formula

$$d\omega(F_1,F_2) = F_1 \cdot \omega(F_2) - F_2 \cdot \omega(F_1) - \omega([F_1,F_2]),$$

where $F_1 \cdot \omega(F_2)$ denotes the vector field F_1 applied as a differential operator to the scalar field $\omega(F_2)$. First, we observe that $[F_1,F_2] = 0$. Furthermore,

$$F_1 \cdot \omega(F_2) = F_1 \cdot \omega_2(F_2) = \omega_2(F_2),$$

where the first equality is due to $e_{P^\natural/G^{\vee\natural}}$ being the zero section of $P^\natural$ over $G^{\vee\natural}$, and the second one due to $e_{P^\natural/G^\natural}$ being the zero section of $P^\natural$ over $G^\natural$. Similarly, we have

$$F_2 \cdot \omega(F_1) = \omega_1(F_1).$$

Therefore, the alternating map on $\mathcal{T}P^\natural \times \mathcal{T}P^\natural$ induced by $d\omega$ satisfies

$$d\omega(F_1(y),F_2(y)) = r_{2,h} \circ \lambda_{P^\natural_{h,G^{\vee\natural}}}(y) - r_{1,h^\vee} \circ \lambda_{P^\natural_{G^\natural,h^\vee}}(y),$$

where $(h,h^\vee) = \pi^\natural(y)$.

Now, let γ be the 2-form on $G^\natural \times_K G^{\vee\natural}$ inducing Deligne's pairing. Since $d\omega = \pi^{\natural *}\gamma$ (see [6, Prop. 2.1]) we have that

$$\gamma((\lambda_{G^\natural}(h),0),(0,\lambda_{G^{\vee\natural}}(h^\vee))) = d\omega(F_1(y),F_2(y)).$$

Finally, note that Deligne's pairing ([8, (10.2.7.3)]) on the pair $(\lambda_{G^\natural}(h), \lambda_{G^{\vee\natural}}(h^\vee))$ is given by the formula

$$(\lambda_{G^\natural}(h), \lambda_{G^{\vee\natural}}(h^\vee))^{\mathrm{Del}}_M = -\gamma((\lambda_{G^\natural}(h),0),(0,\lambda_{G^{\vee\natural}}(h^\vee))).$$

Putting together the last three equalities, we obtain the desired result. ∎

Definition 4.2. Let $\eta : G(K) \to G^\natural(K)$ and $\eta^\vee : G^\vee(K) \to G^{\vee\natural}(K)$ be a pair of splittings of the exact sequences of Lie groups

$$0 \to \omega_{G^\vee}(K) \xrightarrow{\iota} G^\natural(K) \xrightarrow{\theta} G(K) \to 0, \tag{4.1}$$

$$0 \to \omega_G(K) \xrightarrow{\iota^\vee} G^{\vee\natural}(K) \xrightarrow{\theta^\vee} G^\vee(K) \to 0. \tag{4.2}$$

We say that $(\eta, \eta^\vee)$, or also that $(\mathrm{Lie}\,\eta, \mathrm{Lie}\,\eta^\vee)$, are *dual* with respect to Deligne's pairing if

$$(\mathrm{Lie}\,\eta, \mathrm{Lie}\,\eta^\vee)_M^{\mathrm{Del}} = 0.$$

The following result is a slight generalization of [6, Lem. 3.1.1] (see also [16, Thm. 3.1.3]). It implies, in particular, that from any section r of $\mathrm{Lie}\,\theta : \mathrm{Lie}\,G^\natural(K) \to \mathrm{Lie}\,G(K)$ we can always obtain a canonical section η of $\theta : G^\natural(K) \to G(K)$ such that $\mathrm{Lie}\,\eta = r$.

Lemma 4.3. *Let*

$$0 \to V \to X \to Y \to 0$$

be an exact sequence of algebraic K-groups with V a vector group. There is a bijection between splittings of the exact sequence

$$0 \to V(K) \to X(K) \to Y(K) \to 0 \tag{4.3}$$

and splittings of the exact sequence of Lie algebras

$$0 \to \mathrm{Lie}\,V(K) \to \mathrm{Lie}\,X(K) \to \mathrm{Lie}\,Y(K) \to 0. \tag{4.4}$$

Proof. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & V(K) \arrow[r] \arrow[d, equal] & X(K) \arrow[r] \arrow[d, "\lambda_X"] & Y(K) \arrow[r] \arrow[d, "\lambda_Y"] & 0 \\
0 \arrow[r] & \mathrm{Lie}\,V(K) \arrow[r] & \mathrm{Lie}\,X(K) \arrow[r] & \mathrm{Lie}\,Y(K) \arrow[r] & 0.
\end{tikzcd}$$

If $s : X(K) \to V(K)$ is a splitting of (4.3), then $\mathrm{Lie}\,s : \mathrm{Lie}\,X(K) \to \mathrm{Lie}\,V(K)$ is a splitting of (4.4); notice that $\mathrm{Lie}\,s \circ \lambda_X = s$. For the converse, let $r : \mathrm{Lie}\,X(K) \to \mathrm{Lie}\,V(K)$ be a splitting of (4.4). Then

$$s : X(K) \xrightarrow{\lambda_X} \mathrm{Lie}\,X(K) \xrightarrow{r} \mathrm{Lie}\,V(K) = V(K)$$

is a splitting of (4.3). Moreover, by the properties of the logarithm (see [17, §1]), this map is such that $\mathrm{Lie}\,s = r$. We remark that the above also implies that the functor Lie provides a bijection between splittings $s' : Y(K) \to X(K)$ of (4.3) and splittings $r' : \mathrm{Lie}\,Y(K) \to \mathrm{Lie}\,X(K)$ of (4.4). ■

Theorem 4.4. *Let $\eta : G(K) \to G^\natural(K)$ and $\eta^\vee : G^\vee(K) \to G^{\vee\natural}(K)$ be splittings of the exact sequences* (4.1) *and* (4.2)*, respectively. Then:*

(i) *There is a λ-splitting $\psi_1 : P(K) \to K$ of $P(K)$ defined as follows. For $z \in P(K)$ lying above $(g, g^\vee) \in G(K) \times G^\vee(K)$, denote by $s_{g^\vee}$ the rigidification of $P_{G,g^\vee}$ corresponding to $\eta^\vee(g^\vee)$. The map $s_{g^\vee}$ sits in the following diagram:*

$$\begin{tikzcd}
K^* \arrow[rr, "\lambda"] \arrow[d, hook] & & K \arrow[d, hook] \\
P_{G,g^\vee}(K) \arrow[r, "\lambda_{P_{G,g^\vee}}"] \arrow[d] & \mathrm{Lie}\, P_{G,g^\vee}(K) \arrow[d] \arrow[ur, dashed, bend right, "s_{g^\vee}"'] & \\
G(K) \times \{g^\vee\} \arrow[r, "\lambda_G"] & \mathrm{Lie}\, G(K). &
\end{tikzcd}$$

We define the image of z by ψ_1 as

$$\psi_1(z) = s_{g^\vee} \circ \lambda_{P_{G,g^\vee}}(z).$$

(ii) *There is a λ-splitting $\psi_2 : P(K) \to K$ of $P(K)$ defined as follows. For $z \in P(K)$ lying above $(g, g^\vee) \in G(K) \times G^\vee(K)$, denote by s_g the rigidification of $P_{g,G^\vee}$ corresponding to $\eta(g)$. The map s_g sits in the following diagram:*

$$\begin{tikzcd}
K^* \arrow[rr, "\lambda"] \arrow[d, hook] & & K \arrow[d, hook] \\
P_{g,G^\vee}(K) \arrow[r, "\lambda_{P_{g,G^\vee}}"] \arrow[d] & \mathrm{Lie}\, P_{g,G^\vee}(K) \arrow[d] \arrow[ur, dashed, bend right, "s_g"'] & \\
\{g\} \times G^\vee(K) \arrow[r, "\lambda_{G^\vee}"] & \mathrm{Lie}\, G^\vee(K). &
\end{tikzcd}$$

