

Weakly S -semipermutable subgroups and p -nilpotency of groups

HASSAN JAFARIAN DEHKORDY (*) – GHOLAMREZA R. REZAEZADEH (**)

ABSTRACT – A subgroup H of a finite group G is said to be S -semipermutable in G if $HG_p = G_pH$ for every Sylow subgroup G_p of G with $(|H|, p) = 1$. A subgroup H of G is said to be weakly S -semipermutable in G if there exists a normal subgroup T of G such that HT is S -permutable and $H \cap T$ is S -semipermutable in G . In this paper we prove that for a finite group G , if some cyclic subgroups or maximal subgroups of G are weakly S -semipermutable in G , then G is p -nilpotent.

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1. Introduction

All groups considered in this paper are finite. For a group G , let $\pi(G)$ denote the set of all prime divisors of $|G|$; G_p a Sylow p -subgroup of G for some $p \in \pi(G)$; $F(G)$ the Fitting subgroup of G ; $F^*(G)$ the generalized Fitting subgroup of G ; $\Phi(G)$ the Frattini subgroup of G . Let \mathfrak{F} be a class of groups, \mathfrak{F} is said to be a formation, provided that if $G \in \mathfrak{F}$ and $H \trianglelefteq G$, then $G/H \in \mathfrak{F}$, and for any normal subgroups N, M of G , if G/N and G/M are in \mathfrak{F} , then $G/(M \cap N) \in \mathfrak{F}$. The formation \mathfrak{F} is said to be saturated if $G/\Phi(G) \in \mathfrak{F}$ implies that $G \in \mathfrak{F}$. In this paper, \mathfrak{U} will denote the class of all supersoluble groups. As is well known, \mathfrak{U} is a saturated formation. $Z_\infty(G)$

(*) *Indirizzo dell'A.*: Faculty of Mathematical Sciences, Shahrekord University, Shahrekord, Iran; h.jafarian.d@sku.ac.ir

(**) *Indirizzo dell'A.*: Faculty of Mathematical Sciences, Shahrekord University, Shahrekord, Iran; rezaeezadeh@sku.ac.ir

is the hypercenter of G . A normal subgroup H of G is said to be hypercyclically embedded in G if every chief factor of G below H is cyclic. $Z_{\mathcal{U}}(G)$ is said to be a \mathcal{U} -hypercenter of G and is the product of all hypercyclically embedded subgroups of G .

Let H and K be two subgroups of G . The product HK of these subgroups is a subgroup of G if and only if $HK = KH$. The subgroup H is said to be quasinormal or permutable [12] in G if H permutes with all subgroups of G , and S -permutable or S -quasinormal or π -quasinormal [9] in G if H permutes with all Sylow subgroups of G . It is observed by Ore [12] that every permutable subgroup of a finite group G is subnormal in G . By extending this result, Ito and Szép have proved in [8] that for every permutable subgroup H of a finite group G , H/H_G is nilpotent. Here, H_G is the Core of H , which is the largest normal subgroup of G contained in H . Maier and Schmid proved in [11] that for every permutable subgroup H of G it is true that $(H^G/H_G) \subseteq Z_{\infty}(G/H_G)$. Here, H^G is the normal closure of H in G , which is the intersection of all normal subgroups of G that contain H .

A subgroup H of G is said to be S -semipermutable [1] in G , if $HG_p = G_pH$ for every Sylow subgroup G_p of G with $(|H|, p) = 1$.

Clearly, every S -permutable subgroup of G is S -semipermutable in G but the converse does not hold; for example, a Sylow 2-subgroup of S_3 (the symmetric group of degree 3) is semipermutable, thus S -semipermutable in S_3 , but it is not S -permutable in S_3 .

H is said to be an S -embedded subgroup [3] of G if G has normal subgroup T such that HT is S -permutable in G and $H \cap T \leq H_{sG}$, where H_{sG} denotes the subgroup generated by all subgroups of H that are S -permutable in G .

DEFINITION. A subgroup H of a finite group G is called *weakly S -semipermutable* in G if there exists a normal subgroup T of G such that HT is S -permutable and $H \cap T$ is S -semipermutable in G .

