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Classification of p-groups via their 2-nilpotent multipliers

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Abstract – For a p-group of order p^n, it is known that the order of 2-nilpotent multiplier is equal to $|\mathcal{M}^{(2)}(G)| = p^{\frac{1}{2}n(n-1)(n-2)+3-s_2(G)}$, for an integer $s_2(G)$. In this article, we characterize all non-abelian p-groups satisfying $s_2(G) \in \{1, 2, 3\}$.

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1. Preliminaries

The 2-nilpotent multiplier of a group is a generalization of the well-known notion of Schur multiplier. The latter was introduced by J. Schur in his works on projective representations in [15] and plays a considerable role in classifying groups. In fact, 2-nilpotent multiplier is a special case of the more general notion of Baer invariant.

For a group G with a free presentation $G \cong F/R$, the c-nilpotent multiplier of G, $\mathcal{M}^{(c)}(G)$, is defined as

$$\frac{R \cap \gamma_{c+1}(F)}{[R, {}_c F]},$$

in which $\gamma_{c+1}(F)$ is the c-th term of the lower central series of F, and $[R, {}_c F] = [[R, {}_{c-1} F], F]$ (see [4]).

The motivation of studying the 2-nilpotent multiplier comes from [4]. It is the connection to isologism of groups which is an important tool in classifying groups.

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Recall from [6] that a group G which is isomorphic to $H/Z_2(H)$, for some group H, is called 2-capable. Choose a free presentation $G \cong F/R$, and consider the natural epimorphism $\alpha\colon F/[R,F,f] \to G$. We may define $Z_2^*(G) = \alpha(Z_2(F/[R,F,F]))$. Proposition 1.2 in [4] allows us to decide when a group G is 2-capable. More precisely, G is 2-capable if and only if $Z_2^*(G) = 1$. There is a somehow different way for detecting 2-capable groups using the notion of 2-nilpotent multiplier. In more detail, for a group G, the natural epimorphism $\mathcal{M}^{(2)}(G) \to \mathcal{M}^{(2)}(G/N)$ is a monomorphism if and only if N is a subgroup of $Z_2^*(G)$ (see [4, Lemma 2.1]).

Now, we restrict our study to finite p-groups. A famous result of Green shows that for a given finite 2-group G of order p^n, $|\mathcal{M}(G)| = p^{\frac{1}{2}n(n-1)-t(G)}$ for some integer $t(G) \geq 0$. Several authors worked on classifying the structure of G in term of $t(G)$ when $0 \leq t(G) \leq 5$ (see [1, 12–14, 16]). In [10], considering only non-abelian finite p-groups, a Green-type inequality was obtained. The first-named author showed that $|\mathcal{M}(G)| \leq p^{\frac{1}{2}(n-1)(n-2)+1}$, where G is a finite p-group of order p^n, and hence there is an integer $s(G)$ such that $|\mathcal{M}(G)| = p^{\frac{1}{2}(n-1)(n-2)+1-s(G)}$. A similar result for the 2-nilpotent multiplier of finite p-groups appeared in [14]. The authors proved for a non-abelian p-group of order p^n that there exists an integer $s_2(G)$ such that $|\mathcal{M}^{(2)}(G)| = p^{\frac{1}{2}n(n-1)(n-2)+3-s_2(G)}$, and the structure of all p-groups are classified when $s_2(G) = 0$. In the present paper, by the same motivation as in [1, 13, 14, 16], we are interested in characterizing p-groups up to isomorphisms when $s_2(G) \in \{1,2,3\}$.

Let us start by stating some lemmas which are needed for the present work. In the following lemma, $G_1 \otimes G_2$ denotes the non-abelian tensor product of two arbitrary groups G_1 and G_2, and $G_1 \wedge G_2$ denotes the non-abelian exterior product. For more information on these two concepts one may see [2]. It is worth noting that if G_1 and G_2 are two groups acting trivially on each other, then $G_1 \otimes G_2$ coincides with the usual tensor product $G_1/G_1' \otimes G_2/G_2'$ of abelian groups, by [3, Proposition 2.4].

