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G-equivariance of formal models of flag varieties

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Abstract – Let $\mathbb{G}$ be a split connected reductive group scheme over the ring of integers $\mathfrak{o}$ of a finite extension $L|\mathbb{Q}_p$ and $\lambda \in X(\mathbb{T})$ an algebraic character of a split maximal torus $\mathbb{T} \subseteq \mathbb{G}$. Let us also consider the rigid analytic flag variety X^{rig} of $\mathbb{G}$ and $G = \mathbb{G}(L)$. In the first part of this paper, we introduce a family of λ-twisted differential operators on a formal model $\mathcal{Y}$ of X^{rig}. We compute their global sections and we prove coherence together with several cohomological properties. In the second part, we define the category of coadmissible G-equivariant arithmetic $\mathcal{D}(\lambda)$-modules over the family of formal models of the rigid flag variety X^{rig}. We show that if λ is such that $\lambda + \rho$ is dominant and regular (ρ being the Weyl character), then the preceding category is anti-equivalent to the category of admissible locally analytic G-representations, with central character λ. In particular, we generalize the main results from a paper by Huyghe, Patel, Schmidt and Strauch (2019) for algebraic characters.

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1. Introduction

Let $L|\mathbb{Q}_p$ be a finite extension and $\mathfrak{o}$ its ring of integers. Throughout this work, $\mathbb{G}$ will denote a split connected reductive group scheme over $\mathfrak{o}$. We will fix a Borel subgroup $\mathbb{B} \subset \mathbb{G}$ which contains a split maximal torus $\mathbb{T} \subset \mathbb{B}$ of $\mathbb{G}$. We will also denote by $X = \mathbb{G}/\mathbb{B}$ the smooth flag $\mathfrak{o}$-scheme associated to $\mathbb{G}$ and by $\mathfrak{X}$ the smooth formal scheme. In [20] the authors have introduced certain sheaves of differential operators (with congruence level $k \in \mathbb{N}$) $\mathcal{D}^{\dagger}_{\mathcal{Y},k}$ on a family of formal models $\mathcal{Y}$ of X^{rig},

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the rigid analytic flag variety. They study their cohomological properties and show that $\mathcal{Y}$ is $\mathcal{D}^{\dagger}_{\mathcal{Y},k}$-affine. Moreover, it is proved in [20, Theorem 5.3.12] that the category of admissible locally analytic representations with trivial infinitesimal character of the L-analytic group $\mathbb{G}(L)$ can be described in terms of $\mathbb{G}(L)$-equivariant families $(\mathcal{M}_{\mathcal{Y},k})$ of modules over $\mathcal{D}^{\dagger}_{\mathcal{Y},k}$ on the projective system of all formal models $\mathcal{Y}$ of X^{rig}.

Our motivation is to study the preceding equivalence (the localization theorem) in the twisted situation. In this work we will treat the algebraic case, that is, we will only consider characters of the Lie algebra $\mathfrak{t} = \mathrm{Lie}(\mathbb{T})$ arriving from characters $\lambda \in X(\mathbb{T}) = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ via differentiation. In this situation λ induces an invertible sheaf $\mathcal{L}(\lambda)$ on $\mathcal{X}$ and we define $\mathcal{D}^{\dagger}_{\mathcal{X},k}(\lambda)$ as the sheaf of differential operators (with congruence level k) acting on $\mathcal{L}(\lambda)$. We will follow the philosophy described in [20] introducing sheaves of differential operators on each admissible blow-up of $\mathcal{X}$. Let $\mathrm{pr} : \mathcal{Y} \to \mathcal{X}$ be an admissible blow-up, then for $k \gg 0$

$$\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda) = \mathrm{pr}^{*}\mathcal{D}^{\dagger}_{\mathcal{X},k}(\lambda) = \mathcal{O}_{\mathcal{Y}} \otimes_{\mathrm{pr}^{-1}\mathcal{O}_{\mathcal{X}}} \mathrm{pr}^{-1}\mathcal{D}^{\dagger}_{\mathcal{X},k}(\lambda)$$

is a sheaf of rings[1] on $\mathcal{Y}$. Let us denote by ρ the so-called Weyl character and let us assume that $\lambda + \rho \in \mathfrak{t}^{*}_{\mathbb{Q}} = \mathrm{Hom}_L(\mathfrak{t} \otimes_{\mathfrak{o}} L, L)$ is a dominant and regular character of $\mathfrak{t}_{\mathbb{Q}} = \mathfrak{t} \otimes_{\mathfrak{o}} L$. In this situation, we will show that the direct image functor pr_* induces an equivalence of categories between the category of coherent $\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda)$-modules and the category of coherent $\mathcal{D}^{\dagger}_{\mathcal{X},k}(\lambda)$-modules. In addition, we have $\mathrm{pr}_*\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda) = \mathcal{D}^{\dagger}_{\mathcal{X},k}(\lambda)$, which implies that

$$H^0(\mathcal{Y}, \mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda)) = H^0(\mathcal{X}, \mathcal{D}^{\dagger}_{\mathcal{X},k}(\lambda)).$$

It is shown in [21] that $H^0(\mathcal{X}, \mathcal{D}^{\dagger}_{\mathcal{X},k}(\lambda))$ can be identified with the central redaction[2] $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ})_{\lambda}$ of Emerton's analytic distribution algebra $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ})$ of the wide open rigid-analytic k-th congruence group $\mathbb{G}(k)^{\circ}$. Our first result is as follows.

Theorem 1. *Let* $\mathrm{pr} : \mathcal{Y} \to \mathcal{X}$ *be an admissible blow-up. Suppose* $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ *is an algebraic character such that* $\lambda + \rho \in \mathfrak{t}^{*}_{\mathbb{Q}}$ *is a dominant and regular character of* $\mathfrak{t}_{\mathbb{Q}}$. *Then* $H^0(\mathcal{Y}, \bullet)$ *induces an equivalence between the categories of coherent* $\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda)$*-modules and finitely presented* $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ})_{\lambda}$*-modules.*

As in the classical case, the inverse functor is determined by the localization functor

$$\mathcal{L}oc^{\dagger}_{\mathcal{Y},k}(\lambda)(\bullet) = \mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda) \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ})_{\lambda}} (\bullet).$$

(1) The technical condition $k \gg 0$ is clarified in Proposition 4.2. It is also explained in (1.1) below.

(2) Via the classical Harish-Chandra isomorphism, the character λ induces a central character $\chi_{\lambda} : Z(\mathrm{Lie}(\mathbb{G}) \otimes_{\mathfrak{o}} L) \to L$ which allows us to consider the central redaction.

Let us now describe the most important tools in our localization theorem. On the algebraic side, we will first assume that $G_0 = \mathbb{G}(\mathfrak{o})$ and that $D(G_0, L)$ is the algebra of locally analytic distributions of the compact analytic group G_0 (in the sense of [34]). The key point will be to construct the structure of a weak Fréchet–Stein algebra on $D(G_0, L)$ (in the sense of [14, Definition 1.2.6]) that will allow us to localize the coadmissible $D(G_0, L)$-modules (relative to this weak Fréchet–Stein structure). In fact, if $\mathcal{C}^{\mathrm{cont}}(G_0, L)_{\mathbb{G}(k)^\circ\text{-an}}$ is the vector space of locally analytic vectors of the space of continuous L-valued functions and $D(\mathbb{G}(k)^\circ, G_0) = (\mathcal{C}^{\mathrm{cont}}(G_0, L)_{\mathbb{G}(k)^\circ\text{-an}})'_b$ is its strong dual, then we have an isomorphism

$$D(G_0, L) \xrightarrow{\simeq} \varprojlim_{k\in\mathbb{N}} D(\mathbb{G}(k)^\circ, G_0),$$

which defines the structure of a weak Fréchet–Stein algebra and such that

$$D(\mathbb{G}(k)^\circ, G_0) = \bigoplus_{g\in G_0/G_k} \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)\delta_g.$$

Here $G_k = \mathbb{G}(k)(\mathfrak{o})$ is a normal subgroup of G_0, the direct sum runs through a set of representatives of the cosets of G_k in G_0 and δ_g is the Dirac distribution supported in g. We will denote by $\mathcal{C}_{G_0,\lambda}$ the category of coadmissible $D(G_0, L)$-modules with central character λ (coadmissible $D(G_0, L)_\lambda$-modules, where $D(G_0, L)_\lambda$ denotes the central reduction).

Now, on the geometric side, we will consider a G_0-equivariant admissible blow-up $\mathrm{pr} : \mathcal{Y} \to \mathcal{X}$ such that the invertible sheaf $\mathcal{L}(\lambda)$ is equipped with a G_0-action that allows us to define a left G_0-action $T_g : \mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda) \to (\rho_g)_*\mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda)$ on $\mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda)$ (here $g \in G_0$ and $\rho_g : \mathcal{Y} \to \mathcal{Y}$ is the morphism induced by G_0-equivariance), in the sense that for every $g, h \in G_0$ we have the cocycle condition $T_{hg} = (\rho_g)_* T_h \circ T_g$. So, we will say that a coherent $\mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda)$-module $\mathcal{M}$ is *strongly G_0-equivariant* if there is a family $(\varphi_g)_{g\in G_0}$ of isomorphisms $\varphi_g : \mathcal{M} \to (\rho_g)_*\mathcal{M}$ of sheaves of L-vector spaces, which satisfies the following properties (conditions (†)):

- For every $g, h \in G_0$ we have $(\rho_g)_*\varphi_h \circ \varphi_g = \varphi_{hg}$.
- If $\mathcal{U} \subseteq \mathcal{Y}$ is an open subset, $P \in \mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda)(\mathcal{U})$ and $m \in \mathcal{M}(\mathcal{U})$, then

$$\varphi_g(P \bullet m) = T_g(P) \bullet \varphi_g(m).$$

- For any $g \in G_{k+1}$ the application $\varphi_g : \mathcal{M} \to (\rho_g)_*\mathcal{M}$ is equal to the multiplication by $\delta_g \in \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda$.[3]

(3) We identify here $H^0(\mathcal{Y}, \mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda))$ with $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda$ and we use Lemma 4.12 to give a sense to this condition.

A morphism between two strongly G_0-equivariant modules $(\mathcal{M},(\varphi_g^{\mathcal{M}})_{g\in G_0})$ and $(\mathcal{N},(\varphi_g^{\mathcal{N}})_{g\in G_0})$ is a $\mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda)$-linear map $\psi:\mathcal{M}\to\mathcal{N}$ such that $\varphi_g^{\mathcal{N}}\circ\psi=(\rho_g)_*\psi\circ\varphi_g^{\mathcal{M}}$ for every $g\in G_0$. We denote by $\mathrm{Coh}(\mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda),G_0)$ the category of strongly G_0-equivariant $\mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda)$-modules. We have the following result.

Theorem 2. *Let $\lambda\in\mathrm{Hom}(\mathbb{T},\mathbb{G}_m)$ be an algebraic character such that $\lambda+\rho\in\mathfrak{t}^*_{\mathbb{Q}}$ is a dominant and regular character of $\mathfrak{t}_{\mathbb{Q}}$. The functors $\mathcal{L}oc^\dagger_{\mathcal{Y},k}(\lambda)$ and $H^0(\mathcal{Y},\bullet)$ induce equivalences between the categories of finitely presented $D(\mathbb{G}(k)^\circ,G_0)$-modules (with central character λ) and $\mathrm{Coh}(\mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda),G_0)$.*

Still on the geometric side, let us consider the set $\underline{\mathcal{F}}_{\mathcal{X}}$ of couples $(\mathcal{Y},k)$ such that $\mathcal{Y}$ is an admissible blow-up of $\mathcal{X}$ and $k\geq k_{\mathcal{Y}}$, where

$$k_{\mathcal{Y}}=\min_{\mathcal{I}}\min\left\{k\in\mathbb{N}\mid\varpi^k\in\mathcal{I}\right\} \tag{1.1}$$

and $\mathcal{I}$ is an ideal subsheaf of $\mathcal{O}_{\mathcal{X}}$, such that $\mathcal{Y}$ is isomorphic to the blow-up along $V(\mathcal{I})$. This set carries a partial order. As is shown in [20] the group G_0 acts on $\underline{\mathcal{F}}_{\mathcal{X}}$ and this action respects the congruence level. This means that for every couple $(\mathcal{Y},k)\in\underline{\mathcal{F}}_{\mathcal{X}}$ there is a couple $(\mathcal{Y}.g,k_{\mathcal{Y}.g})\in\underline{\mathcal{F}}_{\mathcal{X}}$ with an isomorphism $\rho_g:\mathcal{Y}\to\mathcal{Y}.g$ and such that $k_{\mathcal{Y}}=k_{\mathcal{Y}.g}$. We will say that a family $\mathcal{M}=(\mathcal{M}_{\mathcal{Y},k})_{(\mathcal{Y},k)\in\underline{\mathcal{F}}_{\mathcal{X}}}$ of coherent $\mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda)$-modules is a coadmissible G_0-equivariant $\mathcal{D}(\lambda)$-module on $\underline{\mathcal{F}}_{\mathcal{X}}$ if for any $g\in G_0$, with morphism $\rho_g:\mathcal{Y}\to\mathcal{Y}.g$, there is an isomorphism

$$\varphi:\mathcal{M}_{\mathcal{Y}.g,k}\to(\rho_g)_*\mathcal{M}_{\mathcal{Y},k}$$

that satisfies the conditions (†) and such that, if $(\mathcal{Y}',k')\succeq(\mathcal{Y},k)$ with $\pi:\mathcal{Y}'\to\mathcal{Y}$, then there is a transition morphism $\pi_*\mathcal{M}_{\mathcal{Y}',k'}\to\mathcal{M}_{\mathcal{Y},k}$ which satisfies obvious transitivity conditions. Moreover, a morphism $\mathcal{M}\to\mathcal{N}$ between two such modules is a morphism $\mathcal{M}_{\mathcal{Y},k}\to\mathcal{N}_{\mathcal{Y},k}$ of $\mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda)$-modules which is compatible with the additional structures. We will denote this category by $\mathcal{C}^{G_0}_{\mathcal{X},\lambda}$. For every $\mathcal{M}\in\mathcal{C}^{G_0}_{\mathcal{X},\lambda}$, we will consider the projective limit

$$\Gamma(\mathcal{M})=\varprojlim_{(\mathcal{Y},k)\in\underline{\mathcal{F}}_{\mathcal{X}}}H^0(\mathcal{Y},\mathcal{M}_{\mathcal{Y},k})$$

in the sense of abelian groups.

Now, let M be a coadmissible $D(G_0,L)_\lambda$-module and $V=M'_b$ its associated locally analytic representation. The vector space of $\mathbb{G}(k)^\circ$-analytic vectors $V_{\mathbb{G}(k)^\circ\text{-an}}\subseteq V$ is stable under the action of G_0 and its dual $M_k=(V_{\mathbb{G}(k)^\circ\text{-an}})'_b$ is a finitely presented $D(\mathbb{G}(k)^\circ,G_0)$-module. In this situation, Theorem 2 produces a coherent $\mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda)$-module

$$\mathcal{L}oc^\dagger_{\mathcal{Y},k}(\lambda)(M_k)=\mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda)\otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda}M_k$$

for each element $(\mathcal{Y}, k) \in \underline{\mathcal{F}}_{\mathcal{X}}$. We will denote this family by

$$\mathscr{L}oc_{\lambda}^{G_0}(M) = (\mathscr{L}oc^{\dagger}_{\mathcal{Y},k}(\lambda)(M_k))_{(\mathcal{Y},k)\in\underline{\mathcal{F}}_{\mathcal{X}}}.$$

THEOREM 3. *Let $\lambda \in \operatorname{Hom}(\mathbb{T}, \mathbb{G}_m)$ be an algebraic character such that $\lambda + \rho \in \mathfrak{t}^*_{\mathbb{Q}}$ is a dominant and regular character of $\mathfrak{t}_{\mathbb{Q}}$. The functors $\mathscr{L}oc_{\lambda}^{G_0}(\bullet)$ and $\Gamma(\bullet)$ induce equivalences of categories between the category $\mathcal{C}_{G_0,\lambda}$ (of coadmissible $D(G_0, L)_{\lambda}$-modules) and the category $\mathcal{C}^{G_0}_{\mathcal{X},\lambda}$.*

Finally, the last part of this work is devoted to the study of coadmissible[4] $D(G, L)_{\lambda}$-modules, where $G := \mathbb{G}(L)$. To do this, we will consider the Bruhat–Tits building $\mathcal{B}$ of G (see [9, 10]). It is a simplicial complex equipped with a G-action. For any special vertex $v \in \mathcal{B}$, the theory of Bruhat and Tits associates a reductive group $\mathbb{G}_v$ whose generic fiber is canonically isomorphic to $\mathbb{G} \times_{\operatorname{Spec}(\mathfrak{o})} \operatorname{Spec}(L)$. Let X_v be the flag scheme of $\mathbb{G}_v$, and $\mathcal{X}_v$ its formal completion along its special fiber. We consider the set $\underline{\mathcal{F}}$ composed of triples $(\mathcal{Y}_v, k, v)$ such that v is a special vertex, $\mathcal{Y}_v \to \mathcal{X}_v$ is an admissible blow-up of $\mathcal{X}_v$ and $k \geq k_{\mathcal{Y}_v}$. According to Definition 7.2, $\underline{\mathcal{F}}$ is partially ordered. In addition, for each special vertex $v \in \mathcal{B}$, each element $g \in G$ induces an isomorphism $\rho^v_g : \mathcal{X}_v \to \mathcal{X}_{vg}$, such that if $(\rho^v_g)^{\natural} : \mathcal{O}_{\mathcal{X}_{vg}} \to (\rho^v_g)_*\mathcal{O}_{\mathcal{X}_v}$ is the comorphism map and $\pi : \mathcal{Y}_v \to \mathcal{X}_v$ is an admissible blow-up along $V(\mathcal{I})$, then the (admissible) blow-up along $V((\rho^v_g)^{-1}(\rho^v_g)_*\mathcal{I})$ produces a scheme $\mathcal{Y}_{vg}$ with an isomorphism $\rho^v_g : \mathcal{Y}_v \to \mathcal{Y}_{vg}$, such that $k_{\mathcal{Y}_v} = k_{\mathcal{Y}_{vg}}$ and for every $g, h \in G$ we have $\rho^{vg}_h \circ \rho^v_g = \rho^v_{gh}$.

A coadmissible G-equivariant arithmetic $\mathcal{D}(\lambda)$-module on $\underline{\mathcal{F}}$ consists of a family $(\mathcal{M}_{(\mathcal{Y}_v,k,v)})_{(\mathcal{Y}_v,k,v)\in\underline{\mathcal{F}}}$ of coherent $\mathcal{D}^{\dagger}_{\mathcal{Y}_v,k}(\lambda)$-modules satisfying the conditions (†) plus some compatibility properties (see Definition 7.4) that allow us to form the projective limit

$$\Gamma(\mathcal{M}) = \varprojlim_{(\mathcal{Y}_v,k,v)\in\underline{\mathcal{F}}} H^0(\mathcal{Y}_v, \mathcal{M}_{(\mathcal{Y}_v,k,v)}),$$

which, as we will show, has the structure of a coadmissible $D(G, L)_{\lambda}$-module. On the other hand, given a coadmissible $D(G, L)_{\lambda}$-module M, we consider its continuous dual $V = M'_b$, which is a locally analytic representation of G. Then let $M_{v,k}$ be the dual space of the subspace $V_{\mathbb{G}_v(k)^{\circ}\text{-an}} \subseteq V$ of $\mathbb{G}_v(k)^{\circ}$-analytic vectors. For every $(\mathcal{Y}_v, k, v) \in \underline{\mathcal{F}}$, we have a coherent $\mathcal{D}^{\dagger}_{\mathcal{Y}_v,k}(\lambda)$-module

$$\mathscr{L}oc^{\dagger}_{\mathcal{Y}_v,k}(\lambda)(M_{v,k}) = \mathcal{D}^{\dagger}_{\mathcal{Y}_v,k}(\lambda) \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}_v(k)^{\circ})_{\lambda}} M_{v,k}.$$

We denote this family by $\mathscr{L}oc^{G}_{\lambda}(M)$. We will show the following result (Theorem 7.6).

(4) Here G_0 is a (maximal) compact subgroup of G. This compactness property allows us to define the structure of a weak Fréchet–Stein algebra.

Theorem 4. *Let $\lambda \in \operatorname{Hom}(\mathbb{T}, \mathbb{G}_m)$ be an algebraic character such that $\lambda + \rho \in \mathfrak{t}^*_{\mathbb{Q}}$ is a dominant and regular character of $\mathfrak{t}_{\mathbb{Q}}$. The functors $\mathcal{L}oc^G_\lambda(\bullet)$ and $\Gamma(\bullet)$ give an equivalence between the categories of coadmissible $D(G,L)_\lambda$-modules and coadmissible G-equivariant arithmetic $\mathcal{D}(\lambda)$-modules.*

The last task will be to study the projective limit

$$X_\infty = \varprojlim_{(\mathcal{Y}_v,k,v)} \mathcal{Y}_v.$$

This is the Zariski–Riemann space associated to the rigid flag variety X^{rig}. We can also form the projective limit $\mathcal{D}(\lambda)$ of the sheaves $\mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda)$ which is a sheaf of G-equivariant differential operators on $\mathcal{X}_\infty$. Similarly, if $(\mathcal{M}_{(\mathcal{Y}_v,k,v)})_{(\mathcal{Y}_v,k,v)\in\underline{\mathcal{F}}}$ is a coadmissible G-equivariant arithmetic $\mathcal{D}(\lambda)$-module, then we can form the projective limit $\mathcal{M}_\infty$. The data $\mathcal{M}_{(\mathcal{Y}_v,k,v)\in\underline{\mathcal{F}}} \rightsquigarrow \mathcal{M}_\infty$ induces a faithful functor from the category of coadmissible G-equivariant arithmetic $\mathcal{D}(\lambda)$-modules on $\underline{\mathcal{F}}$ to the category of G-equivariant $\mathcal{D}(\lambda)$-modules on X_∞ (Theorem 7.8). In fact, this is a fully faithful functor as we will briefly explain in Remark 7.9.

We summarize the main results of this work with the following commutative diagram of functors (cf. [30, Theorem 5.4.10]):

$$\begin{array}{ccc}
\left\{\begin{matrix}\text{Coadmissible}\\ D(G,L)_\lambda\text{-modules}\end{matrix}\right\} & \xrightarrow[\simeq]{\mathcal{L}oc^G_\lambda} & \left\{\begin{matrix}\text{Coadmissible } G\text{-equivariant}\\ \text{arithmetic } \mathcal{D}(\lambda)\text{-modules}\end{matrix}\right\} \\
\downarrow & & \downarrow \\
\left\{\begin{matrix}\text{Coadmissible}\\ D(G_0,L)_\lambda\text{-modules}\end{matrix}\right\} & \xrightarrow[\simeq]{\mathcal{L}oc^{G_0}_\lambda} & \left\{\begin{matrix}\text{Coadmissible } G_0\text{-equivariant}\\ \text{arithmetic } \mathcal{D}(\lambda)\text{-modules}\end{matrix}\right\}
\end{array}$$

Here the left-hand vertical arrow is the restriction functor coming from the homomorphism $D(G_0,L)_\lambda \to D(G,L)_\lambda$ and the right-hand vertical arrow is the forgetful functor.

Notation 1.1. Throughout this work, ϖ will denote a uniformizer of $\mathfrak{o}$. Furthermore, if Y is an arbitrary noetherian scheme over $\mathfrak{o}$, then for every $j \in \mathbb{N}$ we will denote by $Y_j := Y \times_{\operatorname{Spec}(\mathfrak{o})} \operatorname{Spec}(\mathfrak{o}/\varpi^{j+1})$ the reduction modulo ϖ^{j+1}, and by

$$\mathcal{Y} = \varinjlim_j Y_j$$

the formal completion of Y along the special fiber. Moreover, if $\mathcal{E}$ is a sheaf of $\mathfrak{o}$-modules on Y then its ϖ-completion

$$\mathscr{E} := \varprojlim_j \mathcal{E}/\varpi^{j+1}\mathcal{E}$$

will be considered as a sheaf on $\mathfrak{Y}$. Finally, the base change of a sheaf of $\mathfrak{o}$-modules on Y (resp. on $\mathfrak{Y}$) to L will always be denoted by the subscript $\mathbb{Q}$. For instance $\mathcal{E}_{\mathbb{Q}} = \mathcal{E} \otimes_{\mathfrak{o}} L$.

2. Arithmetic definitions

2.1 – *p-adic coefficients and divided powers*

Let p be a prime number and let us fix a positive integer m. Throughout this work, we will denote by $\mathbb{Z}_p$ the ring of p-adic integers and by $\mathbb{Z}_{(p)}$ the localization of $\mathbb{Z}$ with respect to the prime ideal (p). Moreover, if $k \in \mathbb{N}$, we will write q_k for the quotient of the euclidean division of k by p^m. Berthelot has introduced in [3] the following coefficients for any two integers k, k' with $k \geq k'$:

$$\left\{ \begin{matrix} k \\ k' \end{matrix} \right\} = \frac{q_k!}{q'_k! q''_k!}, \quad k'' = k - k'.$$

In fact, we can generalize these coefficients for multi-indices $\underline{k} = (k_1, \ldots, k_N) \in \mathbb{N}^N$ by defining $q_{\underline{k}}! = q_{k_1}! \ldots q_{k_N}!$ and

$$\left\{ \begin{matrix} \underline{k} \\ \underline{k'} \end{matrix} \right\} = \frac{q_{\underline{k}}!}{q_{\underline{k'}}! q_{\underline{k''}}!} \in \mathbb{N} \quad \text{and} \quad \left\langle \begin{matrix} \underline{k} \\ \underline{k'} \end{matrix} \right\rangle = \begin{pmatrix} \underline{k} \\ \underline{k'} \end{pmatrix} \left\{ \begin{matrix} \underline{k} \\ \underline{k'} \end{matrix} \right\}^{-1} \in \mathbb{Z}_p.$$

Now, let A be a $\mathbb{Z}_{(p)}$ algebra. We say that a triple (I, J, γ) is an m-PD ideal of A, if γ defines a structure of divided powers on J (a PD-structure in the sense of [5]) and I is endowed with a system of partial divided powers, meaning that for any integer k, which decomposes as $k = p^m q + r$ (with $r < p^m$), there exists an operation defined for every $x \in I$ by

$$x^{\{k\}} = x^r \gamma_q(x^{p^m}).$$

EXAMPLE 2.1. Let $\mathfrak{o}$ be a discrete valuation ring of unequal characteristic $(0, p)$ and uniformizing parameter ϖ. Let us write $p = u\varpi^e$, with u a unit of $\mathfrak{o}$ and e a positive integer (called the absolute ramification index of $\mathfrak{o}$). Let $k \in \mathbb{N}$. Then $\gamma_v(x) := x^v/v!$ defines a PD-structure on $(\varpi)^k$ if and only if $e \leq k(p-1)$ (see [5, Examples 3.2 (3)]). In particular, we dispose of a PD-structure on $(p) \subseteq \mathbb{Z}_{(p)}$. We let $x^{[k]} = \gamma_k(x)$ and we denote by $((p), [\ ])$ this PD-ideal. Moreover, if $k \leq e - 1$ and $m \geq \log_p(k)$, then $(\varpi)^k$ endowed with the preceding PD-structure defines an m-PD-structure on (ϖ) (see [3, Section 1.3, Examples (i)]).

2.2 – *Arithmetic differential operators*

Let us suppose that $\mathfrak{o}$ is endowed with the m-PD-structure $(\mathfrak{a}, \mathfrak{b}, [\,])$ defined in Example 2.1. Let X be a smooth $\mathfrak{o}$-scheme, and $\mathcal{I} \subset \mathcal{O}_X$ a quasi-coherent ideal. Let us consider the sheaf of principal parties $\mathcal{P}_{(m)}(\mathcal{I})$ (see [3, Section 2.1]), which contains an m-PD structure $(\overline{\mathcal{I}}, \widetilde{\mathcal{I}}, [\,])$ and the sequence of ideals $(\overline{\mathcal{I}}^{\{n\}})_{n\in\mathbb{N}}$ defining the m-PD-filtration [4, Section 1.1.3]. For every $n \in \mathbb{N}$, the algebra

$$\mathcal{P}^n_{X,(m)} = \mathcal{P}^n_{(m)}(\mathcal{I})/\overline{\mathcal{I}}^{\{n\}}$$

is quasi-coherent and can be considered as a sheaf on X. Moreover, the projections $p_1, p_2 : X \times_{\mathfrak{o}} X \to X$ induce two morphisms $d_1, d_2 : \mathcal{O}_X \to \mathcal{P}^n_{X,(m)}$ endowing $\mathcal{P}^n_{X,(m)}$ of a *left* and a *right* structure of $\mathcal{O}_X$-algebra, respectively.

Definition 2.2. Let m, n be positive integers. The sheaf of differential operators of level m and order less than or equal to n on X is defined by

$$\mathcal{D}^{(m)}_{X,n} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^n_{X,(m)}, \mathcal{O}_X).$$

If $n \le n'$, then [3, Proposition 1.4.1] gives us a canonical surjection $\mathcal{P}^{n'}_{X,(m)} \to \mathcal{P}^n_{X,(m)}$ which induces the injection $\mathcal{D}^{(m)}_{X,n} \hookrightarrow \mathcal{D}^{(m)}_{X,n'}$ and the sheaf of *differential operators of level m* is defined by

$$\mathcal{D}^{(m)}_X = \bigcup_{n\in\mathbb{N}} \mathcal{D}^{(m)}_{X,n}.$$

We remark for the reader that by definition $\mathcal{D}^{(m)}_X$ is endowed with a natural filtration called the *order filtration*, and like the sheaves $\mathcal{P}^n_{X,(m)}$, the sheaves $\mathcal{D}^{(m)}_{X,n}$ are endowed with two natural structures of $\mathcal{O}_X$-modules.

Moreover, the sheaf $\mathcal{D}^{(m)}_X$ acts on $\mathcal{O}_X$: if $P \in \mathcal{D}^{(m)}_{X,n}$, then this action is given by the composition

$$\mathcal{O}_X \xrightarrow{d_1} \mathcal{P}^n_{X,(m)} \xrightarrow{P} \mathcal{O}_X.$$

Finally, let us give a local description of $\mathcal{D}^{(m)}_{X,n}$. Let U be a smooth open affine subset of X endowed with a family of local coordinates $x_1, \dots, x_N$. Let $dx_1, \dots, dx_N$ be a basis of $\Omega_X(U)$ and $\partial_{x_1}, \dots, \partial_{x_N}$ the dual basis of $\mathcal{T}_X(U)$ (as usual, $\mathcal{T}_X$ and Ω_X denote the tangent and cotangent sheaf on X, respectively). Let $\underline{k} \in \mathbb{N}^N$. Let us use the notation

$$|\underline{k}| = \sum_{i=1}^{\mathbb{N}} k_i \quad \text{and} \quad \partial_i^{[k_i]} = \partial_{x_i}/k_i! \quad \text{for every } 1 \le i \le \mathbb{N}.$$

Then, using the multi-index notation, we have $\underline{\partial}^{[\underline{k}]} = \prod_{i=1}^{\mathbb{N}} \partial_i^{[k_i]}$ and $\underline{\partial}^{\langle \underline{k}\rangle} = q_{\underline{k}}!\underline{\partial}^{[\underline{k}]}$.

In this case, the sheaf $\mathcal{D}^{(m)}_{X,n}$ has the following description on U:

$$\mathcal{D}^{(m)}_{X,n}(U) = \Big\{ \sum_{|\underline{k}|\leq n} a_{\underline{k}}\underline{\partial}^{\langle \underline{k}\rangle} \mid a_{\underline{k}} \in \mathcal{O}_X(U) \text{ and } \underline{k} \in \mathbb{N}^N \Big\}.$$

2.3 – *Symmetric algebra of finite level*

In this subsection, we will focus on introducing the constructions in [19]. As before, let X denote a smooth $\mathfrak{o}$-scheme and let us consider a locally free $\mathcal{O}_X$-module $\mathcal{L}$ of finite rank, the symmetric algebra $\mathbf{S}_X(\mathcal{L})$ associated to $\mathcal{L}$, and the ideal $\mathcal{I}$ of homogeneous elements of degree 1. If $\mathcal{P}_{\mathbf{S}_X(\mathcal{L}),(m)}(\mathcal{I})$ denotes the m-divided power enveloping algebra of $(\mathbf{S}_X(\mathcal{L}), \mathcal{I})$ (see [3, Proposition 1.4.1]), then we can consider the coherent sheaves on X

$$\Gamma_{X,(m)}(\mathcal{L}) = \mathcal{P}_{\mathbf{S}_X(\mathcal{L}),(m)}(\mathcal{I}) \quad \text{and} \quad \Gamma^n_{X,(m)}(\mathcal{L}) = \Gamma_{X,(m)}(\mathcal{L})/\overline{\mathcal{I}}^{\{n+1\}}.$$

Those algebras are graded [19, Proposition 1.3.3] and if $\eta_1, \dots, \eta_N$ is a local basis of $\mathcal{L}$, we have

$$\Gamma^n_{X,(m)}(\mathcal{L}) = \bigoplus_{|\underline{l}|\leq n} \mathcal{O}_X \underline{\eta}^{\{\underline{l}\}}.$$

As before $\underline{\eta}^{\{\underline{l}\}} = \prod_{i=1}^{\mathbb{N}} \eta_i^{\{l_i\}}$ and $q_i!\eta_i^{\{l_i\}} = \eta^{l_i}$. We define by duality

$$\mathrm{Sym}^{(m)}(\mathcal{L}) = \bigcup_{k\in\mathbb{N}} \mathcal{H}om_{\mathcal{O}_X}\big(\Gamma^k_{X,(m)}(\mathcal{L}^\vee), \mathcal{O}_X\big),$$

By [19, Propositions 1.3.3 and 1.3.6] we know that $\mathrm{Sym}^{(m)}(\mathcal{L}) = \bigoplus_{n\in\mathbb{N}} \mathrm{Sym}^{(m)}_n(\mathcal{L})$ is a commutative graded algebra with noetherian sections over any open affine subset. Moreover, locally over a basis $\eta_1, \dots, \eta_N$ of $\mathcal{L}$ we have the following description:

$$\mathrm{Sym}^{(m)}_n(\mathcal{L}) = \bigoplus_{|\underline{l}|=n} \mathcal{O}_X \underline{\eta}^{\langle \underline{l}\rangle}, \quad \text{where } \frac{l_i!}{q_i!}\eta_i^{\langle l_i\rangle} = \eta_i^{l_i}.$$

Remark 2.3. By [5, Proposition A.10] we have that $\mathrm{Sym}^{(0)}(\mathcal{L})$ is the symmetric algebra of $\mathcal{L}$, which justifies the terminology.

We end this subsection by mentioning the following results from [19]. Let $\mathcal{I}$ be the kernel of the comorphism $\Delta^\sharp$ of the diagonal embedding $\Delta : X \to X \times_{\mathrm{Spec}(\mathfrak{o})} X$. In [19, Proposition 1.3.7.3] Huyghe shows that the graded algebra associated to the m-PD-adic filtration of $\mathcal{P}_{X,(m)}$ is identified with the graded m-PD-algebra $\Gamma_{X,(m)}(\mathcal{I}/\mathcal{I}^2) = \Gamma_{X,(m)}(\Omega^1_X)$. More exactly, we dispose of a morphism of $\mathcal{O}_X$-algebras

$$\mathbf{S}_X(\Omega_X) \to \mathrm{gr}_\bullet \mathcal{P}_{X,(m)},$$

which extends, via the universal property [3, Proposition 1.4.1], to a canonical morphism

$$\Gamma^n_{X,(m)}(\Omega^1_X) \xrightarrow{\simeq} \mathrm{gr}_\bullet(\mathcal{P}^n_{X,(m)}).$$

By definition, it induces a graded morphism

$$\mathrm{Sym}^{(m)}(\mathcal{T}_X) \to \mathrm{gr}_\bullet\, \mathcal{D}^{(m)}_X \tag{2.1}$$

which is in fact an isomorphism of $\mathcal{O}_X$-algebras.

2.4 – *Arithmetic distribution algebra of finite level*

As in the introduction, let us consider a split connected reductive group scheme $\mathbb{G}$ over $\mathfrak{o}$ and $m \in \mathbb{N}$ fixed. We give a description of the algebra of distributions of level m introduced in [21]. Let I denote the kernel of the surjective morphism of $\mathfrak{o}$-algebras $\varepsilon_{\mathbb{G}} : \mathfrak{o}[\mathbb{G}] \to \mathfrak{o}$, given by the identity element of $\mathbb{G}$. We know that I/I^2 is a free $\mathfrak{o} = \mathfrak{o}[\mathbb{G}]/I$-module of finite rank. Let $t_1, \dots, t_l \in I$ such that modulo I^2 these elements form a basis of I/I^2. The m-divided power enveloping algebra of $(\mathfrak{o}[\mathbb{G}], I)$, denoted by $P_{(m)}(\mathbb{G})$, is a free $\mathfrak{o}$-module with the elements $\underline{t}^{\{\underline{k}\}} = t_1^{\{k_1\}} \cdots t_l^{\{k_l\}}$ as basis, where

$$q_i!t_i^{\{k_i\}} = t_i^{k_i} \quad \text{for every } k_i = p^m q_i + r_i \text{ and } 0 \le r_i < p^m.$$

These algebras are endowed with a decreasing filtration by ideals $\bar{I}^{\{n\}}$ (the m-PD filtration), such that $\bar{I}^{\{n\}} = \bigoplus_{|\underline{k}|\ge n} \mathfrak{o}\underline{t}^{\{\underline{k}\}}$. The quotients

$$P^n_{(m)}(\mathbb{G}) = P_{(m)}(\mathbb{G})/\bar{I}^{\{n+1\}}$$

are therefore $\mathfrak{o}$-modules generated by the elements $\underline{t}^{\{\underline{k}\}}$ with $|\underline{k}| \le n$ (see [3, Proposition 1.5.3 (ii)]). Moreover, there exists an isomorphism of $\mathfrak{o}$-modules

$$P^n_{(m)}(\mathbb{G}) \simeq \bigoplus_{|\underline{k}|\le n} \mathfrak{o}\underline{t}^{\{\underline{k}\}}$$

and for any two integers n, n' such that $n \le n'$ we have a canonical surjection $\pi^{n',n} : P^{n'}_{(m)}(\mathbb{G}) \to P^n_{(m)}(\mathbb{G})$. The module of distributions of level m and order n is $D^{(m)}_n(\mathbb{G}) = \mathrm{Hom}(P^n_{(m)}(\mathbb{G}), \mathfrak{o})$. The *algebra of distributions of level m* is

$$D^{(m)}(\mathbb{G}) = \varinjlim_n D^{(m)}_n(\mathbb{G}),$$

where the limit is formed with respect to the maps $\mathrm{Hom}_{\mathfrak{o}}(\pi^{n',n}, \mathfrak{o})$. The multiplication is defined as follows. By the universal property (see [3, Proposition 1.4.1]) there exists a canonical map

$$\delta^{n,n'} : P^{n+n'}_{(m)}(\mathbb{G}) \to P^n_{(m)}(\mathbb{G}) \otimes_{\mathfrak{o}} P^{n'}_{(m)}(\mathbb{G}).$$

If $(u, v) \in D^{(m)}_n(\mathbb{G}) \times D^{(m)}_{n'}(\mathbb{G})$, we define $u.v$ as the composition

$$u.v : P^{n+n'}_{(m)}(\mathbb{G}) \xrightarrow{\delta^{n,n'}} P^{n}_{(m)}(\mathbb{G}) \otimes_{\mathfrak{o}} P^{n'}_{(m)}(\mathbb{G}) \xrightarrow{u \otimes v} \mathfrak{o}.$$

Let us denote by $\mathfrak{g} = \mathrm{Hom}_{\mathfrak{o}}(I/I^2, \mathfrak{o})$ the Lie algebra of $\mathbb{G}$. This is a free $\mathfrak{o}$-module with basis $\xi_1, \dots, \xi_l$ defined as the dual basis of the elements $t_1, \dots, t_l$. If for every multi-index $\underline{k} \in \mathbb{N}^l$, $|\underline{k}| \leq n$, we denote by $\underline{\xi}^{\langle \underline{k} \rangle}$ the dual of the element $\underline{t}^{\{\underline{k}\}} \in P^n_{(m)}(\mathbb{G})$, then $D^{(m)}_n(\mathbb{G})$ is a free $\mathfrak{o}$-module of finite rank with a basis given by the elements $\underline{\xi}^{\langle \underline{k} \rangle}$ with $|\underline{k}| \leq n$ (see [21, Proposition 4.1.6]).