Clearly every S -semipermutable subgroup of G is weakly S -semipermutable in G , but the converse does not hold: for example, let $G = S_4$ (symmetric group of degree 4) and $H = \langle (12) \rangle$. Then H is weakly S -semipermutable in G , but H is not S -semipermutable in G .

We prove that for a finite group G , if some cyclic subgroups or maximal subgroups of G are weakly S -semipermutable in G , then G is p -nilpotent.

2. Preliminaries

In this section, we give some results which will be useful in the sequel.

LEMMA 2.1 ([10, Lemma 2.1]). *Suppose that a subgroup H of a group G is S -permutable in G and N is a normal subgroup of G . Then the following hold:*

- (1) *If $H \leq K \leq G$, then H is S -permutable in K .*
- (2) *HN and $H \cap N$ are S -permutable in G , HN/N is S -permutable in G/N .*
- (3) *$H \cap K$ is S -permutable in K .*
- (4) *If H is a p -subgroup of G , then $H \subseteq O_p(G)$ and $O^p(G) \leq N_G(H)$.*
- (5) *H/H_G is nilpotent.*

LEMMA 2.2 ([10, Lemma 2.2]). *Suppose that a subgroup H of a group G is S -semipermutable in G and N is a normal subgroup of G . Then the following hold:*

- (1) *If $H \leq K \leq G$, then H is S -semipermutable in K .*
- (2) *If H is a p -subgroup for some prime $p \in \pi(G)$, then HN/N is S -semipermutable in G/N .*
- (3) *If $(|H|, |N|) = 1$, then HN/N is S -semipermutable in G/N .*
- (4) *If $H \leq O_p(G)$, then H is S -permutable in G .*

LEMMA 2.3 ([7, IV, Theorems 5.4 and 2.8]). *Suppose that G is a minimal non-nilpotent group. Then the following hold:*

- (1) *G has a normal Sylow p -subgroup G_p , for some prime p and $G = G_p Q$, where Q is a non-normal cyclic q -subgroup for some prime $q \neq p$.*
- (2) *$G_p/\Phi(G_p)$ is a minimal normal subgroup of $G/\Phi(G_p)$.*
- (3) *If G_p is non-abelian and $p > 2$, then the exponent of G_p is p ; if G_p is non-abelian and $p = 2$, then the exponent of G_p is 4.*
- (4) *If G_p is abelian, then the exponent of G_p is p .*

LEMMA 2.4 ([16, Lemma 2.2]). *Let G be a group and $p \in \pi(G)$ satisfying $(|G|, p - 1) = 1$. Then the following hold:*

- (1) *If N is a normal subgroup in G of order p , then N lies in $Z(G)$.*
- (2) *If G has cyclic Sylow p -subgroup, then G is p -nilpotent.*
- (3) *If M is a subgroup of G with index p , then M is normal in G .*

LEMMA 2.5 ([2, Lemma 1.2]). *Let H, K and T be subgroups of a group G . Then the following statements are equivalent:*

- (1) $H \cap KT = (H \cap K)(H \cap T)$.
- (2) $HK \cap HT = H(K \cap T)$.

LEMMA 2.6 ([6, Lemma 2.5]). *Let N be an elementary abelian normal subgroup of a group G . If there exists a subgroup D in N satisfying $1 < |D| < |N|$ and every subgroup H of G with $|H| = |D|$ is S -semipermutable in G , then there exists a maximal subgroup L of N which is normal in G .*

LEMMA 2.7. *Suppose that a subgroup H of a group G is weakly S -semipermutable in G and N is a normal subgroup of G . Then the following hold:*

- (1) *If $H \leq K \leq G$, then H is weakly S -semipermutable in K .*
- (2) *If $(|H|, |N|) = 1$, then HN/N is weakly S -semipermutable in G/N .*
- (3) *If $H \leq K \trianglelefteq G$, then G has a normal subgroup L contained in K such that HL is S -permutable in G and $H \cap L$ is S -semipermutable in G .*
- (4) *If H is a p -subgroup of G , then HN/N is weakly S -semipermutable in G/N .*
- (5) *If $N \leq U$ and U/N is weakly S -semipermutable in G/N , then U is weakly S -semipermutable in G .*

PROOF. (1) Since H is weakly S -semipermutable in G , then there exists a normal subgroup T of G such that HT is S -permutable and $H \cap T$ is S -semipermutable in G . Let $T' = T \cap K$. Then $T' \triangleleft K$, $HT' = H(T \cap K) = HT \cap K$ is S -permutable in K by Lemma 2.1, and $H \cap T' = H \cap (T \cap K) = (H \cap T) \cap K = H \cap T$ is S -semipermutable in K by Lemma 2.2. Thus, H is weakly S -semipermutable in K .