Lemma 1.1 ([5, Proposition 2], [7, 9]). *Let G be a finite group and $B \trianglelefteq G$. Set $A = G/B$.*

(i) (a) *If $B \subseteq Z_2(G)$, then*

$$|\mathcal{M}^{(2)}(G)|\,|B \cap \gamma_3(G)| \quad \textit{divides} \quad |\mathcal{M}^{(2)}(A)|\left|\left(B \otimes \frac{G}{\gamma_3(G)}\right) \otimes \frac{G}{\gamma_3(G)}\right|.$$

(b) *The sequence*

$$(B \wedge G) \wedge G \to \mathcal{M}^{(2)}(G) \to \mathcal{M}^{(2)}(G/B) \to B \cap \gamma_3(G) \to 1$$

is exact.

(ii) $|\mathcal{M}^{(2)}(A)|$ *divides* $|\mathcal{M}^{(2)}(G)|\,|B \cap \gamma_3(G)|/|[[B,G],G]|$.

The following result plays an essential role in the rest of the paper.

LEMMA 1.2 ([8]). *Let G be a finite group. Put $G^{ab} = G/G'$. Then there is a natural isomorphism*

$$\mathcal{M}^{(2)}(G \times H) \cong \mathcal{M}^{(2)}(G) \times \mathcal{M}^{(2)}(H) \times (G^{ab} \otimes G^{ab}) \otimes H^{ab} \times (H^{ab} \otimes H^{ab}) \otimes G^{ab}.$$

The following two lemmas are from [14].

LEMMA 1.3. *Let G be an extra-special p-group of order p^{2n+1}.*

(i) *If $n > 1$, then $\mathcal{M}^{(2)}(G)$ is an elementary abelian p-group of order $p^{\frac{1}{3}(8n^3-2n)}$.*

(ii) *Suppose that $|G| = p^3$ and p is odd. Then $\mathcal{M}^{(2)}(G) = \mathbb{Z}_p^{(5)}$ if G is of exponent p and $\mathcal{M}^{(2)}(G) = \mathbb{Z}_p \times \mathbb{Z}_p$ if G is of exponent p^2.*

(iii) *The quaternion group of order 8 has Klein four-group as the 2-nilpotent multiplier, whereas the 2-nilpotent multiplier of the dihedral group of order 8 is $\mathbb{Z}_2 \oplus \mathbb{Z}_4$.*

LEMMA 1.4. *Let $G = \mathbb{Z}_{p^{m_1}} \oplus \mathbb{Z}_{p^{m_2}} \oplus \cdots \oplus \mathbb{Z}_{p^{m_k}}$, where $m_1 \geq m_2 \geq \cdots \geq m_k$ and $\sum_{i=1}^{k} m_i = n$. Then*

(i) *$|\mathcal{M}^{(2)}(G)| = p^{\frac{1}{3}n(n-1)(n+1)}$ if and only if $m_i = 1$ for all i;*

(ii) *$|\mathcal{M}^{(2)}(G)| \leq p^{\frac{1}{3}n(n-1)(n-2)}$ if and only if $m_1 \geq 2$.*

2. Main results

As mentioned above, we know that the order of the 2-nilpotent multiplier of a finite non-abelian p-group of order p^n is bounded by $p^{\frac{1}{3}n(n-1)(n-2)+3}$, therefore for any group G there exists a non-negative integer $s_2(G)$ for which

$$|\mathcal{M}^{(2)}(G)| = p^{\frac{1}{3}n(n-1)(n-2)+3-s_2(G)}.$$

In this paper, we characterize the explicit structures of finite non-abelian p-groups when $s_2(G) \in \{1, 2, 3\}$.

First, we state the following theorem from [14] to prove that the only groups which may have the desired property are those with small derived subgroups.

THEOREM 2.1. *Let G be a p-group of order p^n with $|G'| = p^m$ ($m \geq 1$). Then*

$$|\mathcal{M}^{(2)}(G)| \leq p^{\frac{1}{3}(n-m)((n+2m-2)(n-m-1)+3(m-1))+3}$$

and the equality holds if and only if $G \cong E_1 \times \mathbb{Z}_p^{(n-3)}$.

Lemma 2.2. *Let G be a non-abelian p-group of order p^n with $|G'| \geq p^3$. Then* $|\mathcal{M}^{(2)}(G)| \leq p^{\frac{1}{3}n(n-1)(n-2)-2}$.

Proof. Just use Theorem 2.1 and the fact that n is at least 5. ∎

The following lemma has a completely similar proof to that of Lemma 2.2.