We define the image of z by ψ_2 as

$$\psi_2(z) = s_g \circ \lambda_{P_{g,G^\vee}}(z).$$

(iii) *If $(\eta, \eta^\vee)$ are dual with respect to Deligne's pairing, then $\psi_1 = \psi_2$.*

Proof. By construction, the invariant 1-form $\omega_1 \in \Omega^1_{P^\natural/G^{\vee\natural}}$ is obtained via pullback from an invariant differential

$$\bar{\omega}_1 \in \Omega^1_{P_{G,G^{\vee\natural}}/G^{\vee\natural}},$$

where $P_{G,G^{\vee\natural}}$ denotes the pullback of P along the map $\mathrm{Id} \times \theta^\vee : G \times G^{\vee\natural} \to G \times G^\vee$ (see the proof of [4, Prop. 3.9]). Similarly, $\omega_2 \in \Omega^1_{P^\natural/G^\natural}$ comes from an invariant differential

$$\bar{\omega}_2 \in \Omega^1_{P_{G^\natural,G^\vee}/G^\natural}.$$

Denote by

$$\bar{r}_1 : \mathrm{Lie}(P_{G,G^{\vee\natural}}/G^{\vee\natural}) \to \mathbb{G}_{a,G^{\vee\natural}} \quad \text{and} \quad \bar{r}_2 : \mathrm{Lie}(P_{G^\natural,G^\vee}/G^\natural) \to \mathbb{G}_{a,G^\natural}$$

the homomorphisms corresponding to $\bar{\omega}_1$ and $\bar{\omega}_2$, respectively.

Consider the following diagram, where $\overline{\theta \times \theta^\vee} : P^\natural \to P$ denotes the morphism of biextensions obtained from $\theta \times \theta^\vee$ by pullback:

$$\begin{tikzcd}
P^\natural \arrow[r, "\overline{\theta\times\theta^\vee}"] \arrow[d, "\pi^\natural"'] & P \arrow[d, "\pi"] \\
G^\natural \times_K G^{\vee\natural} \arrow[r, "\theta\times\theta^\vee"] & G \times_K G^\vee.
\end{tikzcd}$$

Let $z \in P(K)$ and $(g, g^\vee) = \pi(z)$. Let $y \in P^\natural(K)$ be the rational point such that

$$\pi^\natural(y) = (\eta(g), \eta^\vee(g^\vee)) \quad \text{and} \quad \overline{\theta \times \theta^\vee}(y) = z.$$

We have the following diagram:

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r, equal] \arrow[d, hook] & \mathbb{G}_m \arrow[r, equal] \arrow[d, hook] & \mathbb{G}_m \arrow[d, hook] \\
P^\natural_{\eta(g),G^{\vee\natural}} \arrow[r, "\bar{\theta}^\vee"] \arrow[d, "\pi^\natural"'] & P_{\eta(g),G^\vee} \arrow[r, "\sim"] \arrow[d] & P_{g,G^\vee} \arrow[d, "\pi"] \\
\{\eta(g)\} \times_K G^{\vee\natural} \arrow[r, "\theta^\vee"] & \{\eta(g)\} \times_K G^\vee \arrow[r, "\sim"] & \{g\} \times_K G^\vee,
\end{tikzcd}$$

where the lower squares are pullback diagrams, so that $\bar{\theta}^\vee$ denotes the morphism of extensions obtained from $\theta^\vee$ by pullback. Notice that the isomorphism

$$P_{\eta(g),G^\vee} \xrightarrow{\sim} P_{g,G^\vee}$$

sends $\bar{\theta}^\vee(y)$ to z. We now consider the corresponding diagram of rigidified extensions of Lie algebras:

$$\begin{tikzcd}
\mathrm{Lie}\,\mathbb{G}_m \arrow[r, equal] \arrow[d, hook] & \mathrm{Lie}\,\mathbb{G}_m \arrow[r, equal] \arrow[d, hook] & \mathrm{Lie}\,\mathbb{G}_m \arrow[d, hook] \\
\mathrm{Lie}\,P^\natural_{\eta(g),G^{\vee\natural}} \arrow[u, dashed, bend right, "r_{2,\eta(g)}"'] \arrow[r, "\mathrm{Lie}\,\bar{\theta}^\vee"] \arrow[d, "\mathrm{Lie}\,\pi^\natural"'] & \mathrm{Lie}\,P_{\eta(g),G^\vee} \arrow[u, dashed, bend right, "\bar{r}_{2,\eta(g)}"'] \arrow[r, "\sim"] \arrow[d] & \mathrm{Lie}\,P_{g,G^\vee} \arrow[u, dashed, bend right, "s_g"'] \arrow[d, "\mathrm{Lie}\,\pi"] \\
\{\eta(g)\} \times_K \mathrm{Lie}\,G^{\vee\natural} \arrow[r, "\mathrm{Lie}\,\theta^\vee"] & \{\eta(g)\} \times_K \mathrm{Lie}\,G^\vee \arrow[r, "\sim"] & \{g\} \times_K \mathrm{Lie}\,G^\vee.
\end{tikzcd}$$

From the commutativity of this diagram and the properties of the logarithm we obtain the following equalities:

$$\begin{aligned} r_{2,\eta(g)} \circ \lambda_{P^{\natural}_{\eta(g),G^{\natural\vee}}}(y) &= \bar{r}_{2,\eta(g)} \circ \operatorname{Lie} \bar{\theta}^{\vee} \circ \lambda_{P^{\natural}_{\eta(g),G^{\natural\vee}}}(y) \\ &= \bar{r}_{2,\eta(g)} \circ \lambda_{P_{\eta(g),G^{\vee}}}(\bar{\theta}^{\vee}(y)) \\ &= s_g \circ \lambda_{P_{g,G^{\vee}}}(z). \end{aligned}$$

Analogously, we have

$$r_{1,\eta^{\vee}(g^{\vee})} \circ \lambda_{P^{\natural}_{G^{\natural},\eta^{\vee}(g^{\vee})}}(y) = s_{g^{\vee}} \circ \lambda_{P_{G,g^{\vee}}}(z).$$

Therefore,

$$\begin{aligned} &\big(\operatorname{Lie} \eta \circ \lambda_G(g), \operatorname{Lie} \eta^{\vee} \circ \lambda_{G^{\vee}}(g^{\vee})\big)^{\mathrm{Del}}_M \\ &\quad = \big(\lambda_{G^{\natural}} \circ \eta(g), \lambda_{G^{\vee\natural}} \circ \eta^{\vee}(g^{\vee})\big)^{\mathrm{Del}}_M \\ &\quad = r_{1,\eta^{\vee}(g^{\vee})} \circ \lambda_{P^{\natural}_{G^{\natural},\eta^{\vee}(g^{\vee})}}(y) - r_{2,\eta(g)} \circ \lambda_{P^{\natural}_{\eta(g),G^{\natural\vee}}}(y) \\ &\quad = s_{g^{\vee}} \circ \lambda_{P_{G,g^{\vee}}}(z) - s_g \circ \lambda_{P_{g,G^{\vee}}}(z) \\ &\quad = \psi_1(z) - \psi_2(z). \end{aligned}$$

Since $z \in P(K)$ was arbitrary, it is clear from the above formula that if $(\eta, \eta^{\vee})$ are dual with respect to Deligne's pairing, then $\psi_1 = \psi_2$.