REMARK 2.4. This remark exemplifies the local situation when $X = \mathrm{Spec}(A)$ with A being a $\mathbb{Z}_{(p)}$-algebra [19, Section 1.3.1].

Let A be an $\mathfrak{o}$-algebra and E a free A-module of finite rank with base $(x_1, \dots, x_N)$. Let $(y_1, \dots, y_N)$ be the dual base of $E^\vee = \mathrm{Hom}_A(E, A)$. As in the preceding subsection, let $\mathbf{S}(E^\vee)$ be the symmetric algebra and $\mathbf{I}(E^\vee)$ the augmentation ideal. Let $\Gamma_{(m)}(E^\vee)$ be the m-divided power enveloping algebra of $(\mathbf{S}(E^\vee), \mathbf{I}(E^\vee))$. We put

$$\Gamma^n_{(m)}(E^\vee) = \Gamma_{(m)}(E^\vee)/\bar{I}^{\{n+1\}}.$$

These are free A-modules with base $y_1^{\{k_1\}} \dots y_N^{\{k_N\}}$ with $\sum k_i \leq n$ (see [19, Rappels 1.1.2 (iii)]). Let $\{\underline{x}^{\langle \underline{k} \rangle}\}_{|\underline{k}| \leq n}$ be the dual base of $\mathrm{Hom}_A(\Gamma^n_{(m)}(E^\vee), A)$. We define

$$\mathrm{Sym}^{(m)}(E) = \bigcup_{n \in \mathbb{N}} \mathrm{Hom}_A\big(\Gamma^n_{(m)}(E^\vee), A\big).$$

This is a free A-module with a base given by all the $\underline{x}^{\langle \underline{k} \rangle}$. The canonical inclusion $\mathrm{Sym}^{(m)}(E) \subseteq \mathbf{S}(E) \otimes_{\mathfrak{o}} L$ gives the relation

$$x_i^{\langle k_i \rangle} = \frac{k_i!}{q_i!} x^{k_i}.$$

Moreover, it also has an algebra structure defined as follows. By [19, Proposition 1.3.1] there exists an application $\Delta_{n,n'} : \Gamma^{n+n'}_{(m)}(E^\vee) \to \Gamma^n_{(m)}(E^\vee) \otimes_A \Gamma^{n'}_{(m)}(E^\vee)$, which allows us to define the product

$$u.v : \Gamma^{n+n'}_{(m)}(E^\vee) \xrightarrow{\Delta_{n,n'}} \Gamma^n_{(m)}(E^\vee) \otimes_A \Gamma^{n'}_{(m)}(E^\vee) \xrightarrow{u \otimes v} A$$

with $u \in \mathrm{Hom}_A(\Gamma^n_{(m)}(E^\vee), A)$ and $v \in \mathrm{Hom}_A(\Gamma^{n'}_{(m)}(E^\vee), A)$. This map endows $\mathrm{Sym}^{(m)}(E)$ with the structure of a graded noetherian $\mathfrak{o}$-algebra [19, Propositions 1.3.3 and 1.3.6].

We have the following important properties [21, Proposition 4.1.15].

Proposition 2.5. (i) *There exists a canonical isomorphism of graded* $\mathfrak{o}$*-algebras* $\mathrm{gr}_\bullet(D^{(m)}(\mathbb{G})) \simeq \mathrm{Sym}^{(m)}(\mathfrak{g})$.

(ii) *The* $\mathfrak{o}$*-algebras* $\mathrm{gr}_\bullet(D^{(m)}(\mathbb{G}))$ *and* $D^{(m)}(\mathbb{G})$ *are noetherian.*

2.5 – *Integral models*

In this subsection, we will assume that X is a smooth $\mathfrak{o}$-scheme endowed with a right $\mathbb{G}$-action.

Definition 2.6. Let A be an L-algebra (resp. a sheaf of L-algebras). We say that an $\mathfrak{o}$-subalgebra A_0 (resp. a subsheaf of $\mathfrak{o}$-algebras) is an integral model of A if $A_0 \otimes_{\mathfrak{o}} L = A$.

Remark 2.7. Let us recall that throughout this paper $\mathfrak{g}$ denotes the Lie algebra of the split connected reductive group $\mathfrak{o}$-scheme $\mathbb{G}$ and $\mathcal{U}(\mathfrak{g})$ its universal enveloping algebra. As we have remarked in the introduction, if $\mathfrak{g}_{\mathbb{Q}}$ denotes the L-Lie algebra of the algebraic group $\mathbb{G}_{\mathbb{Q}} = \mathbb{G} \times_{\mathrm{Spec}(\mathfrak{o})} \mathrm{Spec}(L)$ and $\mathcal{U}(\mathfrak{g}_{\mathbb{Q}})$ its universal enveloping algebra, then $\mathcal{U}(\mathfrak{g})$ is an integral model of $\mathcal{U}(\mathfrak{g}_{\mathbb{Q}})$. Moreover, the algebra of distributions of level m, introduced in the preceding subsection, is also an integral model of $\mathcal{U}(\mathfrak{g}_{\mathbb{Q}})$ (see [21, Section 4.1]). This fact will be especially important in this work.

Proposition 2.8. *The right* $\mathbb{G}$*-action induces a canonical homomorphism of filtered* $\mathfrak{o}$*-algebras*

$$\Phi^{(m)} : D^{(m)}(\mathbb{G}) \to H^0(X, \mathcal{D}_X^{(m)}).$$

Proof. For the proof we refer the reader to the proof of [21, Proposition 4.4.1 (ii)]. Here, we will briefly discuss the construction of $\Phi^{(m)}$. The central idea in the construction is that if $\rho : X \times_{\mathfrak{o}} \mathbb{G} \to X$ denotes the $\mathbb{G}$-action, then the comorphism $\rho^{\natural} : \mathcal{O}_X \to \mathcal{O}_X \otimes_{\mathfrak{o}} \mathfrak{o}[\mathbb{G}]$ induces a morphism

$$\rho_m^{(n)} : \mathcal{P}^n_{X,(m)} \to \mathcal{O}_X \otimes_{\mathfrak{o}} P^n_{(m)}(\mathbb{G})$$

for every $n \in \mathbb{N}$. Those applications are compatible when varying n. Let $u \in D_n^{(m)}(\mathbb{G})$. We define $\Phi^{(m)}(u)$ by

$$\Phi^{(m)}(u) : \mathcal{P}^n_{X,(m)} \xrightarrow{\rho_m^{(n)}} \mathcal{O}_X \otimes_{\mathfrak{o}} P^n_{(m)}(\mathbb{G}) \xrightarrow{\mathrm{id}\otimes u} \mathcal{O}_X.$$

Again, those applications are compatible when varying n and we get the morphism of the proposition. ∎

REMARKS 2.9. (i) If X is endowed with a left $\mathbb{G}$-action, then it turns out that $\Phi^{(m)}$ is an anti-homomorphism.

(ii) In [21, Theorem 4.4.8.3] Huyghe and Schmidt have shown that if $X = \mathbb{G}$ and we consider the right (resp. left) regular action, then the morphism of the preceding proposition is in fact a canonical filtered isomorphism (resp. an anti-isomorphism) between $D^{(m)}(\mathbb{G})$ and $H^0(\mathbb{G}, \mathcal{D}_{\mathbb{G}}^{(m)})^{\mathbb{G}}$, the $\mathfrak{o}$-submodule of (left) $\mathbb{G}$-invariant global sections. This isomorphism induces a bijection between $D_n^{(m)}(\mathbb{G})$ and $H^0(\mathbb{G}, \mathcal{D}_{\mathbb{G},n}^{(m)})^{\mathbb{G}}$, and it is compatible when varying m.

We will denote by

$$\Phi_X^{(m)} : \mathcal{O}_X \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{G}) \to \mathcal{D}_X^{(m)}$$

the morphism of sheaves (of $\mathfrak{o}$-modules) defined as follows: if $U \subseteq X$ is an open subset and $f \in \mathcal{O}_X(U)$, $u \in D^{(m)}(\mathbb{G})$, then

$$\Phi_{X,U}^{(m)}(f \otimes u) = f \cdot \Phi^{(m)}(u)|_U.$$

Let us define $\mathcal{A}_X^{(m)} = \mathcal{O}_X \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{G})$, and let us remark that we can endow this sheaf with the skew ring multiplication coming from the action of $D^{(m)}(\mathbb{G})$ on $\mathcal{O}_X$ via the morphism $\Phi_X^{(m)}$, that is,

$$(f \otimes u) \cdot (g \otimes v) = \big(f \cdot \Phi_X^{(m)}(u)\big) g \otimes v + fg \otimes uv. \tag{2.2}$$

This multiplication defines over $\mathcal{A}_X^{(m)}$ the structure of a sheaf of associative $\mathfrak{o}$-algebras, such that it becomes an integral model of the sheaf $\mathcal{U}^\circ = \mathcal{O}_{X_L} \otimes_L \mathcal{U}(\mathfrak{g}_{\mathbb{Q}})$. To see this, let us recall how the multiplicative structure of the sheaf $\mathcal{U}^\circ$ is defined (cf. [29, Section 5.1] or [27, Section 2]).

Differentiating the right action of $\mathbb{G}_{\mathbb{Q}}$ on $X_{\mathbb{Q}}$, we get a morphism of Lie algebras

$$\tau : \mathfrak{g}_{\mathbb{Q}} \to H^0(X_{\mathbb{Q}}, \mathcal{T}_{X_{\mathbb{Q}}}).$$

This implies that $\mathfrak{g}_{\mathbb{Q}}$ acts on $\mathcal{O}_{X_{\mathbb{Q}}}$ by derivations and we can endow $\mathcal{U}^\circ$ with the skew ring multiplication

$$(f \otimes \eta)(g \otimes \zeta) = \big(f\tau(\eta)\big) g \otimes \zeta + fg \otimes \eta\zeta \tag{2.3}$$

for $\eta \in \mathfrak{g}_{\mathbb{Q}}$, $\zeta \in \mathcal{U}(\mathfrak{g}_{\mathbb{Q}})$ and $f, g \in \mathcal{O}_{X_{\mathbb{Q}}}$. With this product the sheaf $\mathcal{U}^\circ$ becomes a sheaf of associative algebras [27, p. 11].

REMARK 2.10. As in (2.2) we can define a morphism (called the operator-representation) of sheaves of L-algebras

$$\Psi_{X_L} : \mathcal{O}_{X_{\mathbb{Q}}} \otimes_L \mathcal{U}(\mathfrak{g}_{\mathbb{Q}}) \to \mathcal{D}_{X_{\mathbb{Q}}}, \quad f \otimes \eta \mapsto f\tau(\eta) \quad (f \in \mathcal{O}_{X_{\mathbb{Q}}}, \eta \in \mathfrak{g}_{\mathbb{Q}}).$$

We get the commutative diagram

$$\begin{array}{ccc} D^{(m)}(\mathbb{G}) & \xrightarrow{\Phi^{(m)}} & H^0(X, \mathcal{D}_X^{(m)}) \\ \big\downarrow & & \big\downarrow \\ \mathcal{U}(\mathfrak{g}_L) & \xrightarrow{\Psi_{X_{\mathbb{Q}}}} & H^0(X_{\mathbb{Q}}, \mathcal{D}_{X_{\mathbb{Q}}}). \end{array}$$

Given that $D^{(m)}(\mathbb{G})$ is an integral model of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_{\mathbb{Q}})$, then by (2.2) and (2.3) we can conclude that $\mathcal{A}_X^{(m)}$ is also a sheaf of associative $\mathfrak{o}$-algebras being a subsheaf of $\mathcal{U}^{\circ}$.

Proposition 2.11 ([21, Corollary 4.4.6]). (i) *The sheaf* $\mathcal{A}_X^{(m)}$ *is a locally free* $\mathcal{O}_X$*-module.*

(ii) *There exists a unique structure over* $\mathcal{A}_X^{(m)}$ *of filtered* $\mathcal{O}_X$*-rings and there is a canonical isomorphism of graded* $\mathcal{O}_X$*-algebras* $\mathrm{gr}(\mathcal{A}_X^{(m)}) = \mathcal{O}_X \otimes_{\mathfrak{o}} \mathrm{Sym}^{(m)}(\mathfrak{g})$.

(iii) *The sheaf* $\mathcal{A}_X^{(m)}$ *(resp.* $\mathrm{gr}(\mathcal{A}_X^{(m)})$*) is a coherent sheaf of* $\mathcal{O}_X$*-rings (resp. a coherent sheaf of* $\mathcal{O}_X$*-algebras), with noetherian sections over open affine subsets of* X.

3. Twisted arithmetic differential operators with congruence level

In this section, we will introduce congruence levels to the constructions given in Sections 2.2, 2.4 and 2.5. This means, deformations of our (integral) differential operators. This notion will be a fundamental tool to define differential operators on an admissible blow-up of the flag $\mathfrak{o}$-scheme.

3.1 – *Linearization of group actions*

Let us start with the following definition from [17, Chapter II, Exercise 5.18] (cf. [8, Definition 3.1.1]).

Definition 3.1. Let Y be an $\mathfrak{o}$-scheme. A (geometric) line bundle over Y is a scheme $\mathbf{L}$ together with a morphism $\pi : \mathbf{L} \to Y$ such that Y admits an open covering $(U_i)_{i\in I}$ satisfying the following two conditions:

(i) For any $i \in I$ there exists an isomorphism $\psi_i : \pi^{-1}(U_i) \xrightarrow{\simeq} \mathbb{A}^1_{U_i}$.

(ii) For any $i, j \in I$ and for any open affine subset $V = \mathrm{Spec}(A[x]) \subseteq U_i \cap U_i$ the automorphism $\theta_{ij} : \psi_j \circ \psi_i^{-1}|_V : \mathbb{A}^1_V \to \mathbb{A}^1_V$ of $\mathbb{A}^1_V$ is given by a linear automorphism $\theta_{ij}^{\natural}$ of $A[x]$. This means, $\theta_{ij}^{\natural}(a) = a$ for any $a \in A$, and $\theta_{ij}^{\natural}(x) = a_{ij}x$ for a suitable $a_{ij} \in A$.

In the preceding definition, the scheme $\mathbf{L}$ is obtained by gluing the trivial line bundles $p_{1,i} : U_i \times \mathbb{A}^1_{\mathfrak{o}} \to U_i$ via the linear transition functions (a_{ij}). Thus, each fiber $\mathbf{L}_x$ is a line, in the sense that it has a canonical structure of a 1-dimensional affine space.

Definition 3.2. Given a line bundle $\pi : \mathbf{L} \to Y$ and a morphism $\varphi : Y' \to Y$, the pull-back $\varphi^*(\mathbf{L})$ is the fiber product $\mathbf{L} \times_Y Y'$ equipped with its projection to Y'.

Now, let $\pi : \mathbf{L} \to Y$ be a line bundle over Y, then a *section* of π over an open subset $U \subseteq Y$ is a morphism $s : U \to \mathbf{L}$ such that $\pi \circ s = \mathrm{id}_U$. Moreover, the presheaf $\mathcal{L}$ defined by

$$U \subseteq Y \mapsto \{s : U \to \mathbf{L} \mid s \text{ is a section over } U\}$$

is a sheaf called the *sheaf of sections* of the line bundle $\mathbf{L}$. This is an invertible sheaf.

On the other hand, if $\mathcal{E}$ is a locally free sheaf of rank 1 on Y and we let

$$\mathbf{V}(\mathcal{E}) = \underline{\mathrm{Spec}}_Y\left(\mathrm{Sym}_{\mathcal{O}_Y}(\mathcal{E})\right)$$

be the line bundle over Y associated to $\mathcal{E}$ (see [16, Definition 1.7.8]), then we have a one-to-one correspondence between isomorphic classes of locally free sheaves of rank 1 on Y and isomorphic classes of (geometric) line bundles over Y (see [17, Chapter II, Exercises 5.1 (a) and 5.18 (d)]):

$$(3.1)\qquad \begin{cases} \text{Isomorphic classes of} & & \\ \text{locally free sheaves of rank 1} & \leftrightarrow & \text{Isomorphic classes of line bundles,} \\ \mathcal{E} & \mapsto & \mathbf{V}(\mathcal{E}^\vee), \\ \mathcal{L} & \leftarrow\!\shortmid & \mathbf{L}. \end{cases}$$

Let $\pi : \mathbf{L} \to Y$ be a line bundle over Y, let $\mathcal{L}$ be its sheaf of sections and $\varphi : Y' \to Y$ a morphism of schemes; an easy calculation shows that the sheaf of sections of the pull-back line bundle $\varphi^*(\mathbf{L}) = \mathbf{L} \times_Y Y' \to Y'$ is equal to $\varphi^*(\mathcal{L})$.

Let us suppose now that Y is endowed with a right $\mathbb{G}$-action $\alpha : Y \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{G} \to Y$. In particular, for every $g \in \mathbb{G}(\mathfrak{o})$ we dispose of a translation morphism

$$\rho_g : Y = Y \times_{\mathrm{Spec}(\mathfrak{o})} \mathrm{Spec}(\mathfrak{o}) \xrightarrow{\mathrm{id}_Y \times g} Y \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{G} \xrightarrow{\alpha} Y.$$

In the next lines we will study (geometric) line bundles which are endowed with a right $\mathbb{G}$-action.

Definition 3.3. Let $\pi : \mathbf{L} \to Y$ be a line bundle. A $\mathbb{G}$-linearization of $\mathbf{L}$ is a right $\mathbb{G}$-action $\beta : \mathbf{L} \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{G} \to \mathbf{L}$ satisfying the following two conditions:

(i) The diagram

$$\begin{array}{ccc} \mathbf{L} \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{G} & \xrightarrow{\beta} & \mathbf{L} \\ \downarrow{\scriptstyle \pi\times \mathrm{id}_{\mathbb{G}}} & & \downarrow{\scriptstyle \pi} \\ Y \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{G} & \xrightarrow{\alpha} & Y \end{array}$$

is commutative.

(ii) The action on the fibers is $\mathfrak{o}$-linear.

Let $g \in \mathbb{G}(\mathfrak{o})$ and let us suppose that $\Psi : \alpha^*(\mathbf{L}) \to p_1^*(\mathbf{L})$ is a morphism of line bundles over $Y \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{G}$. Let us consider the translation morphism

$$\rho_g : Y = Y \times_{\mathrm{Spec}(\mathfrak{o})} \mathrm{Spec}(\mathfrak{o}) \xrightarrow{\mathrm{id}_Y \times g} Y \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{G} \xrightarrow{\alpha} Y.$$

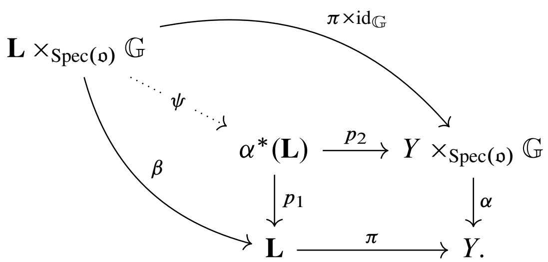
We have the relations $(\mathrm{id}_Y \times g)^* \alpha^*(\mathbf{L}) = \rho_g^*(L)$ and $(\mathrm{id}_Y \times g)^* p_1^*(\mathbf{L}) = \mathbf{L}$. So every morphism of line bundles $\Psi : \alpha^*(\mathbf{L}) \to p_1^*(\mathbf{L})$ induces morphisms $\Psi_g : \rho_g^*(L) \to L$ for all $g \in \mathbb{G}(\mathfrak{o})$. The following reasoning can be found in [12, p. 104] or [8, Lemma 3.2.4].

Proposition 3.4. *Let $\pi : \mathbf{L} \to Y$ be a line bundle over Y endowed with a $\mathbb{G}$-linearization $\beta : \mathbf{L} \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{G} \to \mathbf{L}$. Then there exists an isomorphism*

$$\Psi : \alpha^*(\mathbf{L}) \to p_1^*(\mathbf{L})$$

of line bundles over $\mathbf{L} \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{G}$, such that $\Psi_{gh} = \Psi_g \circ \rho_g^*(\Psi_h)$ for all $g, h \in \mathbb{G}(\mathfrak{o})$.

Proof. By definition of linearization we have the commutative diagram



By the universal property there is a unique morphism of line bundles $\psi : p_1^*(\mathbf{L}) \to \alpha^*(\mathbf{L})$, which is linear on the fibers since so is β. Let $g \in \mathbb{G}(\mathfrak{o})$. To see that ψ is an isomorphism we can use the correspondence (3.1). In this case, if $x \in Y$, $g \in \mathbb{G}(\mathfrak{o})$ and $\psi_{(x,g)} : \mathcal{L}_x \to \mathcal{L}_{xg}$ denotes the respective morphism between the stalks, then $\psi_{(x,g)}$ is an isomorphism, $\psi_{(xg,g^{-1})}$ being the inverse.

Let $g, h \in \mathbb{G}(\mathfrak{o})$. Applying $(\mathrm{id}_X \times g)^*$ to ψ, we get the morphism $\psi_g : \mathbf{L} \to \rho_g^*(\mathbf{L})$ and given that β is a right action ($\rho_h \circ \rho_g = \rho_{gh}$), it fits into the commutative diagram

$$\begin{array}{ccc} \mathbf{L} & \xrightarrow{\psi_g} & \rho_g^*(\mathbf{L}) \\ & {\scriptstyle \psi_{gh}} \searrow & \downarrow {\scriptstyle \rho_g^*(\psi_h)} \\ & & \rho_g^* \rho_h^*(\mathbf{L}) = \rho_{gh}^*(\mathbf{L}). \end{array}$$

Moreover, since $\psi_g : \mathbf{L} \to \rho_g^*(\mathbf{L})$ is an isomorphism for every $g \in \mathbb{G}(\mathfrak{o})$, we can consider the morphism $\Psi_g = \psi_g^{-1} : \rho_g^*(\mathbf{L}) \to \mathbf{L}$ which coincides with the fibers of the morphism

$$\Psi = \psi^{-1} : \alpha^*(\mathbf{L}) \to p_1^*(\mathbf{L}).$$

By construction, these morphisms satisfy the cocycle condition of the proposition. This means that for every $g, h \in \mathbb{G}(\mathfrak{o})$, we have

$$\Psi_{gh} = \Psi_g \circ \rho_g^*(\Psi_h). \qquad \blacksquare$$

3.2 – *Associated Rees rings and differential operators with congruence level*

Throughout this subsection, X will denote a smooth scheme over $\mathfrak{o}$. As usual, we will denote by $\mathcal{D}_X^{(m)}$ the sheaf of level m differential operators on X. As we have remarked in Section 2.2, those sheaves come equipped with a filtration

$$\mathcal{O}_X \subseteq \mathcal{D}_{X,1}^{(m)} \subseteq \cdots \subseteq \mathcal{D}_{X,d}^{(m)} \subseteq \cdots \subseteq \mathcal{D}_X^{(m)},$$

with $\mathcal{D}_{X,d}^{(m)}$ the sheaf of level m differential operators of order less than or equal to d.

Now, let $\mathcal{A}$ be a sheaf of $\mathfrak{o}$-algebras endowed with a positive filtration $(F_d\mathcal{A})_{d\in\mathbb{N}}$ and such that $\mathfrak{o} \subset F_0\mathcal{A}$.[5] The sheaf $\mathcal{A}$ gives rise to a subsheaf of graded rings $R(\mathcal{A})$ of the polynomial algebra $\mathcal{A}[t]$ over $\mathcal{A}$. This is defined by

$$R(\mathcal{A}) = \bigoplus_{i\in\mathbb{N}} F_i\mathcal{A} \cdot t^i,$$

its associated Rees ring. This subsheaf comes equipped with a filtration by the sheaves of subgroups

$$R_d(\mathcal{A}) = \bigoplus_{i=0}^{d} F_i\mathcal{A} \cdot t^i \subseteq R(\mathcal{A}).$$

(5) This digression can be found before the proof of [20, Proposition 3.3.7].

Specializing $R(\mathcal{A})$ in an element $\mu \in \mathfrak{o}$, we get a subsheaf of filtered subrings $\mathcal{A}_\mu$ of $\mathcal{A}$. More exactly, $\mathcal{A}_\mu$ equals the image under the homomorphism of sheaves of rings $\varphi_\mu : R(\mathcal{A}) \to \mathcal{A}$, sending $t \mapsto \mu$, and it is equipped with the filtration induced by $\mathcal{A}$. Moreover, if the sheaf of graded rings $\mathrm{gr}(\mathcal{A})$, associated to the filtration $(F_d\mathcal{A})_{d\in\mathbb{N}}$, is flat over $\mathfrak{o}$, then

$$F_d\mathcal{A}_\mu = \sum_{i=0}^{d} \mu^i F_i\mathcal{A},$$

see [20, Claim 3.3.10.]. If $\psi : \mathcal{A} \to \mathcal{B}$ is a morphism of positive filtered $\mathfrak{o}$-algebras (with $\mathfrak{o} \subseteq F_0\mathcal{A}$ and $\mathfrak{o} \subseteq F_0\mathcal{B}$), then the commutative diagram

$$\begin{array}{ccc} R(\mathcal{A}) & \xrightarrow{a_d t^d \mapsto \psi(a_d) t^d} & R(\mathcal{B}) \\ \downarrow{\scriptstyle \varphi_\mu} & & \downarrow{\scriptstyle \varphi_\mu} \\ \mathcal{A} & \xrightarrow{\psi} & \mathcal{B} \end{array}$$

gives us a filtered morphism of rings $\psi_\mu : \mathcal{A}_\mu \to \mathcal{B}_\mu$. This in particular implies that for $\mu \in \mathfrak{o}$ fixed, the preceding process is functorial.

Remark 3.5. The previous digression is well known for rings. In this setting, we have results completely analogues to the ones presented so far [26, Chapter 12, Section 6]. We will use these results in Section 3.3.

Now, let k be a non-negative integer called a *congruence level* [23, Section 2.1]. By using the order filtration $(\mathcal{D}_X^{(m)})_{d\in\mathbb{N}}$ of the sheaf $\mathcal{D}_X^{(m)}$, we can define the sheaf of *arithmetic differential operators of congruence level* k, $\mathcal{D}_X^{(m,k)}$, as the subsheaf of $\mathcal{D}_X^{(m)}$ given by the specialization of $R(\mathcal{D}_X^{(m)})$ in $\varpi^k \in \mathfrak{o}$. This means

$$\mathcal{D}_X^{(m,k)} = \sum_{d\in\mathbb{N}} \varpi^{kd}\mathcal{D}_{X,d}^{(m)}.$$

By (2.1) and [19, Proposition 1.3.4.2] we can also conclude that, if $(\mathcal{D}_{X,d}^{(m,k)})_{d\in\mathbb{N}}$ denotes the order filtration induced by $\mathcal{D}_X^{(m)}$, then

$$\mathcal{D}_{X,d}^{(m,k)} = \sum_{i=0}^{d} \varpi^{ki}\mathcal{D}_{X,i}^{(m)}.$$

In local coordinates we can describe the sheaf $\mathcal{D}_X^{(m,k)}$ in the following way. Let $U \subseteq X$ be an open affine subset endowed with coordinates $x_1, \dots, x_N$. Let $dx_1, \dots, dx_N$ be a basis of $\Omega_X(U)$ and $\partial_{x_1}, \dots, \partial_{x_N}$ the dual basis of $\mathcal{T}_X(U)$. By using the notation in Section 2.2, one has the following description [23, Section 2.1]:

$$\mathcal{D}_X^{(m,k)}(U) = \left\{ \sum_{\underline{v}}^{<\infty} \varpi^{k|\underline{v}|} a_{\underline{v}} \underline{\partial}^{\langle \underline{v} \rangle} \mid a_{\underline{v}} \in \mathcal{O}_X(U) \right\}.$$

3.3 – *Arithmetic differential operators acting on a line bundle*

Throughout this subsection, $X = \mathbb{G}/\mathbb{B}$ will always denote the flag scheme. For technical reasons (cf. Proposition 2.8) in this work we will always suppose that the group $\mathbb{G}$ and the scheme X are endowed with the right regular $\mathbb{G}$-action. This means that for any $\mathfrak{o}$-algebra A and $g_0, g \in \mathbb{G}(A)$ we have

$$g_0 \bullet g = g^{-1} g_0 \quad \text{and} \quad g_0 \mathbb{B}(A) \bullet g = g^{-1} g_0 \mathbb{B}(A).$$

Under this action, the canonical projection $\mathbb{G} \to X$ is clearly $\mathbb{G}$-equivariant.

Finally, we recall for the reader that the sheaf $\mathcal{D}_X^{(m)}$ is endowed with a left and a right structure of a $\mathcal{O}_X$-module. These structures come from the canonical morphisms of rings $d_1, d_2 : \mathcal{O}_X \to \mathcal{P}^n_{X,(m)}$, which are induced by the universal property and the projections. By construction, these actions also endow the sheaf $\mathcal{D}_X^{(m,k)}$ with a left and a right structure of a $\mathcal{O}_X$-module.

Definition 3.6 (Dominant and regular characters). Let us consider the positive system $\Lambda^+ \subset \Lambda \subset X(\mathbb{T})$ ($X(\mathbb{T}) = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ the group of algebraic characters) associated to the Borel subgroup scheme $\mathbb{B} \subset \mathbb{G}$. The Weyl subgroup $W = N_{\mathbb{G}}(\mathbb{T})/\mathbb{T}$ acts naturally on the space $\mathfrak{t}^*_{\mathbb{Q}} = \mathrm{Hom}_L(\mathfrak{t}_L, L)$, and via differentiation $d : X(\mathbb{T}) \hookrightarrow \mathfrak{t}^*$ we may view $X(\mathbb{T})$ as a subgroup of $\mathfrak{t}^*$ in such a way that $X^*(\mathbb{T}) \otimes_{\mathfrak{o}} L = \mathfrak{t}^*_{\mathbb{Q}}$. Let $\rho = \frac{1}{2} \sum_{\alpha \in \Lambda^+} \alpha$ be the so-called Weyl vector. Let $\check{\alpha}$ be a coroot of $\alpha \in \Lambda$ viewed as an element of $\mathfrak{t}_{\mathbb{Q}}$. An arbitrary weight $\lambda \in \mathfrak{t}^*_{\mathbb{Q}}$ is called *dominant* if $\lambda(\check{\alpha}) \geq 0$ for all $\alpha \in \Lambda^+$. The weight λ is called *regular* if its stabilizer under the W-action is trivial.

Definition 3.7 (Line bundles on the flag scheme). Let us suppose now that $X = \mathbb{G}/\mathbb{B}$ is again the smooth flag $\mathfrak{o}$-scheme. We dispose of a canonical isomorphism $\mathbb{T} \simeq \mathbb{B}/\mathbf{N}$ (here $\mathbf{N}$ is the unipotent radical of $\mathbb{B}$) which in particular implies that every algebraic character $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ induces a character of the Borel subgroup $\lambda : \mathbb{B} \to \mathbb{G}_m$. Let us consider the locally free action of $\mathbb{B}$ on the trivial fiber bundle $\mathbb{G} \times \mathbb{A}^1_{\mathfrak{o}}$ over $\mathbb{G}$ given by

$$b.(g, u) = (gb^{-1}, \lambda(b)u) \quad (g \in \mathbb{G},\ b \in \mathbb{B},\ u \in \mathbb{A}^1_{\mathfrak{o}}).$$

We denote by $\mathbf{L}(\lambda) = \mathbb{B} \setminus (\mathbb{G} \times \mathbb{A}^1_{\mathfrak{o}})$ the quotient space obtained by this action.

Let $\pi : \mathbb{G} \to X$ be the canonical projection. Since the map $\mathbb{G} \times \mathbb{A}^1_{\mathfrak{o}} \to X$, $(g, u) \mapsto \pi(x)$ is constant on $\mathbb{B}$-orbits, it induces a morphism $\pi_\lambda : \mathbf{L}(\lambda) \to X$. Moreover, given that π is locally trivial (see [24, Part II, §1.10 (2)]), $\pi_\lambda : \mathbf{L}(\lambda) \to X$ defines a line bundle over X (see [24, Part I, §5.16]). Furthermore, the right $\mathbb{G}$-action on $\mathbb{G} \times \mathbb{A}^1_{\mathfrak{o}}$ given by

$$(g_0, u) \bullet g \mapsto (g^{-1} g_0, u) \quad (g \in \mathbb{G},\ (g_0, u) \in \mathbb{G} \times \mathbb{A}^1_{\mathfrak{o}})$$

induces a right action on $\mathbf{L}(\lambda)$ for which $\mathbf{L}(\lambda)$ turns out to be a $\mathbb{G}$-linearized line bundle on X. By Proposition 3.4, the sheaf of sections $\mathcal{L}(\lambda)$ of the line bundle $\mathbf{L}(\lambda)$ is a $\mathbb{G}$-equivariant invertible sheaf.

Definition 3.8. Let $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ be an algebraic character. For every congruence level $k \in \mathbb{N}$, we define the sheaf of level m arithmetic differential operators acting on the line bundle $\mathcal{L}(\lambda)$ by

$$\mathcal{D}_X^{(m,k)}(\lambda) = \mathcal{L}(\lambda) \otimes_{\mathcal{O}_X} \mathcal{D}_X^{(m,k)} \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)^\vee.$$

The multiplicative structure of the sheaf $\mathcal{D}_X^{(m,k)}(\lambda)$ is defined as follows. Let us consider $\alpha^\vee, \beta^\vee \in \mathcal{L}(\lambda)^\vee$, $P, Q \in \mathcal{D}_X^{(m,k)}$ and $\alpha, \beta \in \mathcal{L}(\lambda)$, then

$$\alpha \otimes P \otimes \alpha^\vee \bullet \beta \otimes Q \otimes \beta^\vee = \alpha \otimes P \langle \alpha^\vee, \beta \rangle Q \otimes \beta^\vee. \tag{3.2}$$

Moreover, the action of $\mathcal{D}_X^{(m,k)}(\lambda)$ on $\mathcal{L}(\lambda)$ is given by

$$(t \otimes P \otimes t^\vee) \bullet s = (P \bullet \langle t^\vee, s \rangle) t \quad (s, t \in \mathcal{L}(\lambda),\ t^\vee \in \mathcal{L}(\lambda)^\vee).$$

Remark 3.9. Given that the locally free $\mathcal{O}_X$-modules of rank one $\mathcal{L}(\lambda)^\vee$ and $\mathcal{L}(\lambda)$ are in particular flat, the sheaf $\mathcal{D}_X^{(m,k)}(\lambda)$ is filtered by the order of twisted differential operators. That is, the subsheaf $\mathcal{D}_{X,d}^{(m,k)}$ of $\mathcal{D}_X^{(m,k)}$ of differential operators of order less than d induces a subsheaf of twisted differential operators of order less than d by

$$\mathcal{D}_{X,d}^{(m,k)}(\lambda) = \mathcal{L}(\lambda) \otimes_{\mathcal{O}_X} \mathcal{D}_{X,d}^{(m,k)} \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)^\vee. \tag{3.3}$$

Given that the tensor product preserves inductive limits, we obtain

$$\mathcal{D}_X^{(m,k)}(\lambda) = \varinjlim_d \mathcal{D}_{X,d}^{(m,k)}(\lambda).$$

Moreover, the exact sequence

$$0 \to \mathcal{D}_{X,d-1}^{(m,k)} \to \mathcal{D}_{X,d}^{(m,k)} \to \mathcal{D}_{X,d}^{(m,k)} / \mathcal{D}_{X,d-1}^{(m,k)} \to 0$$

and the relation (3.3) give us the isomorphisms

$$\mathrm{gr}(\mathcal{D}_X^{(m,k)}(\lambda)) \simeq \mathcal{L}(\lambda) \otimes_{\mathcal{O}_X} \mathrm{gr}(\mathcal{D}_X^{(m,k)}) \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)^\vee \simeq \mathrm{gr}(\mathcal{D}_X^{(m,k)}).$$

The second isomorphism is defined by $\alpha \otimes P \otimes \alpha^\vee \mapsto \alpha^\vee(\alpha) P$. This is well defined because $\mathrm{gr}(\mathcal{D}_X^{(m,k)})$ is in particular a commutative ring.

Proposition 3.10. *There exists a canonical isomorphism of graded sheaves of algebras*

$$\mathrm{gr}_\bullet(\mathcal{D}_X^{(m,k)}(\lambda)) \xrightarrow{\simeq} \mathrm{Sym}^{(m)}(\varpi^k \mathcal{T}_X).$$

PROOF. By (2.1) and the fact that $\mathcal{D}_X^{(m,k)}$ and $\varpi^k \mathcal{T}_X$ are locally free sheaves (and therefore free ϖ-torsion) we have the short exact sequence

$$0 \to \mathcal{D}_{X,d-1}^{(m,k)} \to \mathcal{D}_{X,d}^{(m,k)} \to \operatorname{Sym}_d^{(m)}(\varpi^k \mathcal{T}_X) \to 0,$$

which gives us the isomorphisms

$$\operatorname{Sym}^{(m)}(\varpi^k \mathcal{T}_X) \simeq \operatorname{gr}_\bullet(\mathcal{D}_X^{(m,k)}) \simeq \operatorname{gr}_\bullet(\mathcal{D}_X^{(m,k)}(\lambda)). \qquad \blacksquare$$

In the next proposition we will use the notation introduced in Sections 2.1 and 2.2.