(2) Since H is weakly S -semipermutable in G , there exists a normal subgroup T of G such that HT is S -permutable and $H \cap T$ is S -semipermutable in G . Let $T'/N = TN/N$. Then $T'/N \triangleleft G/N$, $(H/N)(T'/N) = HTN/N$ is S -permutable, and $(HN/N) \cap (T'/N) = (HN \cap TN)/N = (H \cap TN)N/N = (H \cap T)N/N$ is S -semipermutable in G/N by Lemma 2.2. Hence, HN/N is weakly S -semipermutable in G/N .

(3) Since H is weakly S -semipermutable in G , there exists a normal subgroup T of G such that HT is S -permutable and $H \cap T$ is S -semipermutable in G . Let $T' = T \cap K$. Then $T' \triangleleft G$, $HT' = H(T \cap K) = HT \cap K$ is S -permutable in G by Lemma 2.1, and $H \cap T' = H \cap (T \cap K) = H \cap T$ is S -semipermutable in G .

(4) Since H is weakly S -semipermutable in G , there exists a normal subgroup T of G such that HT is S -permutable and $H \cap T$ is S -semipermutable in G . Since $TN/N \triangleleft G/N$ and HT is S -permutable in G , we have that $(H/N)(TN/N)$ is S -permutable in G/N by Lemma 2.1. Since $(H \cap T)N/N$ is S -semipermutable in G/N , we have that $(HN/N) \cap (TN/N) = (HN \cap TN)/N = (H \cap T)N/N$ is S -semipermutable in G/N .

(5) Apply the same arguments as in part (4). ■

LEMMA 2.8. *Let L be a minimal normal and elementary abelian subgroup of G . Then L has not a proper subgroup D with $1 < |D| < |L|$, such that every subgroup H of L with $|H| = |D|$ is weakly S -semipermutable in G .*

PROOF. Let L have a proper subgroup D with $1 < |D| < |L|$ such that every subgroup H of L with $|H| = |D|$ is weakly S -semipermutable in G . Then there exists a normal subgroup T of G with $T \leq L$ such that HT is S -permutable and $H \cap T$ is S -semipermutable in G by Lemma 2.7. If $T = L$, then $H = H \cap T$ is S -semipermutable in G . If $T = 1$, then $H = HT$ is S -permutable in G . Therefore L has a maximal subgroup which is a normal subgroup of G , a contradiction by Lemma 2.6 and the minimality of L . ■

LEMMA 2.9 ([15, Theorem C]). *Let E be a normal subgroup of a group G . If $F^*(E)$ is hypercyclically embedded in G , then E is hypercyclically embedded in G .*

LEMMA 2.10 ([5, Lemma 2.2]). *Let N be a non-identity normal p -subgroup of a group G . Then the following hold:*

- (1) *If N is elementary and every maximal subgroup of N is S -embedded in G , then some maximal subgroup of N is normal in G .*
- (2) *If N is a group of exponent p and every minimal subgroup of N is S -embedded in G , then $N \leq Z_{\mathcal{U}}(G)$.*

LEMMA 2.11 ([15, Lemma 2.2]). *Let E be a normal p -subgroup of a group G . If $E \leq Z_{\mathcal{U}}(G)$, then $(G/C_G(E))^{\mathcal{A}(p-1)} \leq O_p(G/C_G(E))$.*

LEMMA 2.12 ([14, Lemma 2.16]). *Let \mathfrak{F} be a saturated formation containing \mathcal{U} and G be a group with normal subgroup E such that $G/E \in \mathfrak{F}$. If E is cyclic, then $G \in \mathfrak{F}$.*

3. Main results

THEOREM 3.1. *Let G be a group and p be the smallest prime divisor of $|G|$. If G_p is a Sylow p -subgroup of G such that every maximal subgroup of G_p is weakly S -semipermutable in G , then G is soluble.*