It remains to check that ψ_1 and ψ_2 are indeed λ-splittings. First, notice that for all $c \in K^*$ and $z, z' \in P_{G,g^{\vee}}(K)$ we have

$$\begin{aligned} \psi_1(c + z) &= s_{g^{\vee}} \circ \lambda_{P_{G,g^{\vee}}}(c + z) \\ &= s_{g^{\vee}} \circ \lambda_{P_{G,g^{\vee}}}(c) + s_{g^{\vee}} \circ \lambda_{P_{G,g^{\vee}}}(z) \\ &= \lambda(c) + \psi_1(z), \\ \psi_1(z + z') &= s_{g^{\vee}} \circ \lambda_{P_{G,g^{\vee}}}(z +_2 z') \\ &= s_{g^{\vee}} \circ \lambda_{P_{G,g^{\vee}}}(z) + s_{g^{\vee}} \circ \lambda_{P_{G,g^{\vee}}}(z') \\ &= \psi_1(z) + \psi_1(z'). \end{aligned}$$

In a similar way we prove the compatibility of ψ_2 with the partial group structure $+_1$ of $P(K)$ and the K^*-action. If $(\eta, \eta^{\vee})$ are dual with respect to Deligne's pairing, then $\psi_1 = \psi_2$ and both ψ_1 and ψ_2 are λ-splittings. To prove that ψ_1 is a λ-splitting in the general case (the proof for ψ_2 is done similarly), notice that it is always possible to find a splitting $\tilde{r} : \operatorname{Lie} G(K) \to \operatorname{Lie} G^{\natural}(K)$ of

$$0 \to \omega_{G^{\vee}}(K) \xrightarrow{\operatorname{Lie} \iota} \operatorname{Lie} G^{\natural}(K) \xrightarrow{\operatorname{Lie} \theta} \operatorname{Lie} G(K) \to 0$$

such that $(\tilde{r}, \mathrm{Lie}\,\eta^\vee)$ are dual, due to the fact that Deligne's pairing is perfect (see [4, Thm. 4.3]). Applying Lemma 4.3, we can obtain a splitting $\tilde{\eta}$ of (4.1) such that $\mathrm{Lie}\,\tilde{\eta} = \tilde{r}$. Proceeding as before with $(\tilde{\eta}, \eta^\vee)$, we are able to prove that ψ_1 is a λ-splitting. ∎

Theorem 4.5. *Let $\eta : G(K) \to G^\natural(K)$ and $\eta^\vee : G^\vee(K) \to G^{\vee\natural}(K)$ be a pair of splittings of the exact sequences (4.1) and (4.2), respectively, which are dual with respect to Deligne's pairing. Assume, moreover, that η and $\eta^\vee$ make the following diagrams commute:*

$$\begin{tikzcd} L(K) \arrow[r, equal] \arrow[d, "u"'] & L(K) \arrow[d, "u^\natural"] \\ G(K) \arrow[r, "\eta"] & G^\natural(K), \end{tikzcd} \qquad \begin{tikzcd} L^\vee(K) \arrow[r, equal] \arrow[d, "u^\vee"'] & L^\vee(K) \arrow[d, "u^{\vee\natural}"] \\ G^\vee(K) \arrow[r, "\eta^\vee"] & G^{\vee\natural}(K). \end{tikzcd}$$

Then the λ-splitting $\psi : P(K) \to K$ constructed in Theorem 4.4 is compatible with the $L \times_K L^\vee$-linearization of P. In particular, it induces a λ-splitting of the biextension $Q_M(K)$ of $(M(K), M^\vee(K))$ by K^* in the case that $u(K)$ and $u^\vee(K)$ are injective.

Remark 4.6. The condition $\eta \circ u = u^\natural$ says that, on K-sections, (Id, η) is a splitting of the complex $M^\natural$ seen as an extension of M by $\omega_{G^\vee}$; and similarly for $\eta^\vee$.

Proof. We have to prove that the λ-splitting $\psi : P(K) \to K$ constructed in Theorem 4.4 satisfies that $\psi \circ \tau$ and $\psi \circ \tau^\vee$ vanish on K-sections. We will only prove this for $\psi \circ \tau$ since the proof for $\psi \circ \tau^\vee$ is carried out in a similar way.

We fix a splitting of the following short exact sequence of vector groups:

(4.5) $$\begin{tikzcd} 0 \arrow[r] & \omega_{A^\vee} \arrow[r] & \omega_{G^\vee} \arrow[l, dashed, bend right] \arrow[r] & \omega_{T^\vee} \arrow[l, dashed, bend right] \arrow[r] & 0. \end{tikzcd}$$

This induces by duality a splitting of the following exact sequence of Lie algebras:

(4.6) $$\begin{tikzcd} 0 \arrow[r] & \mathrm{Lie}\,T^\vee \arrow[r] & \mathrm{Lie}\,G^\vee \arrow[l, dashed, bend right, "j"'] \arrow[r] & \mathrm{Lie}\,A^\vee \arrow[l, dashed, bend right] \arrow[r] & 0. \end{tikzcd}$$

Consider the following commutative diagram with exact rows and columns, where the splitting of the middle column is obtained by pushout along ι from the split exact sequence (4.5):

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \omega_{A^\vee} \arrow[r] \arrow[d] & G^\# \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \omega_{G^\vee} \arrow[r, "\iota"] \arrow[d] \arrow[u, dashed, bend right] & G^\natural \arrow[r, "\theta"] \arrow[d, "\sigma"'] \arrow[u, dashed, bend right, "\overline{\sigma}"'] \arrow[ul, phantom, "\ulcorner"] & G \arrow[r] & 0 \\
 & \omega_{T^\vee} \arrow[r, equal] \arrow[d] \arrow[u, dashed, bend right] & \omega_{T^\vee} \arrow[d] \arrow[u, dashed, bend right] & & \\
 & 0 & 0, & &
\end{tikzcd}$$

Let $x \in L(K)$ and denote by $\chi : T^\vee \to \mathbb{G}_m$ the homomorphism corresponding to it. We have the following diagram with exact rows (see [1, §1.2]):

$$\begin{tikzcd}
0 \arrow[r] & T^\vee \arrow[r] \arrow[d, "-\chi"'] & G^\vee \arrow[r] \arrow[d, "\tau'_x"] & A^\vee \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[d, equal] & P_{v(x),A^\vee} \arrow[r] & A^\vee \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{G}_m \arrow[r] & P_{u(x),G^\vee} \arrow[r] \arrow[u] & G^\vee \arrow[r] \arrow[u] & 0,
\end{tikzcd}$$

where v is the composition $L \xrightarrow{u} G \to A$. We also have the corresponding diagram of Lie algebras with exact rows and splittings induced from (4.6) by pushout and pullback:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Lie}\, T^\vee \arrow[r] \arrow[d, "-\,\mathrm{Lie}\,\chi"'] & \mathrm{Lie}\, G^\vee \arrow[r] \arrow[d] \arrow[l, dashed, bend right, "j"'] & \mathrm{Lie}\, A^\vee \arrow[r] \arrow[d, equal] \arrow[l, dashed, bend right] & 0 \\
0 \arrow[r] & \mathrm{Lie}\, \mathbb{G}_m \arrow[r] \arrow[d, equal] & \mathrm{Lie}\, P_{v(x),A^\vee} \arrow[r] \arrow[l, dashed, bend right, "\xi"'] & \mathrm{Lie}\, A^\vee \arrow[r] \arrow[l, dashed, bend right] & 0 \\
0 \arrow[r] & \mathrm{Lie}\, \mathbb{G}_m \arrow[r] & \mathrm{Lie}\, P_{u(x),G^\vee} \arrow[r] \arrow[u] \arrow[l, dashed, bend right] & \mathrm{Lie}\, G^\vee \arrow[r] \arrow[u] \arrow[l, dashed, bend right] & 0.
\end{tikzcd} \tag{4.7}$$