PROPOSITION 3.11. *There exists a covering $\mathcal{S}$ of X by affine open subsets such that over every open subset $U \in \mathcal{S}$ the rings $\mathcal{D}_U^{(m,k)}(\lambda)$ and $\mathcal{D}_U^{(m,k)}$ are isomorphic.*

PROOF. Let us start by considering an affine open subset $U \subseteq X$ endowed with local coordinates $x_1, \dots, x_M$. For every $\underline{v} \in \mathbb{N}^M$ and $f \in \mathcal{O}_X(U)$ we have the following relation [3, Proposition 2.2.4 (iv)]:

$$\underline{\partial}^{\langle \underline{v} \rangle} f = \sum_{\underline{v}'+\underline{v}''=\underline{v}} \left\{ \begin{matrix} \underline{v} \\ \underline{v}' \end{matrix} \right\} \underline{\partial}^{\langle \underline{v}' \rangle}(f) \underline{\partial}^{\langle \underline{v}'' \rangle} \in \mathcal{D}_U^{(m,0)} = \mathcal{D}_U^{(m)}.$$

Now, let us take an affine covering $\mathcal{S}$ of X such that every $U \in \mathcal{S}$ is endowed with local coordinates, and assume that there exists a local section $\alpha \in \mathcal{L}(\lambda)(U)$ such that $\mathcal{L}(\lambda)|_U = \alpha \mathcal{O}_U$ and $\mathcal{L}(\lambda)^\vee|_U = \alpha^\vee \mathcal{O}_U$, where $\alpha^\vee$ denotes the dual element associated to α. Let us show that

$$\mathcal{D}_U^{(m,k)}(\lambda) = \bigoplus_{\underline{v}} \varpi^{k|\underline{v}|} \mathcal{O}_U \cdot (\alpha \otimes \underline{\partial}^{\langle \underline{v} \rangle} \otimes \alpha^\vee). \tag{3.4}$$

To do that, it is enough to show that for every $\underline{v} \in \mathbb{N}^M$ and $f, g \in \mathcal{O}_U$ the section $\alpha \otimes \varpi^{k|\underline{v}|} f \underline{\partial}^{\langle \underline{v} \rangle} \otimes g\alpha^\vee$ belongs to the right side of (3.4). In fact, from the first part of the proof we have

$$\begin{aligned} \alpha \otimes \varpi^{k|\underline{v}|} f \underline{\partial}^{\langle \underline{v} \rangle} \otimes g\alpha^\vee &= \alpha \otimes \varpi^{k|\underline{v}|} f \underline{\partial}^{\langle \underline{v} \rangle} g \otimes \alpha^\vee \\ &= \sum_{\underline{v}'+\underline{v}''=\underline{v}} \varpi^{k|\underline{v}|} f \left\{ \begin{matrix} \underline{v} \\ \underline{v}' \end{matrix} \right\} \underline{\partial}^{\langle \underline{v}' \rangle}(g) \alpha \otimes \underline{\partial}^{\langle \underline{v}'' \rangle} \otimes \alpha^\vee \end{aligned}$$

and we get the relation (3.4). Let us consider the map $\theta : \mathcal{D}_U^{(m,k)}(\lambda) \to \mathcal{D}_U^{(m,k)}$ defined by

$$\theta(\varpi^{k|\underline{v}|} f \alpha \otimes \underline{\partial}^{\langle \underline{v} \rangle} \otimes \alpha^\vee) = \varpi^{k|\underline{v}|} f \underline{\partial}^{\langle \underline{v} \rangle}$$

and let us see that θ is a homomorphism of rings (the multiplication on the left is given by (3.2)). By (3.4), the elements in $\mathcal{D}_U^{(m,k)}(\lambda)$ are linear combinations of the terms

$\varpi^{k|\underline{v}|} f\alpha \otimes \underline{\partial}^{\langle \underline{v}\rangle} \otimes \alpha^{\vee}$; therefore, it is enough to show that θ preserves the multiplicative structure over the elements of this form. So, let us take $\underline{v}, \underline{u} \in \mathbb{N}$ and $f, g \in \mathcal{O}_U$. On the one hand

$$\begin{aligned}
&\theta(\varpi^{k|\underline{v}|} f\alpha \otimes \underline{\partial}^{\langle \underline{v}\rangle} \otimes \alpha^{\vee} \bullet \varpi^{k|\underline{u}|} g\alpha \otimes \underline{\partial}^{\langle \underline{u}\rangle} \otimes \alpha^{\vee}) \\
&\quad = \theta(\varpi^{k|\underline{v}|} f\alpha \otimes \underline{\partial}^{\langle \underline{v}\rangle} \varpi^{k|\underline{u}|} g \underline{\partial}^{\langle \underline{u}\rangle} \otimes \alpha^{\vee}) \\
&\quad = \sum_{\underline{v}'+\underline{v}''=\underline{v}} \varpi^{k|\underline{v}|} f \left\{ \begin{matrix} \underline{v} \\ \underline{v}' \end{matrix} \right\} \underline{\partial}^{\langle \underline{v}'\rangle}(\varpi^{k|\underline{u}|} g) \underline{\partial}^{\langle \underline{v}''\rangle} \underline{\partial}^{\langle \underline{u}\rangle},
\end{aligned}$$

and on the other hand

$$\begin{aligned}
&\theta(\varpi^{k|\underline{v}|} f\alpha \otimes \underline{\partial}^{\langle \underline{v}\rangle} \otimes \alpha^{\vee}) \bullet \theta(\varpi^{k|\underline{u}|} g\alpha \otimes \underline{\partial}^{\langle \underline{u}\rangle} \otimes \alpha^{\vee}) \\
&\quad = \varpi^{k|\underline{v}|} f \underline{\partial}^{\langle \underline{v}\rangle} \bullet \varpi^{k|\underline{u}|} g \underline{\partial}^{\langle \underline{u}\rangle} \\
&\quad = \sum_{\underline{v}'+\underline{v}''=\underline{v}} \varpi^{k|\underline{v}|} f \left\{ \begin{matrix} \underline{v} \\ \underline{v}' \end{matrix} \right\} \underline{\partial}^{\langle \underline{v}'\rangle}(\varpi^{k|\underline{u}|} g) \underline{\partial}^{\langle \underline{v}''\rangle} \underline{\partial}^{\langle \underline{u}\rangle}.
\end{aligned}$$

Both equations show that θ is a ring homomorphism.

Finally, an analogous reasoning shows that the morphism $\theta^{-1} : \mathcal{D}_U^{(m,k)} \to \mathcal{D}_U^{(m,k)}(\lambda)$ defined by

$$\theta^{-1}(\varpi^{k|\underline{v}|} f \underline{\partial}^{\langle \underline{v}\rangle}) = \varpi^{k|\underline{v}|} f\alpha \otimes \underline{\partial}^{\langle \underline{v}\rangle} \otimes \alpha^{\vee}$$

is also a homomorphism of rings and $\theta \circ \theta^{-1} = \theta^{-1} \circ \theta = \mathrm{id}$. ∎

Definition 3.12 (Congruence subgroups and wide open congruence subgroups). Let us denote by $\mathbb{F}_q = \mathfrak{o}/(\varpi)$ the residue field of $\mathfrak{o}$, and let us consider the generic fiber of $\mathbb{G}$,

$$\mathbb{G}_{\mathbb{Q}} = \mathbb{G} \times_{\mathrm{Spec}(\mathfrak{o})} \mathrm{Spec}(L),$$

and the special fiber

$$\mathbb{G}_{\mathbb{F}_q} = \mathbb{G} \times_{\mathrm{Spec}(\mathfrak{o})} \mathrm{Spec}(\mathbb{F}_q).$$

For every $k \in \mathbb{N}$, there exists a smooth model $\mathbb{G}(k)$ of $\mathbb{G}$ such that $\mathrm{Lie}(\mathbb{G}(k)) = \varpi^k \mathfrak{g}$. In fact, we take $\mathbb{G}(0) = \mathbb{G}$ and we construct $\mathbb{G}(1)$ as the dilatation of the trivial subgroup of $\mathbb{G}_{\mathbb{F}_q}$ in $\mathbb{G}$ (see [7, Section 3.2]). This is a flat $\mathfrak{o}$-scheme which is an integral model of $\mathbb{G}_{\mathbb{Q}}$ (see [35, Proposition 1.1]). In general, we let $\mathbb{G}(k+1)$ be the dilatation of the trivial subgroup of $\mathbb{G}(k)_{\mathbb{F}_q}$ in $\mathbb{G}(k)$, in such a way that for every $k \in \mathbb{N}$ we dispose of a canonical morphism $\mathbb{G}(k+1) \to \mathbb{G}(k)$.

Let us describe the distribution algebra $D^{(m)}(\mathbb{G}(k))$ of the congruence group $\mathbb{G}(k)$ (see [20, Section 3.3]). Let us take a triangular decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t} \oplus \overline{\mathfrak{n}}$ and let us consider the basis (f_i), (h_j) and (e_l) of the $\mathfrak{o}$-Lie algebra $\mathfrak{n}$, $\mathfrak{t}$ and $\overline{\mathfrak{n}}$, respectively. Then $D^{(m)}(\mathbb{G}(k))$ equals the $\mathfrak{o}$-subalgebra of $\mathcal{U}(\mathfrak{g}) \otimes_{\mathfrak{o}} L$ generated as an $\mathfrak{o}$-module by the elements

$$q_{\underline{v}}!\varpi^{k|\underline{v}|}\frac{\underline{f}^{\underline{v}}}{\underline{v}!}q_{\underline{v}'}!\varpi^{k|\underline{v}'|}\binom{\underline{h}}{\underline{v}'}q_{\underline{v}''}!\varpi^{k|\underline{v}''|}\frac{\underline{e}^{\underline{v}''}}{\underline{v}''!}. \tag{3.5}$$

An element of the preceding form has order $d = |v| + |v'| + |v''|$. Therefore, the $\mathfrak{o}$-spam of elements of order at most d defines an $\mathfrak{o}$-submodule $D_d^{(m)}(\mathbb{G}(k)) \subset D^{(m)}(\mathbb{G}(k))$. In this way $D^{(m)}(\mathbb{G}(k))$ becomes a filtered $\mathfrak{o}$-algebra, such that by (3.5) and the well-known Poincaré–Birkhoff–Witt theorem we have

$$D^{(m)}(\mathbb{G}(k)) \otimes_{\mathfrak{o}} L = \mathcal{U}(\mathfrak{g}) \otimes_{\mathfrak{o}} L.$$

The preceding discussion also tells us that

$$D^{(m)}(\mathbb{G}(0))_{\varpi^k} = D^{(m)}(\mathbb{G}(k)).$$

Finally, let us introduce a family of certain rigid-analytic "wide-open" groups $\mathbb{G}(k)^{\circ}$, which will be important in our work. To do this, let us first consider the formal completion $\mathfrak{G}(k)$ of the group scheme $\mathbb{G}(k)$ along its special fiber, which is a formal scheme of topological finite type over $\mathrm{Spf}(\mathfrak{o})$. Now, we consider the completion $\widehat{\mathfrak{G}}(k)^{\circ}$ of $\mathfrak{G}(k)^{\circ}$ along its unit section $\mathrm{Spf}(\mathfrak{o}) \to \mathfrak{G}(k)$, and we denote by $\mathbb{G}(k)^{\circ}$ its associated rigid-analytic space [2, (0.2.6)], which is a rigid-analytic group.

We recall for the reader that in Section 2.5 we have introduced the sheaves

$$\mathcal{A}_X^{(m,k)} = \mathcal{O}_X \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{G}(k)),$$

which carries a structure of filtered $\mathcal{O}_X$-rings, such that

$$\mathrm{gr}(\mathcal{A}_X^{(m,k)}) = \mathcal{O}_X \otimes_{\mathfrak{o}} \mathrm{Sym}^{(m)}(\varpi^k \mathfrak{g}).$$

Proposition 3.13. *There exists a canonical surjective homomorphism of sheaves of filtered* $\mathfrak{o}$*-algebras*

$$\Phi_X^{(m,k)} : \mathcal{A}_X^{(m,k)} \to \mathcal{D}_X^{(m,k)}(\lambda).$$

Proof. Let us start by showing the existence of such a morphism. By [21, Corollary 4.5.2], there exists a morphism of sheaves of filtered $\mathfrak{o}$-algebras

$$\mathcal{A}_X^{(m,0)} \to \mathcal{D}_X^{(m,0)}(\lambda). \tag{3.6}$$

Let us first show that after specializing in ϖ^k the Rees ring associated to the twisted order filtration of $\mathcal{D}_X^{(m,0)}(\lambda)$, we get $\mathcal{D}_X^{(m,k)}(\lambda)$. To do that, we consider $\mathcal{D}_X^{(m,0)}$ filtered by the order of differential operators and we define the homomorphisms of $\mathcal{O}_X$-modules

$$\mathcal{L}(\lambda) \otimes_{\mathcal{O}_X} R(\mathcal{D}_X^{(m,0)}) \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)^\vee \underset{\theta^{-1}}{\overset{\theta}{\rightleftarrows}} R(\mathcal{D}_X^{(m,0)}(\lambda)), \tag{3.7}$$

by

$$\theta\Big(\alpha \otimes \sum_i P_i t^i \otimes \alpha^\vee\Big) = \sum_i (\alpha \otimes P_i \otimes \alpha^\vee) t^i$$

with $\mathrm{ord}(P_i) = i$ for every i in the sum, and the obvious definition for θ^{-1} such that $\theta \circ \theta^{-1} = \theta^{-1} \circ \theta = \mathrm{id}$. This shows that (3.7) is an isomorphism of $\mathcal{O}_X$-modules and an easy calculation shows that (3.7) is in fact an isomorphism of rings.

Let us denote by

$$\sigma_1 : R(\mathcal{D}_X^{(m,0)}(\lambda)) \to \mathcal{D}_X^{(m,k)}(\lambda) \quad \text{and} \quad \sigma_2 : R(\mathcal{D}_X^{(m,0)}) \to \mathcal{D}_X^{(m,k)}$$

the morphisms sending $t \mapsto \varpi^k$, and let us consider the diagrams

$$\begin{array}{ccc}
\mathcal{L}(\lambda) \otimes_{\mathcal{O}_X} R(\mathcal{D}_X^{(m,0)}) \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)^\vee & \underset{\theta^{-1}}{\overset{\theta}{\rightleftarrows}} & R(\mathcal{D}_X^{(m,0)}(\lambda)) \\
& \searrow^{\mathrm{id}_{\mathcal{L}(\lambda)} \otimes \sigma_2 \otimes \mathrm{id}_{\mathcal{L}(\lambda)^\vee}} & \downarrow \sigma_1 \\
& & \mathcal{D}_X^{(m,k)}(\lambda).
\end{array}$$

It is straightforward to check that both diagrams are commutative and we can conclude that

$$\begin{aligned}
(\mathcal{D}_{X_0}^{(m,0)}(\lambda))_{\varpi^k} &= \mathrm{im}(\sigma_1) = \mathrm{im}(\mathrm{id}_{\mathcal{L}(\lambda)} \otimes \sigma_2 \otimes \mathrm{id}_{\mathcal{L}(\lambda)^\vee}) \\
&= \mathcal{L}(\lambda) \otimes_{\mathcal{O}_X} \mathrm{im}(\sigma_2) \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)^\vee = \mathcal{D}_X^{(m,k)}(\lambda).
\end{aligned}$$

On the other hand, taking the natural filtration of $\mathcal{A}_X^{(m,0)}$, we have

$$R(\mathcal{A}_X^{(m,0)}) = \mathcal{O}_X \otimes_{\mathfrak{o}} R(D^{(m)}(\mathbb{G}(0))).$$

Therefore, $(\mathcal{A}_X^{(m,0)})_{\varpi^k} = \mathcal{A}_X^{(m,k)}$. The above two calculations tell us that passing to the Rees rings in the map (3.6) and specializing in ϖ^k, we get the desired homomorphism of filtered sheaves of $\mathfrak{o}$-algebras

$$\Phi_X^{(m,k)} : \mathcal{A}_X^{(m,k)} \to \mathcal{D}_X^{(m,k)}(\lambda). \tag{3.8}$$

Let us finally show that this morphism is surjective. To do that, let us recall that the right $\mathbb{G}$-action on X induces a canonical application

$$\mathcal{O}_X \otimes_{\mathfrak{o}} \varpi^k \mathfrak{g} \to \varpi^k \mathcal{T}_X, \tag{3.9}$$

which is surjective by [28, Section 1.6]. Given that $\mathrm{gr}(\mathcal{A}_X^{(m,k)}) = \mathcal{O}_X \otimes_{\mathfrak{o}} \mathrm{Sym}^{(m)}(\varpi^k \mathfrak{g})$, we can conclude from Proposition 3.10 that $\Phi_X^{(m,k)}$ is surjective. ∎

Proposition 3.11 and the same reasoning as in [23, Proposition 2.2.2 (iii)] imply the following meaningful result.[6]

Proposition 3.14. *The sheaf $\mathcal{D}_X^{(m,k)}(\lambda)$ is a sheaf of $\mathcal{O}_X$-rings with noetherian sections over all open affine subsets of X.*

3.4 – *Finiteness properties*

Notation 3.15. To soft the notation in the arguments that we will realize throughout this subsection, from now on we will denote by $\mathcal{D}_{X,\lambda}^{(m,k)}$ the sheaf $\mathcal{D}_X^{(m,k)}(\lambda)$ introduced in Definition 3.8 (see [32, Proposition 3.5.18]).

Throughout this subsection, $\lambda \in X(\mathbb{T})$ will denote an algebraic character. By abuse of notation, we will denote again by λ the character $d\lambda \in \mathrm{Hom}_{\mathfrak{o}\text{-mod}}(\mathfrak{t}, \mathfrak{o})$ induced via differentiation. We will show one important property about the p-torsion of the cohomology groups of coherent $\mathcal{D}_{X,\lambda}^{(m,k)}$-modules, when the character $\lambda + \rho$ is dominant and regular. We will follow the arguments in [28].

Let Y be a projective scheme. There exists a very ample sheaf $\mathcal{O}(1)$ on Y (see [17, Chapter II, Remark 5.16.1]). Therefore, for any arbitrary $\mathcal{O}_Y$-module $\mathcal{E}$ we can consider the twist

$$\mathcal{E}(r) = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}(r),$$

where $r \in \mathbb{Z}$ means the r-th tensor product of $\mathcal{O}(1)$ with itself. We recall that there exists $r_0 \in \mathbb{Z}$, depending of $\mathcal{O}(1)$, such that for every $k \in \mathbb{Z}_{>0}$ and for every $s \geq r_0$, $H^k(Y, \mathcal{O}(s)) = 0$ (see [17, Chapter II, Theorem 5.2 (b)]).

Let us start the results of this subsection with the following proposition which states three important properties of coherent $\mathcal{A}_Y^{(m,k)}$-modules (see [21, Proposition A.2.6.1]). This is a key result for our work. Let $\mathcal{E}$ be a coherent $\mathcal{A}_Y^{(m,k)}$-module.

Proposition 3.16. (i) *$H^0(X, \mathcal{A}_Y^{(m,k)}) = D^{(m)}(\mathbb{G}(k))$ is a noetherian $\mathfrak{o}$-algebra.*

(ii) *There exists a surjection $(\mathcal{A}_Y^{(m,k)}(-r))^{\oplus a} \to \mathcal{E} \to 0$ on $\mathcal{A}_Y^{(m,k)}$-modules for suitable $r \in \mathbb{Z}$ and $a \in \mathbb{N}$.*

(iii) *For any $k \geq 0$ the group $H^k(X, \mathcal{E})$ is a finitely generated $D^{(m)}(\mathbb{G}(k))$-module.*

[6] Of course, this is also an immediately consequence of Proposition 3.13 and [19, Proposition 1.3.6].

Inspired by Proposition 3.13, we will first focus on coherent $\mathcal{A}_Y^{(m,k)}$-modules. The next two results will play an important role when we consider formal completions.

Lemma 3.17. *For every coherent $\mathcal{A}_Y^{(m,k)}$-module $\mathcal{E}$, there exists $r = r(\mathcal{E}) \in \mathbb{Z}$ such that $H^k(X, \mathcal{E}(s)) = 0$ for every $s \geq r$.*

Proof. Let us fix $r_0 \in \mathbb{Z}$ such that $H^k(Y, \mathcal{O}(s)) = 0$ for every $k > 0$ and $s \geq r_0$. We have

$$H^k(Y, \mathcal{A}_Y^{(m,k)}(s)) = H^k(Y, \mathcal{O}(s)) \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{G}(k)) = 0.$$

The rest of the proof follows the inductive argument given in [28, Proposition 2.2.1]. ∎

Let us suppose now that $X = \mathbb{G}/\mathbb{B}$ is the smooth flag $\mathfrak{o}$-scheme of $\mathbb{G}$. From Proposition 3.13 and Lemma 3.17 we have the following result.

Lemma 3.18. *For every coherent $\mathcal{D}_{X,\lambda}^{(m,k)}$-module $\mathcal{E}$, there exist $r = r(\mathcal{E}) \in \mathbb{Z}$, a natural number $a \in \mathbb{N}$ and an epimorphism of $\mathcal{D}_{X,\lambda}^{(m,k)}$-modules*

$$(\mathcal{D}_{X,\lambda}^{(m,k)}(-r))^{\oplus a} \rightarrow \mathcal{E} \rightarrow 0.$$

Proposition 3.19. *Suppose that $\lambda + \rho \in \mathfrak{t}_{\mathbb{Q}}^*$ is a dominant and regular character (cf. Definition 3.6).*

(i) *Let $r \in \mathbb{Z}$ be fixed. For every positive integer $k \in \mathbb{Z}_{>0}$, the cohomology group $H^k(X, \mathcal{D}_{X,\lambda}^{(m,k)}(r))$ has bounded p-torsion.*

(ii) *For every coherent $\mathcal{D}_{X,\lambda}^{(m,k)}$-module $\mathcal{E}$, the cohomology group $H^k(X, \mathcal{E})$ has bounded p-torsion for all $k > 0$.*

Proof. To show (i) we remark that by construction $\mathcal{D}_{X,\lambda,\mathbb{Q}}^{(m,k)} = \mathcal{D}_\lambda$ is the usual sheaf of twisted differential operators on the flag variety X_L (see [24, Part I, §5.17]). As $\mathcal{D}_{X,\lambda,\mathbb{Q}}^{(m,k)}(r)$ is a coherent $\mathcal{D}_\lambda$-module, the classical Beilinson–Bernstein theorem (see [1, p. 2]) allows us to conclude that

$$H^k(X, \mathcal{D}_{X,\lambda}^{(m,k)}(r)) \otimes_{\mathfrak{o}} L = 0$$

for every positive integer $k \in \mathbb{Z}_{>0}$. This in particular implies that the sheaf $\mathcal{D}_{X,\lambda}^{(m,k)}(r)$ has p-torsion cohomology groups $H^k(X, \mathcal{D}_{X,\lambda}^{(m,k)}(r))$, for every $k > 0$ and $r \in \mathbb{Z}$. By Proposition 3.13, we know that $\mathcal{D}_{X,\lambda}^{(m,k)}(r)$ is in particular a coherent $\mathcal{A}_X^{(m,k)}$-module and hence, by the third part of Proposition 3.16 we get that for every $k \geq 0$ the cohomology groups $H^k(X, \mathcal{D}_{X,\lambda}^{(m)}(r))$ are finitely generated $D^{(m)}(\mathbb{G}(k))$-modules. Consequently, they are of finite p-torsion for every integer $0 < k \leq \dim(X)$ and $r \in \mathbb{Z}$.

By Lemma 3.18 we can use the same reasoning as in [28, Corollary 2.2.4] to show (ii). ∎

3.5 – *Passing to formal completions*

We recall for the reader that throughout this work

$$\mathfrak{X} = \varinjlim_{j\in\mathbb{N}} X_j, \quad X_j = X \times_{\mathrm{Spec}(\mathfrak{o})} \mathrm{Spec}(\mathfrak{o}/\varpi^{j+1})$$

denotes the formal completion of X along its special fiber.

Definition 3.20. We will denote by

$$\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda} = \varprojlim_{j\in\mathbb{N}} \mathcal{D}^{(m,k)}_{X,\lambda}/\varpi^{j+1}\mathcal{D}^{(m,k)}_{X,\lambda}$$

the ϖ-adic completion of $\mathcal{D}^{(m,k)}_{X,\lambda}$ and we consider it as a sheaf on $\mathfrak{X}$. Following the notation given at the beginning of this work, the sheaf $\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}$ will denote our sheaf of level m twisted differential operators with congruence level k on the formal flag scheme $\mathfrak{X}$.

Proposition 3.21. (i) *There exists a basis $\mathcal{B}$ of the topology of $\mathfrak{X}$, consisting of open affine subsets, such that for every $\mathfrak{U} \in \mathcal{B}$ the ring $\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda}(\mathfrak{U})$ is two-sided noetherian.*

(ii) *The sheaf of rings $\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}$ is coherent.*

Proof. To show (i) we can take an open affine subset $U \in \mathcal{S}$ and consider its formal completion $\mathfrak{U}$ along the special fiber. We have

$$H^0(\mathfrak{U}, \widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda}) \simeq \widehat{H^0(U, \mathcal{D}^{(m,k)}_{X,\lambda})} \simeq \widehat{H^0(U, \mathcal{D}^{(m,k)}_{X})} \simeq H^0(\mathfrak{U}, \widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X}}).$$

The first and third isomorphisms are given by [15, $(0_I$, 3.2.6)] and the second one arises from Proposition 3.11. By [23, Proposition 2.2.2 (v)] $H^0(\mathfrak{U}, \widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X}})$ is two-sided noetherian. Therefore, we can take $\mathcal{B}$ as the set of affine open subsets of $\mathfrak{X}$ contained in the ϖ-adic completion of an affine open subset $U \in \mathcal{S}$. This proves (i).

By [3, Proposition 3.3.4] we can conclude that (ii) is an immediately consequence of (i) because by [3, (3.4.0.1)]

$$H^0(\mathfrak{U}, \widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}) = H^0(\mathfrak{U}, \widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda}) \otimes_{\mathfrak{o}} L. \qquad \blacksquare$$

From now on, we will always assume that $\lambda + \rho \in \mathfrak{t}^*_{\mathbb{Q}}$ denotes a dominant and regular character, which is induced by an algebraic character $\lambda \in X(\mathbb{T})$. Our next objective is to prove an analogue of Proposition 3.19 for coherent $\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-modules and to conclude that $H^0(\mathfrak{X}, \bullet)$ is an exact functor over the category of coherent $\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-modules.

Proposition 3.22. *Let $\mathcal{E}$ be a coherent $\mathcal{D}^{(m,k)}_{X,\lambda}$-module and $\widehat{\mathcal{E}}$ its ϖ-adic completion, which we consider as a sheaf on $\mathfrak{X}$.*

(i) *For all $k \geq 0$ one has $H^k(\mathfrak{X},\widehat{\mathcal{E}}) = \varprojlim_j H^k(X,\mathcal{E}/\varpi^j\mathcal{E})$.*

(ii) *For all $k > 0$ one has $H^k(\mathfrak{X},\widehat{\mathcal{E}}) = H^k(X,\mathcal{E})$.*

(iii) *The global sections satisfy $H^0(\mathfrak{X},\widehat{\mathcal{E}}) = \varprojlim_j H^0(X,\mathcal{E})/\varpi^j H^0(X,\mathcal{E})$.*

Proof. Let $\mathcal{E}_t$ denote the torsion subpresheaf of $\mathcal{E}$. As X is a noetherian space and $\mathcal{D}^{(m,k)}_{X,\lambda}$ has noetherian ring sections over open affine subsets of X (Proposition 3.14), we can conclude that $\mathcal{E}_t$ is in fact a coherent $\mathcal{D}^{(m,k)}_{X,\lambda}$-module. This is generated by a coherent $\mathcal{O}_X$-module which is annihilated by a power ϖ^c of ϖ, and so is $\mathcal{E}_t$. The quotient $\mathcal{G} := \mathcal{E}/\mathcal{E}_t$ is again a coherent $\mathcal{D}^{(m,k)}_{X,\lambda}$-module; therefore, we can assume, after possibly replacing c by a larger number, that

$$\varpi^c\mathcal{E}_t = 0 \quad \text{and} \quad \varpi^c H^k(X,\mathcal{E}) = \varpi^c H^k(X,\mathcal{G}) = 0 \quad \text{for all } k > 0.$$

From here on the proof of the proposition follows the same reasoning given in [19, Proposition 3.2]. ■

The next proposition is a natural consequence of Lemmas 3.17 and 3.18. The proof is exactly the same as that of [20, Proposition 4.2.2].[7]

Proposition 3.23. *Let $\mathscr{E}$ be a coherent $\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda}$-module.*

(i) *There exists $r_2 = r_2(\mathscr{E}) \in \mathbb{Z}$ such that for all $r \geq r_2$ there are $a \in \mathbb{Z}$ and an epimorphism of $\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda}$-modules*

$$(\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda}(-r))^{\oplus a} \to \mathscr{E} \to 0.$$

(ii) *There exists $r_3 = r_3(\mathscr{E}) \in \mathbb{Z}$ such that for all $r \geq r_3$ one has $H^k(\mathfrak{X},\mathscr{E}) = 0$ for all $k > 0$.*

The same inductive argument exhibited in [19, Proposition 3.4 (i)] shows the following result.

Corollary 3.24. *Let $\mathscr{E}$ be a coherent $\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda}$-module. There exists $c = c(\mathscr{E}) \in \mathbb{N}$ such that for all $k > 0$ the cohomology group $H^k(\mathfrak{X},\mathscr{E})$ is annihilated by ϖ^c.*

(7) We skip the proof here, but the reader can take a look at [31, Proposition 4.1.2] where we have treated the case $k = 0$. The proof for $k \in \mathbb{Z}_{>0}$ is exactly the same.

Now, we want to extend part (i) of the preceding proposition to the sheaves $\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}$. To do that, we need to show that the category of coherent $\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-modules admits integral models (Definition 2.6).

Let $\mathrm{Coh}(\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda})$ be the category of coherent $\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda}$-modules. Let $\mathrm{Coh}(\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda})_{\mathbb{Q}}$ be the category of coherent $\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda}$-modules up to isogeny, whose class of objects is the same as that of $\mathrm{Coh}(\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda})$. For any two objects $\mathcal{M}$ and $\mathcal{N}$ in $\mathrm{Coh}(\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda})_{\mathbb{Q}}$, one has

$$\mathrm{Hom}_{\mathrm{Coh}(\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda})_{\mathbb{Q}}}(\mathcal{M},\mathcal{N}) = \mathrm{Hom}_{\mathrm{Coh}(\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda})}(\mathcal{M},\mathcal{N}) \otimes_{\mathfrak{o}} L.$$

Proposition 3.25. *The functor $\mathcal{M} \mapsto \mathcal{M} \otimes_{\mathfrak{o}} L$ induces an equivalence of categories between $\mathrm{Coh}(\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda})_{\mathbb{Q}}$ and $\mathrm{Coh}(\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}})$.*

Proof. By definition, the sheaf $\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}$ satisfies [3, Conditions 3.4.1] and therefore [3, Proposition 3.4.5] allows us to conclude the proposition. ∎

The proof of the next theorem follows exactly the same lines as that of [20, Theorem 4.2.8].

Theorem 3.26. *Let $\mathcal{E}$ be a coherent $\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-module.*

(i) *There is $r(\mathcal{E}) \in \mathbb{Z}$ such that for every $r \geq r(\mathcal{E})$ there exist $a \in \mathbb{N}$ and an epimorphism of $\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-modules*

$$(\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}(-r))^{\oplus a} \to \mathcal{E} \to 0.$$

(ii) *For all $i > 0$ one has $H^i(\mathfrak{X}, \mathcal{E}) = 0$.*

Proof. By the preceding proposition, there exists a coherent $\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda}$-module $\mathcal{F}$ such that $\mathcal{F} \otimes_{\mathfrak{o}} L \simeq \mathcal{E}$. Therefore, applying Proposition 3.23 to $\mathcal{F}$ gives (i). Moreover, as $\mathfrak{X}$ is a noetherian space, Corollary 3.24 allows us to conclude that

$$H^i(\mathfrak{X}, \mathcal{E}) = H^i(\mathfrak{X}, \mathcal{F}) \otimes_{\mathfrak{o}} L = 0$$

for every $k > 0$ (see [3, (3.4.0.1)]). ∎

3.6 – *The arithmetic Beilinson–Bernstein theorem with congruence level*

3.6.1. Calculation of global sections. Inspired by the arguments exhibited in [22], in this subsection we calculate the global sections of the sheaf $\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}$.

Let us identify the universal enveloping algebra $\mathcal{U}(\mathfrak{t}_{\mathbb{Q}})$ of the Cartan subalgebra $\mathfrak{t}_{\mathbb{Q}}$ with the symmetric algebra $S(\mathfrak{t}_{\mathbb{Q}})$, and let $Z(\mathfrak{g}_{\mathbb{Q}})$ denote the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_{\mathbb{Q}})$ of $\mathfrak{g}_{\mathbb{Q}}$. The classical Harish-Chandra isomorphism $Z(\mathfrak{g}_{\mathbb{Q}}) \simeq$

$S(\mathfrak{t}_{\mathbb{Q}})^W$ (the subalgebra of Weyl invariants) [11, Theorem 7.4.5] allows us to define for every linear form $\lambda \in \mathfrak{t}_{\mathbb{Q}}^*$ a central character [11, §7.4.6]

$$\chi_{\lambda+\rho} : Z(\mathfrak{g}_{\mathbb{Q}}) \to L,$$

which induces the central reduction $\mathcal{U}(\mathfrak{g}_{\mathbb{Q}})_\lambda = \mathcal{U}(\mathfrak{g}_{\mathbb{Q}}) \otimes_{Z(\mathfrak{g}_{\mathbb{Q}}),\chi_{\lambda+\rho}} L$. If we put $\mathrm{Ker}(\chi_{\lambda+\rho})_{\mathfrak{o}} = D^{(m)}(\mathbb{G}(k)) \cap \mathrm{Ker}(\chi_{\lambda+\rho})$, we can consider the central redaction

$$D^{(m)}(\mathbb{G}(k))_\lambda = D^{(m)}(\mathbb{G}(k))/D^{(m)}(\mathbb{G}(k))\,\mathrm{Ker}(\chi_{\lambda+\rho})_{\mathfrak{o}}$$

and its ϖ-adic completion $\widehat{D}^{(m)}(\mathbb{G}(k))_\lambda$. It is clear that $D^{(m)}(\mathbb{G}(k))_\lambda$ is an integral model of $\mathcal{U}(\mathfrak{g}_{\mathbb{Q}})_\lambda$. We denote by $D^{\dagger}(\mathbb{G}(k))_\lambda$ the limit of the inductive system

$$\widehat{D}^{(m)}(\mathbb{G}(k))_\lambda \otimes_{\mathfrak{o}} L \to \widehat{D}^{(m+1)}(\mathbb{G}(k))_\lambda \otimes_{\mathfrak{o}} L.$$

Theorem 3.27. *The morphism $\Phi_\lambda^{(m,k)} : D^{(m)}(\mathbb{G}(k)) \to H^0(X, \mathcal{D}_{X,\lambda}^{(m,k)})$, defined by taking global sections in Proposition 3.13, induces an isomorphism of L-algebras*

$$\widehat{D}^{(m)}(\mathbb{G}(k))_\lambda \otimes_{\mathfrak{o}} L \xrightarrow{\simeq} H^0(\mathfrak{X}, \widehat{\mathcal{D}}_{\mathfrak{X},\lambda,\mathbb{Q}}^{(m,k)}).$$

Proof. The key in the proof is the commutative diagram

$$\begin{array}{ccc} D^{(m)}(\mathbb{G}(k)) & \xrightarrow{\Phi_\lambda^{(m,k)}} & H^0(X, \mathcal{D}_{X,\lambda}^{(m,k)}) \\ \downarrow & & \downarrow \\ \mathcal{U}(\mathfrak{g}_{\mathbb{Q}}) & \xrightarrow{\Phi_\lambda} & H^0(X_{\mathbb{Q}}, \mathcal{D}_\lambda). \end{array}$$

Here Φ_λ is the morphism in [18, (11.2.2)].[8] By the classical Beilinson–Bernstein theorem [1] and the preceding commutative diagram, we have that $\Phi_\lambda^{(m,k)}$ factors through the morphism

$$\overline{\Phi}_\lambda^{(m,k)} : D^{(m)}(\mathbb{G}(k))_\lambda \to H^0(X, \mathcal{D}_{X,\lambda}^{(m,k)}),$$

which becomes an isomorphism after tensoring with L. By [22, Lemma 3.3] we have that $\overline{\Phi}_\lambda^{(m)}$ gives rise to an isomorphism

$$\widehat{D}^{(m)}(\mathbb{G}(k))_\lambda \otimes_{\mathfrak{o}} L \xrightarrow{\simeq} \widehat{H^0(X, \mathcal{D}_{X,\lambda}^{(m,k)})} \otimes_{\mathfrak{o}} L.$$

Proposition 3.22 together with the fact that $\mathfrak{X}$ is in particular a noetherian topological space end the proof of the theorem. ∎

([8]) We recall that $\mathcal{L}(\lambda)$ is a $\mathbb{G}$-equivariant line bundle, which implies the existence of this morphism [18, Section 11.1].

3.6.2. The localization functor. In this subsection, we will introduce the localization functor. Let E be a finitely generated $\widehat{D}^{(m)}(\mathbb{G}(k))_\lambda \otimes_{\mathfrak{o}} L$-module. We define $\mathscr{L}oc^{(m,k)}_{\mathfrak{X},\lambda}(E)$ as the associated sheaf to the presheaf on $\mathfrak{X}$ defined by

$$\mathfrak{U} \subseteq \mathfrak{X} \mapsto \widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}(\mathfrak{U}) \otimes_{\widehat{D}^{(m)}(\mathbb{G}(k))_\lambda \otimes_{\mathfrak{o}} L} E.$$

Then $\mathscr{L}oc^{(m,k)}_{\mathfrak{X},\lambda}$ is a functor from the category of finitely generated $\widehat{D}^{(m)}(\mathbb{G}(k))_\lambda \otimes_{\mathfrak{o}} L$-modules to the category of coherent $\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-modules.

3.6.3. The arithmetic Beilinson–Bernstein theorem. We are finally ready to prove one of the principal results of this work. To start with, we will enunciate the following proposition whose proof can be founded in [31, Proposition 4.4.1].