Lemma 2.3. *Let G be a non-abelian p-group of order p^n with $|G'| = p^2$. Then* $|\mathcal{M}^{(2)}(G)| \leq p^{\frac{1}{3}n(n-1)(n-2)+1}$.

The following theorem gives an upper bound for the order of the 2-nilpotent multiplier of a finite group G. Since B and G/B act trivially on each other, $B \otimes G/B$ is isomorphic to the usual tensor product $B \otimes (G/G'B)$, by [3, Proposition 2.4].

Theorem 2.4. *Let G be a p-group and B be a cyclic central subgroup of G. Then*

$$|\mathcal{M}^{(2)}(G)| \leq |\mathcal{M}^{(2)}(G/B)|\,|(B \otimes G/G'B) \otimes G/G'|.$$

Proof. Let $G = F/R$ and $B = S/R$ be free presentations for G and B, respectively. Since B is central, we have $[S, F] \subseteq R$, and also $R \cap S' = [R, S]$ because B is cyclic. Now $S' \subseteq R$, and so $S' = [R, S]$.

By definition, we have

$$\mathcal{M}^{(2)}(G) \cong \frac{R \cap \gamma_3(F)}{[R, F, F]} \quad \text{and} \quad \mathcal{M}^{(2)}(G/B) \cong \frac{S \cap \gamma_3(F)}{[S, F, F]},$$

and so

$$|\mathcal{M}^{(2)}(G)| \leq |\mathcal{M}^{(2)}(G/B)| \left| \frac{[S, F, F]}{[R, F, F]} \right|.$$

The proof is completed if there exists a well-defined epimorphism

$$\bar{\psi} \colon S/R \otimes F/SF' \otimes F/RF' \longrightarrow \frac{[S, F, F]}{[R, F, F]}.$$

To get this, considering the universal property of the usual tensor product of abelian groups, it is enough to produce a well-defined multi-linear map ψ by the rule

$$\psi(sR, f_1SF', f_2RF') = [s, f_1, f_2][R, F, F].$$

First we show that

$$[sr, f_1s'\gamma', f_2r'\gamma] \equiv [s, f_1, f_2] \pmod{[R, F, F]}$$

where $r, r' \in R$, $s, s' \in S$ and $\gamma, \gamma' \in F'$.

Expanding the commutator on the left hand side we have $[sr, f_1s'\gamma', f_2r'\gamma] = [sr, f_1s'\gamma', r'\gamma][sr, f_1s'\gamma', f_2][sr, f_1s'\gamma', f_2, r'\gamma]$. Trivially, $[sr, f_1s'\gamma', f_2, r'\gamma] \in [S, F, F, F]$, but $[S, F] \subseteq R$, hence $[S, F, F, F] \subseteq [R, F, F]$. On the other hand, $[sr, f_1s'\gamma', r'\gamma] = [sr, f_1s'\gamma', \gamma][sr, f_1s'\gamma', r'][sr, f_1s'\gamma', r', \gamma]$, which is contained in $[S, F, F'][S, F, R]$. A simple use of the three subgroup lemma shows that the latter is contained in $[R, F, F]$. We claim that $[sr, f_1s'\gamma', f_2r'\gamma] \equiv [sr, f_1s'\gamma', f_2] \pmod{[R, F, F]}$. Using commutator calculus again, we get

$$[sr, f_1s'\gamma', f_2] = [sr, s'\gamma', f_2]\big[sr, s'\gamma', f_2, [sr, f_1]^{s'\gamma'}\big]\big[[sr, f_1]^{s'\gamma'}, f_2\big].$$

It is easy to see that

$$[sr, s'\gamma', f_2]\big[sr, s'\gamma', f_2, [sr, f_1]^{s'\gamma'}\big] \in [S, SF', F] = [S, S, F][S, F', F]$$

but we have

$$[S, S, F] = [S', F] = [R, S, F] \subseteq [R, F, F]$$

and

$$[S, F', F] \subseteq [S, F, F, F] = [R, F, F].$$

Finally, $[[sr, f_1]^{s'\gamma'}, f_2] = [sr, f_1, f_2][sr, f_1, f_2, [sr, f_1, s'\gamma']][sr, f_1, s'\gamma', f_2]$, and for the last two we have $[sr, f_1, f_2, [sr, f_1, s'\gamma']][sr, f_1, s'\gamma', f_2] \in [S, F, F, F] \subseteq [R, F, F]$. The first one can be decomposed as