THEOREM 3.2. *Let G be a group, $p \in \pi(G)$ with $(|G|, p-1) = 1$, and G_p be a Sylow p -subgroup of G . If every maximal subgroup of G_p is weakly S -semipermutable in G , then G is p -nilpotent.*

THEOREM 3.3. *Let G be a group, $p \in \pi(G)$ with $(|G|, p - 1) = 1$, and G_p be a Sylow p -subgroup of G . If every cyclic subgroup of G_p which does not have any p -nilpotent supplement in G with order p or 4 (G_2 is a non-abelian 2-group) is weakly S -semipermutable in G , then G is p -nilpotent.*

THEOREM 3.4. *Let G be a group and N be a normal subgroup of G . If there exists a normal subgroup L of G such that $F^*(N) \leq L \leq N$ and for any non-cyclic Sylow p -subgroup L_p of L , either every maximal subgroup of L_p or every cyclic subgroup of L_p with order p or 4 (L_2 is a non-abelian 2-group) is weakly S -semipermutable in G , then $N \leq Z_{\mathfrak{U}}(G)$ (i.e., each G -chief factor below N is cyclic).*

THEOREM 3.5. *Let \mathfrak{F} be a saturated formation containing \mathfrak{U} . If G has a normal subgroup N such that $G/N \in \mathfrak{F}$ and there exists a normal subgroup L of G such that $F^*(N) \leq L \leq N$ and for any non-cyclic Sylow p -subgroup L_p of L , either every maximal subgroup of L_p or every cyclic subgroup of L_p with order p or 4 (L_2 is a non-abelian 2-group) is weakly S -semipermutable in G , then $G \in \mathfrak{F}$.*

PROOF OF THEOREM 3.1. If $p \neq 2$, then G is soluble by Feit–Thompson’s theorem. So let $p = 2$, suppose that the theorem is false and consider a counterexample G of minimal order.

Step 1. $O_p(G) = 1$ and $O_{p'}(G) = 1$.

Suppose that $O_p(G) \neq 1$. It is clear that $|G_p/O_p(G)| \geq 2$. Let $P_1/O_p(G)$ be a maximal subgroup of $G_p/O_p(G)$. Then P_1 is a maximal subgroup of G_p , therefore P_1 is weakly S -semipermutable in G . Then $G/O_p(G)$ is soluble by Lemma 2.7 and the choice of G , hence G is soluble, a contradiction. Suppose that $O_{p'}(G) \neq 1$. Let $M/O_{p'}(G)$ be a maximal subgroup of $G_p O_{p'}(G)/O_{p'}(G)$, $M = P_1 O_{p'}(G)$ for some maximal subgroup P_1 of G_p . So, by the hypothesis of the theorem and Lemma 2.7 (2), $M/O_{p'}(G)$ is weakly S -semipermutable in $G/O_{p'}(G)$. Thus, $G/O_{p'}(G)$ is soluble by our choice of G and hence G is soluble, a contradiction.

Step 2. Final contradiction.

Let N be a minimal normal subgroup of G . If $N \cap G_p \leq \Phi(G_p)$, then N is p -nilpotent by [7, IV, Theorem 4.7]. Let $N_{p'}$ be a normal p -complement of N . Then $N_{p'} \leq O_{p'}(G) = 1$, N is a p -subgroup of G . This is a contradiction as $O_p(G) = 1$ by Step 1. Thus $N \cap G_p \not\leq \Phi(G_p)$. Then there exists a maximal subgroup M of G_p such that $G_p = M(G_p \cap N)$. There exists a normal subgroup T of G such that MT is S -permutable in G and $M \cap T$ is S -semipermutable in G . If $T = 1$, then M is S -permutable in G . Then by Lemma 2.1 (4), $M \leq O_p(G) = 1$, thus $|G_p| = 2$. By [13, Theorem 10.1.9], G is 2-nilpotent. Then G is soluble, a contradiction.