By Lemma 2.2 (i), $u^\natural(x) \in G^\natural(K)$ corresponds to the extension $[L^\vee \to P_{u(x),G^\vee}]$ of $M^\vee$ by $\mathbb{G}_m$ endowed with a $\natural$-structure. By Lemma 2.2 (iv), we know that the invariant differential $\sigma \circ u^\natural(x) \in \omega_{T^\vee}(K)$ is the one associated to the homomorphism $\mathrm{Lie}\,\chi \in \mathrm{Hom}_{\mathscr{O}_K}(\mathrm{Lie}\, T^\vee, \mathbb{G}_a)$. On the other hand, $\overline{\sigma} \circ u^\natural(x) \in G^\#(K)$ is the extension $[L^\vee \to P_{v(x),A^\vee}]$ of $[L^\vee \xrightarrow{v} A^\vee]$ by $\mathbb{G}_m$ endowed with the normal invariant differential associated to $\xi : \mathrm{Lie}\, P_{v(x),A^\vee} \to \mathrm{Lie}\, \mathbb{G}_m$. The above can be summarized in the following diagram:

$$\begin{array}{ccccc} \omega_{T^\vee}(K) & \xleftarrow{\ \sigma\ } & G^\natural(K) & \xrightarrow{\quad \bar{\sigma} \quad} & G^\#(K) \\ \mathrm{Lie}\,\chi & \longleftarrow\!\shortmid & u^\natural(x) & \longmapsto & ([L^\vee \to P_{v(x),A^\vee}], \xi). \end{array}$$

The way in which we obtain an element in $G^\natural(K)$ from a pair of elements in $\omega_{T^\vee}(K)$ and $G^\#(K)$ is by considering the decomposition

$$\mathrm{Lie}\, P_{u(x),G^\vee} \cong \mathrm{Lie}\, T^\vee \times_K \mathrm{Lie}\, P_{v(x),A^\vee}$$

induced by (4.6), as displayed by the following diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathrm{Lie}\,\mathbb{G}_m \arrow[r, equal] \arrow[d] & \mathrm{Lie}\,\mathbb{G}_m \arrow[d] & \\
0 \arrow[r] & \mathrm{Lie}\, T^\vee \arrow[r] \arrow[d, equal] & \mathrm{Lie}\, P_{u(x),G^\vee} \arrow[r] \arrow[d] \arrow[l, bend right, dashed] & \mathrm{Lie}\, P_{v(x),A^\vee} \arrow[r] \arrow[d] \arrow[l, bend right, dashed] & 0 \\
0 \arrow[r] & \{u(x)\} \times_K \mathrm{Lie}\, T^\vee \arrow[r] & \{u(x)\} \times_K \mathrm{Lie}\, G^\vee \arrow[r] \arrow[d] \arrow[l, bend right, dashed, "j"'] & \{v(x)\} \times_K \mathrm{Lie}\, A^\vee \arrow[r] \arrow[d] \arrow[l, bend right, dashed] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

From the decomposition of $\mathrm{Lie}\, P_{u(x),G^\vee}$ and our hypothesis that $\eta \circ u = u^\natural$, it follows that $s_{u(x)}$ can be expressed as

$$s_{u(x)} = \mathrm{Lie}\,\chi + \xi : \mathrm{Lie}\, P_{u(x),G^\vee} \cong \mathrm{Lie}\, T^\vee \times_K \mathrm{Lie}\, P_{v(x),A^\vee} \to \mathbb{G}_a.$$

Observe, moreover, that $\lambda_{P_{u(x),G^\vee}}(\tau(x, g^\vee)) \in \mathrm{Lie}\, P_{u(x),G^\vee}$ corresponds under this isomorphism to

$$(j \circ \lambda_{G^\vee}(g^\vee), \lambda_{P_{v(x),A^\vee}} \circ \tau_x'(g^\vee)) \in \mathrm{Lie}\, T^\vee \times_K \mathrm{Lie}\, P_{v(x),A^\vee}.$$

Furthermore, the middle row in diagram (4.7) allows us to identify $\mathrm{Lie}\, P_{v(x),A^\vee}$ with $\mathrm{Lie}\,\mathbb{G}_m \times_K \mathrm{Lie}\, A^\vee$; under this identification, $\lambda_{P_{v(x),A^\vee}} \circ \tau_x'(g^\vee) \in \mathrm{Lie}\, P_{v(x),A^\vee}$ corresponds to

$$(-\mathrm{Lie}\,\chi \circ j \circ \lambda_{G^\vee}(g^\vee), \lambda_{A^\vee}(a^\vee)) \in \mathrm{Lie}\,\mathbb{G}_m \times_K \mathrm{Lie}\, A^\vee,$$

where $a^\vee \in A^\vee$ is the image of $g^\vee \in G^\vee$ under the canonical projection. Therefore,

$$\begin{aligned}\psi \circ \tau(x, g^\vee) &= s_{u(x)} \circ \lambda_{P_{u(x),G^\vee}}(\tau(x, g^\vee))\\ &= \mathrm{Lie}\,\chi(j \circ \lambda_{G^\vee}(g^\vee)) + \xi(\lambda_{P_{v(x),A^\vee}} \circ \tau'_x(g^\vee))\\ &= \mathrm{Lie}\,\chi \circ j \circ \lambda_{G^\vee}(g^\vee) - \mathrm{Lie}\,\chi \circ j \circ \lambda_{G^\vee}(g^\vee)\\ &= 0.\end{aligned}$$

■

Corollary 4.7. *Let $\rho : K^* \to \mathbb{Q}_p$ be a ramified homomorphism and consider a pair $r : \mathrm{Lie}\,G(K) \to \mathrm{Lie}\,G^\natural(K)$ and $r^\vee : \mathrm{Lie}\,G^\vee(K) \to \mathrm{Lie}\,G^{\vee\natural}(K)$ of splittings of the exact sequences of Lie algebras*

$$0 \to \omega_{G^\vee}(K) \xrightarrow{\mathrm{Lie}\,\iota} \mathrm{Lie}\,G^\natural(K) \xrightarrow{\mathrm{Lie}\,\theta} \mathrm{Lie}\,G(K) \to 0,$$

$$0 \to \omega_{G}(K) \xrightarrow{\mathrm{Lie}\,\iota^\vee} \mathrm{Lie}\,G^{\vee\natural}(K) \xrightarrow{\mathrm{Lie}\,\theta^\vee} \mathrm{Lie}\,G^\vee(K) \to 0,$$

respectively, which are dual with respect to Deligne's pairing. Then:

(i) *There is a ρ-splitting $\psi : P(K) \to \mathbb{Q}_p$.*

(ii) *Let $\eta : G(K) \to G^\natural(K)$ and $\eta^\vee : G^\vee(K) \to G^{\vee\natural}(K)$ be the splittings of* (4.1) *and* (4.2)*, respectively, such that* $\mathrm{Lie}\,\eta = r$ *and* $\mathrm{Lie}\,\eta^\vee = r^\vee$*, as constructed in Lemma 4.3. If the diagrams*

$$\begin{array}{ccc} L(K) & = & L(K) \\ \downarrow u & & \downarrow u^\natural \\ G(K) & \xrightarrow{\eta} & G^\natural(K), \end{array} \qquad \begin{array}{ccc} L^\vee(K) & = & L^\vee(K) \\ \downarrow u^\vee & & \downarrow u^{\vee\natural} \\ G^\vee(K) & \xrightarrow{\eta^\vee} & G^{\vee\natural}(K) \end{array}$$

commute, then the ρ-splitting $\psi : P(K) \to \mathbb{Q}_p$ of (i) *is compatible with the $L \times_K L^\vee$-linearization of P. In particular, if $u(K)$ and $u^\vee(K)$ are injective, then ψ induces a ρ-splitting of the biextension $Q_M(K)$ of $(M(K), M^\vee(K))$ by K^*.*