$$\begin{aligned}[sr, f_1, f_2] = {} & [s, f_1, f_2]\big[s, f_1, f_2, [s, f_1, r]\big] \\ & \cdot [s, f_1, r, f_2]\big[[s, f_1]^r, f_2, [r, f_1]\big][r, f_1, f_2],\end{aligned}$$

and we have

$$\begin{aligned}&\big[s, f_1, f_2, [s, f_1, r]\big][s, f_1, r, f_2] \cdot \big[[s, f_1]^r, f_2, [r, f1]\big][r, f_1, f_2] \\ &\quad \in [S, F, F, F][R, F, F] \subseteq [R, F, F].\end{aligned}$$

The multi-linearity of this mapping follows by a straightforward application of commutator calculus. ∎

Considering Lemmas 2.2 and 2.3, in order to characterize all p-groups with $s_2(G) \in \{1, 2, 3\}$, it is enough to work with p-groups with $|G'| \le p^2$. First we deal with those groups having commutator subgroup of order p. If G/G' is not elementary abelian, we have:

LEMMA 2.5. *Let G be a p-group of order p^n with G' of order p. If G/G' is not elementary abelian, then*

$$|\mathcal{M}^{(2)}(G)| \le p^{\frac{1}{3}n(n-1)(n-2)-2}.$$

PROOF. We use Theorem 2.4 with $B = G'$, to get

$$|\mathcal{M}^{(2)}(G)| \leq |\mathcal{M}^{(2)}(G/G')|\, |G' \otimes G/G' \otimes G/G'|.$$

Since G/G' is not elementary abelian, by using Lemma 1.4 we have

$$|\mathcal{M}^{(2)}(G/G')| \leq p^{\frac{1}{3}(n-1)(n-2)(n-3)}.$$

Since $|G' \otimes G/G' \otimes G/G'| \leq p^{(n-2)^2}$, we get the result. ∎

Now we may assume that G/G' is elementary abelian. In [10, Lemma 2.1] p-groups with $G' = \phi(G)$ (the Frattini subgroup) of order p are classified as the central product of an extra-special p-group H by the center $Z(G)$ of G; that is, $G = H \cdot Z(G)$. Now, depending on how G' embeds into $Z(G)$, we have the following lemma which has a straightforward proof.

LEMMA 2.6. *Let G be a p-group with $G' = \phi(G)$ of order p. Then:*

(i) *If G' is a direct summand of $Z(G)$, then $G = H \times K$ for some finite abelian group K.*

(ii) *If G' is not a direct summand of $Z(G)$, then $G = (H \cdot \mathbb{Z}_{p^2}) \times K$ where K is a finite abelian p-group.*

PROOF. As G is a p-group and $|G'| = p$, we have $G' \subseteq Z(G)$. Consider G/G' as a vector space over $\mathbb{Z}_p$ and let H/G' be a complement to $Z(G)/G'$ in it. It is easy to see that $G = H \cdot Z(G)$ and $H \cap Z(G) = G'$. Now, if G' is a direct summand of $Z(G)$, then we have $Z(G) = G' \times K$ for some abelian subgroup K of $Z(G)$ and hence $G = H \times K$. If G' is not a direct summand of $Z(G)$, we have $\exp(Z(G)) = p^2$, because G/G' is an elementary abelian p-group and $G' \subseteq Z(G)$. Now it is easy to see that $Z(G) = \mathbb{Z}_{p^2} \times K$ and $G' \subseteq \mathbb{Z}_{p^2}$, so we can write $G = (H \cdot \mathbb{Z}_{p^2}) \times K$. ∎

As we consider the groups for which G/G' is elementary abelian, we have only the following two cases:

(1) $G = H \times T$,

(2) $G = H \cdot \mathbb{Z}_{p^2} \times T$,

where T is an elementary abelian p-group. By Lemma 1.2, without loss of generality we can assume that $Z(G) = \mathbb{Z}_{p^2}$. For the groups of type (1) we have the following theorem.