Proposition 3.28. *Let $\mathcal{E}$ be a coherent $\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-module. Then $\mathcal{E}$ is generated by its global sections as $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-module. Furthermore, every coherent $\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-module admits a resolution by finite free $\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-modules.*

Theorem 3.29. *Let us suppose that $\lambda \in X(\mathbb{T})$ is an algebraic character such that $\lambda + \rho \in \mathfrak{t}^*_{\mathbb{Q}}$ is a dominant and regular character of $\mathfrak{t}_{\mathbb{Q}}$. The functors $\mathscr{L}oc^{(m,k)}_{\mathfrak{X},\lambda}$ and $H^0(\mathfrak{X},\bullet)$ are quasi-inverse equivalences of categories between the abelian categories of finitely generated $\widehat{D}^{(m)}(\mathbb{G}(k))_\lambda \otimes_{\mathfrak{o}} L$-modules and coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-modules.*

Proof. The proof of [19, Proposition 5.2.1] carries over word by word. ∎

3.7 – *The sheaves $\mathcal{D}^\dagger_{\mathfrak{X},k}(\lambda)$*

In this subsection, we will study the problem of passing to the inductive limit when m varies, this means

$$\mathcal{D}^\dagger_{\mathfrak{X},k}(\lambda) = \Big(\varinjlim_{m\in\mathbb{N}} \widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda}\Big) \otimes_{\mathfrak{o}} L, \quad D^\dagger(\mathbb{G}(k))_\lambda = \Big(\varinjlim_{m\in\mathbb{N}} \widehat{D}^{(m)}(\mathbb{G}(k))_\lambda\Big) \otimes_{\mathfrak{o}} L.$$

As in Section 3.6.2 let us consider the following localization functor $\mathscr{L}oc^\dagger_{\mathfrak{X},k}(\lambda)$ from the category of finitely presented $D^\dagger(\mathbb{G}(k))_\lambda$-modules to the category of coherent $\mathcal{D}^\dagger_{\mathfrak{X},k}(\lambda)$. Let E be a finitely presented $D^\dagger(\mathbb{G}(k))_\lambda$-module, then $\mathscr{L}oc^\dagger_{\mathfrak{X},k}(\lambda)(E)$ denotes the associated sheaf to the presheaf on $\mathfrak{X}$ defined by

$$\mathfrak{U} \subseteq \mathfrak{X} \mapsto \mathcal{D}^\dagger_{\mathfrak{X},k}(\lambda) \otimes_{D^\dagger(\mathbb{G}(k))_\lambda} E.$$

As before, it is clear that $\mathscr{L}oc^\dagger_{\mathfrak{X},k}(\lambda)$ is a functor from the category of finitely presented $D^\dagger(\mathbb{G}(k))_\lambda$-modules to the category of coherent $\mathcal{D}^\dagger_{\mathfrak{X},k}(\lambda)$-modules.

Definition 3.30 (Analytic distribution algebra). The wide-open rigid analytic groups, defined in Definition 3.12, play an important role in the work developed by Emerton in [13], to treat locally analytic representations of p-adic groups. The analytic distribution algebra of $\mathbb{G}(k)^{\circ}$ is defined to be the continuous dual space of the space of rigid-analytic functions on $\mathbb{G}(k)^{\circ}$. That is,

$$\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ}) = \big(\mathcal{O}_{\mathbb{G}(k)^{\circ}}(\mathbb{G}(k)^{\circ})\big)'_b = \mathrm{Hom}_L^{\mathrm{cont}}\big(\mathcal{O}_{\mathbb{G}(k)^{\circ}}(\mathbb{G}(k)^{\circ}), L\big)_b,$$

which is a topological L-algebra of compact type.

In [21, Proposition 5.2.1] Huyghe and Schmidt have shown that

$$D^{\dagger}(\mathbb{G}(k)) \simeq \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ}).$$

As $\mathfrak{X}$ is a noetherian space, Theorem 3.27 and the preceding relation tell us that

$$\begin{aligned} H^0(\mathfrak{X}, \mathcal{D}^{\dagger}_{\mathfrak{X},k}(\lambda)) &= \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ})_{\lambda} \\ &= \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ})/\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ})(\mathrm{Ker}(\chi_{\lambda+\rho})). \end{aligned} \tag{3.10}$$

We will concentrate our efforts to prove the following Beilinson–Bernstein theorem for the sheaves $\mathcal{D}^{\dagger}_{\mathfrak{X},k}(\lambda)$.

Theorem 3.31. *Let $\lambda \in X(\mathbb{T})$ be an algebraic character, such that $\lambda + \rho \in \mathfrak{t}^*_{\mathbb{Q}}$ is dominant and regular. The functors $\mathcal{L}oc^{\dagger}_{\mathfrak{X},k}(\lambda)$ and $H^0(\mathfrak{X}, \bullet)$ are quasi-inverse equivalences of categories between the abelian categories of finitely presented (left) $D^{\dagger}(\mathbb{G}(k))_{\lambda}$-modules and coherent $\mathcal{D}^{\dagger}_{\mathfrak{X},k}(\lambda)$-modules.*

Let us start by recalling the following proposition [3, Proposition 3.6.1].

Proposition 3.32. *Let Y be a topological space, and $\{\mathcal{D}_i\}_{i\in J}$ be a filtered inductive system of coherent sheaves of rings on Y, such that for any $i \leq j$ the morphisms $\mathcal{D}_i \to \mathcal{D}_j$ are flat. Then the sheaf $\mathcal{D}^{\dagger} = \varinjlim_{i\in J} \mathcal{D}_i$ is a coherent sheaf of rings.*

Proposition 3.33. *The sheaf of rings $\mathcal{D}^{\dagger}_{\mathfrak{X},k}(\lambda)$ is coherent.*

Proof. The previous proposition tells us that we only need to show that the transition morphisms $\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}} \to \widehat{\mathcal{D}}^{(m+1,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}$ are flat. As this is a local property we can take $U \in \mathcal{S}$ (see the notation in Proposition 3.11) and verify this property over the formal completion $\mathfrak{U}$. In this case, the argument used in the proof of the first part of Proposition 3.21 gives us the following commutative diagram:

$$\begin{array}{ccc} \widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}(\mathfrak{U}) & \longrightarrow & \widehat{\mathcal{D}}^{(m+1,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}(\mathfrak{U}) \\ \downarrow \wr & & \downarrow \wr \\ \widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\mathbb{Q}}(\mathfrak{U}) & \longrightarrow & \widehat{\mathcal{D}}^{(m+1,k)}_{\mathfrak{X},\mathbb{Q}}(\mathfrak{U}). \end{array}$$

The flatness theorem [23, Proposition 2.2.11 (iii)] states that the lower morphism is flat and so is the morphism on the top. ■

LEMMA 3.34. *For every coherent* $\mathcal{D}^{\dagger}_{\mathfrak{X},k}(\lambda)$*-module* $\mathcal{E}$ *there exist* $m \geq 0$*, a coherent* $\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}$*-module* $\mathcal{E}_m$ *and an isomorphism of* $\mathcal{D}^{\dagger}_{\mathfrak{X},k}(\lambda)$*-modules*

$$\tau : \mathcal{D}^{\dagger}_{\mathfrak{X},k}(\lambda) \otimes_{\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}} \mathcal{E}_m \xrightarrow{\simeq} \mathcal{E}.$$

Moreover, if $(m', \mathcal{E}_{m'}, \tau')$ *is another such triple, then there exist* $l \geq \max\{m, m'\}$ *and an isomorphism of* $\widehat{\mathcal{D}}^{(l,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}$*-modules*

$$\tau_l : \widehat{\mathcal{D}}^{(l,k)}_{\mathfrak{X},\lambda,\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}} \mathcal{E}_m \xrightarrow{\simeq} \widehat{\mathcal{D}}^{(l,k)}_{\mathfrak{X},\lambda,\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m',k)}_{\mathfrak{X},\lambda,\mathbb{Q}}} \mathcal{E}_{m'}$$

such that $\tau' \circ (\mathrm{id}_{\mathcal{D}^{\dagger}_{\mathfrak{X},k}(\lambda)} \otimes \tau_l) = \tau$*.*

PROOF. This is [3, Proposition 3.6.2 (ii)]. We remark that $\mathfrak{X}$ is quasi-compact and separated, and the sheaf $\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}$ satisfies the conditions in [3, §3.4.1]. ■

PROPOSITION 3.35. *Let* $\mathcal{E}$ *be a coherent* $\mathcal{D}^{\dagger}_{\mathfrak{X},k}(\lambda)$*-module.*

(i) *There exists an integer* $r(\mathcal{E})$ *such that for all* $r \geq r(\mathcal{E})$ *there are* $a \in \mathbb{N}$ *and an epimorphism of* $\mathcal{D}^{\dagger}_{\mathfrak{X},k}(\lambda)$*-modules*

$$(\mathcal{D}^{\dagger}_{\mathfrak{X},k}(\lambda)(-r))^{\oplus a} \to \mathcal{E} \to 0.$$

(ii) *For all* $i > 0$ *one has* $H^i(\mathfrak{X}, \mathcal{E}) = 0$*.*

The proof is exactly as the one of [20, Theorem 4.2.8].

PROOF. Let $\mathcal{E}$ be a coherent $\mathcal{D}^{\dagger}_{\mathfrak{X},k}(\lambda)$-coherent module. The preceding proposition tells us that there exist $m \in \mathbb{N}$, a coherent $\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-module $\mathcal{E}_m$ and an isomorphism of $\mathcal{D}^{\dagger}_{\mathfrak{X},k}(\lambda)$-modules

$$\tau : \mathcal{D}^{\dagger}_{\mathfrak{X},k}(\lambda) \otimes_{\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda,Q}} \mathcal{E}_m \xrightarrow{\simeq} \mathcal{E}.$$

Now we use Proposition 3.26 for $\mathcal{E}_m$ and we get the desired surjection in (i) after tensoring with $\mathcal{D}^{\dagger}_{\mathfrak{X},k}(\lambda)$. To show (ii) we may use the fact that, as $\mathfrak{X}$ is a noetherian topological space, cohomology commutes with direct limits. Therefore, given that $\widehat{\mathcal{D}}^{(l,k)}_{\mathfrak{X},\lambda,\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}} \mathcal{E}_m$ is a coherent $\mathcal{D}^{(l,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-module for every $l \geq m$, we have

$$H^i(\mathfrak{X},\mathcal{E}) = \varinjlim_{l\geq m} H^i\left(\mathfrak{X}, \widehat{\mathcal{D}}^{(l,k)}_{\mathfrak{X},\lambda,\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}} \mathcal{E}_m\right) = 0$$

for every $i > 0$. ∎

Proposition 3.36. *Let $\mathcal{E}$ be a coherent $\mathcal{D}^{\dagger}_{\mathfrak{X},k}(\lambda)$-module. Then $\mathcal{E}$ is generated by its global sections as $\mathcal{D}^{\dagger}_{\mathfrak{X},k}(\lambda)$-module. Moreover, $\mathcal{E}$ has a resolution by finite free $\mathcal{D}^{\dagger}_{\mathfrak{X},k}(\lambda)$-modules and $H^0(\mathfrak{X},\mathcal{E})$ is a $D^{\dagger}(\mathbb{G}(k))_{\lambda} \otimes_{\mathfrak{o}} L$-module of finite presentation.*

The proof is exactly as the one of [19, Theorem 5.1].

Proof. Theorem 3.34 gives us a coherent $\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-module $\mathcal{E}_m$ such that

$$\mathcal{E} \simeq \mathcal{D}^{\dagger}_{\mathfrak{X},k}(\lambda) \otimes_{\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}} \mathcal{E}_m.$$

Moreover, $\mathcal{E}_m$ has a resolution by finite free $\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-modules (Proposition 3.28). Both results clearly imply the first and the second part of the proposition. The final part is therefore a consequence of the first part and the acyclicity of the functor $H^0(\mathfrak{X},\bullet)$. ∎

Proof of Theorem 3.31. All in all, we can follow the arguments of [28, Corollary 2.3.7]. We start by taking

$$(D^{\dagger}(\mathbb{G}(k))_{\lambda} \otimes_{\mathfrak{o}} L)^{\oplus a} \to (D^{\dagger}(\mathbb{G}(k))_{\lambda} \otimes_{\mathfrak{o}} L)^{\oplus b} \to E \to 0,$$

a finitely presented $D^{\dagger}(\mathbb{G}(k))_{\lambda} \otimes_{\mathfrak{o}} L$-module. By localizing and applying the global sections functor, we obtain a commutative diagram

$$\begin{array}{ccccc}
(D^{\dagger}(\mathbb{G}(k))_{\lambda} \otimes_{\mathfrak{o}} L)^{\oplus a} & \longrightarrow & (D^{\dagger}(\mathbb{G}(k))_{\lambda} \otimes_{\mathfrak{o}} L)^{\oplus b} & \twoheadrightarrow & E \\
\downarrow & & \downarrow & & \downarrow \\
(D^{\dagger}(\mathbb{G}(k))_{\lambda} \otimes_{\mathfrak{o}} L)^{\oplus a} & \longrightarrow & (D^{\dagger}(\mathbb{G}(k))_{\lambda} \otimes_{\mathfrak{o}} L)^{\oplus b} & \twoheadrightarrow & H^0(\mathfrak{X}, \mathcal{L}oc^{\dagger}_{\mathfrak{X},k}(\lambda)(E)),
\end{array}$$

which tells us that $E \to H^0(\mathfrak{X}, \mathcal{L}oc^{\dagger}_{\mathfrak{X},k}(\lambda)(E))$ is an isomorphism. The reader can follow the same arguments to show that if $\mathcal{E}$ is a coherent $\mathcal{D}^{\dagger}_{\mathfrak{X},k}(\lambda)$-module, then the canonical morphism $\mathcal{D}^{\dagger}_{\mathfrak{X},k}(\lambda) \otimes_{D^{\dagger}(\mathbb{G}(k))_{\lambda}\otimes_{\mathfrak{o}} L} H^0(\mathfrak{X},\mathcal{E}) \to \mathcal{E}$ is an isomorphism. The second assertion follows because any equivalence between abelian categories is exact. ∎

4. Twisted differential operators on formal models of flag varieties

Throughout this section, $X = \mathbb{G}/\mathbb{B}$ will denote the smooth flag $\mathfrak{o}$-scheme and $\lambda \in X(\mathbb{T}) = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ will always denote an algebraic character. As before, we will denote by $\mathcal{L}(\lambda)$ the (algebraic) line bundle on X induced by λ (Section 3.1). In this section, we will generalize the construction given by Huyghe–Patel–Strauch–Schmidt in [20] by introducing sheaves of twisted differential operators on an admissible blow-up of the smooth formal flag $\mathfrak{o}$-scheme $\mathfrak{X}$. The reader will figure out that some reasoning is inspired by the results in [20].

4.1 – *Differential operators on admissible blow-ups*

We start with the following definition.

Definition 4.1. Let $\mathcal{I} \subseteq \mathcal{O}_X$ be a coherent ideal sheaf. We say that a blow-up $\mathrm{pr} : Y \to X$ along the closed subset $V(\mathcal{I})$ is admissible if there is $k \in \mathbb{N}$ such that $\varpi^k \mathcal{O}_X \subseteq \mathcal{I}$.

Let us fix an open ideal $\mathcal{I} \subseteq \mathcal{O}_X$ and an admissible blow-up $\mathrm{pr} : Y \to X$ along $V(\mathcal{I})$. We point out to the reader that $\mathcal{I}$ is not uniquely determined by the space Y. In the sequel we will use the notation

$$k_Y := \min_{\mathcal{I}} \min\{k \in \mathbb{N} \mid \varpi^k \in \mathcal{I}\},$$

where the first minimum runs over all open ideal sheaves $\mathcal{I}$ such that the blow-up along $V(\mathcal{I})$ is isomorphic to Y.

Now, as $\mathcal{I}$ is an open ideal sheaf, the blow-up induces a canonical isomorphism $Y_{\mathbb{Q}} \simeq X_{\mathbb{Q}}$ between the generic fibers. Moreover, as ϖ is invertible on $X_{\mathbb{Q}}$, we have

$$\mathcal{D}_X^{(m,k)}|_{X_{\mathbb{Q}}} = \mathcal{D}_X|_{X_{\mathbb{Q}}} = \mathcal{D}_{X_{\mathbb{Q}}},$$

the usual sheaf of (algebraic) differential operators on $X_{\mathbb{Q}}$. Therefore

$$\mathrm{pr}^{-1}(\mathcal{D}_X^{(m,k)})|_{Y_{\mathbb{Q}}} = \mathcal{D}_{Y_{\mathbb{Q}}}.$$

In particular, $\mathcal{O}_{Y_{\mathbb{Q}}}$ has the natural structure of a (left) $\mathrm{pr}^{-1}(\mathcal{D}_X^{(m,k)})|_{Y_{\mathbb{Q}}}$-module. The idea is to find those congruence levels $k \in \mathbb{N}$ such that the preceding structure extends to a module structure on $\mathcal{O}_Y$ over $\mathrm{pr}^{-1}(\mathcal{D}_X^{(m,k)})$. Let us denote

$$\mathcal{D}_Y^{(m,k)} = \mathrm{pr}^*(\mathcal{D}_X^{(m,k)}) = \mathcal{O}_Y \otimes_{\mathrm{pr}^{-1}\mathcal{O}_X} \mathrm{pr}^{-1}\mathcal{D}_X^{(m,k)}.$$

The problem to find those congruence levels was studied in [20, 23]. In fact, we have the following result (see [20, Corollary 2.1.18]).

Proposition 4.2. *Let $k \geq k_Y$. The sheaf $\mathcal{D}_Y^{(m,k)}$ is a sheaf of rings on Y. Moreover, it is locally free over $\mathcal{O}_Y$.*

Explicitly, if ∂_1, ∂_2 are both local sections of $\mathrm{pr}^{-1}(\mathcal{D}_X^{(m,k)})$, and if f_1, f_2 are local sections of $\mathcal{O}_Y$, then

$$(f_1 \otimes \partial_1) \bullet (f_2 \otimes \partial_2) = f_1 \partial_1(f_2) \otimes \partial_2 + f_1 f_2 \otimes \partial_1 \partial_2.$$

We have all the ingredients that allow us to construct the desired sheaves over Y, that is, to extend the sheaves of rings defined in the preceding section to an admissible blow-up Y of X. Let $k \geq k_Y$ be fixed. Let us first recall that taking arbitrary sections $P, Q \in \mathcal{D}_X^{(m,k)}$, $s, t \in \mathcal{L}(\lambda)$ and $s^\vee, t^\vee \in \mathcal{L}(\lambda)^\vee$ (the last two are not necessarily the duals of s and t) over an arbitrary open subset $U \subseteq X$, the multiplicative structure of the sheaf $\mathcal{D}_{X,\lambda}^{(m,k)}$ is defined by (cf. (3.2))

$$s \otimes P \otimes s^\vee \bullet t \otimes Q \otimes t^\vee = s \otimes P \langle s^\vee, t \rangle Q \otimes t^\vee.$$

Now, if $\mathrm{pr} : Y \to X$ denotes the projection, we put

$$\mathcal{D}_Y^{(m,k)}(\lambda) = \mathrm{pr}^*(\mathcal{D}_{X,\lambda}^{(m,k)}) = \mathrm{pr}^*\mathcal{L}(\lambda) \otimes_{\mathcal{O}_Y} \mathrm{pr}^*\mathcal{D}_X^{(m,k)} \otimes_{\mathcal{O}_Y} \mathrm{pr}^*\mathcal{L}(\lambda)^\vee.$$

Proposition 4.2 allows us to endow the sheaf of $\mathcal{O}_Y$-modules $\mathcal{D}_Y^{(m,k)}(\lambda)$ with a multiplicative structure for every $k \geq k_Y$. On local sections we have

$$s \otimes P \otimes s^\vee \bullet t \otimes Q \otimes t^\vee = s \otimes P \langle s^\vee, t \rangle Q \otimes t^\vee,$$

where $s, t \in \mathrm{pr}^*\mathcal{L}(\lambda)$, $s^\vee, t^\vee \in \mathrm{pr}^*\mathcal{L}(\lambda)^\vee$ and $P, Q \in \mathcal{D}_Y^{(m,k)}$ are local sections.

Let $\mathcal{Y}$ be the completion of Y along its special fiber $Y_{\mathbb{F}_q} = Y \times_{\mathrm{Spec}(\mathfrak{o})} \mathrm{Spec}(\mathfrak{o}/\varpi)$.

Notation 4.3. In this work we will only consider formal blow-ups $\mathcal{Y}$ arising from the formal completion along the special fiber of an admissible blow-up $Y \to X$ (see [20, Proposition 2.2.9]). Under this assumption we will identify $k_Y = k_{\mathcal{Y}}$.

Definition 4.4. Let $\mathrm{pr} : Y \to X$ be an admissible blow-up of the flag variety X and let $k \geq k_Y$. The sheaves

$$\widehat{\mathcal{D}}_{\mathcal{Y},\mathbb{Q}}^{(m,k)}(\lambda) = \Big(\varprojlim_{i \in \mathbb{N}} \mathcal{D}_Y^{(m,k)}(\lambda)/\varpi^{i+1}\mathcal{D}_Y^{(m,k)}(\lambda)\Big) \otimes_{\mathfrak{o}} L,$$

$$\mathcal{D}_{\mathcal{Y},k}^{\dagger}(\lambda) = \varinjlim_{m \in \mathbb{N}} \widehat{\mathcal{D}}_{\mathcal{Y},\mathbb{Q}}^{(m,k)}(\lambda).$$

are called sheaves of λ-twisted arithmetic differential operators on $\mathcal{Y}$.

Proposition 4.5. (i) *The sheaves* $\mathcal{D}_Y^{(m,k)}(\lambda)$ *are filtered by the order of twisted differential operators and there is a canonical isomorphism of graded sheaves of algebras*

$$\mathrm{gr}(\mathcal{D}_Y^{(m,k)}(\lambda)) \simeq \mathrm{Sym}^{(m)}(\varpi^k \mathrm{pr}^*\mathcal{T}_X),$$

where $k \geq k_Y$.

(ii) *There is a basis for the topology of* Y*, consisting of affine open subsets, such that for any open subset* $U \in Y$ *in this basis, the ring* $\mathcal{D}_Y^{(m,k)}(\lambda)(U)$ *is noetherian. In particular, the sheaf of rings* $\mathcal{D}_Y^{(m,k)}(\lambda)$ *is coherent.*

(iii) *The sheaf* $\widehat{\mathcal{D}}_{\mathcal{Y},\mathbb{Q}}^{(m,k)}(\lambda)$ *is coherent.*

Proof. By (2.1), we have an exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{D}_{X,d-1}^{(m,k)} \to \mathcal{D}_{X,d}^{(m,k)} \to \mathrm{Sym}_d^{(m)}(\varpi^k \mathcal{T}_X) \to 0.$$

Taking the tensor product with $\mathcal{L}(\lambda)$ and $\mathcal{L}(\lambda)^\vee$ on the left and on the right, respectively, and applying pr^*, we obtain the exact sequence (since $\mathrm{Sym}_d^{(m)}(\varpi^k \mathcal{T}_X)$ is a locally free $\mathcal{O}_X$-module of finite rank)

$$\begin{aligned} 0 &\to \mathcal{D}_{Y,d-1}^{(m,k)}(\lambda) \to \mathcal{D}_{Y,d}^{(m,k)}(\lambda) \\ &\to \mathrm{pr}^*\mathcal{L}(\lambda) \otimes_{\mathcal{O}_Y} \mathrm{Sym}_d^{(m)}(\varpi^k \mathrm{pr}^*\mathcal{T}_X) \otimes_{\mathcal{O}_Y} \mathrm{pr}^*\mathcal{L}(\lambda)^\vee \to 0, \end{aligned}$$

which implies (i) because

$$\mathrm{pr}^*\mathcal{L}(\lambda) \otimes_{\mathcal{O}_Y} \mathrm{Sym}^{(m)}(\varpi^k \mathrm{pr}^*\mathcal{T}_X) \otimes_{\mathcal{O}_Y} \mathrm{pr}^*\mathcal{L}(\lambda)^\vee \simeq \mathrm{Sym}^{(m)}(\varpi^k \mathrm{pr}^*\mathcal{T}_X)$$

by commutativity of the symmetric algebra.

Let $U \subseteq X$ be an affine open subset endowed with local coordinates $x_1, \ldots, x_M$ and such that $\mathcal{L}(\lambda)|_U = s\mathcal{O}_U$ for some $s \in \mathcal{L}(\lambda)(U)$. Then, by Lemma 3.11 we have the following local description for $\mathcal{D}_Y^{(m,k)}(\lambda)$ on $V = \mathrm{pr}^{-1}(U)$:

$$\mathcal{D}_Y^{(m,k)}(\lambda)(V) = \Big\{ \sum_{\underline{v}}^{<\infty} \varpi^{k|\underline{v}|} a_{\underline{v}} \underline{\partial}^{\langle \underline{v} \rangle} \mid \underline{v} = (v_1, \ldots, v_M) \in \mathbb{N}^M,\ a_{\underline{v}} \in \mathcal{O}_Y(V) \Big\}.$$

By (i), the graded algebra $\mathrm{gr}_\bullet(\mathcal{D}_Y^{(m,k)}(\lambda)(V))$ is isomorphic to $\mathrm{Sym}^{(m)}(\varpi^k \mathrm{pr}^*\mathcal{T}_X(V))$ which is known to be noetherian [19, Proposition 1.3.6]. Therefore, taking as a basis the set of affine open subsets of Y that are contained in some $\mathrm{pr}^{-1}(U)$, we get (ii).

As $\mathcal{D}_Y^{(m,k)}(\lambda)$ is $\mathcal{O}_Y$-quasi-coherent and has, by (ii) and [23, Proposition 2.2.2 (iii)], noetherian sections over the affine open subsets of Y, it is certainly a sheaf of coherent rings by [3, Proposition 3.1.3]. Finally, by definition, we see that $\widehat{\mathcal{D}}_{\mathcal{Y}}^{(m,k)}(\lambda)$ satisfies the conditions (a) and (b) of [3, (3.3.3)] and hence [3, Proposition 3.3.4] gives us (iii). ∎

Let us briefly study the problem of passing to the inductive limit when m varies.

Let $U \subseteq X$ be such that $\mathcal{D}_X^{(m,k)}(\lambda)|_U \simeq \mathcal{D}_X^{(m,k)}|_U$ and let us take an affine open subset $V \subseteq Y$ such that $V \subseteq \mathrm{pr}^{-1}(U)$. We have the commutative diagram

$$\begin{array}{ccc} V & \xhookrightarrow{i_V} & Y \\ \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow{\scriptstyle \mathrm{pr}} \\ U & \xhookrightarrow{i_U} & X, \end{array}$$

which implies that $\mathcal{D}_Y^{(m,k)}(\lambda)|_V \simeq \mathcal{D}_Y^{(m,k)}|_V$, as sheaves of rings. In particular, if $\mathfrak{V}$ denotes the formal p-adic completion of V along the special fiber $V_{\mathbb{F}_q}$, we have the commutative diagram (cf. Proposition 3.33)

$$\begin{array}{ccc} \widehat{\mathcal{D}}_{\mathcal{Y},\mathbb{Q}}^{(m,k)}(\lambda)(\mathfrak{V}) & \longrightarrow & \widehat{\mathcal{D}}_{\mathcal{Y},\mathbb{Q}}^{(m+1,k)}(\lambda)(\mathfrak{V}) \\ \downarrow \wr & & \downarrow \wr \\ \widehat{\mathcal{D}}_{\mathcal{Y},\mathbb{Q}}^{(m,k)}(\mathfrak{V}) & \longrightarrow & \widehat{\mathcal{D}}_{\mathcal{Y},\mathbb{Q}}^{(m+1,k)}(\mathfrak{V}). \end{array} \tag{4.1}$$

Given that the morphism of sheaves $\widehat{\mathcal{D}}_{\mathcal{Y},\mathbb{Q}}^{(m,k)} \to \widehat{\mathcal{D}}_{\mathcal{Y},\mathbb{Q}}^{(m+1,k)}$ is left and right flat [23, Proposition 2.2.11 (iii)], the preceding diagram allows us to conclude that the morphism $\widehat{\mathcal{D}}_{\mathcal{Y},\mathbb{Q}}^{(m,k)}(\lambda) \to \widehat{\mathcal{D}}_{\mathcal{Y},\mathbb{Q}}^{(m+1,k)}(\lambda)$ is also left and right flat. By Proposition 3.32 we have the following result.

Proposition 4.6. *The sheaf of rings $\mathcal{D}_{\mathcal{Y},k}^{\dagger}(\lambda)$ is coherent.*

As we will explain later, there exists a canonical epimorphism of sheaves of filtered $\mathfrak{o}$-algebras[9]

$$\mathcal{A}_Y^{(m,k)} = \mathcal{O}_Y \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{G}(k)) \to \mathcal{D}_Y^{(m,k)}(\lambda),$$

which allows us to conclude the following proposition exactly as we have done in the proof of Proposition 3.28 (cf. [20, Proposition 4.3.1]).

Proposition 4.7. *Let $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ be an algebraic character such that $\lambda + \rho \in \mathfrak{t}_{\mathbb{Q}}^*$ is a dominant and regular character of $\mathfrak{t}_{\mathbb{Q}}$.*

(i) *Let $\mathcal{E}$ be a coherent $\widehat{\mathcal{D}}_{\mathcal{Y},\mathbb{Q}}^{(m,k)}(\lambda)$-module. Then $\mathcal{E}$ is generated by its global sections as $\widehat{\mathcal{D}}_{\mathcal{Y},\mathbb{Q}}^{(m,k)}(\lambda)$-module. Furthermore, $\mathcal{E}$ has a resolution by finite free $\widehat{\mathcal{D}}_{\mathcal{Y},\mathbb{Q}}^{(m,k)}(\lambda)$-modules.*

(ii) *Let $\mathcal{E}$ be a coherent $\mathcal{D}_{\mathcal{Y},k}^{\dagger}(\lambda)$-module. Then $\mathcal{E}$ is generated by its global sections as $\mathcal{D}_{\mathcal{Y},k}^{\dagger}(\lambda)$-module. Furthermore, $\mathcal{E}$ has a resolution by finite free $\mathcal{D}_{\mathcal{Y},k}^{\dagger}(\lambda)$-modules.*

(9) We construct this morphism in (7.3). The arguments given there are independent and we will not introduce a circular argument.

4.2 – *An invariance theorem for admissible blow-ups*

Let $\mathrm{pr} : \mathfrak{Y} \to \mathfrak{X}$ be an admissible blow-up along a closed subset $\mathbf{V}(\mathcal{I})$ defined by an open ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_{\mathfrak{X}}$. Using Notation 4.3, we can suppose that $\mathfrak{Y}$ is obtained as the formal completion of an admissible blow-up[10] $Y \to X$ along a closed subset $\mathbf{V}(\mathcal{J})$ defined by an open ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_X$, such that $\mathcal{I}$ is the formal ϖ-adic completion of $\mathcal{J}$. Let us denote by $Y_i := Y \times_{\mathrm{Spec}(\mathfrak{o})} \mathrm{Spec}(\mathfrak{o}/\varpi^{i+1})$ the redaction module ϖ^{i+1} and by $\gamma_i : Y_i \to Y$ the canonical closed embedding. In [23] the authors have studied the cohomological properties of the sheaves

$$\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{Y},\mathbb{Q}} = \varprojlim_{i\in\mathbb{N}} \gamma_i^* \mathcal{D}^{(m,k)}_Y \otimes_{\mathfrak{o}} L \quad \text{and} \quad \mathcal{D}^{\dagger}_{\mathfrak{Y},k} = \varinjlim_{m\in\mathbb{N}} \widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{Y},\mathbb{Q}}.$$

Let us consider the commutative diagram

$$\begin{array}{ccc} Y_i & \xrightarrow{\mathrm{pr}_i} & X_i \\ \downarrow{\scriptstyle \gamma_i} & & \downarrow{\scriptstyle \gamma_i} \\ Y & \xrightarrow{\mathrm{pr}} & X. \end{array}$$

Here $\mathrm{pr}_i : Y_i \to X_i$ denotes the redaction modulo ϖ^{i+1} of the morphism pr. We put

$$\underline{\mathcal{L}(\lambda)^\vee} = \varprojlim_i \gamma_i^* \mathrm{pr}^* \mathcal{L}(\lambda)^\vee \quad \text{and} \quad \underline{\mathcal{L}(\lambda)} = \varprojlim_i \gamma_i^* \mathrm{pr}^* \mathcal{L}(\lambda).$$

By using the preceding commutative diagram, we have

$$\begin{aligned} \gamma_i^* \mathcal{D}^{(m,k)}_Y(\lambda) &= \gamma_i^* \big(\mathrm{pr}^* \mathcal{L}(\lambda) \otimes_{\mathcal{O}_Y} \mathcal{D}^{(m,k)}_Y \otimes_{\mathcal{O}_Y} \mathrm{pr}^* \mathcal{L}(\lambda)^\vee\big) \\ &= \gamma_i^* (\mathrm{pr}^* \mathcal{L}(\lambda)) \otimes_{\mathcal{O}_{Y_i}} \gamma_i^* \mathcal{D}^{(m,k)}_Y \otimes_{\mathcal{O}_{Y_i}} \gamma_i^* (\mathrm{pr}^* \mathcal{L}(\lambda)^\vee). \end{aligned}$$

Taking the projective limit and tensoring with L, we get the following description of the sheaves $\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{Y},\mathbb{Q}}(\lambda)$:

$$\widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{Y},\mathbb{Q}}(\lambda) = \underline{\mathcal{L}(\lambda)}_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}} \widehat{\mathcal{D}}^{(m,k)}_{\mathfrak{Y},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}} \underline{\mathcal{L}(\lambda)^\vee}_{\mathbb{Q}}.$$

Taking the inductive limit, we get the characterization

$$\mathcal{D}^{\dagger}_{\mathfrak{Y},k}(\lambda) = \underline{\mathcal{L}(\lambda)}_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}} \mathcal{D}^{\dagger}_{\mathfrak{Y},k} \otimes_{\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}} \underline{\mathcal{L}(\lambda)^\vee}_{\mathbb{Q}}.$$

([10]) By abuse of notation we will denote again by $\mathrm{pr} : Y \to X$ the canonical morphism of this algebraic blow-up.

As in the preceding subsection, the sheaf $\underline{\mathcal{L}(\lambda)}_{\mathbb{Q}}$ is endowed with the following (left) $\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda)$-action:

$$(t \otimes P \otimes t^{\vee}) \bullet s = (P \bullet \langle t^{\vee}, s\rangle)t \quad (s, t \in \underline{\mathcal{L}(\lambda)}_{\mathbb{Q}} \text{ and } t^{\vee} \in \underline{\mathcal{L}(\lambda)^{\vee}}_{\mathbb{Q}}).$$

We end this first discussion by remarking that the relation $\mathrm{pr}_i^* \circ \gamma_i^* = \gamma_i^* \circ \mathrm{pr}^*$, coming from the preceding commutative diagram, implies that

$$\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda) = \mathrm{pr}^* \mathcal{D}^{\dagger}_{\mathfrak{X},k}(\lambda).$$

Let us suppose that $\pi : Y' \to Y$ is a morphism of admissible blow-ups. By abuse of notation, we will denote by $\pi : \mathcal{Y}' \to \mathcal{Y}$ also the respective morphism obtained by functoriality by completing along the special fiber. This is a morphism of formal admissible blow-ups in the sense of [6, Part II, Section 8.2, Definition 3]. We have commutative diagrams

$$\begin{array}{ccc} Y' & \xrightarrow{\pi} & Y \\ & \searrow^{\mathrm{pr}'} & \downarrow \mathrm{pr} \\ & & X. \end{array} \quad \text{resp.} \quad \begin{array}{ccc} \mathcal{Y}' & \xrightarrow{\pi} & \mathcal{Y} \\ & \searrow^{\mathrm{pr}'} & \downarrow \mathrm{pr} \\ & & \mathfrak{X}. \end{array}$$

Let $k \geq \max\{k_{Y'}, k_Y\}$. Let us write

$$\mathcal{D}^{(m,k)}_{X,i}(\lambda) = \mathcal{D}^{(m,k)}_{X}(\lambda)/\varpi^{i+1}\mathcal{D}^{(m,k)}_{X}(\lambda),$$

considered as a sheaf over X_i. Let $\pi_i : Y'_i \to Y_i$ denote the redactions module ϖ^{i+1}. The preceding commutative diagram implies that

$$\mathcal{D}^{(m,k)}_{Y',i}(\lambda) = (\mathrm{pr}'_i)^* \mathcal{D}^{(m,k)}_{X_i}(\lambda) = \pi_i^* \mathcal{D}^{(m,k)}_{Y_i}(\lambda). \tag{4.2}$$

In this way, the sheaf $\mathcal{D}^{(m,k)}_{Y',i}(\lambda)$ can be endowed with the structure of a right $\pi_i^{-1}\mathcal{D}^{(m,k)}_{Y_i}(\lambda)$-module. Passing to the projective limit, the sheaf $\widehat{\mathcal{D}}^{(m,k)}_{\mathcal{Y}'}(\lambda)$ is a sheaf of right $\pi^{-1}\widehat{\mathcal{D}}^{(m,k)}_{\mathcal{Y}}(\lambda)$-modules. So, passing to the inductive limit over m, we can conclude that $\mathcal{D}^{\dagger}_{\mathcal{Y}',k}(\lambda)$ is a right $\pi^{-1}\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda)$-module. For a $\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda)$-module $\mathcal{E}$, we define

$$\pi^{!}\mathcal{E} = \mathcal{D}^{\dagger}_{\mathcal{Y}',k}(\lambda) \otimes_{\pi^{-1}\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda)} \pi^{-1}\mathcal{E},$$

with analogous definitions for $\widehat{\mathcal{D}}^{(m,k)}_{\mathcal{Y},\mathbb{Q}}(\lambda)$.

THEOREM 4.8. *Let $\pi : Y' \to Y$ be a morphism over X of admissible blow-ups. Let $k \geq \max\{k_{Y'}, k_Y\}$.*

(i) *If $\mathcal{E}$ is a coherent $\mathcal{D}^{\dagger}_{\mathcal{Y}',k}(\lambda)$-module, then $R^j\pi_*\mathcal{E} = 0$ for every $j > 0$. Moreover,*

$$\pi_*\mathcal{D}^{\dagger}_{\mathcal{Y}',k}(\lambda) = \mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda),$$

so π_* induces an exact functor between coherent modules over $\mathcal{D}^{\dagger}_{\mathcal{Y}',k}(\lambda)$ and $\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda)$, respectively.

(ii) *The formation $\pi^!$ is an exact functor from the category of coherent $\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda)$-modules to the category of coherent $\mathcal{D}^{\dagger}_{\mathcal{Y}',k}(\lambda)$-modules.*

(iii) *The functors π_* and $\pi^!$ are quasi-inverse equivalences between the categories of coherent $\mathcal{D}^{\dagger}_{\mathcal{Y}',k}(\lambda)$-modules and coherent $\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda)$-modules.*

We remark for the reader that this theorem has an equivalent version for the sheaves $\widehat{\mathcal{D}}^{(m,k)}_{\mathcal{Y},\mathbb{Q}}(\lambda)$ and $\widehat{\mathcal{D}}^{(m,k)}_{\mathcal{Y}',\mathbb{Q}}(\lambda)$.