Hence $T \neq 1$ and $N \leq T$, $N \cap M = N \cap M \cap T$ is S -semipermutable in G , thus for every Sylow q -subgroup N_q of N , $N_q(N \cap M) = (N \cap M)N_q$ and therefore $(N \cap M)N_q$ is a subgroup of G . Then there exists a proper normal subgroup N' of N such that either $N \cap M \leq N'$ or $N_q < N'$ by Lemma 2.1. If $N_q \leq N'$, then $N = N'$ by [7, I, Theorem 9.12], a contradiction. Thus $N \cap M \leq N' \cap M$ and $|N/N'|_2 = |N|_2/|N'|_2 = |G_2 \cap N|/|G_2 \cap N'| \leq |G_2 \cap N|/|M \cap N| \leq 2$. Hence N/N' is p -nilpotent by [13, Theorem 10.1.9]. Then N/N' is soluble, so N/N' is an elementary p -group, therefore N is a p -group, a contradiction. ■

PROOF OF THEOREM 3.2. Suppose that the theorem is false and consider a counterexample G of minimal order.

Step 1. G is soluble.

G is soluble by Feit–Thompson’s theorem and Theorem 3.1.

Step 2. G_p is not cyclic and G is not a non-abelian simple group.

G has a non-cyclic Sylow p -subgroup G_p by Lemma 2.4, thus G_p is not cyclic. Let G be a non-abelian simple group and G_p^* be a maximal subgroup of G_p . Then there exists a normal subgroup T of G such that G_p^*T is S -permutable in G and $G_p^* \cap T$ is S -semipermutable in G . If $T = 1$, then G_p^* is a proper subnormal subgroup of G , a contradiction. If $T = G$, we also get a contradiction.

Step 3. There exists a unique minimal normal subgroup N of G such that G/N is p -nilpotent and $G = N \rtimes T$ where T is a maximal subgroup of G .

Assume that N is a minimal normal subgroup of G and $G_p \in \text{Syl}_p(G)$. Hence G_pN/N is a Sylow p -subgroup of G/N . Let M/N be a maximal subgroup of G_pN/N . Then there exists a maximal subgroup P_1 of G_p such that $M = P_1N$ and $G_p \cap N = P_1 \cap N$ is a Sylow p -subgroup of N . Since P_1 is weakly S -semipermutable in G , there exists a normal subgroup K of G such that P_1K is S -permutable and $P_1 \cap K$ is S -semipermutable in G . KN/N is a normal subgroup of G/N . P_1KN/N is S -permutable in G/N by Lemma 2.1, so $(M/N)(KN/N) = (P_1N/N)(KN/N) = P_1KN/N$ is S -permutable in G/N . $(P_1 \cap K)N/N$ is S -semipermutable in G/N by Lemma 2.2. $(P_1 \cap N)(K \cap N) = P_1K \cap N$, hence $P_1N \cap KN = (P_1 \cap K)N$ by Lemma 2.5, thus $M/N \cap K/N = P_1N/N \cap KN/N = (P_1N \cap KM)/N$ is S -semipermutable in G/N . Then M/N is weakly S -semipermutable in G/N , therefore G/N is p -nilpotent by the choice of G . Since the class of all p -nilpotent groups is a saturated formation, N is a unique minimal normal subgroup of G and $N \not\leq \Phi(G)$. Then there exists a maximal subgroup T of G such that $G = N \rtimes T$ and $C_G(N) \cap T = 1$, thus $N = C_G(N)$.

Step 4. $O_{p'}(G) = 1$.

Let $O_{p'}(G) \neq 1$. Then $G/O_{p'}(G)$ is p -nilpotent by the choice of G , hence G is p -nilpotent, which is a contradiction.

Step 5. $O_p(G) \neq 1$ and $O_p(G) = N$.

Let $O_p(G) = 1$. Then $N \leq O_{p'}(G) = 1$, thus $G = G/N$ is p -nilpotent by Step 3, a contradiction. There exists a normal subgroup L of G such that $G = LN$ and $L \cap N = 1$. Since $O_p(G) \cap L$ is normalized by N and L , thus $O_p(G) \cap L$ is normalized by $G = LN$. This implies $O_p(G) = N$.

Step 6. Final contradiction.