Theorem 2.7. *Let $G = H \times T$, where H is an extra-special p-group and T is an elementary abelian p-group. Then:*

(i) *If $H = E_1$ then $|\mathcal{M}^{(2)}(G)| = p^{\frac{1}{3}n(n-1)(n-2)+3}$.*

(ii) *If $H = D_8$ then $|\mathcal{M}^{(2)}(G)| = 2^{\frac{1}{3}n(n-1)(n-2)+1}$.*

(iii) *In all other cases, $|\mathcal{M}^{(2)}(G)| = p^{\frac{1}{3}n(n-1)(n-2)}$.*

Proof. It is just straightforward computations using Lemmas 1.2 and 1.3. ∎

For the groups of type (2), first we compute the order of the 2-nilpotent multiplier of $H \cdot \mathbb{Z}_{p^2}$. It should be noted that, as mentioned before Theorem 2.7, we may assume that $Z(G) = \mathbb{Z}_{p^2}$.

Theorem 2.8. *With the above notation and assumptions, let $G = H \cdot \mathbb{Z}_{p^2}$ be of order p^n. Then $|\mathcal{M}^{(2)}(G)| = p^{\frac{1}{3}n(n-1)(n-2)}$.*

Proof. Using Theorem 2.4 with $B = \mathbb{Z}_{p^2}$, we get

$$|\mathcal{M}^{(2)}(G)| \leq |\mathcal{M}^{(2)}(G/\mathbb{Z}_{p^2})|\,|\mathbb{Z}_{p^2} \otimes G/\mathbb{Z}_{p^2} \otimes G/G'|.$$

In order to compute $|\mathcal{M}^{(2)}(G/\mathbb{Z}_{p^2})|$, we have

$$\frac{G}{\mathbb{Z}_{p^2}} = \frac{H \cdot \mathbb{Z}_{p^2}}{\mathbb{Z}_{p^2}} \cong \frac{H}{H \cap \mathbb{Z}_{p^2}}.$$

But as we had in the proof of Lemma 2.6, $H \cap \mathbb{Z}_{p^2} = G'$. Therefore, $G/\mathbb{Z}_{p^2} \cong H/G'$. By assumption, $|H| = p^{2m+1}$, so H/G' is an elementary abelian p-group of order p^{2m}, hence using Lemma 1.2 and the multi-linearity of the tensor product of abelian groups, we have

$$|\mathcal{M}^{(2)}(G/\mathbb{Z}_{p^2})| = p^{\frac{1}{3}2m(2m+1)(2m-1)} \quad \text{and} \quad |\mathbb{Z}_{p^2} \otimes G/\mathbb{Z}_{p^2} \otimes G/G'| = p^{(2m+1)^2}.$$

After some computations, one gets $|\mathcal{M}^{(2)}(G)| \leq p^{\frac{1}{3}n(n-1)(n-2)}$. Now, Lemma 1.1(a) with $B = G'$ shows that $|\mathcal{M}^{(2)}(G/G')| \leq |\mathcal{M}^{(2)}(G)|$. The result now follows by using Lemma 1.4. ∎

Now the following theorem, whose proof is completely similar to the last two ones, completes the groups of type (2).

Theorem 2.9. *Let $G = H \cdot \mathbb{Z}_{p^2} \times T$ be of order p^n, where T is an elementary abelian p-group and H is an extra-special p-groups. Then $|\mathcal{M}^{(2)}(G)| = p^{\frac{1}{3}n(n-1)(n-2)}$.*

In the rest we concentrate on the groups with the derived subgroup of order p^2.

Lemma 2.10. *Let G be a p-group of order p^n with G' of order p^2. If $Z(G)$ is not elementary abelian, then* $|\mathcal{M}^{(2)}(G)| \le p^{\frac{1}{3}n(n-1)(n-2)-2}$.

Proof. Choose $B \subseteq Z(G)$ cyclic of order p^2 and use Theorem 2.4 to obtain

$$|\mathcal{M}^{(2)}(G)| \le |\mathcal{M}^{(2)}(G/B)|\,|B \otimes G/B \otimes G/G'|.$$

Since

$$|\mathcal{M}^{(2)}(G/B)| \le p^{\frac{1}{3}(n-1)(n-2)(n-3)} \quad \text{and} \quad |B \otimes G/B \otimes G/G'| \le p^{(n-2)^2},$$

we have $|\mathcal{M}^{(2)}(G)| \le p^{\frac{1}{3}n(n-1)(n-2)-2}$, and the result follows. ■

In the class of groups with an elementary abelian center we must consider the following two lemmas.