Proof. Let us first assume that $\mathcal{E} = \mathcal{D}^{\dagger}_{\mathcal{Y}',k}(\lambda)$. Let us consider the covering $\mathcal{B}$ of $\mathcal{X}$, defined in Proposition 3.21 and let us take $\mathcal{U} \in \mathcal{B}$. We put $\mathcal{V}' = \mathrm{pr}'^{-1}(\mathcal{U})$ and $\mathcal{V} = \mathrm{pr}^{-1}(\mathcal{U})$. By assumption $\mathcal{V}' = \pi^{-1}(\mathcal{V})$ in such a way that

$$R^j\pi_*(\mathcal{D}^{\dagger}_{\mathcal{Y}',k}(\lambda))|_{\mathcal{V}} = R^j\pi_*(\mathcal{D}^{\dagger}_{\mathcal{Y}',k}(\lambda)|_{\mathcal{V}'}) = R^j\pi_*(\mathcal{D}^{\dagger}_{\mathcal{Y}',k}|_{\mathcal{V}'}) = R^j\pi_*(\mathcal{D}^{\dagger}_{\mathcal{Y}',k})|_{\mathcal{V}}.$$

Now we can use [23, Theorem 2.3.8 (i)] to conclude that $R^j\pi_*\mathcal{D}^{\dagger}_{\mathcal{Y}',k}(\lambda) = 0$ for every $j > 0$. Furthermore, by (4.2) there exists a canonical map

$$\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda) \to \pi_*\mathcal{D}^{\dagger}_{\mathcal{Y}',k}(\lambda),$$

which is an isomorphism by the preceding reasoning and [23, Theorem 2.3.8 (i)].

To handle with the second part let us define the assertion a_j for every $j \geq 1$ as follows:

$$R^l\pi_*\mathcal{E} = 0 \text{ for any coherent } \mathcal{D}^{\dagger}_{\mathcal{Y}',k}(\lambda)\text{-module } \mathcal{E} \text{ and for all } l \geq j.$$

The assertion a_j is true for $j = \dim(\mathcal{Y}) + 1$. Let us suppose that a_{j+1} is true and let us take a coherent $\mathcal{D}^{\dagger}_{\mathcal{Y}',k}(\lambda)$-module $\mathcal{E}$. By Proposition 4.7 there exist $b \in \mathbb{N}$ and a short exact sequence of coherent $\mathcal{D}^{\dagger}_{\mathcal{Y}',k}(\lambda)$-modules

$$0 \to \mathcal{F} \to (\mathcal{D}^{\dagger}_{\mathcal{Y}',k}(\lambda))^{\oplus b} \to \mathcal{E} \to 0.$$

Since $R^j\pi_*\mathcal{D}^{\dagger}_{\mathcal{Y}',k}(\lambda) = 0$ for every $j > 0$, the long exact sequence for π_* gives us

$$R^j\pi_*\mathcal{E} \simeq R^{j+1}\pi_*\mathcal{F},$$

which is 0 by induction hypothesis. This ends the proof of (i).

Let us show (ii) for the sheaves $\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda)$. The proof for the sheaves $\widehat{\mathcal{D}}^{(m,k)}_{\mathcal{Y},\mathbb{Q}}(\lambda)$ follows the same argument. Given that

$$\pi^!\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda) = \mathcal{D}^{\dagger}_{\mathcal{Y}',k}(\lambda),$$

and since the tensor product is right exact, we can conclude that $\pi^!$ preserves coherence.

Now, let $\mathcal{M}$ be a coherent $\mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda)$-module. We have a morphism $\pi^{-1}\mathcal{M} \to \pi^!\mathcal{M}$ sending $m \mapsto 1 \otimes m$. This map induces the morphism $\mathcal{M} \to \pi_*\pi^!\mathcal{M}$. To show that this is an isomorphism is a local question on $\mathcal{Y}$. If $\mathcal{V} \subseteq \mathcal{Y}$ is the formal completion of an affine open subset $V \subseteq \mathrm{pr}^{-1}(U)$, and $U \subseteq X$ is an affine open subset such that $\mathcal{D}^{(m,k)}_X(\lambda)|_U \simeq \mathcal{D}^{(m,k)}_X|_U$ (Lemma 3.11), then by (4.1) and [23, Corollary 2.2.15] we can conclude that the previous map is in fact an isomorphism over $\mathcal{V}$. Finally, if $\mathcal{F}$ is a coherent $\mathcal{D}^\dagger_{\mathcal{Y}',k}(\lambda)$-module, then we have the map $\pi^!\pi_*\mathcal{F} \to \mathcal{F}$, sending $P \otimes m \mapsto Pm$. To see that this is an isomorphism, we can use the preceding reasoning. ∎

Let us recall that if $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ is an algebraic character such that $\lambda + \rho \in \mathfrak{t}^*_{\mathbb{Q}}$ is a dominant and regular character of $\mathfrak{t}_{\mathbb{Q}}$, then by (3.10) we have

$$H^0(\mathcal{X}, \mathcal{D}^\dagger_{\mathcal{X},k}(\lambda)) = \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda = \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)/\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)(\mathrm{Ker}\, \chi_{\lambda+\rho}).$$

Theorem 4.8 has the following corollary.

Corollary 4.9. *Let $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ be an algebraic character such that $\lambda + \rho \in \mathfrak{t}^*_{\mathbb{Q}}$ is a dominant and regular character of $\mathfrak{t}_{\mathbb{Q}}$. In the situation of Theorem 4.8 one has*

$$H^0(\mathcal{Y}, \mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda)) = H^0(\mathcal{X}, \mathcal{D}^\dagger_{\mathcal{X},k}(\lambda)) = D^\dagger(\mathbb{G}(k))_\lambda = H^0(\mathcal{Y}', \mathcal{D}^\dagger_{\mathcal{Y}',k}(\lambda)).$$

Theorem 4.10. *Let $\mathrm{pr} : Y \to X$ be an admissible blow-up. Let us suppose that $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ is an algebraic character such that $\lambda + \rho \in \mathfrak{t}^*_{\mathbb{Q}}$ is a dominant and regular character of $\mathfrak{t}_{\mathbb{Q}}$.*

(i) *For any coherent $\mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda)$-module $\mathcal{E}$ and for all $q > 0$ one has $H^q(\mathcal{Y}, \mathcal{E}) = 0$.*

(ii) *The functor $H^0(\mathcal{Y}, \bullet)$ is an equivalence between the category of coherent $\mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda)$-modules and the category of finitely presented $D^\dagger(\mathbb{G}(k))_\lambda$-modules.*

The same statement holds for coherent modules over $\widehat{\mathcal{D}}^{(m,k)}_{\mathcal{Y},\mathbb{Q}}(\lambda)$.

Proof. The first part of the theorem follows from $H^0(\mathcal{Y}, \bullet) = H^0(\mathcal{X}, \bullet) \circ \pi_*$. Now we only have to apply Theorems 4.8 and 3.31.

Let us consider the category $\mathrm{Mod}_{\mathrm{fp}}(D^\dagger(\mathbb{G}(k))_\lambda)$ of finitely presented $D^\dagger(\mathbb{G}(k))_\lambda$-modules, and the category $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathcal{X},k}(\lambda))$ of coherent $\mathcal{D}^\dagger_{\mathcal{X},k}(\lambda)$-modules (with analogous notation on $\mathcal{Y}$). We denote by $\mathcal{L}oc^\dagger_{\mathcal{Y},k}(\lambda)$ the exact functor defined by the composition

$$\mathrm{Mod}_{\mathrm{fp}}(D^\dagger(\mathbb{G}(k))_\lambda) \xrightarrow{\mathcal{L}oc^\dagger_{\mathcal{X},k}(\lambda)} \mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathcal{X},k}(\lambda)) \xrightarrow{\pi^!} \mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda)).$$

Fixing a finitely presented $D^\dagger(\mathbb{G}(k))_\lambda$-module E, we see that

$$\begin{aligned}\pi^!(\mathcal{L}oc^\dagger_{\mathcal{X},k}(\lambda)(E)) &= \mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda) \otimes_{\pi^{-1}\mathcal{D}^\dagger_{\mathcal{X},k}(\lambda)} \pi^{-1}\mathcal{D}^\dagger_{\mathcal{X},k}(\lambda) \otimes_{D^\dagger(\mathbb{G}(k))_\lambda} E \\ &= \mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda) \otimes_{D^\dagger(\mathbb{G}(k))_\lambda} E.\end{aligned}$$

Now, to show that

$$H^0\big(\mathcal{Y}, \pi^!(\mathcal{L}oc^{\dagger}_{\mathcal{X},k}(\lambda)(E))\big) = E,$$

we can take a resolution

$$(D^{\dagger}(\mathbb{G}(k))_{\lambda})^{\oplus b} \to (D^{\dagger}(\mathbb{G}(k))_{\lambda})^{\oplus a} \to E \to 0,$$

to get the diagram

$$\begin{array}{ccccc} (D^{\dagger}(\mathbb{G}(k))_{\lambda})^{\oplus b} & \longrightarrow & (D^{\dagger}(\mathbb{G}(k))_{\lambda})^{\oplus a} & \twoheadrightarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ (D^{\dagger}(\mathbb{G}(k))_{\lambda})^{\oplus b} & \longrightarrow & (D^{\dagger}(\mathbb{G}(k))_{\lambda})^{\oplus a} & \twoheadrightarrow & H^0(\mathcal{Y}, \mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda) \otimes_{D^{\dagger}(\mathbb{G}(k))_{\lambda}} E), \end{array}$$

where the sequence on the top is clearly exact. By definition $\mathcal{L}oc^{\dagger}_{\mathcal{Y},k}(\lambda)(\bullet)$ is an exact functor and by (i) the global section functor $H^0(\mathcal{Y}, \bullet)$ is also exact. This shows that the sequence at the bottom is also exact and we end the proof of the theorem. ∎

In the sequel we will denote by G_0 the compact locally L-analytic group $G_0 := \mathbb{G}(\mathfrak{o})$.

4.3 – *Group action on blow-ups*

Let $\mathfrak{G}$ be the ϖ-completion of $\mathbb{G}$, along its special fiber $\mathbb{G}_{\mathbb{F}_p} = \mathbb{G} \times_{\operatorname{Spec}(\mathfrak{o})} \operatorname{Spec}(\mathfrak{o}/\varpi)$. Let us denote by $\alpha : \mathcal{X} \times_{\operatorname{Spf}(\mathfrak{o})} \mathfrak{G} \to \mathcal{X}$ the induced right $\mathfrak{G}$-action on the formal flag $\mathfrak{o}$-scheme $\mathcal{X}$ (cf. Section 3.3). For every $g \in \mathfrak{G}(\mathfrak{o}) = G_0$ we have an automorphism ρ_g of $\mathcal{X}$ given by

$$\rho_g : \mathcal{X} = \mathcal{X} \times_{\operatorname{Spf}(\mathfrak{o})} \operatorname{Spf}(\mathfrak{o}) \xrightarrow{\mathrm{id}_{\mathcal{X}} \times g} \mathcal{X} \times_{\operatorname{Spf}(\mathfrak{o})} \mathfrak{G} \xrightarrow{\alpha} \mathcal{X}.$$

As $\mathfrak{G}$ acts on the right, we have the relation

$$(\rho_g)_*(\rho_h^{\natural}) \circ \rho_g^{\natural} = \rho_{hg}^{\natural} \quad (g, h \in G_0). \tag{4.3}$$

Here $\rho_g^{\natural} : \mathcal{O}_{\mathcal{X}} \to (\rho_g)_*\mathcal{O}_{\mathcal{X}}$ denotes the comorphism of ρ_g.

Let $H \subseteq G_0$ be an open subgroup. We say that an open ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_{\mathcal{X}}$ is *H-stable* if for all $g \in H$ the comorphism $\rho_g^{\natural}$ maps $\mathcal{I} \subseteq \mathcal{O}_{\mathcal{X}}$ into $(\rho_g)_*\mathcal{I} \subseteq (\rho_g)_*\mathcal{O}_{\mathcal{X}}$. In this case $\rho_g^{\natural}$ induces a morphism of sheaves of graded rings

$$\bigoplus_{d \in \mathbb{N}} \mathcal{I}^d \to (\rho_g)_*\Big(\bigoplus_{d \in \mathbb{N}} \mathcal{I}^d\Big)$$

on $\mathcal{X}$. This morphism induces an automorphism of the blow-up $\mathcal{Y} = \operatorname{Proj}(\bigoplus_{d \in \mathbb{N}} \mathcal{I}^d)$, let us say ρ_g by abuse of notation, and the action of H on $\mathcal{X}$ lifts to a right action of H

on $\mathcal{Y}$, in the sense that for every $g, h \in H$ the relation (4.3) is verified. We have the commutative diagram

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\rho_g} & \mathcal{Y} \\ \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow{\scriptstyle \mathrm{pr}} \\ \mathcal{X} & \xrightarrow{\rho_g} & \mathcal{X}. \end{array} \tag{4.4}$$

Definition 4.11. Let $H \subseteq G_0$ be an open subgroup and $\mathrm{pr} : \mathcal{Y} \to \mathcal{X}$ an admissible blow-up defined by an open ideal subsheaf $\mathcal{I} \subseteq \mathcal{O}_{\mathcal{X}}$. We say that $\mathcal{Y}$ is H-equivariant if $\mathcal{I}$ is H-stable.

We will need the following result in the next sections. The reader can find its proof in [20, Lemma 5.2.3].

Lemma 4.12. *Let* $\mathrm{pr} : \mathcal{Y} \to \mathcal{X}$ *be an admissible blow-up, and let us assume that* $k \geq k_Y = k_{\mathcal{Y}}$. *Then* $\mathcal{Y}$ *is* $G_k = \mathbb{G}(k)(\mathfrak{o})$*-equivariant and the induced action of every* $g \in G_{k+1}$ *on the special fiber of* Y *is the identity. Therefore,* G_{k+1} *acts trivially on the underlying topological space of* $\mathcal{Y}$.

As noted, for every $g \in \mathfrak{G}(\mathfrak{o}) = \mathbb{G}(\mathfrak{o}) = G_0$ there exists an isomorphism

$$\rho_g : \mathcal{X} \xrightarrow{\mathrm{id}_{\mathcal{X}} \times g} \mathcal{X} \times_{\mathrm{Spec}(\mathfrak{o})} \mathfrak{G} \xrightarrow{\alpha} \mathcal{X},$$

which induces an $\mathcal{O}_{\mathcal{X}}$-linear isomorphism $\Phi_g : \mathcal{L}(\lambda) \to (\rho_g)_*(\mathcal{L}(\lambda))$ (Proposition 3.4) verifying the cocycle condition

$$\Phi_{hg} = (\rho_g)_*(\Phi_h) \circ \Phi_g \quad (g, h \in \mathbb{G}(\mathfrak{o})). \tag{4.5}$$

In particular, we have an induced G_0-action on the sheaf $\mathcal{D}^{\dagger}_{\mathcal{X},k}(\lambda)$:

$$T_g : \mathcal{D}^{\dagger}_{\mathcal{X},k}(\lambda) \to (\rho_g)_*\mathcal{D}^{\dagger}_{\mathcal{X},k}(\lambda), \quad P \mapsto \Phi_g \circ P \circ (\Phi_g)^{-1}. \tag{4.6}$$

Locally, if $\mathcal{U} \subseteq \mathcal{X}$ and $P \in \mathcal{D}^{\dagger}_{\mathcal{X},k}(\lambda)(\mathcal{U})$, then the cocycle condition (4.5) tells that the diagram

$$\begin{array}{ccc} \mathcal{L}(\lambda)(\mathcal{U}.(hg)^{-1}) = \mathcal{L}(\lambda)(\mathcal{U}.g^{-1}h^{-1}) & \xrightarrow{T_{gh,\mathcal{U}}(P)} & \mathcal{L}(\lambda)(\mathcal{U}.g^{-1}h^{-1}) \\ \downarrow{\scriptstyle \Phi^{-1}_{h,\mathcal{U}.g^{-1}} = (\rho_g)_*\Phi^{-1}_{h,\mathcal{U}}} & & \uparrow{\scriptstyle \Phi_{h,\mathcal{U}.g^{-1}} = (\rho_g)_*\Phi_{h,\mathcal{U}}} \\ \mathcal{L}(\lambda)(\mathcal{U}.g^{-1}) & & \mathcal{L}(\mathcal{U}.g^{-1}) \\ \downarrow{\scriptstyle \Phi^{-1}_{g,\mathcal{U}}} & & \uparrow{\scriptstyle \Phi_{g,\mathcal{U}}} \\ \mathcal{L}(\lambda)(\mathcal{U}) & \xrightarrow{P} & \mathcal{L}(\lambda)(\mathcal{U}) \end{array} \tag{4.7}$$

is commutative and we get the relation

$$T_{hg} = (\rho_g)_* T_h \circ T_g \quad (g, h \in G_0).$$

Let us suppose that $H \subseteq G_0$ is an open subgroup and that $\mathrm{pr} : \mathcal{Y} \to \mathcal{X}$ is an H-equivariant admissible blow-up. Using the commutative diagram (4.4), we get

$$\mathrm{pr}^*(\rho_g)^* \mathcal{L}(\lambda) = (\rho_g)^* \mathrm{pr}^* \mathcal{L}(\lambda) = (\rho_g)^* \underline{\mathcal{L}(\lambda)}$$

(notation given at the beginning of the preceding subsection). Pulling back the isomorphism $(\rho_g)^* \mathcal{L}(\lambda) \to \mathcal{L}(\lambda)$, via $(\mathrm{pr})^*$, we get by adjointness the map

$$L_g : \underline{\mathcal{L}(\lambda)} \xrightarrow{\simeq} (\rho_g)_* \underline{\mathcal{L}(\lambda)},$$

which satisfies, by functoriality, the cocycle condition

$$L_{hg} = (\rho_g)_* L_h \circ L_g \quad (g, h \in H). \tag{4.8}$$

As in (4.6) we can define (from now on we will work on admissible blow-ups of $\mathcal{Y}$ so we will use the same notation)

$$T_g : \mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda) \to (\rho_g)_* \mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda), \quad P \to L_g \circ P \circ L_g^{-1}. \tag{4.9}$$

Exactly as we have done in (4.7) we can conclude that

$$T_{hg} = (\rho_g)_* T_h \circ T_g,$$

for every $g, h \in H$.

5. Localization of locally analytic representations

We recall for the reader that G_0 denotes the compact locally L-analytic group $G_0 = \mathbb{G}(\mathfrak{o})$. In this section, we will show how to localize admissible locally analytic representations of G_0. We will denote by $\mathcal{C}^{\mathrm{an}}(G_0, L)$ the space of L-valued locally L-analytic functions on G_0 and by $D(G_0, L)$ its strong dual (the space of *locally analytic distributions* in the sense of [34, Section 11]). This space contains a set of delta distributions $\{\delta_g\}_{g \in G_0}$ defined by $\delta_g(f) = f(g)$, if $f \in \mathcal{C}^{\mathrm{an}}(G_0, L)$, in such a way that the map $g \mapsto \delta_g$ is an injective group homomorphism from G_0 into $D(G_0, L)^\times$. We also recall that given that G_0 is compact, this space carries the structure of a nuclear Fréchet–Stein algebra [34, Theorem 24.1]. For our work it will be enough to define a weak Fréchet–Stein structure (in the sense of [14, Definition 1.2.8]) on the algebra $D(G_0, L)$.

We finally recall that in Definition 3.30 we have introduced Emerton's distribution algebra as the continuous dual space of the space of rigid-analytic functions on $\mathbb{G}(k)^\circ$:

$$\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ) = \mathrm{Hom}_L^{\mathrm{cont}}\big(\mathcal{O}_{\mathbb{G}(k)^\circ}(\mathbb{G}(k)^\circ), L\big).$$

5.1 – *Coadmissible modules*

Let us start by recalling that G_0 acts on the space $\mathcal{C}^{\mathrm{cts}}(G_0, L)$, of continuous L-valued functions, by the formula

$$(g \bullet f)(x) := f(g^{-1}x) \quad (g, x \in G_0,\ f \in \mathcal{C}^{\mathrm{cts}}(G_0, L)).$$

Moreover, given an admissible locally analytic representation V of G_0 (see [34, first definition of Lecture VI]) then, by definition, its strong dual $M := (V)'_b$ is a coadmissible module[11] over $D(G_0, L)$.

Given a continuous representation W of G_0, we can consider the subspace $W_{\mathbb{G}(k)^\circ} \subseteq W$ of $\mathbb{G}(k)^\circ$-analytic vectors [14, Definition 3.4.1]. In particular, the G_0-action on $\mathcal{C}^{\mathrm{cts}}(G_0, L)$, defined at the beginning of this subsection, allows us to consider the subspace $\mathcal{C}^{\mathrm{cts}}(G_0, L)_{\mathbb{G}(k)^\circ\text{-an}}$ and we have a canonical isomorphism of topological L-vector spaces

$$\varinjlim_{k} \mathcal{C}^{\mathrm{cts}}(G_0, L)_{\mathbb{G}(k)^\circ\text{-an}} \xrightarrow{\simeq} \mathcal{C}^{\mathrm{an}}(G_0, L). \tag{5.1}$$

As in [14, Proposition 5.3.1], for each $k \in \mathbb{Z}_{>0}$ we denote the strong dual of the space of $\mathbb{G}(k)^\circ$-analytic vectors of $\mathcal{C}^{\mathrm{cts}}(G_0, L)$ by

$$D(\mathbb{G}(k)^\circ, G_0) = (\mathcal{C}^{\mathrm{cts}}(G_0, L)_{\mathbb{G}(k)^\circ\text{-an}})'_b.$$

The ring structure on $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)$ extends naturally to a ring structure on $D(\mathbb{G}(k)^\circ, G_0)$, such that

$$D(\mathbb{G}(k)^\circ, G_0) = \bigoplus_{g \in G_0/G_k} \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)\delta_g. \tag{5.2}$$

Dualizing the isomorphism (5.1) yields an isomorphism of topological L-algebras

$$D(G_0, L) \xrightarrow{\simeq} \varprojlim_{k \in \mathbb{Z}_{>0}} D(\mathbb{G}(k)^\circ, G_0). \tag{5.3}$$

(11) We recall for the reader that the category of coadmissible $D(G_0, L)$-modules is a full abelian subcategory of the category of $D(G_0, L)$-modules and the "strong dual" functor induces an anti-equivalence of categories to the category of admissible locally analytic representations [34, Theorem 20.1].

This is the weak Fréchet–Stein structure on the locally analytic distribution algebra $D(G_0, L)$ (see [14, Proposition 5.3.1]).

Let V be an admissible locally analytic representation and $M := V'_b$. By [14, Lemma 6.1.6] the subspace $V_{\mathbb{G}(k)^\circ\text{-an}} \subseteq V$ is a nuclear Fréchet space; therefore, its strong dual $M_k := (V_{\mathbb{G}(k)^\circ\text{-an}})'_b$ is a space of compact type and a finitely generated topological $D(\mathbb{G}(k)^\circ, G_0)$-module (see [14, Lemma 6.1.13]). By [14, Theorem 6.1.20] the module M is a coadmissible $D(G_0, L)$-module relative to the weak Fréchet–Stein structure of $D(G_0, L)$ defined in the previous paragraph.

We have the following result from [20, Lemma 5.1.7].

Lemma 5.1. (i) *The $D(\mathbb{G}(k)^\circ, G_0)$-module M_k is finitely generated.*

(ii) *There are natural isomorphisms*

$$D(\mathbb{G}(k-1)^\circ, G_0) \otimes_{D(\mathbb{G}(k)^\circ, G_0)} M_k \xrightarrow{\simeq} M_{k-1}.$$

(iii) *The natural map $D(\mathbb{G}(k-1)^\circ, G_0) \otimes_{D(G_0, L)} M \to M_k$ is bijective.*

Now, let $\lambda \in \operatorname{Hom}(\mathbb{T}, \mathbb{G}_m)$ be an algebraic character such that $\lambda + \rho \in \mathfrak{t}^*_{\mathbb{Q}}$ is a dominant and regular character of $\mathfrak{t}_{\mathbb{Q}}$. Let us recall that we have identifications

$$\mathcal{D}^{\text{an}}(\mathbb{G}(k)^\circ)_\lambda = D^\dagger(\mathbb{G}(k))_\lambda = \varinjlim_{m \in \mathbb{N}} (\widehat{D}^{(m)}(\mathbb{G}(k))_\lambda) \otimes_{\mathfrak{o}} L.$$

The preceding relation and the fact that the ring structure of $\mathcal{D}^{\text{an}}(\mathbb{G}(k)^\circ)$ extends naturally to a ring structure on $D(\mathbb{G}(k)^\circ, G_0)$ allow us to consider the ring

$$D(\mathbb{G}(k)^\circ, G_0)_\lambda = D(\mathbb{G}(k)^\circ, G_0)/\operatorname{Ker}(\chi_{\lambda+\rho}) D(\mathbb{G}(k)^\circ, G_0).$$

From now on, we will denote by $\mathcal{C}_{G_0}$ the full subcategory of $\operatorname{Mod}(D(G_0, L))$ consisting of coadmissible modules, with respect to the preceding weak Fréchet–Stein structure on $D(G_0, L)$.

Definition 5.2. We define the category $\mathcal{C}_{G_0,\lambda}$ of coadmissible $D(G_0, L)$-modules with central character $\lambda \in \operatorname{Hom}(\mathbb{T}, \mathbb{G}_m)$ by

$$\mathcal{C}_{G_0,\lambda} := \operatorname{Mod}\big(D(G_0, L)/\operatorname{Ker}(\chi_\lambda) D(G_0, L)\big) \cap \mathcal{C}_{G_0}.$$

We point out that the preceding definition is completely legal because the center $Z(\mathfrak{g}_{\mathbb{Q}})$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_{\mathbb{Q}})$ lies in the center of $D(G_0, L)$ (see [32, Proposition 3.7]). We also recall that the group $G_k := \mathbb{G}(k)(\mathfrak{o})$ is contained in $\mathcal{D}^{\text{an}}(\mathbb{G}(k)^\circ)$ as a set of Dirac distributions. For each $g \in G_k$ we will write δ_g for the image of the Dirac distribution supported at g in

$$H^0(\mathcal{Y}, \mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda)) = \mathcal{D}^{\text{an}}(\mathbb{G}(k)^\circ)_\lambda.$$

Inspired by [20, Definition 5.2.7] we have the following definition.

Definition 5.3. Let $H \subset G_0$ be an open subgroup and $\mathcal{Y}$ an H-equivariant admissible blow-up of $\mathcal{X}$. Let us suppose that $k \geq k_{\mathcal{Y}}$ (notation as in Notation 4.3). A strongly H-equivariant $\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda)$-module is a $\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda)$-module $\mathcal{M}$ together with a family $(\varphi_g)_{g\in H}$ of isomorphisms

$$\varphi_g : \mathcal{M} \to (\rho_g)_*\mathcal{M}$$

of sheaves of L-vector spaces, satisfying the following conditions:

(i) For all $g, h \in H$ one has $(\rho_g)_*(\varphi_h) \circ \varphi_g = \varphi_{hg}$.

(ii) For all open subsets $\mathcal{U} \subset \mathcal{Y}$, all $P \in \mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda)(\mathcal{U})$, and all $m \in \mathcal{M}(\mathcal{U})$ one has $\varphi_g(P \bullet m) = T_g(P) \bullet \varphi_g(m)$.

(iii) For all $g \in H \cap G_{k+1}$ the map $\varphi_g : \mathcal{M} \to (\rho_g)_*\mathcal{M} = \mathcal{M}$ is equal to multiplication by $\delta_g \in H^0(\mathcal{Y}, \mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda))$.[12]

A morphism between two strongly H-equivariant $\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda)$-modules $(\mathcal{M}, (\varphi_g^{\mathcal{M}})_{g\in H})$ and $(\mathcal{N}, (\varphi_g^{\mathcal{N}})_{g\in H})$ is a $\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda)$ linear morphism $\psi : \mathcal{M} \to \mathcal{N}$ such that for all $g \in H$

$$\varphi_g^{\mathcal{N}} \circ \psi = (\rho_g)_*(\psi) \circ \varphi_g^{\mathcal{M}}.$$

We denote the category of strongly H-equivariant coherent $\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda)$-modules by $\mathrm{Coh}(\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda), G_0)$.

Remark 5.4. Let $\mathcal{M} \in \mathrm{Coh}(\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda), G_0)$. In what follows we will use the notation $gm = \varphi_{g,\mathcal{U}}(m) \in \mathcal{M}(\mathcal{U}.g^{-1})$, for $\mathcal{U} \subseteq \mathcal{Y}$ an open subset, $g \in G_0$ and $m \in \mathcal{M}(\mathcal{U})$. This notation is inspired by property (ii) of the previous definition. In fact, if $g, h \in G_0$, then by (ii) we have $h(gm) = (hg)m$.

Theorem 5.5. *Let $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ be an algebraic character such that $\lambda + \rho \in \mathfrak{t}^*_{\mathbb{Q}}$ is a dominant and regular character of $\mathfrak{t}_{\mathbb{Q}}$. Let $\mathrm{pr} : \mathcal{Y} \to \mathcal{X}$ be a G_0-equivariant admissible blow-up, and let $k \geq k_{\mathcal{Y}}$. The functors $\mathcal{L}oc^{\dagger}_{\mathcal{Y},k}(\lambda)$ and $H^0(\mathcal{Y}, \bullet)$ induce quasi-inverse equivalences between the category of finitely presented $D(\mathbb{G}(k)^{\circ}, G_0)_{\lambda}$-modules and* $\mathrm{Coh}(\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda), G_0)$.

Before starting the proof, we recall that the functor $\mathcal{L}oc^{\dagger}_{\mathcal{Y},k}(\lambda)$ has been defined in the proof of Theorem 4.10. An explicit expression is given in (5.4) below.

([12]) This condition makes sense because the elements $g \in G_{k+1}$ act trivially on the underlying topological space of $\mathcal{Y}$, cf. Lemma 4.12.

Proof. If $\mathcal{M} \in \operatorname{Coh}(\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda), G_0)$, then in particular $\mathcal{M}$ is a coherent $\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda)$-module. Since by Corollary 4.9 and Theorem 4.10 we have that $H^0(\mathcal{Y},\mathcal{M})$ is a finitely presented $\mathcal{D}^{\text{an}}(\mathbb{G}(k)^{\circ})_{\lambda}$-module, we can conclude by (5.2) that $H^0(\mathcal{Y},\mathcal{M})$ is a finitely presented $D(\mathbb{G}(k)^{\circ}, G_0)_{\lambda}$-module.

On the other hand, let us suppose that M is a finitely presented $D(\mathbb{G}(k)^{\circ}, G_0)_{\lambda}$-module. By (5.2) we can consider

$$\mathcal{M} = \mathcal{L}oc^{\dagger}_{\mathcal{Y},k}(\lambda)(M) = \mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda) \otimes_{\mathcal{D}^{\text{an}}(\mathbb{G}(k)^{\circ})_{\lambda}} M. \tag{5.4}$$

For every $g \in G_0$ we want to define an isomorphism of sheaves of L-vector spaces

$$\varphi_g : \mathcal{M} \to (\rho_g)_*\mathcal{M}$$

satisfying the conditions (i), (ii) and (iii) of Definition 5.3. As we have remarked, the Dirac distributions induce an injective morphism from G_0 to the group of units of $D(G_0, L)$. Since by (5.3) M is in particular a G_0-module, we have an isomorphism

$$\mathcal{M} \to ((\rho_g)_*\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda)) \otimes_{\mathcal{D}^{\text{an}}(\mathbb{G}(k)^{\circ})_{\lambda}} M.$$

On local sections it is defined by $\varphi_{g,\mathcal{U}}(P \otimes m) = T_{g,\mathcal{U}}(P) \otimes gm$. Here T_g is the isomorphism defined in (4.9).

One has an isomorphism

$$(\rho_g)_*(\mathcal{M}) \xrightarrow{\simeq} ((\rho_g)_*\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda)) \otimes_{\mathcal{D}^{\text{an}}(\mathbb{G}(k)^{\circ})_{\lambda}} M.$$

Indeed, $(\rho_g)_*$ is exact and so choosing a finite presentation of M as $\mathcal{D}^{\text{an}}(\mathbb{G}(k)^{\circ})_{\lambda}$-module reduces to the case $M = \mathcal{D}^{\text{an}}(\mathbb{G}(k)^{\circ})_{\lambda}$ which is trivially true. This implies that the preceding isomorphism extends to an isomorphism

$$\varphi_g : \mathcal{M} \to (\rho_g)_*\mathcal{M}.$$

Let $g, h \in G_0$, $\mathcal{U} \subseteq \mathcal{Y}$ an open subset, $P, Q \in \mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda)(\mathcal{U})$ and $m \in M$. Then

$$\begin{aligned} \varphi_{h,\mathcal{U}.g^{-1}}(\varphi_{g,\mathcal{U}})(P \otimes m) &= T_{h,\mathcal{U}.g^{-1}}(T_{g,\mathcal{U}}(P)) \otimes hg\,m \\ &= T_{hg,\mathcal{U}}(P) \otimes (hg)m \\ &= \varphi_{hg,\mathcal{U}}(P \otimes m), \end{aligned}$$

and the family of isomorphisms $(\varphi_g)_{g\in G_0}$ verifies condition (i). Now, by definition $T_{g,\mathcal{U}}(PQ) = T_{g,\mathcal{U}}(P)T_{g,\mathcal{U}}(Q)$ and thus

$$\varphi_{g,\mathcal{U}}(PQ \otimes m) = T_{g,\mathcal{U}}(P)\varphi_{g,\mathcal{U}}(Q \otimes m),$$

which gives (ii). Finally, given that the delta distributions δ_g for g in the normal subgroup G_{k+1} of G_0 are contained in $\mathcal{D}^{an}(\mathbb{G}(k)^{\circ})$, we have $g.P = T_g(P) = \delta_g P \delta_{g^{-1}}$, therefore

$$\varphi_{g,\mathcal{U}}(P \otimes m) = g.P \otimes g.m = \delta_g P \delta_{g^{-1}} \delta_g \otimes m = \delta_g P \otimes m,$$

and condition (iii) follows. ■

REMARK 5.6. If $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ denotes the trivial character, then $\mathcal{D}^{\dagger}_{\mathfrak{X},k}(\lambda) = \mathcal{D}^{\dagger}_{\mathfrak{X},k}$ is the sheaf of arithmetic differential operators introduced in [20]. Moreover, by construction, if $\mathrm{pr} : \mathcal{Y} \to \mathfrak{X}$ denotes an H-equivariant admissible blow-up, then $\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda) = \mathcal{D}^{\dagger}_{\mathcal{Y},k}$ and for every $g \in H$ the isomorphism T_g equals the isomorphism $\mathrm{Ad}(g)$ defined in [20, (5.2.6)].