We have $G_p = N \rtimes (G_p \cap T)$ by Steps 3 and 4. Then there exists a maximal subgroup P' of G_p such that $G_p \cap T \leq P'$, $G_p = NP'$. There is a normal subgroup L of G such that $P'L$ is S -permutable and $P' \cap L$ is S -semipermutable in G . Since N is minimal normal in G , we have $N \leq (P')^G = (P')^{O^p(G)G_p}$ by $G = O^p(G)G_p$. If $L = 1$, then P' is S -permutable in G and $N \leq (P')^{G_p} = P'$ by Lemma 2.1. Since $G_p = NP'$, we have $G_p = P'$, a contradiction. Hence $L \neq 1$ and $N \leq L$. Then $N \cap P' = N \cap (P' \cap L)$ is S -semipermutable in G by Lemma 2.2. If $N \cap P' \neq 1$, then $N \leq (N \cap P')^G = (N \cap P')^{O^p(G)G_p} \leq (P')^{G_p} = P'$, a contradiction. Thus $N \cap P' = 1$ and $|N| = p$. Then $G/N = G/C_G(N) \cong A \leq \text{Aut}(N)$ and $(|G|, p-1) = 1$ lead to $G = N$, a contradiction. ■

PROOF OF THEOREM 3.3. Suppose that the theorem is false and consider a counterexample G of minimal order.

Step 1. G is a minimal non-nilpotent group and G_p is non-cyclic.

Clearly G is non-nilpotent. If G_p is cyclic, then G is p -nilpotent by Lemma 2.4, a contradiction. Hence G_p is non-cyclic. Let $M \not\leq G$ and L be a cyclic subgroup of $G_p \cap M$ with order p or 4 (G_2 is a non-abelian 2-group) which does not have any p -nilpotent supplement in M , hence L has no p -nilpotent supplement in G . Therefore L is weakly S -semipermutable in G by hypothesis. Thus L is weakly S -semipermutable in M by Lemma 2.7. Hence M is p -nilpotent, by the choice of G . Then G is minimal non-nilpotent and G satisfies the results of Lemma 2.3.

Step 2. There exists a minimal normal subgroup $L/\Phi(G_p)$ of $G_p/\Phi(G_p)$ such that $L/\Phi(G_p)$ is not S -permutable in $G/\Phi(G_p)$.

If all minimal normal subgroups of $G_p/\Phi(G_p)$ are S -permutable in $G/\Phi(G_p)$, we get a contradiction by Lemma 2.8. Hence $G_p/\Phi(G_p)$ has a minimal normal subgroup $L/\Phi(G_p)$ such that $L/\Phi(G_p)$ is not S -permutable in $G/\Phi(G_p)$.

Step 3. Final contradiction.

Let $x \in L \setminus \Phi(G_p)$. Then $\langle x \rangle$ is a cyclic group of order p or 4. Suppose that T is an arbitrary supplement of $\langle x \rangle$ in G . Then $G_p = G_p \cap \langle x \rangle T = \langle x \rangle (G_p \cap T)$. Since $G_p/\Phi(G_p)$ is a normal subgroup of $G/\Phi(G_p)$, we have $((G_p \cap T)\Phi(G_p))/\Phi(G_p) \triangleleft G/\Phi(G_p)$, then $(G_p \cap T)\Phi(G_p) \triangleleft G$. The group $G_p/\Phi(G_p)$ is a minimal normal subgroup of $G/\Phi(G_p)$ by Lemma 2.3, hence $(G_p \cap T) \leq \Phi(G_p)$ or $(G_p \cap T) = G_p$, as $G_p/\Phi(G_p)$ is a chief factor of G . If $(G_p \cap T) \leq \Phi(G_p)$, then $G_p = \langle x \rangle$, a contradiction. If $(G_p \cap T) = G_p$, then $T = G$ and T is a unique supplement of $\langle x \rangle$ in G . A cyclic subgroup $\langle x \rangle$ is weakly S -semipermutable in G by Step 2. Then there exists a normal subgroup K of G with $K \leq G_p$ such that $K\langle x \rangle$ is S -permutable and $\langle x \rangle \cap K$ is S -semipermutable in G by Lemma 2.7. Then $\langle x \rangle \cap K$ is S -permutable in G by Lemma 2.2. Since $G_p/\Phi(G_p)$ is a chief factor of G , we have $K \leq \Phi(G_p)$ or $K = G_p$. If $K \leq \Phi(G_p)$, then $L/\Phi(G_p) = (\langle x \rangle K \Phi(G_p))/\Phi(G_p)$ is S -permutable in $G/\Phi(G_p)$, a contradiction. If $K = G_p$, then $\langle x \rangle = \langle x \rangle \cap K$ is S -permutable in G , and $L/\Phi(G_p) = (\langle x \rangle \Phi(G_p))/\Phi(G_p)$ is S -permutable in $G/\Phi(G_p)$, a contradiction. ■