Proof. (i) By [16, p. 319], there exist a branch $\lambda : K^* \to K$ of the p-adic logarithm and a $\mathbb{Q}_p$-linear map $\delta : K \to \mathbb{Q}_p$ such that $\rho = \delta \circ \lambda$. Let $\psi : P(K) \to K$ be the λ-splitting constructed as in Theorem 4.4 from the splittings $\eta, \eta^\vee$ of (4.1), (4.2), respectively, satisfying $\mathrm{Lie}\,\eta = r$ and $\mathrm{Lie}\,\eta^\vee = r^\vee$. Then $\psi_\rho := \delta \circ \psi : P(K) \to \mathbb{Q}_p$ is a ρ-splitting of $P(K)$.

(ii) If $\eta \circ u = u^\natural$ and $\eta^\vee \circ u^\vee = u^{\vee\natural}$, then $\psi_\rho \circ \tau = \delta \circ \psi \circ \tau$ is zero on K-sections, and similarly for $\psi_\rho \circ \tau^\vee$. Therefore, ψ_ρ is compatible with the $L \times_K L^\vee$-linearization of P and thus induces a ρ-splitting of $Q_M(K)$, in the case that $u(K)$ and $u^\vee(K)$ are injective.

■

5. Local pairing between zero-cycles

In this section we construct a pairing between disjoint zero-cycles of degree zero on a curve over a local field and its regular locus, which generalizes the local pairing defined in [12, p. 212] in the case of an elliptic curve (see also [7]).

Let K be a finite extension of $\mathbb{Q}_p$ and C a semi-normal irreducible curve over K. Consider the commutative diagram

$$\begin{tikzcd} C' \arrow[r, hook, "j'"] \arrow[d, two heads, "\pi"'] & \overline{C}' \arrow[d, two heads, "\overline{\pi}"] \\ C \arrow[r, hook, "j"] & \overline{C}, \end{tikzcd}$$

where C' is the normalization of C, $\overline{C}'$ is a smooth compactification of C', and $\overline{C}$ (resp. C) is the curve obtained from $\overline{C}'$ (resp. C') by contracting each of the finite sets $\pi^{-1}(x)$, for $x \in C$. Let S be the set of singular points of C, $S' := \pi^{-1}(S)$, and $F := \overline{C}' - C' = \overline{C} - C$. We recall from Section 2.2.3 the homological Picard 1-motive of C,

$$\mathrm{Pic}^-(C) = \big[u : \mathrm{Div}^0_{S'/S}(\overline{C}', F) \to \mathrm{Pic}^0(\overline{C}', F)\big],$$

and the cohomological Albanese 1-motive of C,

$$\mathrm{Alb}^+(C) = \mathrm{Pic}^-(C)^\vee = \big[u^\vee : \mathrm{Div}^0_F(\overline{C}) \to \mathrm{Pic}^0(\overline{C})\big].$$

Denote by $\overline{C}_{\mathrm{reg}}$ the set of smooth points of $\overline{C}$ and let $a_x^+ : \overline{C}_{\mathrm{reg}} \to \mathrm{Pic}^0(\overline{C})$ be the Albanese mapping, which depends on a base point $x \in \overline{C}_{\mathrm{reg}}$ which we assume to be rational over K (see [3, p. 50]). Extending by linearity, one obtains a mapping $a^+_{\overline{C}} : Z^0(\overline{C}_{\mathrm{reg}}/K) \to \mathrm{Pic}^0(\overline{C})$ on the group of zero-cycles of degree zero on $\overline{C}_{\mathrm{reg}}$ defined over K; notice that $a^+_{\overline{C}}$ does not depend on any base point. Finally, we denote by P the Poincaré biextension of $(\mathrm{Pic}^-(C), \mathrm{Alb}^+(C))$ by $\mathbb{G}_m$.

We consider a homomorphism $\rho : K^* \to \mathbb{Q}_p$ and a ρ-splitting $\psi : P(K) \to \mathbb{Q}_p$ which is compatible with the $\mathrm{Div}^0_{S'/S}(\overline{C}', F) \times_K \mathrm{Div}^0_F(\overline{C})$-linearization of P. Our aim is to construct a pairing

$$[\cdot, \cdot]_C : (Z^0(C/K) \times Z^0(C_{\mathrm{reg}}/K))' \to \mathbb{Q}_p,$$

where $(Z^0(C/K) \times Z^0(C_{\mathrm{reg}}/K))'$ denotes the subset of $Z^0(C/K) \times Z^0(C_{\mathrm{reg}}/K)$ consisting of pairs of zero-cycles of degree zero defined over K with disjoint support.

First, we define a pairing

$$[\cdot, \cdot]'_C : (\mathrm{Div}^0(\overline{C}', F) \times Z^0(\overline{C}_{\mathrm{reg}}/K))' \to \mathbb{Q}_p$$

on the set of all pairs (D, z), with D a divisor on $\overline{C}'$ algebraically equivalent to 0 whose support is contained in $\overline{C}' - F$, and z a zero-cycle of degree zero on $\overline{C}_{\text{reg}}$ defined over K, satisfying $\operatorname{supp} D \cap \operatorname{supp} z = \emptyset$. Notice that a divisor $D \in \operatorname{Div}^0(\overline{C}', F) \subset \operatorname{Div}^0(\overline{C}')$ corresponds to a line bundle $L(D)$ over $\overline{C}'$ together with a rational section $s_D : \overline{C}' \dashrightarrow L(D)$ which is defined on the open subset $\overline{C}' - \operatorname{supp} D \subset \overline{C}'$; in particular, s_D is defined on F since $\operatorname{supp} D \cap F = \emptyset$. Moreover, the pullback along a_x^+ of $P_{\mathcal{O}(D)}$, the fiber of the Poincaré bundle P over $\mathcal{O}(D) \in \operatorname{Pic}^0(\overline{C}', F)$, is the restriction of $L(D)$ to $\overline{C}_{\text{reg}}$, and so a_x^+ induces a map $a_{x,D}^+ : L(D)|_{\overline{C}_{\text{reg}}} \to P_{\mathcal{O}(D)}$ by pullback:

$$\begin{tikzcd} L(D)|_{\overline{C}_{\text{reg}}} \arrow[r, "a_{x,D}^+"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & P_{\mathcal{O}(D)} \arrow[d] \\ \overline{C}_{\text{reg}} \arrow[r, "a_x^+"] \arrow[u, bend left, dashed, "s_D|_{\overline{C}_{\text{reg}}}"] & \{\mathcal{O}(D)\} \times_K \operatorname{Pic}^0(\overline{C}). \end{tikzcd}$$

Therefore, we can define

$$[D, z]'_C := \sum n_j \psi \circ a_{x,D}^+ \circ s_D(x_j),$$

where $z = \sum n_j x_j \in Z^0(\overline{C}_{\text{reg}}/K)$. Notice that since z has degree zero, $[D, z]'_C$ does not depend on the base point x.