Now, let us take a morphism $\pi : \mathcal{Y}' \to \mathcal{Y}$ of G_0-equivariant admissible blow-ups of $\mathfrak{X}$ (whose lifted actions we denote by $\rho^{\mathcal{Y}'}$ and $\rho^{\mathcal{Y}}$), and let us suppose that $k \geq k_{\mathcal{Y}}$ and $k' \geq \max\{k'_{\mathcal{Y}}, k\}$. By Theorem 4.8 we have an injective morphism of sheaves

$$\Psi : \pi_* \mathcal{D}^{\dagger}_{\mathcal{Y}',k'}(\lambda) = \mathcal{D}^{\dagger}_{\mathcal{Y},k'}(\lambda) \hookrightarrow \mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda). \tag{5.5}$$

Moreover, this inclusion is G_0-equivariant in the sense that if $g \in G_0$, then we have

$$T^{\mathcal{Y}}_g \circ \Psi = (\rho^{\mathcal{Y}}_g)_*(\Psi) \circ \pi_*(T^{\mathcal{Y}'}_g).$$

Now, let us consider $\mathcal{M}_{\mathcal{Y}'} \in \mathrm{Coh}(\mathcal{D}^{\dagger}_{\mathcal{Y}',k'}(\lambda), G_0)$ and $\mathcal{M}_{\mathcal{Y}} \in \mathrm{Coh}(\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda), G_0)$ together with a morphism $\psi : \pi_* \mathcal{M}_{\mathcal{Y}'} \to \mathcal{M}_{\mathcal{Y}}$ linear relative to Ψ and which is G_0-equivariant, i.e. satisfying

$$\varphi^{\mathcal{M}_{\mathcal{Y}}}_g \circ \psi = (\rho^{\mathcal{Y}}_g)_* \psi \circ \pi_*(\varphi^{\mathcal{M}_{\mathcal{Y}'}}_g)$$

for all $g \in G_0$. Using Ψ, we obtain a morphism of $\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda)$-modules

$$\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda) \otimes_{\pi_* \mathcal{D}^{\dagger}_{\mathcal{Y}',k'}(\lambda)} \pi_* \mathcal{M}_{\mathcal{Y}'} \to \mathcal{M}_{\mathcal{Y}}.$$

Let us denote by $\mathcal{K}$ the submodule of $\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda) \otimes_{\pi_* \mathcal{D}^{\dagger}_{\mathcal{Y}',k'}(\lambda)} \pi_* \mathcal{M}_{\mathcal{Y}'}$ locally generated by all the elements of the form $P\delta_h \otimes m - P \otimes (h \bullet m)$, where $h \in G_{k+1}$, m is a local section of $\pi_* \mathcal{M}_{\mathcal{Y}'}$ and P is a local section of $\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda)$. As in [20, p. 35] we will consider the quotient

$$\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda) \otimes_{\pi_* \mathcal{D}^{\dagger}_{\mathcal{Y}',k'}(\lambda), G_{k+1}} \pi_* \mathcal{M}_{\mathcal{Y}'} := \mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda) \otimes_{\pi_* \mathcal{D}^{\dagger}_{\mathcal{Y}',k'}(\lambda)} \pi_* \mathcal{M}_{\mathcal{Y}'} / \mathcal{K}. \tag{5.6}$$

Let us see that this module lies in $\mathrm{Coh}(\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda), G_0)$. To do that let us first show that

$$\begin{aligned}(\rho_g)_* \mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda) \otimes_{(\rho_g)_* \pi_* \mathcal{D}^{\dagger}_{\mathcal{Y}',k'}(\lambda)} (\rho_g)_* \pi_* \mathcal{M}_{\mathcal{Y}'} \\ \simeq (\rho_g)_* \big(\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda) \otimes_{\pi_* \mathcal{D}^{\dagger}_{\mathcal{Y}',k'}(\lambda)} \pi_* \mathcal{M}_{\mathcal{Y}'}\big).\end{aligned}$$

As $\mathcal{M}_{\mathcal{Y}'}$ is a coherent $\mathcal{D}^{\dagger}_{\mathcal{Y}',k'}(\lambda)$-module, by Proposition 3.36 and Theorem 4.8 we can find a finite presentation of $\mathcal{M}_{\mathcal{Y}'}$,

$$(\mathcal{D}^{\dagger}_{\mathcal{Y}',k'}(\lambda))^{\oplus a} \to (\mathcal{D}^{\dagger}_{\mathcal{Y}',k'}(\lambda))^{\oplus b} \to \mathcal{M}_{\mathcal{Y}'} \to 0,$$

which induces, by exactness of $(\rho_g)_*$ and π_* (Theorem 4.8.), the exact sequence

$$((\rho_g)_*\mathcal{D}^{\dagger}_{\mathcal{Y},k'}(\lambda))^{\oplus a} \to ((\rho_g)_*\mathcal{D}^{\dagger}_{\mathcal{Y},k'}(\lambda))^{\oplus b} \to (\rho_g)_*\pi_*\mathcal{M}_{\mathcal{Y}'} \to 0.$$

Tensoring the previous exact sequence with $(\rho_g)_*\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda)$ over $(\rho_g)_*\pi_*\mathcal{D}^{\dagger}_{\mathcal{Y}',k'}(\lambda)$ and using the relation $\pi_*\mathcal{D}^{\dagger}_{\mathcal{Y}',k'}(\lambda) = \mathcal{D}^{\dagger}_{\mathcal{Y},k'}(\lambda)$, we see that the canonical map

$$\begin{aligned}(\rho_g)_*\big(\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda) \otimes_{\pi_*\mathcal{D}^{\dagger}_{\mathcal{Y}',k'}(\lambda)} \pi_*\mathcal{M}_{\mathcal{Y}'}\big)& \\ \to (\rho_g)_*\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda) \otimes_{(\rho_g)_*\pi_*\mathcal{D}^{\dagger}_{\mathcal{Y}',k'}(\lambda)} (\rho_g)_*\pi_*\mathcal{M}_{\mathcal{Y}'}&\end{aligned}$$

is an isomorphism. We dispose of a diagonal action

$$\varphi_g : \mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda) \otimes_{\pi_*\mathcal{D}^{\dagger}_{\mathcal{Y}',k'}(\lambda)} \pi_*\mathcal{M}_{\mathcal{Y}'} \to (\rho_g)_*\big(\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda) \otimes_{\pi_*\mathcal{D}^{\dagger}_{\mathcal{Y}',k'}(\lambda)} \pi_*\mathcal{M}_{\mathcal{Y}'}\big)$$

defined on simple tensor products by

$$g \bullet (P \otimes m) = g \bullet P \otimes g \bullet m,$$

for $g \in G_0$, and P and m local sections of $\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda)$ and $\pi_*\mathcal{M}_{\mathcal{Y}'}$, respectively (in order to soft the notation we use the accord introduced in Remark 5.4). Now to see that (5.6) is a strongly G_0-equivariant $\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda)$-module, we only need to check that the diagonal action fixes the submodule $\mathcal{K}$, i.e., $\varphi_g(\mathcal{K}) \subset \mathcal{K}$. We have

$$\begin{aligned}&g \bullet (P\delta_h \otimes m - P \otimes h \bullet m)\\ &\quad = g \bullet (P\delta_h) \otimes g \bullet m - g \bullet P \otimes g \bullet (h \bullet m)\\ &\quad = (g \bullet P)(g \bullet \delta_h) \otimes g \bullet m - g \bullet P \otimes (ghg^{-1}) \bullet (g \bullet m)\\ &\quad = (g \bullet P)\delta_{ghg^{-1}} \otimes g \bullet m - g \bullet P \otimes (ghg^{-1}) \bullet (g \bullet m).\end{aligned}$$

As G_{k+1} is a normal subgroup, we can conclude that $ghg^{-1} \in G_{k+1}$ and G_0 fixes $\mathcal{K}$. Moreover, since the target of the preceding morphism is strongly G_0-equivariant, this factors through the quotient and we thus obtain a morphism of $\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda)$-modules

$$\overline{\psi} : \mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda) \otimes_{\pi_*\mathcal{D}^{\dagger}_{\mathcal{Y}',k'}(\lambda),G_{k+1}} \pi_*\mathcal{M}_{\mathcal{Y}'} \to \mathcal{M}_{\mathcal{Y}}. \tag{5.7}$$

By construction $\overline{\psi} \in \mathrm{Coh}(\mathcal{D}^{\dagger}_{\mathcal{Y},k}(\lambda), G_0)$.

6. Admissible blow-ups and formal models

The following discussion is given in [20, (3.1.1) and (5.2.10)]. Let us start by considering the generic fiber $X_{\mathbb{Q}} := X \times_{\mathrm{Spec}(\mathfrak{o})} \mathrm{Spec}(L)$ of the flag scheme X (the flag *variety*). For the rest of this work X^{rig} will denote the rigid-analytic space associated via the GAGA functor to $X_{\mathbb{Q}}$ (see [6, Part I, Section 5.4, Definition and Proposition 3]). Any admissible formal $\mathfrak{o}$-scheme $\mathcal{Y}$ (in the sense of [6, Part II, Section 7.4, Definitions 1 and 4]) whose associated rigid-analytic space is isomorphic to X^{rig} will be called a *formal model* of X^{rig}. For any two formal models $\mathcal{Y}_1$ and $\mathcal{Y}_2$ there exist a formal model $\mathcal{Y}'$ and admissible formal blow-up morphisms $\mathcal{Y}' \to \mathcal{Y}_1$ and $\mathcal{Y}' \to \mathcal{Y}_2$ (see [6, Part II, Section 8.2, Remark 10]).

Now, let us denote by $\mathcal{F}_{\mathfrak{X}}$ the set of admissible formal blow-ups $\mathcal{Y} \to \mathfrak{X}$. This set is ordered by $\mathcal{Y}' \succeq \mathcal{Y}$ if the blow-up morphism $\mathcal{Y}' \to \mathfrak{X}$ factors as the composition of a morphism $\mathcal{Y}' \to \mathcal{Y}$ and the blow-up morphism $\mathcal{Y} \to \mathfrak{X}$. In this case, the morphism $\mathcal{Y}' \to \mathcal{Y}$ is unique (see [6, Part II, Section 8.2, Proposition 9]) and it is itself a blow-up morphism (see [25, Section 8.1.3, Proposition 1.12 (d) and Theorem 1.24]). By [6, Part II, Section 8.2, Remark 10] the set $\mathcal{F}_{\mathfrak{X}}$ is directed and it is cofinal in the set of all formal models. Furthermore, any formal model $\mathcal{Y}$ of X^{rig} is dominated by one which is a G_0-equivariant admissible blow-up of $\mathfrak{X}$ (see [20, Proposition 5.2.14]). In particular, if $\mathfrak{X}_\infty$ denotes the projective limit of all formal models of X^{rig}, then

$$\mathfrak{X}_\infty = \varprojlim_{\mathcal{F}_{\mathfrak{X}}} \mathcal{Y}.$$

We will be interested in the following directed subset of $\mathcal{F}_{\mathfrak{X}}$.

Definition 6.1. We denote by $\underline{\mathcal{F}}_{\mathfrak{X}}$ the set of pairs $(\mathcal{Y}, k)$, where $\mathcal{Y} \in \mathcal{F}_{\mathfrak{X}}$ and $k \in \mathbb{N}$ satisfies $k \geq k_{\mathcal{Y}}$. This set is ordered by $(\mathcal{Y}', k') \succeq (\mathcal{Y}, k)$ if and only if $\mathcal{Y} \succeq \mathcal{Y}$ and $k' \geq k$.

We will need the following auxiliary result.

Lemma 6.2. *Let $\mathcal{Y}'$, $\mathcal{Y} \in \mathcal{F}_{\mathfrak{X}}$ be G_0-equivariant admissible blow-ups. Suppose $(\mathcal{Y}', k') \succeq (\mathcal{Y}, k)$ with canonical morphism $\pi : \mathcal{Y}' \to \mathcal{Y}$ over $\mathfrak{X}$ and let M be a coherent $D(\mathbb{G}(k')^\circ, G_0)_\lambda$-module with*

$$\mathscr{M} = \mathscr{L}oc^\dagger_{\mathcal{Y}',k'}(\lambda)(M) \in \mathrm{Coh}(\mathcal{D}^\dagger_{\mathcal{Y}',k'}(\lambda), G_0).$$

Then there exists a canonical isomorphism in $\mathrm{Coh}(\mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda), G_0)$ given by

$$\mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda) \otimes_{\pi_*\mathcal{D}^\dagger_{\mathcal{Y}',k'}(\lambda), G_{k+1}} \pi_*\mathscr{M} \xrightarrow{\simeq} \mathscr{L}oc^\dagger_{\mathcal{Y},k}(\lambda)\big(D(\mathbb{G}(k)^\circ, G_0) \otimes_{D(\mathbb{G}(k')^\circ, G_0)} M\big).$$

Proof. The proof follows word for word the reasoning given in [20, Lemma 5.2.12] when $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ is equal to the trivial character. Let Σ be a system of representatives in G_{k+1} for the cosets in $G_{k+1}/G_{k'+1}$. By (5.2) we have a canonical map

$$\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda \to D(\mathbb{G}(k)^\circ, G_0)_\lambda,$$

which is compatible with variation in k. Now, let us take a $D(\mathbb{G}(k')^\circ, G_0)_\lambda$-module M and let us consider the free $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda$-module

$$\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda^{M\times\Sigma} = \bigoplus_{(m,h)\in M\times\Sigma} \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda e_{m,h},$$

whose formation is functorial in M and it comes with a linear map

$$\begin{aligned} f_M : \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda^{M\times\Sigma} &\to \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k')^\circ)_\lambda} M,\\ \lambda_{m,h}e_{m,h} &\mapsto (\lambda_{m,h}\delta_h)\otimes m - \lambda_{m,h}\otimes(\delta_h.m), \end{aligned}$$

which fits into an exact sequence

$$\begin{aligned} \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda^{M\times\Sigma} &\xrightarrow{f_M} \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k')^\circ)_\lambda} M\\ &\twoheadrightarrow D(\mathbb{G}(k)^\circ, G_0)_\lambda \otimes_{D(\mathbb{G}(k')^\circ, G_0)_\lambda} M \end{aligned}$$

if M is a finitely presented $D(\mathbb{G}(k')^\circ, G_0)_\lambda$-module; see Claim 1 in the proof of [20, Lemma 5.2.12].

Now, let M be a finitely presented $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k')^\circ)_\lambda$-module and

$$\mathcal{M} = \mathcal{L}oc^\dagger_{\mathcal{Y}',k'}(\lambda)(M).$$

Then the natural morphism

$$\mathcal{L}oc^\dagger_{\mathcal{Y},k}(\lambda)\big(\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k')^\circ)_\lambda} M\big) \to \mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda) \otimes_{\pi_*\mathcal{D}^\dagger_{\mathcal{Y}',k'}(\lambda)} \pi_*\mathcal{M} \tag{6.1}$$

is bijective. In fact, by Theorem 4.8 we know that the functor π_* is exact on coherent $\mathcal{D}^\dagger_{\mathcal{Y}',k'}(\lambda)$-modules, so taking a finite presentation of M, we reduce to the case $M = \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k')^\circ)_\lambda$ which is clear.

Finally, let M be a finitely presented $D(\mathbb{G}(k')^\circ, G_0)_\lambda$-module. Let $m_1, \ldots, m_a$ be generators for M as a $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k')^\circ)_\lambda$-module. We have a sequence of $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda$-modules

$$\begin{aligned} \bigoplus_{(i,h)} \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda e_{m_i,h} &\xrightarrow{f_a} \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k')^\circ)_\lambda} M\\ &\twoheadrightarrow D(\mathbb{G}(k)^\circ, G_0)_\lambda \otimes_{D(\mathbb{G}(k')^\circ, G_0)_\lambda} M, \end{aligned}$$

where f_a denotes the restriction of the morphism f_M to the free submodule of $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda^{M\times\Sigma}$ generated by finitely many vectors $e_{m_i,h}$, with $1 \leq i \leq a$ and $h \in \Sigma$. Since $\mathrm{im}(f_a) = \mathrm{im}(f_M)$, the sequence is exact. Since it consists of finitely presented $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda$-modules, we can apply the localization functor $\mathscr{L}oc^\dagger_{\mathcal{Y},k}(\lambda)$ to it. Given that

$$\mathscr{L}oc^\dagger_{\mathcal{Y},k}(\lambda)\Big(\bigoplus_{(i,h)} \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda e_{m_i,h}\Big)$$
$$= \mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda) \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda} \bigoplus_{(i,h)} \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda e_{m_i,h} = \mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda)^{\oplus a|\Sigma|},$$

the morphism in (6.1) gives us the exact sequence

$$\mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda)^{\oplus a|\Sigma|} \to \mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda) \otimes_{\pi_*\mathcal{D}_{\mathcal{Y}',k'}(\lambda)} \pi_*\mathcal{M} \to \mathscr{L}oc^\dagger_{\mathcal{Y},k}(\lambda)(M_{k,\lambda}) \to 0,$$
$$e_{m_i,h} \otimes P \mapsto (P\delta_h \otimes m_i - P \otimes \delta_h m),$$

where $\mathcal{M} = \mathscr{L}oc^\dagger_{\mathcal{Y}',k'}(\lambda)(M)$ and $M_{k,\lambda} = D(\mathbb{G}(k)^\circ, G_0)_\lambda \otimes_{D(\mathbb{G}(k')^\circ,G_0)_\lambda} M$. The cokernel of the first map in this sequence equals by definition

$$\mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda) \otimes_{\pi_*\mathcal{D}^\dagger_{\mathcal{Y}',k'}(\lambda),G_{k+1}} \pi_*\mathcal{M},$$

and we get the desired isomorphism. ∎

Now, let $\mathcal{I}$ be an open ideal sheaf on $\mathfrak{X}$, and let $g \in G_0$. Then

$$\mathcal{J} := (\rho_g^\natural)^{-1}((\rho_g)_*(\mathcal{I}))$$

is again an open ideal sheaf on $\mathfrak{X}$. Let $\mathcal{Y}$ be the blow-up of $\mathcal{I}$ and $\mathcal{Y}.g$ the blow-up of $\mathcal{J}$, with canonical morphism $\mathrm{pr}_g : \mathcal{Y}.g \to \mathfrak{X}$. We have the following result from [20, Lemma 5.2.16].

Lemma 6.3. *There exists a morphism $\rho_g : \mathcal{Y} \to \mathcal{Y}.g$ such that the diagram*

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\rho_g} & \mathcal{Y}.g \\ \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow{\scriptstyle \mathrm{pr}_g} \\ \mathfrak{X} & \xrightarrow{\rho_g} & \mathfrak{X} \end{array}$$

is commutative. Moreover, we have $k_{\mathcal{Y}.g} = k_{\mathcal{Y}}$ and for any two elements $g, h \in G_0$, we have a canonical isomorphism $(\mathcal{Y}.g).h \simeq \mathcal{Y}.(gh)$, such that the composition morphism $\mathcal{Y} \to \mathcal{Y}.g \to (\mathcal{Y}.g).h \simeq \mathcal{Y}.(gh)$ is equal to ρ_{gh}. This gives a right action of the group G_0 on the family $\mathfrak{F}_{\mathfrak{X}}$.

Let $\mathrm{pr} : \mathcal{Y} \to \mathcal{X}$ be an admissible blow-up and let us denote by $\underline{\mathcal{L}(\lambda)}$ the invertible sheaf on $\mathcal{Y}$ induced by pulling back the invertible sheaf on $\mathcal{X}$ induced by the character λ. This is $\underline{\mathcal{L}(\lambda)} = \mathrm{pr}^*\mathcal{L}(\lambda)$. Furthermore, for $g \in G_0$ if $\rho_g : \mathcal{Y} \to \mathcal{Y}.g$ is the morphism given by the previous lemma and $\mathrm{pr}_g : \mathcal{Y}.g \to \mathcal{X}$ is the blow-up morphism, then we will denote

$$\underline{\mathcal{L}_g(\lambda)} = \mathrm{pr}_g^*\mathcal{L}(\lambda).$$

The notation being fixed, we advise the reader that, in order to simplify the notation, we will avoid underlining these sheaves in the rest of this work if the context is clear and there is not risk to any confusion.

Let us recall that in Section 4.3 we have built for any $g \in G_0$ an $\mathcal{O}_{\mathcal{X}}$-linear isomorphism $\Phi_g : \mathcal{L}(\lambda) \to (\rho_g)_*\mathcal{L}(\lambda)$, where $\rho_g = \alpha \circ (\mathrm{id}_{\mathcal{X}} \times g)$ is the translation morphism and α the right $\mathfrak{G}$-action on $\mathcal{X}$. By pulling back this morphism and using the commutative diagram in the previous lemma ($\rho_g^* \circ \mathrm{pr}_g^* = \mathrm{pr}^* \circ \rho_g^*$), we have an $\mathcal{O}_{\mathcal{Y}}$-linear isomorphism $(\rho_g)^*\mathrm{pr}_g^*\mathcal{L}(\lambda) \to \mathrm{pr}^*\mathcal{L}(\lambda)$. By adjointness and following the accord established in the previous paragraph, we get an $\mathcal{O}_{\mathcal{Y}.g}$-linear morphism

$$L_g : \mathcal{L}_g(\lambda) \to (\rho_g)_*\mathcal{L}(\lambda).$$

By construction L_g satisfies the cocycle condition (4.8). This means that for every $g, h \in G_0$ we have

$$L_{hg} = \mathcal{L}_{hg}(\lambda) \xrightarrow{L_g} (\rho_g)_*\mathcal{L}_h(\lambda) \xrightarrow{(\rho_g)_*L_h} (\rho_{hg})_*\mathcal{L}(\lambda).$$

In particular, L_g is an isomorphism for every $g \in G_0$.

Exactly as we have done in (4.9), and given that by construction $\mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda)$ acts on $\mathcal{L}(\lambda)$ (resp. $\mathcal{D}^\dagger_{\mathcal{Y}.g,k}(\lambda)$ acts on $\mathcal{L}_g(\lambda)$), we can build an isomorphism

$$T_g : \mathcal{D}^\dagger_{\mathcal{Y}.g,k}(\lambda) \to (\rho_g)_*\mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda), \quad P \mapsto L_g \circ P \circ L_g^{-1},$$

satisfying the cocycle condition

$$T_{hg} = (\rho_g)_*T_h \circ T_g \quad (g, h \in G_0).$$

From the previous lemma we get [20, Corollary 5.2.18]:

Corollary 6.4. *Let us suppose that $(\mathcal{Y}', k') \succeq (\mathcal{Y}, k)$ for $\mathcal{Y}, \mathcal{Y}' \in \mathcal{F}_{\mathcal{X}}$ and let $\pi : \mathcal{Y}' \to \mathcal{Y}$ be the unique morphism over $\mathcal{X}$. Let $g \in G_0$. Then $(\mathcal{Y}'.g, k') \succeq (\mathcal{Y}.g, k)$ and if we denote by $\pi.g : \mathcal{Y}'.g \to \mathcal{Y}.g$ the unique morphism over $\mathcal{X}$, we have a commutative diagram*

$$\begin{array}{ccc} \mathcal{Y}' & \xrightarrow{\rho_g} & \mathcal{Y}'.g \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi.g} \\ \mathcal{Y} & \xrightarrow{\rho_g} & \mathcal{Y}.g. \end{array}$$

Based on [20, Definition 5.2.19] we introduce the following definition.

Definition 6.5. A coadmissible G_0-equivariant $\mathcal{D}(\lambda)$-module on $\mathcal{F}_{\mathfrak{X}}$ consists of a family $\mathcal{M} := (\mathcal{M}_{\mathfrak{Y},k})_{(\mathfrak{Y},k)}$ of coherent $\mathcal{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$-modules $\mathcal{M}_{\mathfrak{Y},k}$ for all $(\mathfrak{Y},k) \in \underline{\mathcal{F}}_{\mathfrak{X}}$, with the following properties:

(a) For any $g \in G_0$ with morphism $\rho_g : \mathfrak{Y} \to \mathfrak{Y}.g$, there exists an isomorphism

$$\varphi_g : \mathcal{M}_{\mathfrak{Y}.g,k} \to (\rho_g)_* \mathcal{M}_{\mathfrak{Y},k}$$

of sheaves of L-vector spaces, satisfying the following properties:

(i) For all $g, h \in G_0$ one has $(\rho_g)_*(\varphi_h) \circ \varphi_g = \varphi_{hg}$.

(ii) For all open subsets $\mathfrak{U} \subseteq \mathfrak{Y}.g$, all $P \in \mathcal{D}^{\dagger}_{\mathfrak{Y}.g,k}(\lambda)(\mathfrak{U})$, and all $m \in \mathcal{M}_{\mathfrak{Y}.g,k}(\mathfrak{U})$ one has $\varphi_g(P \bullet m) = T_{g,\mathfrak{U}}(P) \bullet \varphi_{g,\mathfrak{U}}(m)$.

(iii) For all $g \in G_{k+1}$ the map $\varphi_g : \mathcal{M}_{\mathfrak{Y}.g,k} = \mathcal{M}_{\mathfrak{Y},k} \to (\rho_g)_* \mathcal{M}_{\mathfrak{Y},k} = \mathcal{M}_{\mathfrak{Y},k}$ is equal to multiplication by $\delta_g \in H^0(\mathfrak{Y}, \mathcal{D}^{\dagger}_{\mathfrak{Y},k}(\lambda))$.[13]

(b) Suppose $\mathfrak{Y}, \mathfrak{Y}' \in \mathcal{F}_{\mathfrak{X}}$ are both G_0-equivariant, and assume further that $(\mathfrak{Y}', k') \succeq (\mathfrak{Y}, k)$, and that $\pi : \mathfrak{Y}' \to \mathfrak{Y}$ is the unique morphism over $\mathfrak{X}$. We require the existence of a transition morphism $\psi_{\mathfrak{Y}',\mathfrak{Y}} : \pi_* \mathcal{M}_{\mathfrak{Y}',k'} \to \mathcal{M}_{\mathfrak{Y},k}$, linear relative to the canonical morphism $\Psi : \pi_* \mathcal{D}^{\dagger}_{\mathfrak{Y}',k'}(\lambda) \to \mathcal{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$. By using the commutative diagram in the preceding corollary, we require

$$\varphi_g \circ \psi_{\mathfrak{Y}'.g,\mathfrak{Y}.g} = (\rho_g)_*(\psi_{\mathfrak{Y}',\mathfrak{Y}}) \circ (\pi.g)_*(\varphi_g). \tag{6.2}$$

The morphism induced by $\psi_{\mathfrak{Y}',\mathfrak{Y}}$,

$$\overline{\psi}_{\mathfrak{Y}',\mathfrak{Y}} : \mathcal{D}^{\dagger}_{\mathfrak{Y},k}(\lambda) \otimes_{\pi_* \mathcal{D}^{\dagger}_{\mathfrak{Y}',k'}(\lambda), G_{k+1}} \pi_* \mathcal{M}_{\mathfrak{Y}'} \to \mathcal{M}_{\mathfrak{Y}},$$

is required to be an isomorphism of $\mathcal{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$-modules. Additionally, the morphisms $\psi_{\mathfrak{Y}',\mathfrak{Y}}$ are required to satisfy the transitivity rule

$$\psi_{\mathfrak{Y}',\mathfrak{Y}} \circ \pi_*(\psi_{\mathfrak{Y}'',\mathfrak{Y}'}) = \psi_{\mathfrak{Y}'',\mathfrak{Y}}$$

for $(\mathfrak{Y}'', k'') \succeq (\mathfrak{Y}', k') \succeq (\mathfrak{Y}, k)$ in $\underline{\mathcal{F}}_{\mathfrak{X}}$. Moreover, $\psi_{\mathfrak{Y},\mathfrak{Y}} = \mathrm{id}_{\mathcal{M}_{\mathfrak{Y},k}}$.

A morphism $\mathcal{M} \to \mathcal{N}$ between such modules consists of morphisms $\mathcal{M}_{\mathfrak{Y},k} \to \mathcal{N}_{\mathfrak{Y},k}$ of $\mathcal{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$-modules, which is compatible with the extra structures imposed in (a) and (b). We denote the resulting category by $\mathcal{C}^{G_0}_{\mathfrak{X},\lambda}$.

([13]) As is remarked in [20, Definition 5.2.19 (iii)], if $g \in G_{k+1}$, then $\mathfrak{Y}.g = \mathfrak{Y}$ and g acts trivially on the underlying topological space $|\mathfrak{Y}|$.

Let us build now the bridge to the category $\mathcal{C}_{G_0,\lambda}$ of coadmissible $D(G_0, L)_\lambda$-modules. Given such a module M, we have its associated admissible locally analytic G_0-representation $V := M'_b$ together with its subspace of $\mathbb{G}(k)^\circ$-analytic vectors $V_{\mathbb{G}(k)^\circ\text{-an}} \subseteq V$. As we have remarked, this is stable under the G_0-action and its dual $M_k = (V_{\mathbb{G}(k)^\circ\text{-an}})'_b$ is a finitely presented $D(\mathbb{G}(k)^\circ, G_0)_\lambda$-module. In this situation we produce a coherent $\mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda)$-module

$$\mathcal{L}oc^\dagger_{\mathcal{Y},k}(\lambda)(M_k) = \mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda) \otimes_{\mathcal{D}^{\text{an}}(\mathbb{G}(k)^\circ)_\lambda} M_k$$

for any element $(\mathcal{Y}, k) \in \underline{\mathcal{F}}_{\mathcal{X}}$. We will denote the resulting family by

$$\mathcal{L}oc^{G_0}_\lambda(M) = (\mathcal{L}oc^\dagger_{\mathcal{Y},k}(\lambda)(M_k))_{(\mathcal{Y},k)\in\underline{\mathcal{F}}_{\mathcal{X}}}.$$

On the other hand, let $\mathcal{M}$ be an arbitrary coadmissible G_0-equivariant arithmetic $\mathcal{D}(\lambda)$-module on $\mathcal{F}_{\mathcal{X}}$. The transition morphisms $\psi_{\mathcal{Y}',\mathcal{Y}} : \pi_* \mathcal{M}_{\mathcal{Y}',k'} \to \mathcal{M}_{\mathcal{Y},k}$ induce maps $H^0(\mathcal{Y}', \mathcal{M}_{\mathcal{Y}',k'}) \to H^0(\mathcal{Y}, \mathcal{M}_{\mathcal{Y},k})$ on global sections. We let

$$\Gamma(\mathcal{M}) = \varprojlim_{(\mathcal{Y},k)\in\underline{\mathcal{F}}_{\mathcal{X}}} H^0(\mathcal{Y}, \mathcal{M}_{\mathcal{Y},k}).$$

The projective limit is taken in the sense of abelian groups. We have the following theorem. Except for some technical details the proof follows word for word the reasoning given in [20, Theorem 5.2.23].

THEOREM 6.6. *Let us suppose that $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ is an algebraic character such that $\lambda + \rho \in \mathfrak{t}^*_{\mathbb{Q}}$ is a dominant and regular character of $\mathfrak{t}_{\mathbb{Q}}$. The functors $\mathcal{L}oc^{G_0}_\lambda$ and $\Gamma(\bullet)$ induce quasi-inverse equivalences between the categories $\mathcal{C}_{G_0,\lambda}$ (of coadmissible $D(G_0, L)_\lambda$-modules) and $\mathcal{C}^{G_0}_{\mathcal{X},\lambda}$.*

PROOF. Let us take $M \in \mathcal{C}_{G_0,\lambda}$ and $\mathcal{M} \in \mathcal{C}^{G_0}_{\mathcal{X},\lambda}$. As in the proof of [20, Theorem 5.2.23] we will organize the proof in four steps.

Claim 1. We have $\mathcal{L}oc^{G_0}_\lambda(M) \in \mathcal{C}^{G_0}_{\mathcal{X},\lambda}$ and $\mathcal{L}oc^{G_0}_\lambda(M)$ is functorial in M.

Let us start by defining

$$\varphi_g : \mathcal{L}oc^\dagger_{\mathcal{Y}.g,k}(\lambda)(M_k) \to (\rho_g)_* \mathcal{L}oc^\dagger_{\mathcal{Y},k}(\lambda)(M_k) \quad (g \in G_0)$$

satisfying (i), (ii) and (iii) in the preceding definition. Let $\widetilde{\varphi}_g : M_k \to M_k$ denote the map dual to the map $V_{\mathbb{G}(k)^\circ\text{-an}} \to V_{\mathbb{G}(k)^\circ\text{-an}}$ given by $w \mapsto g^{-1}w$. By definition $\widetilde{\varphi}_h \circ \widetilde{\varphi}_g = \widetilde{\varphi}_{hg}$. Let $\mathcal{U} \subseteq \mathcal{Y}.g$ be an open subset, $P \in \mathcal{D}^\dagger_{\mathcal{Y}.g,k}(\lambda)(\mathcal{U})$ and $m \in M_k$. We define

$$\varphi_{g,\mathcal{U}}(P \otimes m) = T_{g,\mathcal{U}}(P) \otimes \widetilde{\varphi}_g(m).$$

Given that $(\rho_g)_*$ is exact, we can choose a finite presentation of M_k as a $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda$-module to conclude that we have a canonical isomorphism

$$(\rho_g)_*(\mathcal{L}oc^\dagger_{\mathcal{Y},k}(\lambda)(M_k)) \xrightarrow{\simeq} ((\rho_g)_*\mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda)) \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda} M_k.$$

This means that the above definition extends to a map

$$\varphi_g : \mathcal{D}^\dagger_{\mathcal{Y}.g,k}(\lambda) \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda} M_k \to (\rho_g)_*(\mathcal{L}oc^\dagger_{\mathcal{Y},k}(\lambda)(M_k)).$$

The family $\{\varphi_g\}_{g\in G_0}$ satisfies (i), (ii) and (iii) in (a). Let us verify condition (b). We suppose that $\mathcal{Y}', \mathcal{Y}$ are G_0-equivariant and that $(\mathcal{Y}', k) \succeq (\mathcal{Y}, k)$ with canonical morphism $\pi : \mathcal{Y}' \to \mathcal{Y}$ over $\mathcal{X}$. As π_* is exact, we have an isomorphism

$$\pi_*(\mathcal{L}oc^\dagger_{\mathcal{Y}',k'}(\lambda)(M_{k'})) \xrightarrow{\simeq} \pi_*(\mathcal{D}^\dagger_{\mathcal{Y}',k'}(\lambda)) \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda} M_{k'}.$$

(This is an argument already given in the text for the functor $(\rho_g)_*$). On the other hand, we have that $\mathbb{G}(k')^\circ \subseteq \mathbb{G}(k)^\circ$ and we have a map $\widetilde{\psi}_{\mathcal{Y}',\mathcal{Y}} : M_{k'} \to M_k$ obtained as the dual map of the natural inclusion $V_{\mathbb{G}(k)^\circ\text{-an}} \hookrightarrow V_{\mathbb{G}(k')^\circ\text{-an}}$. Let $\mathcal{U} \subseteq \mathcal{Y}$ be an open subset, $P \in \pi_*\mathcal{D}^\dagger_{\mathcal{Y}',k'}(\lambda)(\mathcal{U})$ and $m \in M_{k'}$. We define

$$\psi_{\mathcal{Y}',\mathcal{Y}}(P \otimes m) = \Psi_{\mathcal{Y}',\mathcal{Y}}(P) \otimes \widetilde{\psi}_{\mathcal{Y}',\mathcal{Y}}(m),$$

where Ψ is the canonical injection $\pi_*\mathcal{D}^\dagger_{\mathcal{Y}',k'}(\lambda) \hookrightarrow \mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda)$. By using the preceding isomorphism, we can conclude that this morphism extends naturally to a map

$$\psi_{\mathcal{Y}',\mathcal{Y}} : \pi_*(\mathcal{L}oc^\dagger_{\mathcal{Y}',k'}(\lambda)(M_{k'})) \to \mathcal{L}oc^\dagger_{\mathcal{Y},k}(\lambda)(M_k).$$

The cocycle condition (6.2) translates into the diagram

$$\begin{array}{ccc} (\pi.g)_*(\rho_g^{\mathcal{Y}})_*(\mathcal{L}oc^\dagger_{\mathcal{Y}',k'}(\lambda)(M_{k'})) & \xrightarrow{(\rho_g^{\mathcal{Y}})_*\psi_{\mathcal{Y}',\mathcal{Y}}} & (\rho_g^{\mathcal{Y}})_*(\mathcal{L}oc^\dagger_{\mathcal{Y},k}(\lambda)(M_k)) \\ {\scriptstyle (\pi.g)_*\varphi_g}\uparrow & & \uparrow{\scriptstyle \varphi_g} \\ (\pi.g)_*(\mathcal{L}oc^\dagger_{\mathcal{Y}'.g,k'}(\lambda)(M_{k'})) & \xrightarrow{\psi_{\mathcal{Y}',\mathcal{Y}}} & \mathcal{L}oc^\dagger_{\mathcal{Y}.g,k}(\lambda)(M_k). \end{array} \tag{6.3}$$

We have used

$$(\rho_g^{\mathcal{Y}})_*\pi_*(\mathcal{L}oc^\dagger_{\mathcal{Y}',k'}(\lambda)(M_{k'})) = (\pi.g)_*(\rho_g^{\mathcal{Y}})_*(\mathcal{L}oc^\dagger_{\mathcal{Y}',k'}(\lambda)(M_{k'})).$$

By construction, the diagrams

$$\begin{array}{ccc} (\rho_g^{\mathcal{Y}})_*\pi_*\mathcal{D}^\dagger_{\mathcal{Y}',k'}(\lambda) & \xrightarrow{(\rho_g^{\mathcal{Y}})_*\Psi_{\mathcal{Y}',\mathcal{Y}}} & (\rho_g^{\mathcal{Y}})_*\mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda) \\ {\scriptstyle (\pi.g)_*T_g}\uparrow & & \uparrow{\scriptstyle T_g} \\ (\pi_g)_*\mathcal{D}^\dagger_{\mathcal{Y}'.g,k'}(\lambda) & \xrightarrow{\Psi_{\mathcal{Y}'.g,\mathcal{Y}.g}} & \mathcal{D}^\dagger_{\mathcal{Y}.g,k}(\lambda), \end{array} \qquad \begin{array}{ccc} M_{k'} & \xrightarrow{\widetilde{\psi}_{\mathcal{Y}',\mathcal{Y}}} & M_k \\ \downarrow{\scriptstyle \widetilde{\varphi}_g} & & \downarrow{\scriptstyle \widetilde{\varphi}_g} \\ M_{k'} & \xrightarrow{\widetilde{\psi}_{\mathcal{Y}',\mathcal{Y}}} & M_k \end{array} \tag{6.4}$$

are commutative; therefore, (6.3) is also a commutative diagram. As before, we have used the relation

$$(\pi.g)_*(\rho_g^{\mathcal{Y}'})_*\mathcal{D}^\dagger_{\mathcal{Y}',k'}(\lambda) = (\rho_g^{\mathcal{Y}})_*\pi_*\mathcal{D}^\dagger_{\mathcal{Y}',k'}(\lambda).$$

The transitivity properties are clear. Let us see that the induced morphism $\overline{\psi}_{\mathcal{Y}',\mathcal{Y}}$ is in fact an isomorphism. The morphism $\overline{\psi}_{\mathcal{Y}',\mathcal{Y}}$ corresponds under the isomorphism of Lemma 6.2 to the linear extension

$$D(\mathbb{G}(k)^\circ, G_0) \otimes_{D(\mathbb{G}(k')^\circ, G_0)} M_{k'} \to M_k$$

of $\widetilde{\psi}_{\mathcal{Y}',\mathcal{Y}}$ via functoriality of $\mathcal{L}oc^\dagger_{\mathcal{Y},k}(\lambda)$. By Lemma 5.1 this linear extension is an isomorphism and hence, so is $\overline{\psi}_{\mathcal{Y}',\mathcal{Y}}$. We conclude that $\mathcal{L}oc_\lambda^{G_0}(M) \in \mathcal{C}^{G_0}_{\mathcal{X},\lambda}$. Given a morphism $M \to N$ in $\mathcal{C}_{G_0,\lambda}$, we get, by definition, morphisms $M_k \to N_k$ for any $k \in \mathbb{Z}_{>0}$ compatible with $\widetilde{\varphi}_g$ and $\widetilde{\psi}_{\mathcal{Y}',\mathcal{Y}}$. By functoriality of $\mathcal{L}oc^\dagger_{\mathcal{Y},k}(\lambda)$, they give rise to linear maps

$$\mathcal{L}oc^\dagger_{\mathcal{Y},k}(\lambda)(M_k) \to \mathcal{L}oc^\dagger_{\mathcal{Y},k}(\lambda)(N_k),$$

which are compatible with the maps φ_g and $\psi_{\mathcal{Y}',\mathcal{Y}}$.

Claim 2. $\Gamma(\mathcal{M})$ *is an object in* $\mathcal{C}_{G_0,\lambda}$.

For $k \in \mathbb{N}$ we choose $(\mathcal{Y}, k) \in \mathcal{F}_{\mathcal{X}}$ and we put $N_k := H^0(\mathcal{Y}, \mathcal{M}_{(\mathcal{Y},k)})$. By (5.7), Lemma 6.2 and the fact that $\mathcal{M} \in \mathcal{C}^{G_0}_{\mathcal{X},\lambda}$ we get linear isomorphisms

$$D(\mathbb{G}(k)^\circ, G_0) \otimes_{D(\mathbb{G}(k')^\circ, G_0)} N_{k'} \to N_k$$

for $k' \geq k$. This implies that the modules N_k form a $(D(\mathbb{G}(k)^\circ, G_0))_{k\in\mathbb{N}}$-sequence and the projective limit is a coadmissible module.

Claim 3. $\Gamma \circ \mathcal{L}oc_\lambda^{G_0}(M) \simeq M$.

If $V := M'_b$, then we have by definition compatible isomorphisms

$$H^0(\mathcal{Y}, \mathcal{L}oc_\lambda^{G_0}(M)_{(\mathcal{Y},k)}) = H^0(\mathcal{Y}, \mathcal{L}oc^\dagger_{\mathcal{Y},k}(\lambda)(M_k)) = (V_{\mathbb{G}(k)^\circ\text{-an}})'_b,$$

which imply that the coadmissible modules $\Gamma \circ \mathcal{L}oc_\lambda^{G_0}(M)$ and M have isomorphic $(D(\mathbb{G}(k)^\circ, G_0))_{k\in\mathbb{N}}$-sequences.

Claim 4. $\mathcal{L}oc_\lambda^{G_0} \circ \Gamma(\mathcal{M}) \simeq \mathcal{M}$.