Lemma 2.11. *Let G be a p-group of order p^n with G' of order p^2. Let $Z(G)$ be elementary abelian. If $|Z(G)| \ge p^3$ or $|Z(G)| = p^2$, and $G' \ne Z(G)$, then* $|\mathcal{M}^{(2)}(G)| \le p^{\frac{1}{3}n(n-1)(n-2)-2}$.

Proof. Let K be a central subgroup of order p with $K \cap G' = 1$. By Lemma 1.1(a), we have $|\mathcal{M}^{(2)}(G)| \le |\mathcal{M}^{(2)}(G/K)|\,|K \otimes G/\gamma_3(G) \otimes G/\gamma_3(G)|$. But G/K is a non-abelian p-group with $|(G/K)'| = p^2$, thus $|\mathcal{M}^{(2)}(G/K)| \le p^{\frac{1}{3}(n-1)(n-2)(n-3)+1}$ by Lemma 2.9. Since $|K \otimes G/\gamma_3(G) \otimes G/\gamma_3(G)| \le p^{(n-2)^2}$, the result follows. ■

Lemma 2.12. *Let G be a p-group of order p^n with G' of order p^2. If G/G' is not elementary, then* $|\mathcal{M}^{(2)}(G)| = p^{\frac{1}{3}n(n-1)(n-2)-2}$.

Proof. The result is obtained by a similar argument used in the proof of Lemma 2.5 and Theorems 2.7 and 2.8. ■

The next lemma shows that the same upper bound in Lemma 2.11 works when $Z(G)$ is of order p.

Lemma 2.13. *Let G be a p-group of order p^n with G' of order p^2. If $|Z(G)| = p$, then* $|\mathcal{M}^{(2)}(G)| \le p^{\frac{1}{3}n(n-1)(n-2)-2}$.

Proof. By using Lemma 1.1(a) when $B = Z(G)$, and Theorems 2.7 and 2.8, the result follows. ■

The last case is the one for which $G' = Z(G) \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$.

THEOREM 2.14. *There is no finite p-group of order p^n with $G' = Z(G) \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$ such that $|\mathcal{M}^{(2)}(G)| = p^{\frac{1}{3}n(n-1)(n-2)}$.*

PROOF. By contradiction, assume that there is a finite p-group G of order p^n such that $|\mathcal{M}^{(2)}(G)| = p^{\frac{1}{3}n(n-1)(n-2)}$ and $G' = Z(G) \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$. Let K be a central subgroup of order p in G'; by Lemma 1.1(a), we have $|\mathcal{M}^{(2)}(G)| \leq |\mathcal{M}^{(2)}(G/K)|\,|K \otimes G/G' \otimes G/G'|$. Now Theorems 2.7 and 2.8 show that

$$|\mathcal{M}^{(2)}(G/K)| \leq p^{\frac{1}{3}(n-1)(n-2)(n-3)+3},$$

whereas G/G' is elementary abelian by Lemma 2.12. Therefore, $p^{\frac{1}{3}n(n-1)(n-2)} = |\mathcal{M}^{(2)}(G)| \leq p^{\frac{1}{3}(n-1)(n-2)(n-3)+3}\, p^{(n-2)^2}$, whence $n \leq 5$. Since $n \neq 4$, we have $n = 5$. Now [11, page 345] shows that $G \cong \mathbb{Z}_p^{(4)} \rtimes \mathbb{Z}_p$. By a similar argument used in the proof of [14, Theorem 3.5], we have $|\mathcal{M}^{(2)}(G)| = p^{18}$, which is a contradiction. Hence, the assumption is false and the result follows. ∎

We conclude summarizing the achieved results.

THEOREM 2.15. *Let G be a non-abelian p-group of order p^n. Then:*

(i) *There is no group G with $|\mathcal{M}^{(2)}(G)| = p^{\frac{1}{3}n(n-1)(n-2)+2}$.*

(ii) $|\mathcal{M}^{(2)}(G)| = p^{\frac{1}{3}n(n-1)(n-2)+1}$ *if and only if $p = 2$ and $G \cong D_8 \times \mathbb{Z}_2^{(n-3)}$.*

(iii) $|\mathcal{M}^{(2)}(G)| = p^{\frac{1}{3}n(n-1)(n-2)}$ *if and only if $G \cong H_m \times \mathbb{Z}_p^{(n-2m-1)}$, where H_m is an extra-special p-group of order p^{2m+1} and $m \geq 2$ or $G \cong H_m \cdot \mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(n-2m-2)}$.*

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