When $D \in \operatorname{Div}^0_{S'/S}(\overline{C}', F) \subset \operatorname{Div}^0(\overline{C}', F)$, we have that $a_{x,D}^+ \circ s_D = \tau \circ a_x^+$ on $\overline{C}_{\text{reg}}$:

$$\begin{tikzcd} L(D)|_{\overline{C}_{\text{reg}}} \arrow[r, "a_{x,D}^+"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & P_{u(D)} \arrow[d] \\ \overline{C}_{\text{reg}} \arrow[r, "a_x^+"] \arrow[u, bend left, dashed, "s_D|_{\overline{C}_{\text{reg}}}"] & \{D\} \times_K \operatorname{Pic}^0(\overline{C}). \arrow[u, bend right, dashed, "\tau"'] \end{tikzcd}$$

This implies that $[D, z]'_C = 0$, for all $D \in \operatorname{Div}^0_{S'/S}(\overline{C}', F)$. Notice that, since every closed point in C' is also closed in $\overline{C}'$, the subgroup of divisors in $\operatorname{Div}^0(\overline{C}', F)$ that are defined over K is $Z^0(C'/K)$. Moreover, since $\overline{C}'$ is irreducible, the subgroup of divisors in $\operatorname{Div}^0_{S'/S}(\overline{C}', F)$ that are defined over K is the free abelian subgroup generated by cycles of the form $x_0 - x_1$, where $\pi(x_0) = \pi(x_1)$; denote this group by $Z^0((S'/S)/K)$. Recalling that the pushforward of cycles along π preserves the degree, we obtain the following exact sequence:

$$0 \to Z^0((S'/S)/K) \to Z^0(C'/K) \xrightarrow{\pi_*} Z^0(C/K) \to 0.$$

Therefore, $[\cdot, \cdot]'$ is a pairing on $(Z^0(C'/K) \times Z^0(\overline{C}_{\text{reg}}/K))'$ which is zero when restricted to $(Z^0((S'/S)/K) \times Z^0(\overline{C}_{\text{reg}}/K))'$, yielding a pairing

$$[\cdot, \cdot]''_C : (Z^0(C/K) \times Z^0(\overline{C}_{\text{reg}}/K))' \to \mathbb{Q}_p.$$

By restricting to $Z^0(C_{\mathrm{reg}}/K) \subset Z^0(\bar{C}_{\mathrm{reg}}/K)$, we get the desired pairing

$$[\cdot, \cdot]_C : (Z^0(C/K) \times Z^0(C_{\mathrm{reg}}/K))' \to \mathbb{Q}_p.$$

We make the observation that $[D, z]'_C = 0$ whenever $z \in Z^0(F/K)$ (notice that $F = \bar{C}_{\mathrm{reg}} - C_{\mathrm{reg}}$). Indeed, since $\bar{C}'$ is irreducible, the subgroup of divisors in $\mathrm{Div}^0_F(\bar{C})$ defined over K is $Z^0(F/K)$, and so the restriction of $a^+_{\bar{C}}$ to $Z^0(F/K)$ equals $u^\vee$:

$$\begin{tikzcd}
Z^0(F/K) \arrow[r, hook] \arrow[d, hook] & \mathrm{Div}^0_F(\bar{C}) \arrow[d, "u^\vee"] \\
Z^0(\bar{C}_{\mathrm{reg}}/K) \arrow[r, "a^+_{\bar{C}}"] & \mathrm{Pic}^0(\bar{C}).
\end{tikzcd}$$

Moreover, we have that the trivialization $\tau^\vee$ is given by the formula

$$\tau^\vee\Big(\mathcal{O}(D), \sum n_j x_j\Big) = \sum n_j a^+_{x,D} \circ s_D(x_j),$$

for $D \in \mathrm{Div}^0(\bar{C}', F)$ and $\sum n_j x_j \in Z^0(F/K)$, which implies that

$$\Big[D, \sum n_j x_j\Big]'_C = \psi \circ \tau^\vee\Big(\mathcal{O}(D), \sum n_j x_j\Big) = 0.$$

6. Global pairing on rational points

In this section we define a global pairing between the rational points of a 1-motive over a number field and its dual. The construction, which is given in Proposition 6.3, generalizes the global pairing defined in [12, Lem. 3.1] in the case of abelian varieties (see also [16, p. 337]).

Let F be a number field endowed with a set of places $\mathcal{V}$. For each place v, let F_v denote the completion of F with respect to v. For v discrete, denote by $\mathcal{O}_{F_v}$ the ring of integers of F_v, and let π_v be a uniformizer of $\mathcal{O}_{F_v}$ such that $\pi_v \in F$. Let $M_F = [L_F \xrightarrow{u_F} G_F]$ be a 1-motive over F, where G_F is an extension of A_F by T_F. For each place v, denote by $M_{F_v} = [L_{F_v} \xrightarrow{u_{F_v}} G_{F_v}]$ its base change to F_v, so that G_{F_v} is an extension of A_{F_v} by T_{F_v}. Denote by P_F the Poincaré biextension of $(M_F, M_F^\vee)$ and by P_{F_v} its base change to F_v, which coincides with the Poincaré biextension of $(M_{F_v}, M_{F_v}^\vee)$. Finally, denote by

$$\tau_{F_v} : L_{F_v} \times_{F_v} G^\vee_{F_v} \to P_{F_v}, \quad \tau^\vee_{F_v} : G_{F_v} \times_{F_v} L^\vee_{F_v} \to P_{F_v}$$

the trivializations associated to the 1-motive M_{F_v} and its dual. Observe that M_{F_v} has good reduction over $\mathcal{O}_{F_v}$ for almost all discrete places v (see [2, Lem. 3.3]). When this is the case, there exists an $\mathcal{O}_{F_v}$-1-motive

$$M_{\mathcal{O}_{F_v}} = [L_{\mathcal{O}_{F_v}} \xrightarrow{u_{\mathcal{O}_{F_v}}} G_{\mathcal{O}_{F_v}}]$$

with $G_{\mathcal{O}_{F_v}}$ an extension of an abelian scheme $A_{\mathcal{O}_{F_v}}$ by a torus $T_{\mathcal{O}_{F_v}}$, whose generic fiber is M_{F_v}. Furthermore, the Poincaré biextension $P_{\mathcal{O}_{F_v}}$ of $(M_{\mathcal{O}_{F_v}}, M^\vee_{\mathcal{O}_{F_v}})$ has generic fiber equal to P_{F_v} and its trivializations

$$\tau_{\mathcal{O}_{F_v}} : L_{\mathcal{O}_{F_v}} \times_{\mathcal{O}_{F_v}} G^\vee_{\mathcal{O}_{F_v}} \to P_{\mathcal{O}_{F_v}}, \quad \tau^\vee_{\mathcal{O}_{F_v}} : G_{\mathcal{O}_{F_v}} \times_{\mathcal{O}_{F_v}} L^\vee_{\mathcal{O}_{F_v}} \to P_{\mathcal{O}_{F_v}}$$

extend τ_{F_v} and $\tau^\vee_{F_v}$, respectively.

Consider a family $\rho = (\rho_v)_{v\in\mathcal{V}}$ of homomorphisms $\rho_v : F_v^* \to \mathbb{Q}_p$ and, for every v, a ρ_v-splitting $\psi_v : P_{F_v}(F_v) \to \mathbb{Q}_p$ of $P_{F_v}(F_v)$ such that

(i) $\rho_v(\mathcal{O}^*_{F_v}) = 0$ for almost all discrete places v,

(ii) the "sum formula"

$$\sum_{v\in\mathcal{V}} \rho_v(c) = 0$$

holds for all $c \in F^*$, and

(iii) $\psi_v(P_{\mathcal{O}_{F_v}}(\mathcal{O}_{F_v})) = 0$ for almost all discrete places v for which M_{F_v} has good reduction.