Let $N := \Gamma(\mathcal{M})$ and $V := N'_b$ the corresponding admissible representation. Let $\mathcal{N} := \mathcal{L}oc_\lambda^{G_0}(N)$. According to Lemma 5.1

$$N_k = D(\mathbb{G}(k)^\circ, G_0) \otimes_{D(G_0,L)} N$$

produces a $(D(\mathbb{G}(k)^\circ, G_0))_{k\in\mathbb{N}}$-sequence for the coadmissible module N which is isomorphic to its constituting sequence $(H^0(\mathcal{Y}, \mathcal{M}_{\mathcal{Y},k}))_{(\mathcal{Y},k)\in\mathcal{F}_\mathcal{X}}$ from Claim 2. Now let $(\mathcal{Y}, k) \in \mathcal{F}_\mathcal{X}$. We have the following isomorphisms:

$$\mathcal{N}_{\mathcal{Y},k} = \mathcal{L}\mathrm{oc}^\dagger_{\mathcal{Y},k}(\lambda)(N_k) \simeq \mathcal{L}\mathrm{oc}^\dagger_{\mathcal{Y},k}(\lambda)(H^0(\mathcal{Y}, \mathcal{M}_{\mathcal{Y},k})) \simeq \mathcal{M}_{\mathcal{Y},k}.$$

By T_g-linearity the action maps $\varphi_g^{\mathcal{M}_{\mathcal{Y},k}}$ and $\varphi_g^{\mathcal{N}_{\mathcal{Y},k}}$, constructed in Claim 1, are the same. Similarly if $(\mathcal{Y}', k') \succeq (\mathcal{Y}, k)$ are G_0-equivariant, then the transition maps $\psi^{\mathcal{M}_{\mathcal{Y}',\mathcal{Y}}}$ and $\psi^{\mathcal{N}_{\mathcal{Y}',\mathcal{Y}}}$ coincide, by $\Psi_{\mathcal{Y}',\mathcal{Y}}$-linearity. Hence $\mathcal{N} \simeq \mathcal{M}$ in $\mathcal{C}^{G_0}_{\mathcal{X},\lambda}$.

This ends the proof of the theorem. ∎

6.1 – *Coadmissible G_0-equivariant $\mathcal{D}(\lambda)$-modules on the Zariski–Riemann space*

Let us recall that $\mathcal{X}_\infty$ denotes the projective limit of all formal models of X^{rig} (the rigid-analytic space associated by the GAGA functor to the flag variety $X_\mathbb{Q}$). The set $\mathcal{F}_\mathcal{X}$ of admissible formal blow-ups $\mathcal{Y} \to \mathfrak{X}$ is ordered by setting $\mathcal{Y}' \succeq \mathcal{Y}$ if the blow-up morphism $\mathcal{Y}' \to \mathcal{X}$ factors as $\mathcal{Y}' \xrightarrow{\pi} \mathcal{Y} \to \mathfrak{X}$, with π a blow-up morphism. The set $\mathcal{F}_\mathcal{X}$ is directed in the sense that any two elements have a common upper bound, and it is cofinal in the set of all formal models. In particular,

$$\mathcal{X}_\infty = \varprojlim_{\mathcal{F}_\mathcal{X}} \mathcal{Y}.$$

The space $\mathcal{X}_\infty$ is also known as the Zariski–Riemann space [6, Part II, Section 9.3][14]. In this subsection, we indicate how to realize coadmissible G_0-equivariant $\mathcal{D}(\lambda)$-modules on $\mathcal{F}_\mathcal{X}$ as sheaves on the Zariski–Riemann space $\mathcal{X}_\infty$. We start with the following proposition whose proof can be found in [20, Proposition 5.2.14].

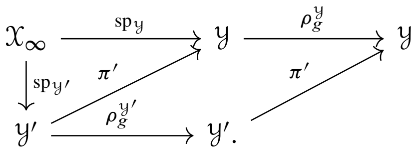
Proposition 6.7. *Any formal model $\mathcal{Y}$ of X^{rig} is dominated by one which is a G_0-equivariant admissible blow-up of $\mathcal{X}$.*

Remark 6.8. As $\mathcal{F}_\mathcal{X}$ is cofinal in the set of all formal models, the preceding proposition tells us that the set of all G_0-equivariant admissible blow-ups of $\mathcal{X}$ is also cofinal in the set of all formal models of $\mathcal{X}$. From now on, we will assume that if $\mathcal{Y} \in \mathcal{F}_\mathcal{X}$, then $\mathcal{Y}$ is also G_0-equivariant, and we will denote by $\rho_g^\mathcal{Y} : \mathcal{Y} \to \mathcal{Y}$ the morphism induced by every $g \in G_0$.

For every $\mathcal{Y} \in \mathcal{F}_\mathcal{X}$ we denote by $\mathrm{sp}_\mathcal{Y} : \mathcal{X}_\infty \to \mathcal{Y}$ the canonical projection map. Let $\mathcal{Y}' \succeq \mathcal{Y}$ with blow-up morphism $\pi' : \mathcal{Y}' \to \mathcal{Y}$ and $g \in G_0$. Let us consider the following

([14]) In this reference, this space is denoted by $\langle \mathcal{X} \rangle$, cf. [23, Section 3.2].

commutative diagram coming from the G_0-equivariance of the family $\mathcal{F}_{\mathcal{X}}$:



This diagram allows us to define a continuous function

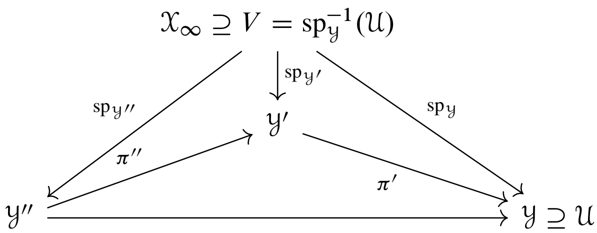
$$\rho_g : \mathcal{X}_\infty \to \mathcal{X}_\infty, \quad (a_{\mathcal{Y}})_{\mathcal{Y}\in\mathcal{F}_{\mathcal{X}}} \mapsto (\rho_g^{\mathcal{Y}}(a_{\mathcal{Y}}))_{\mathcal{Y}\in\mathcal{F}_{\mathcal{X}}}, \tag{6.5}$$

which defines a G_0-action on the space $\mathcal{X}_\infty$.

Let $\mathcal{U} \subseteq \mathcal{Y}$ be an open subset and let us take $V := \mathrm{sp}_{\mathcal{Y}}^{-1}(\mathcal{U}) \subset \mathcal{X}_\infty$. Using the relation $\mathrm{sp}_{\mathcal{Y}} = \pi \circ \mathrm{sp}_{\mathcal{Y}'}$, we see that

$$\mathrm{sp}_{\mathcal{Y}'}(V) = \mathrm{sp}_{\mathcal{Y}'}(\mathrm{sp}_{\mathcal{Y}}^{-1}(\mathcal{U})) = \mathrm{sp}_{\mathcal{Y}'}(\mathrm{sp}_{\mathcal{Y}'}^{-1}(\pi'^{-1}(\mathcal{U}))) = \pi'^{-1}(\mathcal{U}),$$

which implies that $\mathrm{sp}_{\mathcal{Y}'}(V)$ is an open subset of $\mathcal{Y}'$. Let $\mathcal{Y}'' \xrightarrow{\pi''} \mathcal{Y}' \xrightarrow{\pi'} \mathcal{Y}$ be morphisms over $\mathcal{Y}$. The commutative diagram



implies that

$$\pi''^{-1}(\mathrm{sp}_{\mathcal{Y}'}(V)) = \pi''^{-1}(\pi''(\mathrm{sp}_{\mathcal{Y}''}(V))) = \mathrm{sp}_{\mathcal{Y}''}(V). \tag{6.6}$$

In this situation, the morphism $\Psi_{\mathcal{Y}'',\mathcal{Y}'} : \pi''_*\mathcal{D}^\dagger_{\mathcal{Y}'',k''}(\lambda) \to \mathcal{D}^\dagger_{\mathcal{Y}',k'}(\lambda)$ (defined in (5.5)) induces the ring homomorphism

$$\mathcal{D}^\dagger_{\mathcal{Y}'',k''}(\lambda)(\mathrm{sp}_{\mathcal{Y}''}(V)) = \pi''_*\mathcal{D}^\dagger_{\mathcal{Y}'',k''}(\lambda)(\mathrm{sp}_{\mathcal{Y}'}(V)) \xrightarrow{\Psi_{\mathcal{Y}'',\mathcal{Y}'}} \mathcal{D}^\dagger_{\mathcal{Y}',k'}(\lambda)(\mathrm{sp}_{\mathcal{Y}'}(V))$$

and we can form the projective limit as in [20, (5.2.25)]:

$$\mathcal{D}(\lambda)(V) = \varprojlim_{\mathcal{Y}'\to\mathcal{Y}} \mathcal{D}^\dagger_{\mathcal{Y}',k'}(\lambda)(\mathrm{sp}_{\mathcal{Y}'}(V)).$$

By definition, the open subsets of the form $V = \mathrm{sp}_{\mathcal{Y}}^{-1}(\mathcal{U})$ form a basis for the topology of $\mathcal{X}_\infty$ and $\mathcal{D}(\lambda)$ is a presheaf on this basis. The associated sheaf on $\mathcal{X}_\infty$ to this presheaf will also be denoted by $\mathcal{D}(\lambda)$.

Since $(\rho_g^{\mathcal{Y}'})_* \circ \pi''_* = \pi''_* \circ (\rho_g^{\mathcal{Y}''})_*$, relation (6.6) and the commutativity of the first diagram in (6.4) tell us that the following diagram is also commutative:

$$\begin{array}{ccc} \pi''_*\mathcal{D}^\dagger_{\mathcal{Y}'',k''}(\lambda)(\mathrm{sp}_{\mathcal{Y}'}(V)) & \xrightarrow{\Psi_{\mathrm{sp}_{\mathcal{Y}'}(V)}} & \mathcal{D}^\dagger_{\mathcal{Y}',k'}(\lambda)(\mathrm{sp}_{\mathcal{Y}'}(V)) \\ \Big\downarrow T^{\mathcal{Y}''}_{g,\mathrm{sp}_{\mathcal{Y}''}(V)} & & \Big\downarrow T^{\mathcal{Y}'}_{g,\mathrm{sp}_{\mathcal{Y}'}(V)} \\ (\rho_g^{\mathcal{Y}'})_*\pi''_*\mathcal{D}^\dagger_{\mathcal{Y}'',k''}(\lambda)(\mathrm{sp}_{\mathcal{Y}'}(V)) & \xrightarrow{\Psi_{\mathrm{sp}_{\mathcal{Y}'}(\rho_g^{-1}(V))}} & (\rho_g^{\mathcal{Y}'})_*\mathcal{D}^\dagger_{\mathcal{Y}',k'}(\lambda)(\mathrm{sp}_{\mathcal{Y}'}(V)). \end{array}$$

We have used the relations

$$\mathcal{D}^\dagger_{\mathcal{Y}'',k''}(\lambda)(\mathrm{sp}_{\mathcal{Y}''}(V)) = \pi''_*\mathcal{D}^\dagger_{\mathcal{Y}'',k''}(\lambda)(\mathrm{sp}_{\mathcal{Y}'}(V)),$$
$$\mathcal{D}^\dagger_{\mathcal{Y}'',k''}(\lambda)((\rho_g^{\mathcal{Y}''})^{-1}(\mathrm{sp}_{\mathcal{Y}''}(V))) = (\rho_g^{\mathcal{Y}'})_*\pi''_*\mathcal{D}^\dagger_{\mathcal{Y}'',k''}(\lambda)(\mathrm{sp}_{\mathcal{Y}'}(V)).$$

Let us identify

$$\begin{aligned}&\mathcal{D}(\lambda)(V)\\ &= \Big\{P = (P_{\mathcal{Y}',k'})_{(\mathcal{Y}',k')} \in \prod_{\mathcal{F}_{\mathfrak{X}}} \mathcal{D}^\dagger_{\mathcal{Y}',k'}(\lambda)(\mathrm{sp}_{\mathcal{Y}'}(V)) \mid \Psi_{\mathcal{Y}'',\mathcal{Y}'}(P_{\mathcal{Y}'',k''}) = P_{\mathcal{Y}',k'}\Big\}\end{aligned}$$

and let us consider the sequence

$$g.P = (T^{\mathcal{Y}''}_{g,\mathrm{sp}_{\mathcal{Y}''}(V)}(P_{\mathcal{Y}'',k''}))_{(\mathcal{Y}'',k'')} \in \prod_{\mathcal{F}_{\mathfrak{X}}} \mathcal{D}^\dagger_{\mathcal{Y}'',k''}(\lambda)((\rho_g^{\mathcal{Y}''})^{-1}\mathrm{sp}_{\mathcal{Y}''}(V)).$$

Using the commutativity of the preceding diagram, we see that

$$\begin{aligned}\Psi_{\mathrm{sp}_{\mathcal{Y}'}(\rho_g^{-1}(V))}(T^{\mathcal{Y}''}_{g,\mathrm{sp}_{\mathcal{Y}''}(V)}(P_{\mathcal{Y}'',k''})) &= T^{\mathcal{Y}'}_{g,\mathrm{sp}_{\mathcal{Y}'}(V)}(\Psi_{\mathrm{sp}_{\mathcal{Y}'}(V)}(P_{\mathcal{Y}'',k''}))\\ &= T^{\mathcal{Y}'}_{g,\mathrm{sp}_{\mathcal{Y}'}(V)}(P_{\mathcal{Y}',k'}).\end{aligned}$$

Therefore, for $g \in G_0$, the morphisms $T_g^{\mathcal{Y}}$ assemble to a G_0-action

$$T_g : \mathcal{D}(\lambda) \xrightarrow{\simeq} (\rho_g)_*\mathcal{D}(\lambda).$$

This action is on the left, in the sense that if $g, h \in G_0$, then $(\rho_g)_*T_h \circ T_g = T_{hg}$. Let us suppose now that $\mathcal{M} = (\mathcal{M}_{\mathcal{Y},k}) \in \mathcal{C}^{G_0}_{\mathfrak{X},\lambda}$. We have the transition maps

$$\psi_{\mathcal{Y}'',\mathcal{Y}'} : \pi''_*\mathcal{M}_{\mathcal{Y}'',k''} \to \mathcal{M}_{\mathcal{Y}',k'},$$

which are linear relative to the morphism (5.5). As before, we have the map

$$\mathcal{M}_{\mathcal{Y}'',k''}(\mathrm{sp}_{\mathcal{Y}''}(V)) = \pi''_*\mathcal{M}_{\mathcal{Y}'',k''}(\mathrm{sp}_{\mathcal{Y}'}(V)) \xrightarrow{\psi_{\mathrm{sp}_{\mathcal{Y}'}(V)}} \mathcal{M}_{\mathcal{Y}',k'}(\mathrm{sp}_{\mathcal{Y}'}(V)),$$

which allows us to define $\mathscr{M}_\infty$ as the sheaf on $\mathfrak{X}_\infty$ associated to the presheaf

$$\mathscr{M}_\infty(V) = \varprojlim_{\mathcal{Y}'\to\mathcal{Y}} \mathscr{M}_{\mathcal{Y}',k'}(\mathrm{sp}_{\mathcal{Y}'}(V)).$$

By definition, we have the commutative diagram

$$\begin{array}{ccc}
\pi''_*\mathscr{M}_{\mathcal{Y}'',k''}(\mathrm{sp}_{\mathcal{Y}'}(V)) & \xrightarrow{\psi_{\mathrm{sp}_{\mathcal{Y}'}(V)}} & \mathscr{M}_{\mathcal{Y}',k'}(\mathrm{sp}_{\mathcal{Y}'}(V)) \\
\downarrow \varphi^{\mathcal{Y}''}_{g,\mathrm{sp}_{\mathcal{Y}''}(V)} & & \downarrow \varphi^{\mathcal{Y}'}_{g,\mathrm{sp}_{\mathcal{Y}'}(V)} \\
(\rho^{\mathcal{Y}'}_g)_*\pi''_*\mathscr{M}_{\mathcal{Y}'',k''}(\mathrm{sp}_{\mathcal{Y}'}(V)) & \xrightarrow{\psi_{\mathrm{sp}_{\mathcal{Y}'}(\rho_g^{-1}(V))}} & (\rho^{\mathcal{Y}'}_g)_*\mathscr{M}_{\mathcal{Y}',k'}(\mathrm{sp}_{\mathcal{Y}'}(V)).
\end{array}$$

Identifying

$$\mathscr{M}_\infty(V) = \Big\{m = (m_{\mathcal{Y}',k'})_{(\mathcal{Y}',k')} \in \prod_{\mathcal{F}_\mathcal{X}} \mathscr{M}_{\mathcal{Y}',k'}(\mathrm{sp}_{\mathcal{Y}'}(V)) \mid \psi_{\mathcal{Y}'',\mathcal{Y}'}(m_{\mathcal{Y}'',k''}) = m_{\mathcal{Y}',k'}\Big\},$$

we see as before that if

$$g.m = (\varphi^{\mathcal{Y}''}_{g,\mathrm{sp}_{\mathcal{Y}''}(V)}(m_{\mathcal{Y}'',k''}))_{(\mathcal{Y}'',k'')} \in \prod_{\mathcal{F}_\mathcal{X}} \mathscr{M}_{\mathcal{Y}'',k''}((\rho^{\mathcal{Y}''}_g)^{-1}\mathrm{sp}_{\mathcal{Y}''}(V)),$$

then the preceding commutative diagram implies that

$$\begin{aligned}
\psi_{\mathrm{sp}_{\mathcal{Y}'}(\rho_g^{-1}(V))}(\varphi^{\mathcal{Y}''}_{g,\mathrm{sp}_{\mathcal{Y}''}(V)}(m_{\mathcal{Y}'',k''})) &= \varphi^{\mathcal{Y}'}_{g,\mathrm{sp}_{\mathcal{Y}'}(V)}(\psi_{\mathrm{sp}_{\mathcal{Y}'}(V)}(m_{\mathcal{Y}'',k''})) \\
&= \varphi^{\mathcal{Y}'}_{g,\mathrm{sp}_{\mathcal{Y}'}(V)}(m_{\mathcal{Y}',k'}).
\end{aligned}$$

Therefore, we get a family $(\varphi_g)_{g\in G_0}$ of isomorphisms

$$\varphi_g : \mathscr{M}_\infty \to (\rho_g)_*\mathscr{M}_\infty \tag{6.7}$$

of sheaves of L-vector spaces. By Definition 6.5, if $g, h \in G_0$, then $\varphi_{hg} = (\rho_g)_*\varphi_h \circ \varphi_g$. Furthermore, under the preceding identifications, if $P = (P_{\mathcal{Y}',k'}) \in \mathscr{D}(\lambda)(V)$ and $m = (m_{\mathcal{Y}',k'}) \in \mathscr{M}_\infty(V)$, then $P.m = (P_{\mathcal{Y}',k'}.m_{\mathcal{Y}',k'})_{(\mathcal{Y}',k')\in\mathcal{F}_\mathcal{X}}$. Therefore,

$$\begin{aligned}
\varphi_{g,V}(P.m) &= (\varphi^{\mathcal{Y}'}_{g,\mathrm{sp}_{\mathcal{Y}'}(V)}(P_{\mathcal{Y}',k'}.m_{\mathcal{Y}',k'}))_{(\mathcal{Y}',k')\in\mathcal{F}_\mathcal{X}} \\
&= \big(T^{\mathcal{Y}'}_{g,\mathrm{sp}_{\mathcal{Y}'}(V)}(P_{\mathcal{Y}',k'}).\varphi^{\mathcal{Y}'}_{g,\mathrm{sp}_{\mathcal{Y}'}(V)}(m_{\mathcal{Y}',k'})\big)_{(\mathcal{Y}',k')\in\mathcal{F}_\mathcal{X}} \\
&= T_{g,V}(P).\varphi_{g,V}(m).
\end{aligned}$$

In particular, $\mathscr{M}_\infty$ is a G_0-equivariant $\mathcal{D}(\lambda)$-module on the topological G_0-space $\mathfrak{X}_\infty$. Let us see that the formation of $\mathscr{M}_\infty$ is functorial. Let $\gamma : \mathscr{M} \to \mathscr{N}$ be a morphism in $\mathcal{C}^{G_0}_{\mathcal{X},\lambda}$. We have the following commutative diagram:

$$\begin{array}{ccc} \pi''_*\mathcal{M}_{\mathcal{Y}'',k''} & \xrightarrow{\psi^{\mathcal{M}}_{\mathcal{Y}'',\mathcal{Y}'}} & \mathcal{M}_{\mathcal{Y}',k'} \\ \downarrow{\scriptstyle \pi''_*(\gamma_{\mathcal{Y}'',k''})} & & \downarrow{\scriptstyle \gamma_{\mathcal{Y}',k'}} \\ \pi_*\mathcal{N}_{\mathcal{Y}'',k''} & \xrightarrow{\psi^{\mathcal{N}}_{\mathcal{Y}'',\mathcal{Y}'}} & \mathcal{N}_{\mathcal{Y}',k'}. \end{array}$$

Let $m = (m_{\mathcal{Y},k})_{(\mathcal{Y},k)\in\mathcal{F}_{\mathcal{X}}} \in \mathcal{M}_\infty(V)$ and

$$s := (\gamma_{\mathcal{Y}'',k''}(m_{\mathcal{Y}'',k''}))_{(\mathcal{Y}'',k'')\in\mathcal{F}_{\mathcal{X}}} \in \prod_{(\mathcal{Y}'',k'')\in\mathcal{F}_{\mathcal{X}}} \mathcal{N}_{\mathcal{Y}'',k''}(\mathrm{sp}_{\mathcal{Y}''}(V)).$$

Commutativity in the preceding diagram implies that

$$\begin{aligned} \psi^{\mathcal{N}}_{\mathrm{sp}_{\mathcal{Y}''}(V)}(s_{\mathcal{Y}'',k''}) &= \psi^{\mathcal{N}}_{\mathrm{sp}_{\mathcal{Y}''}(V)}(\gamma_{\mathrm{sp}_{\mathcal{Y}''}(V)}(m_{\mathcal{Y}'',k''})) \\ &= \gamma_{\mathrm{sp}_{\mathcal{Y}'}(V)}(\psi^{\mathcal{M}}_{\mathrm{sp}_{\mathcal{Y}'}(V)}(m_{\mathcal{Y}'',k''})) \\ &= \gamma_{\mathrm{sp}_{\mathcal{Y}'}(V)}(m_{\mathcal{Y}',k'}) = s_{\mathcal{Y}',k'}. \end{aligned}$$

Therefore, $s \in \mathcal{N}_\infty(V)$ and γ induces a morphism $\gamma_\infty : \mathcal{M}_\infty \to \mathcal{N}_\infty$.This shows that the preceding construction is functorial. The next proposition is the twisted analogue of [20, Proposition 5.2.29]. We follow their proof word by word.

Proposition 6.9. *Let $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ be an algebraic character which induces, via derivation, a dominant and regular character $\lambda + \rho$ of $\mathfrak{t}^*_{\mathbb{Q}}$. The functor $\mathcal{M} \rightsquigarrow \mathcal{M}_\infty$ from the category $\mathcal{C}^{G_0}_{\mathcal{X},\lambda}$ to G_0-equivariant $\mathcal{D}(\lambda)$-modules is a faithful functor.*

Proof. We start the proof by remarking that $\mathrm{sp}_{\mathcal{Y}}(\mathcal{X}_\infty) = \mathcal{Y}$ for every $\mathcal{Y} \in \mathcal{F}_{\mathcal{X}}$. By Remark 6.8, the global sections of $\mathcal{M}_\infty$ are equal to

$$H^0(\mathcal{X}_\infty, \mathcal{M}_\infty) = \varprojlim_{(\mathcal{Y},k)\in\underline{\mathcal{F}}_{\mathcal{X}}} H^0(\mathcal{Y}, \mathcal{M}_{\mathcal{Y},k}) = \Gamma(\mathcal{M}).$$

Now, let $f, h : \mathcal{M} \to \mathcal{N}$ be two morphisms in $\mathcal{C}^{G_0}_{\mathcal{X},\lambda}$ such that $f_\infty = h_\infty$. By Theorem 6.6, it is enough to verify $\Gamma(f) = \Gamma(h)$ which is clear since $H^0(\mathcal{X}_\infty, f_\infty) = H^0(\mathcal{X}_\infty, h_\infty)$. ∎

Let $(\bullet)_\infty$ denote the previous functor. Then we denote by $\mathcal{L}oc^{G_0}_\infty(\lambda)$ the composition of the functor $\mathcal{L}oc^{G_0}_\lambda$ with $(\bullet)_\infty$, i.e.,

$$\{\text{Coadmissible } D(G_0, L)_\lambda\text{-modules}\} \xrightarrow{\mathcal{L}oc^{G_0}_\infty(\lambda)} \{G_0\text{-equivariant } \mathcal{D}(\lambda)\text{-modules}\}.$$

Since $\mathcal{L}oc^{G_0}_\lambda$ is an equivalence of categories, the preceding proposition implies that $\mathcal{L}oc^{G_0}_\infty(\lambda)$ is a faithful functor.

7. G-equivariant modules

Throughout this section, we will use the notation $G = \mathbb{G}(L)$ and denote by $\mathcal{B}$ the semi-simple Bruhat–Tits building of the p-adic group G (see [9, 10]). This is a simplicial complex endowed with a natural right G-action.

The purpose of this section is to extend the above results from G_0-equivariant objects to objects equivariant for the whole group G.

We start by fixing some notation.[15] To each special vertex $v \in \mathcal{B}$ the Bruhat–Tits theory associates a connected reductive group $\mathfrak{o}$-scheme $\mathbb{G}_v$, whose generic fiber $(\mathbb{G}_v)_{\mathbb{Q}} := \mathbb{G}_v \times_{\mathrm{Spec}(\mathfrak{o})} \mathrm{Spec}(L)$ is canonically isomorphic to $\mathbb{G}_{\mathbb{Q}}$. We denote by X_v the smooth flag scheme of $\mathbb{G}_v$ whose generic fiber $(X_v)_{\mathbb{Q}}$ is canonically isomorphic to the flag variety $X_{\mathbb{Q}}$. We will distinguish the next constructions by adding the corresponding vertex to them. For instance, we will write Y_v for an (algebraic) admissible blow-up of the smooth model X_v, $G_{v,0}$ for the group of points $\mathbb{G}_v(\mathfrak{o})$ and $G_{v,k}$ for the group of points $\mathbb{G}_v(k)(\mathfrak{o})$. We will use the same conventions if we deal with formal completions. For instance, $\mathcal{Y}_v$ will always denote an admissible formal blow-up of $\mathcal{X}_v$. We point out to the reader that the morphism $\mathcal{Y}_v \to \mathfrak{X}_v$ will make part of the blow-up $\mathcal{Y}_v$. Moreover, even if for another special vertex $v' \neq v$ the formal $\mathfrak{o}$-scheme $\mathcal{Y}_v$ is also a blow-up of the smooth formal model $\mathcal{X}_{v'}$, we will only consider it as a blow-up of $\mathcal{X}_v$. We will denote by $\mathcal{F}_v := \mathcal{F}_{\mathcal{X}_v}$ the set of all admissible formal blow-ups $\mathcal{Y}_v \to \mathfrak{X}_v$ of $\mathcal{X}_v$, and by $\underline{\mathcal{F}}_v = \underline{\mathcal{F}}_{\mathcal{X}_v}$ the respective directed system of Definition 6.1. By the preceding accord, the sets $\mathcal{F}_v$ and $\mathcal{F}_{v'}$ are disjoint if $v \neq v'$. Let

$$\mathcal{F} = \bigsqcup_v \mathcal{F}_v,$$

where v runs over all special vertices of $\mathcal{B}$. We recall for the reader that $\mathcal{X}_\infty$ is equal to the projective limit of all formal models of X^{rig}.

Remark 7.1. The set $\mathcal{F}$ is partially ordered in the following way: $\mathcal{Y}_{v'} \succeq \mathcal{Y}_v$ if the projection $\mathrm{sp}_{\mathcal{Y}_{v'}} : \mathcal{X}_\infty \to \mathcal{Y}_{v'}$ factors through the projection $\mathrm{sp}_{\mathcal{Y}_v} : \mathcal{X}_\infty \to \mathcal{Y}_v$ so that we have the commutative diagram

$$\begin{array}{ccc} & \mathcal{X}_\infty & \\ \swarrow^{\mathrm{sp}_{\mathcal{Y}_{v'}}} & & \searrow^{\mathrm{sp}_{\mathcal{Y}_v}} \\ \mathcal{Y}_{v'} & \longrightarrow & \mathcal{Y}_v. \end{array}$$

(15) This is exactly as in [20, (5.3.1)].

Definition 7.2. Define $\underline{\mathcal{F}} = \bigsqcup_v \underline{\mathcal{F}}_v$, where v runs over all the special vertices of $\mathcal{B}$. This set is partially ordered as follows. We say that $(\mathcal{Y}_{v'}, k') \succeq (\mathcal{Y}_v, k)$ if $\mathcal{Y}_{v'} \succeq \mathcal{Y}_v$ and $\mathrm{Lie}(\mathbb{G}_{v'}(k')) \subset \mathrm{Lie}(\mathbb{G}_v(k))$ (or equivalent $\varpi^{k'} \mathrm{Lie}(\mathbb{G}_{v'}) \subset \varpi^k \mathrm{Lie}(\mathbb{G}_v)$) as lattices in $\mathfrak{g}_{\mathbb{Q}}$.

For any special vertex $v \in \mathcal{B}$, any element $g \in G$ induces an isomorphism

$$\rho_g^v : X_v \to X_{v.g}.$$

The isomorphism induced by ρ_g^v on the generic fibers $(X_v)_{\mathbb{Q}} \simeq X_{\mathbb{Q}} \simeq (X_{v.g})_{\mathbb{Q}}$ coincides with right translation by g on $X_{\mathbb{Q}}$:

$$\rho_g : X_{\mathbb{Q}} = X_{\mathbb{Q}} \times_{\mathrm{Spec}(L)} \mathrm{Spec}(L) \xrightarrow{\mathrm{id}_{X_L} \times g} X_{\mathbb{Q}} \times_{\mathrm{Spec}(L)} \mathrm{Spec}(\mathbb{G}_{\mathbb{Q}}) \xrightarrow{\alpha_{\mathbb{Q}}} X_{\mathbb{Q}},$$

where we have used $\mathbb{G}(L) = \mathbb{G}_{\mathbb{Q}}(L)$. Moreover, ρ_g^v induces a morphism $\mathcal{X}_v \to \mathcal{X}_{v.g}$, which we denote again by ρ_g^v, and which coincides with the right translation on $\mathcal{X}_v$ if $g \in G_{v,0}$ (of course in this case $vg = v$). Let $(\rho_g^v)^\natural : \mathcal{O}_{\mathcal{X}_{vg}} \to (\rho_g^v)_* \mathcal{O}_{\mathcal{X}_v}$ be the comorphism of ρ_g^v. If $\pi : \mathcal{Y}_v \to \mathfrak{X}_v$ is an admissible blow-up of an ideal $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}_v}$, then blowing-up $((\rho_g^v)^\natural)^{-1}((\rho_g^v)_* \mathcal{I})$ produces a formal scheme $\mathcal{Y}_{vg}$ (cf. Lemma 6.3), together with an isomorphism $\rho_g^v : \mathcal{Y}_v \to \mathcal{Y}_{vg}$. As in Lemma 6.3 we have $k_{\mathcal{Y}_v} = k_{\mathcal{Y}_{vg}}$. For any $g, h \in G$ and any admissible formal blow-up $\mathcal{Y}_v \to \mathfrak{X}_v$, we have

$$\rho_h^{vg} \circ \rho_g^v = \rho_{gh}^v : \mathcal{Y}_v \to \mathcal{Y}_{vgh}.$$

This gives a right G-action on the family $\mathcal{F}$ and on the projective limit $\mathcal{X}_\infty$. Finally, if $\mathcal{Y}_{v'} \succeq \mathcal{Y}_v$ with morphism $\pi : \mathcal{Y}_{v'} \to \mathcal{Y}_v$ and $g \in G$, then $\mathcal{Y}_{v'g} \succeq \mathcal{Y}_{vg}$, and we have the relation $\rho_g^v \circ \pi = \pi g \circ \rho_g^{v'}$ (here $\pi g : \mathcal{Y}_{v'g} \to \mathcal{Y}_{vg}$). Now, over every special vertex $v \in \mathcal{B}$ the algebraic character λ induces an invertible sheaf $\mathcal{L}_v(\lambda)$ on X_v, such that for every $g \in G$ there exists an isomorphism

$$L_g^v : \mathcal{L}_{vg}(\lambda) \to (\rho_g^v)_* \mathcal{L}_v(\lambda),$$

satisfying the cocycle condition

$$L_{hg}^{vhg} = (\rho_g^{vh})_* L_h^v \circ L_g^{vh} \quad (h, g \in G). \tag{7.1}$$

As usual, for every special vertex $v \in \mathcal{B}$, we will denote by $\mathscr{L}_v(\lambda)$ the p-adic completion of the sheaf $\mathcal{L}_v(\lambda)$, which is considered as an invertible sheaf on $\mathcal{X}_v$. Let $(\mathcal{Y}_v, k) \in \mathcal{F}$ with blow-up morphism $\mathrm{pr} : \mathcal{Y}_v \to \mathfrak{X}_v$. At the level of differential operators, we will denote by $\mathcal{D}^\dagger_{\mathcal{Y}_v,k}(\lambda)$ the sheaf of arithmetic differential operators on $\mathcal{Y}_v$ acting on the line bundle $\mathcal{L}_v(\lambda)$.[16] We have the following important properties. Let $g \in G$. As in (4.9) the isomorphism (7.1) induces a left action

$$T_g^v : \mathcal{D}^\dagger_{\mathcal{Y}_{vg},k}(\lambda) \xrightarrow{\simeq} (\rho_g^v)_* \mathcal{D}^\dagger_{\mathcal{Y}_v,k}(\lambda), \quad P \mapsto L_g^v P (L_g^v)^{-1}.$$

Now, we identify the global sections $H^0(\mathcal{Y}_v, \mathcal{D}^\dagger_{\mathcal{Y}_v,k}(\lambda))$ with $\mathcal{D}^{\mathrm{an}}(\mathbb{G}_v(k)^\circ)_\lambda$ and obtain the group homomorphism

$$G_{v,k+1} \to H^0(\mathcal{Y}_v, \mathcal{D}^\dagger_{\mathcal{Y}_v,k}(\lambda))^\times, \quad g \mapsto \delta_g,$$

where $G_{v,k+1} = \mathbb{G}_v(k)^\circ(L)$ denotes the group of L-rational points (or $\mathfrak{o}$-points of $\mathbb{G}_v(k+1)$). The proof of the following proposition is in much the same way as the proof of [20, Proposition 5.3.2].