PROOF OF THEOREM 3.4. Since $F^*(N) \leq L$, it is sufficient to show that $L \leq Z_{\mathbb{U}}(G)$ by Lemma 2.9. Suppose that the theorem is false and consider a counterexample (G, L) for which $|G||L|$ is minimal. Let $L_p \in \text{Syl}_p(L)$ where p is the smallest prime number dividing $|L|$. By [7, IV, Theorem 2.8], L_p is not cyclic.

Step 1. L is p -nilpotent.

Since L_p is not cyclic, either every maximal subgroup of L_p or any cyclic subgroup of L_p , with order p or 4 (L_2 is a non-abelian 2-group) is weakly S -semipermutable in L by Lemma 2.7. Then L is p -nilpotent by Theorems 3.2 and 3.3.

Step 2. $L = L_p$.

L is p -nilpotent by Step 1. Therefore there exists a normal p -complement subgroup K of L , thus $K \triangleleft G$. If $K \neq 1$, then $L/K \leq Z_{\mathbb{U}}(G/K)$. Since $K \leq Z_{\mathbb{U}}(G)$, we have $L \leq Z_{\mathbb{U}}(G)$ by the choice of (G, L) , thus $N \leq Z_{\mathbb{U}}(G)$, which is a contradiction. So $K = 1$ and then $L = L_p$.

Step 3. L_p is not a minimal normal subgroup of G and every cyclic subgroup of L_p with order p or 4 (L_2 is a non-abelian 2-group) is S -embedded in G .

Assume that L_p is a minimal normal subgroup of G . Thus L_p is a normal elementary abelian p -subgroup of G by Step 2. Let M be a maximal subgroup of L_p . Thus ML_p is S -permutable in G and $M \cap L_p = M \leq L_p = (L_p)_{sG}$. Then M is S -embedded in G . Therefore L_p has a maximal subgroup such that it is normal in G by Lemma 2.10, a contradiction. This contradiction shows that L_p is not a minimal normal subgroup of G . Assume that any maximal subgroup M of L_p is weakly S -

semipermutable in G . Hence, for any minimal normal subgroup H of G with $H \leq L_p$, $L_p/H \leq Z_{\mathcal{U}}(G/H)$ by Lemma 2.7 and the choice of (G, L) . Therefore H is a unique minimal normal subgroup of G with $H \leq L_p$ and $|H| > p$. Now we consider two cases: $\Phi(L_p) = 1$, and $\Phi(L_p) \neq 1$. If $\Phi(L_p) = 1$, then L_p is an elementary abelian p -subgroup of G . Let H_1 be a maximal subgroup of H , let B be the complement of H in L_p , and let $L_1 = H_1B$. Then L_1 is a maximal subgroup of L_p , therefore L_1 is weakly S -semipermutable in G , thus there exists $T \triangleleft G$ with $T \leq L_p$ such that TL_1 is S -permutable and $T \cap L_p$ is S -semipermutable in G by Lemma 2.7. If $T = 1$, then $L_1 = TL_1$ is S -permutable in G , hence $H_1 = H_1(B \cap H) = H_1B \cap H = L_1 \cap H$ is S -permutable in G by Lemma 2.1. If $T = L_p$, then $L_1 = T \cap L_1$ is S -semipermutable in G , hence L_1 is S -permutable in G by Lemma 2.2, hence $H_1 = H_1(B \cap H) = H_1B \cap H = L_1 \cap H$ is S -permutable in G by Lemma 2.1. If $1 < T < L_p$, then $H \leq T$, hence $H_1 \leq T$, thus $H_1 \leq L_1 \cap T \leq (L_1)_{sG}$, since $H \cap L_1 = H_1$ and $H_1 = H \cap (L_1)_{sG}$ is S -permutable in G . Therefore every maximal subgroup of H is S -permutable in G . It is a contradiction by Lemma 2.8. Hence $\Phi(L_p) \neq 1$ and $H \leq \Phi(L_p)$. Thus $L_p/H \leq Z_{\mathcal{U}}(G/H)$ by the choice of (G, L) . Then $(G/C_G(L_p/\Phi(L_p)))^{\mathcal{A}(p-1)}$ is a p -group by Lemma 2.11. Hence $(G/C_G(H))^{\mathcal{A}(p-1)}$ is a p -group by [4, Theorem 5.1.4]. Therefore $O_p(G/C_G(H)) = 1$ by [17, Corollary 6.4] and $(G/C_G(H)) \in \mathcal{A}(p-1)$. Then $|H| = p$ by [17, I, Theorem 1.4]. This is a contradiction by $|H| > p$. Therefore every cyclic subgroup K of L_p with order p or 4 (L_2 is a non-abelian 2-group) is weakly S -semipermutable in G , thus K is S -embedded in G .