Denote by $\mathcal{V}'$ the set of discrete places v satisfying condition (iii); then this condition is equivalent to $\mathcal{V} - \mathcal{V}'$ being a finite set. Notice that $\rho_v(\mathcal{O}^*_{F_v}) = 0$ for all $v \in \mathcal{V}'$. We have the following result:

Proposition 6.1. *There is a pairing*

$$\langle \cdot, \cdot \rangle : G_F(F) \times G^\vee_F(F) \to \mathbb{Q}_p$$

such that if $y \in P_F(F)$ lies above $(g, g^\vee) \in G_F(F) \times G^\vee_F(F)$, then

$$\langle g, g^\vee \rangle = \sum_{v\in\mathcal{V}} \psi_v(y). \tag{6.1}$$

Proof. First, we prove that the right-hand side of (6.1) is a finite sum. For this, we use the fact that the 1-motive M_F has good reduction over $\mathcal{O}_F[1/N]$, for N sufficiently divisible (see [2, Lem. 3.3]). This implies that M_F extends to a 1-motive $M_{\mathcal{O}_F[1/N]} = [L_{\mathcal{O}_F[1/N]} \to G_{\mathcal{O}_F[1/N]}]$ over $\mathcal{O}_F[1/N]$, and similarly for $M^\vee_F$. Moreover, the Poincaré biextension P_F extends as well to a biextension $P_{\mathcal{O}_F[1/N]}$ over $\mathcal{O}_F[1/N]$. We then obtain a tower of two biextensions as follows:

$$
\begin{tikzcd}
\mathcal{O}_F[1/N]^* \arrow[r, hook] \arrow[d, hook] & F^* \arrow[d, hook] \\
P_{\mathcal{O}_F[1/N]}(\mathcal{O}_F[1/N]) \arrow[r, hook] \arrow[d, two heads] & P_F(F) \arrow[d, two heads] \\
G_{\mathcal{O}_F[1/N]}(\mathcal{O}_F[1/N]) \times G^\vee_{\mathcal{O}_F[1/N]}(\mathcal{O}_F[1/N]) \arrow[r, hook] & G_F(F) \times G^\vee_F(F).
\end{tikzcd}
$$

Consider a pair of F-points $(g, g^\vee) \in G_F(F) \times G^\vee_F(F)$. We have that, for S sufficiently divisible, $(g, g^\vee)$ belongs to the image of

$$G_{\mathcal{O}_F[1/S]}(\mathcal{O}_F[1/S]) \times G^\vee_{\mathcal{O}_F[1/S]}(\mathcal{O}_F[1/S]) \hookrightarrow G_F(F) \times G^\vee_F(F).$$

So, up to multiplying N by a factor, we can assume that $(g, g^\vee)$ is in the image of

$$G_{\mathcal{O}_F[1/N]}(\mathcal{O}_F[1/N]) \times G^\vee_{\mathcal{O}_F[1/N]}(\mathcal{O}_F[1/N]) \hookrightarrow G_F(F) \times G^\vee_F(F)$$

(notice that now N also depends on the pair $(g, g^\vee)$). Let $y \in P_{\mathcal{O}_F[1/N]}(\mathcal{O}_F[1/N])$ be an element lying above $(g, g^\vee)$; observe that $y \in P_{\mathcal{O}_{F_v}}(\mathcal{O}_{F_v})$ for almost all v. From this we get that $\psi_v(y) = 0$ for almost all v, thus proving that

$$\sum_{v \in \mathcal{V}} \psi_v(y)$$

is a finite sum.

Observe that if $y \in P_F(F)$ lies above $(g, g^\vee)$, then any other element lying above $(g, g^\vee)$ is of the form $c + y$, for $c \in F^*$. From (ii) and the fact that each ψ_v is a ρ_v-splitting we obtain the equalities

$$\sum_{v \in \mathcal{V}} \psi_v(c + y) = \sum_{v \in \mathcal{V}} \rho_v(c) + \sum_{v \in \mathcal{V}} \psi_v(y) = \sum_{v \in \mathcal{V}} \psi_v(y),$$

which proves that the right-hand side of (6.1) indeed defines a map on $G_F(F) \times G^\vee_F(F)$. It remains to check that this map is bilinear. Let $y_1, y_2 \in P_F(F)$ mapping to $(g_1, g^\vee), (g_2, g^\vee) \in G_F(F) \times G^\vee_F(F)$, respectively. Since the ψ_v are ρ_v-splittings, we get that

$$
\begin{aligned}
\langle g_1 + g_2, g^\vee \rangle &= \sum_{v \in \mathcal{V}} \psi_v(y_1 +_2 y_2) \\
&= \sum_{v \in \mathcal{V}} \psi_v(y_1) + \sum_{v \in \mathcal{V}} \psi_v(y_2) \\
&= \langle g_1, g^\vee \rangle + \langle g_2, g^\vee \rangle.
\end{aligned}
$$

In a similar way we verify linearity in $G^\vee_F$. ∎

From now on we will assume that L_F and T_F are split. We assume, moreover, that any ψ_v factors through a ρ_v-splitting $\psi_{A,v}$ of $P_{A_{F_v}}(F_v)$:

$$\psi_v : P_{F_v}(F_v) \to P_{A_{F_v}}(F_v) \xrightarrow{\psi_{A,v}} \mathbb{Q}_p.$$

Lemma 6.2. *For every $x^\vee \in L_F^\vee(F)$ and $g \in G_F(F)$ there exists $t \in T_F(F)$ such that*

$$\sum_{v\in\mathcal{V}} \psi_v \circ \tau_{F_v}^\vee(g, x^\vee) = \sum_{v\in\mathcal{V}-\mathcal{V}'} \psi_v \circ \tau_{F_v}^\vee(t^{-1}g, x^\vee),$$

and similarly for every $x \in L_F(F)$ and $g^\vee \in G_F^\vee(F)$.

Proof. Fix $x^\vee \in L_F^\vee(F)$ and $g \in G_F(F)$. Suppose that $L_F^\vee \cong \mathbb{Z}_F^r$ and let $(m_1, \ldots, m_r) \in \mathbb{Z}_F^r$ be the element corresponding to $x^\vee$. Notice that this induces an isomorphism $T_F \cong \mathbb{G}_{m,F}^r$. Consider a discrete place v in $\mathcal{V}'$. Since G_{F_v} has good reduction, we have $A_{F_v}(F_v) = A_{\mathcal{O}_{F_v}}(\mathcal{O}_{F_v})$, which induces isomorphisms

$$\frac{G_{F_v}(F_v)}{G_{\mathcal{O}_{F_v}}(\mathcal{O}_{F_v})} \cong \frac{T_{F_v}(F_v)}{T_{\mathcal{O}_{F_v}}(\mathcal{O}_{F_v})} \cong \mathbb{Z}^r. \tag{6.2}$$

Moreover, since M_{F_v} has good reduction, the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[rr, hook] & & \mathbb{Q}_p \\
P_{\mathcal{O}_{F_v}}(\mathcal{O}_{F_v}) \arrow[u, "\psi_v|_{P_{\mathcal{O}_{F_v}}}"] \arrow[rr, hook] \arrow[d] & & P_{F_v}(F_v) \arrow[u, "\psi_v"'] \arrow[d] \\
G_{\mathcal{O}_{F_v}}(\mathcal{O}_{F_v}) \times G^\vee_{\mathcal{O}_{F_v}}(\mathcal{O}_{F_v}) \arrow[rr, hook] & & G_{F_v}(F_v) \times G^\vee_{F_v}(F_v) \\
G_{\mathcal{O}_{F_v}}(\mathcal{O}_{F_v}) \times L^\vee_{\mathcal{O}_{F_v}}(\mathcal{O}_{F_v}) \arrow[u, "\mathrm{Id}\times u^\vee_{\mathcal{O}_{F_v}}"'] \arrow[rr, hook] \arrow[uu, bend left=60, "\tau^\vee_{\mathcal{O}_{F_v}}"] & & G_{F_v}(F_v) \times L^\vee_{F_v}(F_v). \arrow[u, "\mathrm{Id}\times u^\vee_{F_v}"] \arrow[uu, bend right=60, "\tau^\vee_{F_v}"']
\end{tikzcd}$$