Proposition 7.3. *Suppose $(\mathcal{Y}_{v'}, k') \succeq (\mathcal{Y}_v, k)$ for pairs $(\mathcal{Y}_{v'}, k'), (\mathcal{Y}_v, k) \in \underline{\mathcal{F}}$ with morphism $\pi : \mathcal{Y}_{v'} \to \mathcal{Y}_v$. There exists a canonical morphism of sheaves of rings*

$$\Psi_{\mathcal{Y}_{v'},\mathcal{Y}_v} : \pi_* \mathcal{D}^\dagger_{\mathcal{Y}_{v'},k'}(\lambda) \to \mathcal{D}^\dagger_{\mathcal{Y}_v,k}(\lambda),$$

which is G-equivariant in the sense that for every $g \in G$ we have[17]

$$T_g^v \circ \Psi_{\mathcal{Y}_{v'g},\mathcal{Y}_{vg}} = (\rho_g^v)_* \Psi_{\mathcal{Y}_{v'},\mathcal{Y}_v} \circ (\pi g)_* T_g^{v'}.$$

Proof. Let us denote by $\mathrm{pr}' : \mathcal{Y}_{v'} \to \mathcal{X}_{v'}$ and $\mathrm{pr} : \mathcal{Y}_v \to \mathcal{X}_v$ the blow-up morphisms, and let us put $\widetilde{\mathrm{pr}} = \mathrm{pr} \circ \pi$. We have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{Y}_{v'} & \xrightarrow{\pi} & \mathcal{Y}_v \\ \downarrow{\scriptstyle \mathrm{pr}'} & \searrow^{\widetilde{\mathrm{pr}}} & \downarrow{\scriptstyle \mathrm{pr}} \\ \mathcal{X}_{v'} & & \mathcal{X}_v. \end{array}$$

Let us fix $m \in \mathbb{N}$. As in [20, Proposition 5.3.6] we show first the existence of a canonical morphism of sheaves of $\mathfrak{o}$-algebras

$$\mathcal{D}^{(m,k)}_{Y_{v'}}(\lambda) \to \widetilde{\mathrm{pr}}^* \mathcal{D}^{(m,k)}_{X_v}(\lambda). \tag{7.2}$$

Here $Y_{v'}$, Y_v, $X_{v'}$ and X_v denote the $\mathfrak{o}$-scheme of finite type whose completions are $\mathcal{Y}_{v'}$, $\mathcal{Y}_v$, $\mathcal{X}_{v'}$ and $\mathcal{X}_v$, respectively. The morphisms between these schemes will be denoted

(16) By abuse of notation, we denote again by $\mathcal{L}_v(\lambda)$ the invertible sheaf $\mathrm{pr}^* \mathcal{L}_v(\lambda)$ on $\mathcal{Y}_v$.
(17) In order to simplify the notation we will avoid the indices. For instance, we will write Ψ for the morphisms $\Psi_{\mathcal{Y}_{v'},\mathcal{Y}_v}$ and $\Psi_{\mathcal{Y}_{v'g},\mathcal{Y}_{vg}}$.

by the same letters, for instance $\mathrm{pr} : Y_v \to X_v$. We recall for the reader that the sheaf $\mathcal{D}^{(m,k')}_{Y_{v'}}(\lambda)$ is filtered by locally free sheaves of finite rank

$$\begin{aligned}\mathcal{D}^{(m,k')}_{Y_{v'},d}(\lambda) &= \mathrm{pr}'^{*}\mathcal{L}_{v'}(\lambda) \otimes_{\mathcal{O}_{Y_{v'}}} \mathrm{pr}'^{*}\mathcal{D}^{(m,k')}_{X_{v'},d} \otimes_{\mathcal{O}_{Y_{v'}}} \mathrm{pr}'^{*}\mathcal{L}_{v'}(\lambda)^{\vee} \\ &= \mathrm{pr}'^{*}(\mathcal{D}^{(m,k')}_{X_{v'},d}(\lambda)).\end{aligned}$$

Therefore, by the projection formula [17, Part II, Exercise 5.1 (d)] and given that $\mathrm{pr}'_{*}\mathcal{O}_{Y_{v'}} = \mathcal{O}_{X_{v'}}$ (cf. [20, Lemma 3.2.3 (iii)]) we have for every $d \in \mathbb{N}$

$$\begin{aligned}\mathrm{pr}'_{*}(\mathcal{D}^{(m,k')}_{Y_{v'},d}(\lambda)) &= \mathrm{pr}'_{*}\big(\mathcal{O}_{Y_{v'}} \otimes_{\mathcal{O}_{Y_{v'}}} \mathrm{pr}'^{*}\mathcal{D}^{(m,k')}_{X_{v'},d}(\lambda)\big) \\ &= \mathrm{pr}'_{*}(\mathcal{O}_{Y_{v'}}) \otimes_{\mathcal{O}_{X_{v'}}} \mathcal{D}^{(m,k')}_{X_{v'},d}(\lambda) = \mathcal{D}^{(m,k')}_{X_{v'},d}(\lambda),\end{aligned}$$

which implies that

$$\mathrm{pr}'_{*}(\mathcal{D}^{(m,k')}_{Y_{v'}}(\lambda)) = \mathcal{D}^{(m,k')}_{X_{v'}}(\lambda)$$

because the direct image commutes with inductive limits on a noetherian space. By Proposition 3.13 and the preceding relation we have a canonical map of filtered $\mathfrak{o}$-algebras

$$D^{(m)}(\mathbb{G}_{v'}(k')) \to H^0(X_{v'}, \mathcal{D}^{(m,k')}_{X_{v'}}(\lambda)) = H^0(Y_{v'}, \mathcal{D}^{(m,k')}_{Y_{v'}}(\lambda)),$$

in particular we get a morphism of sheaves of filtered $\mathfrak{o}$-algebras (this is exactly as we have done in (3.8)):

$$\Phi^{(m,k')}_{Y_{v'}} : \mathcal{A}^{(m,k')}_{Y_{v'}} = \mathcal{O}_{Y_{v'}} \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{G}_{v'}(k')) \to \mathcal{D}^{(m,k')}_{Y_{v'}}(\lambda). \tag{7.3}$$

Applying $\mathrm{Sym}^{(m)}(\bullet) \circ \varpi^{k'}\mathrm{pr}'^{*}(\bullet)$ to the surjection (3.9), we obtain a surjection

$$\mathcal{O}_{Y_{v'}} \otimes_{\mathfrak{o}} \mathrm{Sym}^{(m)}(\mathrm{Lie}(\mathbb{G}_{v'}(k'))) \to \mathrm{Sym}^{(m)}(\varpi^{k'}\mathrm{pr}'^{*}\mathcal{T}_{X_{v'}}),$$

which equals the associated graded morphism of (7.3) by Proposition 4.5. Hence $\Phi^{(m,k')}_{Y_{v'}}$ is surjective. On the other hand, if we apply $\widetilde{\mathrm{pr}}^{*}$ to the surjection

$$\Phi^{(m,k)}_{X_v} : \mathcal{A}^{(m,k)}_{X_v} = \mathcal{O}_{X_v} \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{G}_v(k)) \to \mathcal{D}^{(m,k)}_{X_v}(\lambda),$$

we obtain the surjection

$$\mathcal{O}_{Y_{v'}} \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{G}_v(k)) \to \widetilde{\mathrm{pr}}^{*}\mathcal{D}^{(m,k)}_{X_v}(\lambda).$$

Recall that $(\mathcal{Y}_{v'}, k') \succeq (\mathcal{Y}_v, k)$ implies, in particular, that $\mathrm{Lie}(\mathbb{G}_{v'}(k')) \subseteq \mathrm{Lie}(\mathbb{G}_v(k))$ and thus $\varpi^{k'}\,\mathrm{Lie}(\mathbb{G}_{v'}) \subset \varpi^{k}\,\mathrm{Lie}(\mathbb{G}_v)$. By (3.5), the preceding inclusion gives rise to an injective ring homomorphism $D^{(m)}(\mathbb{G}_{v'}(k')) \hookrightarrow D^{(m)}(\mathbb{G}_v(k))$. Let us see that the composition

$$\mathcal{O}_{Y_{v'}} \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{G}_{v'}(k')) \hookrightarrow \mathcal{O}_{Y_{v'}} \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{G}_{v}(k)) \twoheadrightarrow \widetilde{\mathrm{pr}}^{*}\mathcal{D}^{(m,k)}_{X_v}(\lambda)$$

factors through $\mathcal{D}^{(m,k')}_{Y_{v'}}(\lambda)$:

$$\begin{array}{ccc} \mathcal{O}_{Y_{v'}} \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{G}_{v'}(k')) & \longrightarrow & \widetilde{\mathrm{pr}}^{*}\mathcal{D}^{(m,k)}_{X_v}(\lambda) \\ \downarrow & \nearrow & \\ \mathcal{D}^{(m,k')}_{Y_{v'}}(\lambda). & & \end{array}$$

Since by Lemma 3.11 all those sheaves are ϖ-torsion free, this can be checked after tensoring with L in which case we have that $\mathcal{D}^{(m,k')}_{Y_{v'}}(\lambda) \otimes_{\mathfrak{o}} L \simeq \widetilde{\mathrm{pr}}^{*}\mathcal{D}^{(m,k)}_{X_v}(\lambda) \otimes_{\mathfrak{o}} L$ is the (push-forward of the) sheaf of algebraic twisted differential operators on the generic fiber of $Y_{v'}$ (cf. the discussion given at the beginning of Section 4.1). We thus get the canonical morphism of sheaves (7.2). Passing to completions, we get a canonical morphism

$$\widehat{\mathcal{D}}^{(m,k')}_{\mathcal{Y}_{v'}}(\lambda) \to \widetilde{\mathrm{pr}}^{*}\widehat{\mathcal{D}}^{(m,k)}_{\mathcal{X}_v}(\lambda).$$

Taking the inductive limit over all m and inverting ϖ gives a canonical morphism

$$\mathcal{D}^{\dagger}_{\mathcal{Y}_{v'},k'}(\lambda) \to \widetilde{\mathrm{pr}}^{*}\mathcal{D}^{\dagger}_{\mathcal{X}_v,k}(\lambda).$$

Now, let us consider the formal scheme $\mathcal{Y}_{v'}$ as a blow-up of $\mathcal{X}_v$ via $\widetilde{\mathrm{pr}}$. Then π becomes a morphism of formal schemes over $\mathcal{X}_v$ and we consider $\widetilde{\mathrm{pr}}^{*}\mathcal{D}^{\dagger}_{\mathfrak{X}_v,k}(\lambda)$ as the sheaf of arithmetic differential operators with congruence level k defined on $\mathcal{Y}_{v'}$ via $\widetilde{\mathrm{pr}}^{*}$. Using the invariance theorem (Theorem 4.8), we get $\pi_{*}(\widetilde{\mathrm{pr}}^{*}\mathcal{D}^{\dagger}_{\mathcal{X}_v,k}(\lambda)) = \mathcal{D}^{\dagger}_{\mathcal{Y}_v,k}$. Then applying π_* to the morphism $\mathcal{D}^{\dagger}_{\mathcal{Y}_{v'},x'}(\lambda) \to \widetilde{\mathrm{pr}}^{*}\mathcal{D}^{\dagger}_{\mathcal{X}_v,k}(\lambda)$ gives the morphism

$$\Psi : \pi_{*}\mathcal{D}^{\dagger}_{\mathcal{Y}_{v'},k'}(\lambda) \to \mathcal{D}^{\dagger}_{\mathcal{Y}_v,k}$$

of the statement. As in [20, Proposition 5.3.8], making use of the maps $\Phi^{(m,k)}_{Y_v}$, the assertion about the G-equivariance is reduced to the functorial properties of the rings $D^{(m)}(\mathbb{G}_v(k))$. ■

Definition 7.4. A coadmissible G-equivariant arithmetic $\mathcal{D}(\lambda)$-module on $\mathcal{F}$ consists of a family $\mathcal{M} = (\mathcal{M}_{\mathcal{Y}_v,k})_{(\mathcal{Y}_v,k)\in\mathcal{F}}$ of coherent $\mathcal{D}^{\dagger}_{\mathcal{Y}_v,k}(\lambda)$-modules with the following properties:

(a) For any special vertex $v \in \mathcal{B}$ and $g \in G$ with isomorphism $\rho^{v}_{g} : \mathcal{Y}_v \to \mathcal{Y}_{vg}$, there exists an isomorphism

$$\varphi^{v}_{g} : \mathcal{M}_{\mathcal{Y}_{vg},k} \to (\rho^{v}_{g})_{*}\mathcal{M}_{\mathcal{Y},k}$$

of sheaves of L-vector spaces, satisfying the following conditions:

(i) For all $h, g \in G$ one has $(\rho_g^{vh})_* \varphi_h^v \circ \varphi_g^{vh} = \varphi_{hg}^v$.[18]

(ii) For all open subsets $\mathcal{U} \subseteq \mathcal{Y}_{vg}$, all $P \in \mathcal{D}^\dagger_{\mathcal{Y}_{vg},k}(\lambda)(\mathcal{U})$, and all $m \in \mathcal{M}_{\mathcal{Y}_{vg},k}(\mathcal{U})$ one has $\varphi^v_{g,\mathcal{U}}(P.m) = T^v_{g,\mathcal{U}}(P).\varphi^v_{g,\mathcal{U}}(m)$.

(iii) For all $g \in G_{k+1,v}$ the map $\varphi^v_g : \mathcal{M}_{\mathcal{Y},k} \to (\rho^v_g)_* \mathcal{M}_{\mathcal{Y},k} = \mathcal{M}_{\mathcal{Y},k}$ is equal to the multiplication by $\delta_g \in H^0(\mathcal{Y}_v, \mathcal{D}^\dagger_{\mathcal{Y}_v,k}(\lambda))$.

(b) For any two pairs $(\mathcal{Y}_{v'}, k') \succeq (\mathcal{Y}_v, k)$ in $\mathcal{F}$ with morphism $\pi : \mathcal{Y}_{v'} \to \mathcal{Y}_v$ there exists a transition morphism $\psi_{\mathcal{Y}_{v'},\mathcal{Y}_v} : \pi_* \mathcal{M}_{\mathcal{Y}_{v'}} \to \mathcal{M}_{\mathcal{Y}_v}$, linear relative to the canonical morphism $\Psi : \pi_* \mathcal{D}^\dagger_{\mathcal{Y}_{v'},k'}(\lambda) \to \mathcal{D}^\dagger_{\mathcal{Y}_v,k}(\lambda)$ (in the preceding proposition) such that

$$\varphi^v_g \circ \psi_{\mathcal{Y}_{v'g},\mathcal{Y}_{vg}} = (\rho^v_g)_* \psi_{\mathcal{Y}_{v'},\mathcal{Y}_v} \circ (\pi.g)_* \varphi^{v'}_g \tag{7.4}$$

for any $g \in G$ (where we have used the relation $(\rho^v_g)_* \circ \pi_* = (\pi.g)_* \circ (\rho^{v'}_g)_*$). If $v' = v$, and $(\mathcal{Y}'_v, k') \succeq (\mathcal{Y}_v, k)$ in $\underline{\mathcal{F}}_v$, and if $\mathcal{Y}'_v$, $\mathcal{Y}_v$ are $G_{v,0}$-equivariant, then we require additionally that the morphism induced by $\psi_{\mathcal{Y}'_v,\mathcal{Y}_v}$ (cf. (5.7)),

$$\overline{\psi}_{\mathcal{Y}'_v,\mathcal{Y}_v} : \mathcal{D}^\dagger_{\mathcal{Y}_v,k}(\lambda) \otimes_{\pi_* \mathcal{D}^\dagger_{\mathcal{Y}'_v,k'}(\lambda), G_{v,k+1}} \pi_* \mathcal{M}_{\mathcal{Y}'_v,k'} \to \mathcal{M}_{\mathcal{Y}_v,k},$$

is an isomorphism of $\mathcal{D}^\dagger_{\mathcal{Y}_v,k}(\lambda)$-modules. As in Theorem 6.6, the morphisms $\psi_{\mathcal{Y}_{v'},\mathcal{Y}_v} : \pi_* \mathcal{M}_{\mathcal{Y}_{v'},k'} \to \mathcal{M}_{\mathcal{Y}_v,k}$ are required to satisfy the transitive condition

$$\psi_{\mathcal{Y}_{v'},\mathcal{Y}_v} \circ \pi_*(\psi_{\mathcal{Y}_{v''},\mathcal{Y}_{v'}}) = \psi_{\mathcal{Y}_{v''},\mathcal{Y}_v},$$

whenever $(\mathcal{Y}_{v''}, k'') \succeq (\mathcal{Y}_{v'}, k') \succeq (\mathcal{Y}_v, k)$ in $\underline{\mathcal{F}}$. Moreover, $\psi_{\mathcal{Y}_v,\mathcal{Y}_v} = \mathrm{id}_{\mathcal{M}_{\mathcal{Y}_v,k}}$.

A morphism $\mathcal{M} \to \mathcal{N}$ between two coadmissible G-equivariant arithmetic $\mathcal{D}(\lambda)$-modules consists in a family of morphisms $\mathcal{M}_{\mathcal{Y},k} \to \mathcal{N}_{\mathcal{Y},k}$ of $\mathcal{D}^\dagger_{\mathcal{Y},k}(\lambda)$-modules, which respect the extra conditions imposed in (a) and (b). We denote the resulting category by $\mathcal{C}^{\mathcal{F}}_{G,\lambda}$.

We recall for the reader that $D(G_0, L)$ is a Fréchet–Stein algebra [34, Theorem 24.1]. Moreover, a $D(G, L)$-module is called coadmissible if it is coadmissible as a $D(H, L)$-module for every compact open subgroup $H \subseteq G$ (cf. the first definition in [33, Section 6]). Given that for any two compact open subgroups $H \subseteq H' \subseteq G$ the algebra $D(H', L)$ is finitely generated free and hence coadmissible as a $D(H, L)$-module, it follows from [33, Lemma 3.8] that the preceding condition needs to be tested only for a single compact open subgroup $H \subseteq G$. This motivates the following definition where we will consider the weak Fréchet–Stein structure of $D(G_0, L)$ defined in (5.3).

([18]) Here we use that the action of G on $\mathcal{B}$ is on the right, thus $(\rho_g^{vh})_* \circ (\rho_h^v)_* = (\rho_{hg}^v)_*$.

Definition 7.5. We say that M is a coadmissible $D(G,L)$-module if M is coadmissible as a $D(G_0,L)$-module.

Let us construct now the bridge to the category of coadmissible $D(G,L)_\lambda$-modules. Let M be such a coadmissible $D(G,L)_\lambda$-module and let $V=M'_b$. We fix a special vertex $v\in\mathcal{B}$. Let $V_{\mathbb{G}_v(k)^\circ\text{-an}}$ be the subspace of $\mathbb{G}_v(k)^\circ$-analytic vectors and let $M_{v,k}$ be its continuous dual.[19] For any $(\mathcal{Y}_v,k)\in\underline{\mathcal{F}}$ we have a coherent $\mathcal{D}^\dagger_{\mathcal{Y}_v,k}(\lambda)$-module

$$\mathcal{L}oc^\dagger_{\mathcal{Y}_v,k}(\lambda)(M_{v,k})=\mathcal{D}^\dagger_{\mathcal{Y}_v,k}(\lambda)\otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}_v(k)^\circ)_\lambda}M_{v,k}$$

and we can consider the family

$$\mathcal{L}oc^G_\lambda(M)=(\mathcal{L}oc^\dagger_{\mathcal{Y}_v,k}(\lambda)(M_{v,k}))_{(\mathcal{Y}_v,k)\in\underline{\mathcal{F}}}.$$

On the other hand, given an object $\mathcal{M}\in\mathcal{C}^{\mathcal{F}}_{G,\lambda}$, we may consider the projective limit

$$\Gamma(\mathcal{M})=\varprojlim_{(\mathcal{Y},k)\in\underline{\mathcal{F}}}H^0(\mathcal{Y},\mathcal{M}_{\mathcal{Y},k})$$

with respect to the transition maps $\psi_{\mathcal{Y}',\mathcal{Y}}$. Here the projective limit is taken in the sense of abelian groups and over the cofinal family of pairs $(\mathcal{Y}_v,k)\in\underline{\mathcal{F}}$ with $G_{v,0}$-equivariant $\mathcal{Y}_v$, cf. Remark 6.8.

Theorem 7.6. *Let us suppose that $\lambda\in\mathrm{Hom}(\mathbb{T},\mathbb{G}_m)$ is an algebraic character such that $\lambda+\rho\in\mathfrak{t}^*_{\mathbb{Q}}$ is a dominant and regular character of $\mathfrak{t}_{\mathbb{Q}}$ (and therefore, a dominant and regular character on every special vertex of $\mathcal{B}$). The functors $\mathcal{L}oc^G_\lambda(\bullet)$ and $\Gamma(\bullet)$ induce quasi-inverse equivalences between the category of coadmissible $D(G,L)_\lambda$-modules and $\mathcal{C}^{\mathcal{F}}_{G,\lambda}$.*

Proof. The proof (which is similar to the one of [20, Theorem 5.3.12]) is an extension of the proof of Theorem 6.6, taking into account the additional G-action. Let M be a coadmissible $D(G,L)_\lambda$-module and let $\mathcal{M}\in\mathcal{C}^{\mathcal{F}}_{G,\lambda}$. The proof of the theorem follows the following steps.

Claim 1. One has $\mathcal{L}oc^G_\lambda(M)\in\mathcal{C}^{\mathcal{F}}_{G,\lambda}$ and $\mathcal{L}oc^G_\lambda(\bullet)$ is functorial.

Let $g\in G$, $v\in\mathcal{B}$ a special vertex and $\rho^v_g:\mathcal{Y}_v\to\mathcal{Y}_{vg}$ the respective isomorphism. For conditions (a) for $\mathcal{L}oc^G_\lambda(M)$ we need the maps

$$\varphi_g:\mathcal{L}oc^G_\lambda(M)_{\mathcal{Y}_{vg},k}=\mathcal{L}oc^\dagger_{\mathcal{Y}_{vg},k}(\lambda)(M_{vg,k})\to(\rho^v_g)_*\,\mathcal{L}oc^G_\lambda(M)_{\mathcal{Y}_v,k}$$

([19]) Here we use the fact that $(\mathbb{G}_v)_L=\mathbb{G}_L$.

satisfying the properties (i), (ii) and (iii). Let $\widetilde{\varphi}^v_g : M_{vg,k} \to M_{v,k}$ denote the dual map to $V_{\mathbb{G}_v(k)^\circ\text{-an}} \to V_{\mathbb{G}_{vg}(k)^\circ\text{-an}}; w \mapsto g^{-1}w$.[20] Let $\mathfrak{U} \subseteq \mathcal{Y}_{vg}$ be an open subset and $P \in \mathcal{D}^\dagger_{\mathcal{Y}_{vg},k}(\lambda)(\mathfrak{U})$, $m \in M_{vg,k}$. We define

$$\varphi^v_{g,\mathfrak{U}}(P \otimes m) = T^v_{g,\mathfrak{U}}(P) \otimes \widetilde{\varphi}^v_g(m). \tag{7.5}$$

Exactly as we have done in Theorem 6.6, the family (φ^v_g) satisfies the requirements (i), (ii) and (iii). Let us verify now condition (b). Given $(\mathcal{Y}_{v'}, k') \succeq (\mathcal{Y}_v, k)$ in $\underline{\mathcal{F}}$, we have $\mathbb{G}_{v'}(k')^\circ \subseteq \mathbb{G}_v(k)^\circ$ in $\mathbb{G}^{\text{rig}}$ and we denote by $\widetilde{\psi}_{\mathcal{Y}_{v'},\mathcal{Y}_v} : M_{v',k'} \to M_{v,k}$ the map dual to the natural inclusion $V_{\mathbb{G}_v(k)^\circ\text{-an}} \subseteq V_{\mathbb{G}_{v'}(k')^\circ\text{-an}}$. Let $\mathfrak{U} \subseteq \mathcal{Y}_{v'}$ be an open subset and $P \in \pi_*\mathcal{D}^\dagger_{\mathcal{Y}_{v'},k'}(\lambda)(\mathfrak{U})$, $m \in M_{v',k'}$. We then define[21]

$$\psi_{\mathcal{Y}_{v'},\mathcal{Y}_v}(P \otimes m) = \Psi_{\mathcal{Y}_{v'},\mathcal{Y}_v}(P) \otimes \widetilde{\psi}_{\mathcal{Y}_{v'},\mathcal{Y}_v}(m),$$

where $\Psi_{\mathcal{Y}_{v'},\mathcal{Y}_v} : \pi_*\mathcal{D}^\dagger_{\mathcal{Y}_{v'},k'}(\lambda) \to \mathcal{D}^\dagger_{\mathcal{Y}_v,k}(\lambda)$ is the canonical morphism given by the preceding proposition. This definition extends to a map

$$\psi_{\mathcal{Y}_{v'},\mathcal{Y}_v} : \pi_* \mathcal{L}oc^G_\lambda(M)_{\mathcal{Y}_{v'},k'} \to \mathcal{L}oc^G_\lambda(M)_{\mathcal{Y}_v,k},$$

which satisfies all the required conditions. The functoriality of $\mathcal{L}oc^G_\lambda(\bullet)$ can be verified exactly as we have done for the functor $\mathcal{L}oc^{G_0}_\lambda(\bullet)$.

Claim 2. *$\Gamma(\mathcal{M})$ is a coadmissible $D(G,L)_\lambda$-module.*

We already know that $\Gamma(\mathcal{M})$ is a coadmissible $D(G_{v,0},L)_\lambda$-module for any v (Theorem 6.6). So it suffices to exhibit a compatible G-action on $\Gamma(\mathcal{M})$. Let $g \in G$. The isomorphisms $\varphi^v_g : \mathcal{M}_{\mathcal{Y}_{vg},k} \to (\rho^v_g)_*\mathcal{M}_{\mathcal{Y},k}$ induce isomorphisms at the level of global sections (which we denote again by φ^v_g to soft the notation):

$$\varphi^v_g : H^0(\mathcal{Y}_{vg,k}, \mathcal{M}_{\mathcal{Y}_{vg},k}) \to H^0(\mathcal{Y}_v, \mathcal{M}_{\mathcal{Y},k}).$$

Let us identify

$$\begin{aligned}
&\Gamma(\mathcal{M}) \\
&= \varprojlim_{(\mathcal{Y}_{vg},k)\in\underline{\mathcal{F}}_{vg}} H^0(\mathcal{Y}_{vg,k}, \mathcal{M}_{\mathcal{Y}_{vg},k}) \\
&= \Big\{(m_{\mathcal{Y}_{vg},k})_{(\mathcal{Y}_{vg},k)} \in \prod_{\underline{\mathcal{F}}_{vg}} H^0(\mathcal{Y}_{vg,k}, \mathcal{M}_{\mathcal{Y}_{vg},k}) \mid \psi_{\mathcal{Y}'_{vg},\mathcal{Y}_{vg}}(m_{\mathcal{Y}'_{vg},k}) = m_{\mathcal{Y}_{vg},k}\Big\},
\end{aligned}$$

(20) Here we use $\mathbb{G}_{vg}(k)^\circ = g^{-1}\mathbb{G}_v(k)^\circ g$ in $\mathbb{G}^{\text{rig}}$.
(21) We avoid the subscript $\mathfrak{U}$ in order to soft the notation.

where, by abuse of notation, we have denoted by $\psi_{\mathcal{Y}'_{vg},\mathcal{Y}_{vg}}$ the morphism obtained by taking global sections on $\psi_{\mathcal{Y}'_{vg},\mathcal{Y}_{vg}} : (\pi.g)_*\mathcal{D}^{\dagger}_{\mathcal{Y}'_{vg},k'}(\lambda) \to \mathcal{D}^{\dagger}_{\mathcal{Y}_{vg},k}(\lambda)$. For $g \in G$ and

$$m := (m_{\mathcal{Y}_{vg},k})_{(\mathcal{Y}_{vg},k)\in\underline{\mathcal{F}}_{vg}} \in \Gamma(\mathcal{M}),$$

we define

$$g.m = (\varphi^v_g(m_{\mathcal{Y}_{vg},k}))_{(\mathcal{Y}_{vg},k)} \in \prod_{\underline{\mathcal{F}}_v} H^0(\mathcal{Y}_v, \mathcal{M}_{\mathcal{Y}_v,k}),$$
$$g.m_{(\mathcal{Y}_v,k)\in\underline{\mathcal{F}}_v} = \varphi^v_g(m_{\mathcal{Y}_{vg},k}).$$

We want to see that

$$g.m \in \Gamma(\mathcal{M}) = \varprojlim_{(\mathcal{Y}_v,k)\in\underline{\mathcal{F}}_v} H^0(\mathcal{Y}_v, \mathcal{M}_{\mathcal{Y}_v,k})$$

and that this assignment defines a left G-action on $\Gamma(\mathcal{M})$. Taking global sections on (7.4), we get the relation $\varphi^v_g \circ \psi_{\mathcal{Y}'_{vg},\mathcal{Y}_{vg}} = \psi_{\mathcal{Y}'_v,\mathcal{Y}_v} \circ \varphi^v_g$, which implies that

$$\begin{aligned}\psi_{\mathcal{Y}'_v,\mathcal{Y}_v}(g.m_{\mathcal{Y}'_v,k'}) &= \psi_{\mathcal{Y}'_v,\mathcal{Y}_v}(\varphi^v_g(m_{\mathcal{Y}'_{vg},k'}))\\ &= \varphi^v_g(\psi_{\mathcal{Y}'_{vg},\mathcal{Y}_{vg}}(m_{\mathcal{Y}'_{vg},k'}))\\ &= \varphi^v_g(m_{\mathcal{Y}_{vg},k}) = g.m_{\mathcal{Y}_v,k}.\end{aligned}$$

We obtain an isomorphism

$$\Gamma(\mathcal{M}) = \varprojlim_{\underline{\mathcal{F}}_{vg}} H^0(\mathcal{Y}_{vg}, \mathcal{M}_{\mathcal{Y}_{vg},k}) \xrightarrow{g} \varprojlim_{\underline{\mathcal{F}}_v} H^0(\mathcal{Y}_v, \mathcal{M}_{\mathcal{Y}_v,k}) = \Gamma(\mathcal{M}).$$

According to (i) in (a) we have the sequence

$$\varphi^v_{hg} : H^0(\mathcal{Y}_{vhg}, \mathcal{M}_{\mathcal{Y}_{vhg},k}) \xrightarrow{\varphi^{vh}_g} H^0(\mathcal{Y}_{vh}, \mathcal{M}_{\mathcal{Y}_{vh},k}) \xrightarrow{\varphi^v_h} H^0(\mathcal{Y}_v, \mathcal{M}_{\mathcal{Y}_v,k}),$$

which tells us that $h.(g.m) = (hg).m$ for $h, g \in G$ and $m \in \Gamma(\mathcal{M})$. This gives a G-action on $\Gamma(\mathcal{M})$ which, by construction, is compatible with its various $D(G_{v,0}, L)$-module structures.

Claim 3. $\Gamma \circ \mathcal{L}oc^G_\lambda(M) \simeq M$.

By Theorem 6.6 we know that this holds as a coadmissible $D(G_0, L)_\lambda$-module, so we need to identify the G-action on both sides. Let v be a special vertex. According to (7.5), the action

$$\Gamma \circ \mathcal{L}oc^G_\lambda(M) \simeq \varprojlim_k M_{vg,k} \to \varprojlim_v M_{v,k} \simeq \Gamma \circ \mathcal{L}oc^G_\lambda(M)$$

of an element $g \in G$ on $\Gamma \circ \mathcal{L}oc_\lambda^G(M)$ is induced by $\widetilde{\varphi}_g^v : M_{vg,k} \to M_{v,k}$. By dualizing

$$V = \bigcup_{k \in \mathbb{N}} V_{\mathbb{G}_{vg}(k)^\circ\text{-an}} = \bigcup_{k \in \mathbb{N}} V_{\mathbb{G}_v(k)^\circ\text{-an}},$$

we obtain the identification

$$M \simeq \varprojlim_k M_{vg,k} \simeq \varprojlim_k M_{v,k}.$$

Therefore, we get back the original action of g on M.

Claim 4. $\mathcal{L}oc_\lambda^G \circ \Gamma(\mathcal{M}) \simeq \mathcal{M}$.

We know that

$$\mathcal{L}oc_\lambda^G(\Gamma(\mathcal{M}))_{\mathcal{Y}_v,k} = \mathcal{M}_{\mathcal{Y}_v,k}$$

as $\mathcal{D}^\dagger_{\mathcal{Y}_v,k}(\lambda)$-modules for any $(\mathcal{Y}_v, k) \in \underline{\mathcal{F}}$, cf. Theorem 6.6. It remains to verify that these isomorphisms are compatible with the maps φ_g^v and $\psi_{\mathcal{Y}_{v'},\mathcal{Y}_v}$ on both sides. To do that, let us see that the maps φ_g^v on the left-hand side are induced by the maps of the right-hand side. Given

$$\varphi_g^v : \mathcal{M}_{\mathcal{Y}_v,k} \to (\rho_g^v)_* \mathcal{M}_{\mathcal{Y}_v,k},$$

the corresponding map

$$\varphi_g^v : \mathcal{L}oc_\lambda^G(\Gamma(\mathcal{M}))_{\mathcal{Y}_{vg},k} \to (\rho_g^v)_*(\mathcal{L}oc_\lambda^G(\Gamma(\mathcal{M}))_{\mathcal{Y}_v,k})$$

equals the map

$$\mathcal{D}^\dagger_{\mathcal{Y}_{vg},k}(\lambda) \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}_{vg}(k)^\circ)_\lambda} M_{vg,k} \to (\rho_g^v)_*(\mathcal{D}^\dagger_{\mathcal{Y}_v,k}(\lambda) \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}_v(k)^\circ)_\lambda} M_{v,k}),$$

where $M_{vg,k} = H^0(\mathcal{Y}_{vg}, \mathcal{M}_{\mathcal{Y}_{vg},k})$ and $M_{v,k} = H^0(\mathcal{Y}_v, \mathcal{M}_{\mathcal{Y}_v,k})$. Locally, the preceding morphism is given by $T^v_{g,\mathcal{Y}_{gv}} \otimes H^0(\mathcal{Y}_{vg}, \varphi_g^v)$, cf. (7.5). Let $\mathfrak{U} \subseteq \mathcal{Y}_v$ be an open subset, $P \in \mathcal{D}^\dagger_{\mathcal{Y}_v,k}(\lambda)(\mathfrak{U})$ and $m \in M_{v,k} = H^0(\mathcal{Y}_{vg}, \mathcal{M}_{\mathcal{Y}_{vg},k})$. The isomorphisms

$$\mathcal{L}oc_\lambda^G(\Gamma(\mathcal{M}))_{\mathcal{Y}_v,k} \simeq \mathcal{M}_{\mathcal{Y}_v,k}$$

are induced (locally) by $P \otimes m \mapsto P.(m|_{\mathfrak{U}})$. Condition (ii) tells us that these morphisms interchange the maps φ_g^v, as desired. The compatibility with the transition maps $\psi_{\mathcal{Y}_{v'},\mathcal{Y}_v}$ for two models $(\mathcal{Y}_{v'}, k') \succeq (\mathcal{Y}_v, k)$ in $\underline{\mathcal{F}}$ follows the arguments given in Theorem 6.6, by using the fact that the morphisms $\psi_{\mathcal{Y}_{v'},\mathcal{Y}_v}$ are compatible with the canonical map $\Psi : \pi_* \mathcal{D}^\dagger_{\mathcal{Y}_{v'},k'}(\lambda) \to \mathcal{D}^\dagger_{\mathcal{Y}_v,k}$.

This ends the proof of the theorem. ∎

As in the case of the group G_0, we now indicate how objects from $\mathcal{C}^{\mathcal{F}}_{G,\lambda}$ can be realized as G-equivariant sheaves on the G-space $\mathcal{X}_\infty$. The following discussion is an adaptation of the discussion given in [30, (5.4.3) and Proposition 5.4.5] to our case.

Proposition 7.7. *The G_0-equivariant structure of the sheaf $\mathcal{D}(\lambda)$ extends to a G-equivariant structure.*

Proof. Let $g \in G$ and let $v, v' \in \mathcal{B}$ be special vertices. Let us suppose that $(\mathcal{Y}_{v'}, k') \succeq (\mathcal{Y}_v, k)$ in $\underline{\mathcal{F}}$. The isomorphism $\rho_g^{v'} : \mathcal{Y}_{v'} \to \mathcal{Y}_{v'g}$ induces a ring isomorphism

$$T_g^{v'} : \mathcal{D}^\dagger_{\mathcal{Y}_{v'g},k'}(\lambda) \to (\rho_g^{v'})_* \mathcal{D}^\dagger_{\mathcal{Y}_{v'},k'}(\lambda).$$

On the other hand, and exactly as we have done in (6.5), the commutative diagram

$$\begin{array}{ccccc} \mathcal{X}_\infty & \xrightarrow{\mathrm{sp}_{\mathcal{Y}_v}} & \mathcal{Y}_v & \xrightarrow{\rho_g^v} & \mathcal{Y}_{vg} \\ \big\downarrow{\scriptstyle \mathrm{sp}_{\mathcal{Y}_{v'}}} & \nearrow{\scriptstyle \pi} & & \nearrow{\scriptstyle \pi.g} & \\ \mathcal{Y}_{v'} & \xrightarrow{\rho_g^{v'}} & \mathcal{Y}_{v'g} & & \end{array}$$

defines a continuous function

$$\rho_g : \mathcal{X}_\infty \to \mathcal{X}_\infty, \quad (a_v) \mapsto (\rho_g^v(a_v)),$$

which satisfies

$$\mathrm{sp}_{\mathcal{Y}_{v'g}} \circ \rho_g = \rho_g^{v'} \circ \mathrm{sp}_{\mathcal{Y}_{v'}}.$$

In particular, if $V \subseteq \mathcal{X}_\infty$ is the open subset $V := \mathrm{sp}^{-1}_{\mathcal{Y}_v}(\mathcal{U})$ with $\mathcal{U} \subseteq \mathcal{Y}_v$ an open subset, then

$$(\rho_g^{v'})^{-1}(\mathrm{sp}_{\mathcal{Y}_{v'g}}(V)) = \mathrm{sp}_{\mathcal{Y}_{v'}}(\rho_g^{-1}(V))$$

and so the map $T_g^{v'}$ induces the morphism

$$\mathcal{D}^\dagger_{\mathcal{Y}_{v'g},k'}(\lambda)(\mathrm{sp}_{\mathcal{Y}_{v'g}}(V)) \to \mathcal{D}^\dagger_{\mathcal{Y}_{v'},k'}(\lambda)(\mathrm{sp}_{\mathcal{Y}_{v'}}(\rho_g^{-1}(V))). \tag{7.6}$$

Moreover, if $(\mathcal{Y}_{v''}, k'') \succeq (\mathcal{Y}_{v'}, k') \succeq (\mathcal{Y}_v, k)$ in $\underline{\mathcal{F}}$, and as before $V = \mathrm{sp}^{-1}_{\mathcal{Y}_v}(\mathcal{U}) \subseteq \mathcal{X}_\infty$ with $\mathcal{U} \subseteq \mathcal{Y}_v$ an open subset, then the commutative diagram

$$\begin{array}{ccc} \mathcal{D}^\dagger_{\mathcal{Y}_{v''g},k''}(\lambda)(\mathrm{sp}_{\mathcal{Y}_{v''g}}(V)) & \longrightarrow & \mathcal{D}^\dagger_{\mathcal{Y}_{v''},k''}(\lambda)(\mathrm{sp}_{\mathcal{Y}_{v''}}(\rho_g^{-1}(V))). \\ \big\downarrow & & \big\downarrow \\ \mathcal{D}^\dagger_{\mathcal{Y}_{v'g},k'}(\lambda)(\mathrm{sp}_{\mathcal{Y}_{v'g}}(V)) & \longrightarrow & \mathcal{D}^\dagger_{\mathcal{Y}_{v'},k'}(\lambda)(\mathrm{sp}_{\mathcal{Y}_{v'}}(\rho_g^{-1}(V))). \end{array}$$

implies that if we identify

$$\mathcal{D}(\lambda)(V) = \varprojlim_{(\mathcal{Y}_{vg},k)\in\mathcal{F}_{vg}} \mathcal{D}^{\dagger}_{\mathcal{Y}_{vg},k}(\lambda)(\mathrm{sp}_{\mathcal{Y}_{vg}}(V))$$

and we take projective limits in (7.6), then we get a ring homomorphism

$$T_{g,V} : \mathcal{D}(\lambda)(V) \to (\rho_g)_*\mathcal{D}(\lambda)(V),$$

which implies that the sheaf $\mathcal{D}(\lambda)$ is G-equivariant. Furthermore, by construction, this G-equivariant structure extends the G_0-structure defined in Section 6.1. ■

Finally, let us recall the faithful functor

$$\mathcal{M} \rightsquigarrow \mathcal{M}_\infty$$

from coadmissible G_0-equivariant arithmetic $\mathcal{D}(\lambda)$-modules on $\mathcal{F}_{\mathcal{X}}$ to G_0-equivariant $\mathcal{D}(\lambda)$-modules on $\mathcal{X}_\infty$. If $\mathcal{M}$ comes from a coadmissible G-equivariant $\mathcal{D}(\lambda)$-module on $\mathcal{F}$, then $\mathcal{M}_\infty$ is in fact G-equivariant (as in (6.7), this can be proved by using the family of L-linear isomorphisms $(\varphi_g^v)_{g\in G}$). As in Proposition 6.9, Theorem 7.6 gives us the following result.

Theorem 7.8. *Let us suppose that $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ is an algebraic character such that $\lambda + \rho \in \mathfrak{t}^*_{\mathbb{Q}}$ is a dominant and regular character of $\mathfrak{t}_{\mathbb{Q}}$. The functor $\mathcal{M} \rightsquigarrow \mathcal{M}_\infty$ from the category $\mathcal{C}^{\mathcal{F}}_{G,\lambda}$ to G-equivariant $\mathcal{D}(\lambda)$-modules on $\mathcal{X}_\infty$ is a faithful functor.*

Remark 7.9. We end this work by remarking to the reader that the functors in Proposition 6.9 and Theorem 7.8 become *fully faithful* functors if we required that the objects in the target category carry a structure of locally convex topological $\mathcal{D}(\lambda)$-modules (cf. [20, Propositions 5.2.31 and 5.3.16]). In fact, following [20, (5.2.30)], we can see that $\mathcal{D}(\lambda)$ carries a natural structure of a sheaf of locally convex topological L-algebras and, more generally, if $\mathcal{M} \in \mathcal{C}^{G_0}_{\mathcal{X},\lambda}$ (resp. $\mathcal{M} \in \mathcal{C}^{\mathcal{F}}_{G,\lambda}$), then $\mathcal{M}_\infty$ becomes a G_0-equivariant (resp. G-equivariant) sheaf of locally convex topological L-vector spaces, endowed with the structure of a topological $\mathcal{D}(\lambda)$-module.

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