Step 4. There exists $R \triangleleft G$ with $R \leq L_p$, L_p/R is a non-cyclic chief factor of G , $R \leq Z_{\mathcal{U}}(G)$ and $V \leq R$ for any normal subgroup $V \neq R$ of G contained in L_p .

Let L_p/R be a chief factor of G . Then $R \neq 1$ by Step 3, hence $R \leq Z_{\mathcal{U}}(G)$ by the choice of (G, L) . Therefore $L \leq Z_{\mathcal{U}}(G)$ and we know that L_p/R is not cyclic. Let $V \triangleleft G$ with $V \not\leq L_p$. Then $V \leq Z_{\mathcal{U}}(G)$ by the choice of (G, L) . If $V \not\leq R$, then $L_p \leq Z_{\mathcal{U}}(G)$ by the G -isomorphism $L_p/R = VR/R \cong V/(V \cap R)$, a contradiction by the choice of (G, L) , therefore $V \leq R$.

Let L_p be a non-abelian 2-subgroup of G . We use $\Omega := \Omega_2(L_p)$, otherwise, let $\Omega := \Omega_1(L_p)$.

Step 5. $C_G(\Omega)/C_G(H)$ is a p -group.

It is true by [2, A, Lemma 1.2].

Step 6. $\Omega \not\leq Z_{\mathcal{U}}(G)$.

Let $\Omega \leq Z_{\mathcal{U}}(G)$. Then $(G/C_G(\Omega))^{\mathcal{A}(p-1)}$ is a p -group by Lemma 2.11. Hence $(G/C_G(L_p/R)) \in \mathcal{A}(p-1)$ and so $|L_p/R| = p$ by Step 5, a contradiction. Hence $\Omega \not\leq Z_{\mathcal{U}}(G)$.

Step 7. Final contradiction.

$\Omega = L_p$ by Steps 4 and 6. Let W_1, W_2, \dots, W_m be the set of all cyclic subgroups of L_p with order p or 4 (L_2 is a non-abelian 2-group) and $W_i \not\leq R$. Then W_i is S -embedded in G for $i = 1, 2, \dots, m$. Then there exists $T_i < G$ with $T_i \leq L_p$ such that $W_i T_i$ is S -permutable in G and $W_i \cap T_i \leq (W_i)_{sG}$ for $i = 1, 2, \dots, m$. Since $R < G$, if W_i is S -permutable in G , then $W_i R/R$ is S -permutable in G/R . Suppose that W_i is not S -permutable in G , then $1 < T_i < L_p$. Thus $T_i \leq R$ and $W_i R = W_i T_i R$ is S -permutable in G . Therefore $W_i R$ is S -permutable in G in all of the cases, thus $W_i R/R$ is S -permutable in G/R in all of the cases. This is a contradiction by Lemma 2.8, since $(L_p/R) = (W_1 R/R)(W_2 R/R) \dots (W_m R/R)$ is an elementary abelian group. ■

PROOF OF THEOREM 3.5. We have $N \leq Z_{\mathfrak{U}}(G)$ by Theorem 3.4. Since $G/N \in \mathfrak{F}$, Lemma 2.12 implies $G \in \mathfrak{F}$. ■

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