This implies that the map

$$\psi_v \circ \tau_{F_v}^\vee(\,\cdot\,, x^\vee) : G_{F_v}(F_v) \to \mathbb{Q}_p$$

factors through the quotient $G_{F_v}(F_v)/G_{\mathcal{O}_{F_v}}(\mathcal{O}_{F_v})$. Thus, any $t_v \in T_{F_v}(F_v)$ whose class in $T_{F_v}(F_v)/T_{\mathcal{O}_{F_v}}(\mathcal{O}_{F_v})$ equals that of g satisfies

$$\psi_v \circ \tau_{F_v}^\vee(g, x^\vee) = \psi_v \circ \tau_{F_v}^\vee(t_v, x^\vee), \tag{6.3}$$

where we identify t_v with the corresponding point in $G_{F_v}(F_v)$. If the class of g corresponds to $(n_1, \ldots, n_r) \in \mathbb{Z}^r$ under the isomorphism (6.2), we may choose t_v of the form $t_v := (\pi_v^{n_1}, \dots, \pi_v^{n_r})$; in this way, t_v belongs to $T_F(F)$ and

$$\psi_w \circ \tau_{F_w}^\vee(t_v, x^\vee) = 0, \tag{6.4}$$

for all $w \in \mathcal{V}'$ such that $w \neq v$. To prove this last assertion, start by considering any place $w \in \mathcal{V}$. We have the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & & \mathbb{G}_{m,F_w} \arrow[r, equal] \arrow[d] & \mathbb{G}_{m,F_w} \arrow[d] & \\
0 \arrow[r] & T_{F_w} \arrow[r, "i"] \arrow[d, "\cong"] & P_{G_{F_w},\{x^\vee\}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P_{A_{F_w},a^\vee} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_{F_w} \times_{F_w} \{x^\vee\} \arrow[r] & G_{F_w} \times_{F_w} \{x^\vee\} \arrow[r] \arrow[u, dashed, bend left, "\tau_{F_w}^\vee"] & A_{F_w} \times_{F_w} \{a^\vee\} \arrow[r] & 0,
\end{tikzcd}$$

where $a^\vee \in A_{F_w}^\vee(F_w)$ denotes the image of $x^\vee$ under the composition

$$L_{F_w}^\vee \xrightarrow{u_{F_w}} G_{F_w}^\vee \to A_{F_w}^\vee.$$

The map i is the one that when composed with $P_{G_{F_w},\{x^\vee\}} \to G_{F_w} \times_{F_w} \{x^\vee\}$ equals the natural injection and when composed with $P_{G_{F_w},\{x^\vee\}} \to P_{A_{F_w},a^\vee}$ equals zero. Let $\chi : T_F \to \mathbb{G}_{m,F}$ be the map corresponding to $x^\vee \in L_F^\vee$. With this notation we have

$$\tau_{F_w}^\vee(t, x^\vee) = \chi(t) + i(t),$$

for all $t \in T_{F_w}$. In particular, for $w \neq v$ in $\mathcal{V}'$ and $t = t_v$ we get

$$\begin{aligned}
\psi_w \circ \tau_{F_w}^\vee(t_v, x^\vee) &= \psi_w(\chi(t_v) + i(t_v)) \\
&= \rho_w(\chi(t_v)) \\
&= \rho_w(\pi_v^{\sum n_i m_i}) \\
&= (n_1 m_1 + \cdots + n_r m_r)\rho_w(\pi_v) \\
&= 0,
\end{aligned}$$

where the second equality is deduced from $\psi_w(i(t_v)) = 0$ (since ψ_w is obtained from a ρ_w-splitting of $P_{A_{F_w}}$), and the last one from the fact that $\pi_v \in \mathcal{O}_{F_w}^*$. Define

$$t := \prod_{v \in \mathcal{V}'} t_v \in T_F(F).$$

Notice that this is a finite product, since $g \in G_{\mathcal{O}_{F_v}}(\mathcal{O}_{F_v})$ for almost all $v \in \mathcal{V}'$. From (6.3) and (6.4), we get that t satisfies

$$\psi_v \circ \tau_{F_v}^\vee(t, x^\vee) = \psi_v \circ \tau_{F_v}^\vee(g, x^\vee),$$

for every $v \in \mathcal{V}'$. Therefore, we obtain

$$\begin{aligned}\sum_{v\in\mathcal{V}} \psi_v \circ \tau_{F_v}^\vee(g, x^\vee) &= \sum_{v\in\mathcal{V}-\mathcal{V}'} \psi_v \circ \tau_{F_v}^\vee(g, x^\vee) + \sum_{v\in\mathcal{V}'} \psi_v \circ \tau_{F_v}^\vee(g, x^\vee)\\ &= \sum_{v\in\mathcal{V}-\mathcal{V}'} \psi_v \circ \tau_{F_v}^\vee(g, x^\vee) + \sum_{v\in\mathcal{V}'} \psi_v \circ \tau_{F_v}^\vee(t, x^\vee)\\ &= \sum_{v\in\mathcal{V}-\mathcal{V}'} \psi_v \circ \tau_{F_v}^\vee(g, x^\vee) - \sum_{v\in\mathcal{V}-\mathcal{V}'} \psi_v \circ \tau_{F_v}^\vee(t, x^\vee)\\ &= \sum_{v\in\mathcal{V}-\mathcal{V}'} \psi_v \circ \tau_{F_v}^\vee(t^{-1}g, x^\vee),\end{aligned}$$

where the third equality is derived from

$$\sum_{v\in\mathcal{V}} \psi_v \circ \tau_{F_v}^\vee(t, x^\vee) = \sum_{v\in\mathcal{V}} \rho_v(\chi(t)) = 0. \qquad \blacksquare$$

Proposition 6.3. *Suppose that $u_F(K)$ and $u_F^\vee(K)$ are injective, and that the ρ_v-splittings ψ_v are compatible with the $L_{F_v} \times_{F_v} L_{F_v}^\vee$-linearization of P_{F_v}, for every place $v \in \mathcal{V} - \mathcal{V}'$. Then the pairing $\langle\cdot,\cdot\rangle$ of Proposition 6.1 descends to a pairing*

$$\langle\cdot,\cdot\rangle_M : M_F(F) \times M_F^\vee(F) \to \mathbb{Q}_p.$$

Proof. Fix $g \in G_F(F)$ and $x^\vee \in L_F^\vee(F)$, and let $t \in T_F(F)$ be the element constructed in Lemma 6.2. We have

$$\sum_{v\in\mathcal{V}} \psi_v \circ \tau_{F_v}^\vee(g, x^\vee) = \sum_{v\in\mathcal{V}-\mathcal{V}'} \psi_v \circ \tau_{F_v}^\vee(t^{-1}g, x^\vee) = 0.$$

Since we have the analogous equality for every $x \in L_F(F)$ and $g^\vee \in G_F^\vee(F)$, the pairing $\langle\cdot,\cdot\rangle$ is zero on $G(F) \times \mathrm{Im}(u^\vee(F))$ and $\mathrm{Im}(u(F)) \times G^\vee(F)$, inducing a pairing

$$\langle\cdot,\cdot\rangle_M : M_F(F) \times M_F^\vee(F) \to \mathbb{Q}_p. \qquad \blacksquare$$

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