Rend. Sem. Mat. Univ. Padova 150 (2023), 137–148
DOI 10.4171/RSMUP/112

Weakly S-semipermutable subgroups and p-nilpotency of groups

Hassan Jafarian Dehkordy (*) – Gholamreza R. Rezaeezadeh (**)

Abstract – A subgroup H of a finite group G is said to be S-semipermutable in G if $HG_p = G_p H$ for every Sylow subgroup G_p of G with $(|H|, p) = 1$. A subgroup H of G is said to be weakly S-semipermutable in G if there exists a normal subgroup T of G such that HT is S-permutable and $H \cap T$ is S-semipermutable in G. In this paper we prove that for a finite group G, if some cyclic subgroups or maximal subgroups of G are weakly S-semipermutable in G, then G is p-nilpotent.

Mathematics Subject Classification (2020) – Primary 20D15; Secondary 20D20, 20F19, 20D10.

Keywords – S-permutable, S-semipermutable, weakly S-semipermutable, p-nilpotent.

1. Introduction

All groups considered in this paper are finite. For a group G, let $\pi(G)$ denote the set of all prime divisors of $|G|$; G_p a Sylow p-subgroup of G for some $p \in \pi(G)$; $F(G)$ the Fitting subgroup of G; $F^*(G)$ the generalized Fitting subgroup of G; $\Phi(G)$ the Frattini subgroup of G. Let $\mathfrak{F}$ be a class of groups, $\mathfrak{F}$ is said to be a formation, provided that if $G \in \mathfrak{F}$ and $H \trianglelefteq G$, then $G/H \in \mathfrak{F}$, and for any normal subgroups N, M of G, if G/N and G/M are in $\mathfrak{F}$, then $G/(M \cap N) \in \mathfrak{F}$. The formation $\mathfrak{F}$ is said to be saturated if $G/\Phi(G) \in \mathfrak{F}$ implies that $G \in \mathfrak{F}$. In this paper, $\mathfrak{U}$ will denote the class of all supersoluble groups. As is well known, $\mathfrak{U}$ is a saturated formation. $Z_\infty(G)$ is the

(*) *Indirizzo dell'A.*: Faculty of Mathematical Sciences, Shahrekord University, Shahrekord, Iran; h.jafarian.d@sku.ac.ir

(**) *Indirizzo dell'A.*: Faculty of Mathematical Sciences, Shahrekord University, Shahrekord, Iran; rezaeezadeh@sku.ac.ir

hypercenter of G. A normal subgroup H of G is said to be hypercyclically embedded in G if every chief factor of G below H is cyclic. $Z_{\mathfrak{U}}(G)$ is said to be a $\mathfrak{U}$-hypercenter of G and is the product of all hypercyclically embedded subgroups of G.

Let H and K be two subgroups of G. The product HK of these subgroups is a subgroup of G if and only if $HK = KH$. The subgroup H is said to be quasinormal or permutable [12] in G if H permutes with all subgroups of G, and S-permutable or S-quasinormal or π-quasinormal [9] in G if H permutes with all Sylow subgroups of G. It is observed by Ore [12] that every permutable subgroup of a finite group G is subnormal in G. By extending this result, Ito and Szép have proved in [8] that for every permutable subgroup H of a finite group G, H/H_G is nilpotent. Here, H_G is the Core of H, which is the largest normal subgroup of G contained in H. Maier and Schmid proved in [11] that for every permutable subgroup H of G it is true that $(H^G/H_G) \subseteq Z_\infty(G/H_G)$. Here, H^G is the normal closure of H in G, which is the intersection of all normal subgroups of G that contain H.

A subgroup H of G is said to be S-semipermutable [1] in G, if $HG_p = G_pH$ for every Sylow subgroup G_p of G with $(|H|, p) = 1$.

Clearly, every S-permutable subgroup of G is S-semipermutable in G but the converse does not hold; for example, a Sylow 2-subgroup of S_3 (the symmetric group of degree 3) is semipermutable, thus S-semipermutable in S_3, but it is not S-permutable in S_3.

H is said to be an S-embedded subgroup [3] of G if G has normal subgroup T such that HT is S-permutable in G and $H \cap T \leq H_{sG}$, where H_{sG} denotes the subgroup generated by all subgroups of H that are S-permutable in G.

Definition. A subgroup H of a finite group G is called *weakly S-semipermutable* in G if there exists a normal subgroup T of G such that HT is S-permutable and $H \cap T$ is S-semipermutable in G.

Clearly every S-semipermutable subgroup of G is weakly S-semipermutable in G, but the converse does not hold: for example, let $G = S_4$ (symmetric group of degree 4) and $H = \langle (12) \rangle$. Then H is weakly S-semipermutable in G, but H is not S-semipermutable in G.

We prove that for a finite group G, if some cyclic subgroups or maximal subgroups of G are weakly S-semipermutable in G, then G is p-nilpotent.

2. Preliminaries

In this section, we give some results which will be useful in the sequel.

Lemma 2.1 ([10, Lemma 2.1]). *Suppose that a subgroup H of a group G is S-permutable in G and N is a normal subgroup of G. Then the following hold:*

(1) *If $H \leq K \leq G$, then H is S-permutable in K.*

(2) *HN and $H \cap N$ are S-permutable in G, HN/N is S-permutable in G/N.*

(3) *$H \cap K$ is S-permutable in K.*

(4) *If H is a p-subgroup of G, then $H \subseteq O_p(G)$ and $O^p(G) \leq N_G(H)$.*

(5) *H/H_G is nilpotent.*

Lemma 2.2 ([10, Lemma 2.2]). *Suppose that a subgroup H of a group G is S-semipermutable in G and N is a normal subgroup of G. Then the following hold:*

(1) *If $H \leq K \leq G$, then H is S-semipermutable in K.*

(2) *If H is a p-subgroup for some prime $p \in \pi(G)$, then HN/N is S-semipermutable in G/N.*

(3) *If $(|H|, |N|) = 1$, then HN/N is S-semipermutable in G/N.*

(4) *If $H \leq O_p(G)$, then H is S-permutable in G.*

Lemma 2.3 ([7, IV, Theorems 5.4 and 2.8]). *Suppose that G is a minimal non-nilpotent group. Then the following hold:*

(1) *G has a normal Sylow p-subgroup G_p, for some prime p and $G = G_pQ$, where Q is a non-normal cyclic q-subgroup for some prime $q \neq p$.*

(2) *$G_p/\Phi(G_p)$ is a minimal normal subgroup of $G/\Phi(G_p)$.*

(3) *If G_p is non-abelian and $p > 2$, then the exponent of G_p is p; if G_p is non-abelian and $p = 2$, then the exponent of G_p is 4.*

(4) *If G_p is abelian, then the exponent of G_p is p.*

Lemma 2.4 ([16, Lemma 2.2]). *Let G be a group and $p \in \pi(G)$ satisfying $(|G|, p - 1) = 1$. Then the following hold:*

(1) *If N is a normal subgroup in G of order p, then N lies in $Z(G)$.*

(2) *If G has cyclic Sylow p-subgroup, then G is p-nilpotent.*

(3) *If M is a subgroup of G with index p, then M is normal in G.*

Lemma 2.5 ([2, Lemma 1.2]). *Let H, K and T be subgroups of a group G. Then the following statements are equivalent:*

(1) *$H \cap KT = (H \cap K)(H \cap T)$.*

(2) *$HK \cap HT = H(K \cap T)$.*

Lemma 2.6 ([6, Lemma 2.5]). *Let N be an elementary abelian normal subgroup of a group G. If there exists a subgroup D in N satisfying $1 < |D| < |N|$ and every subgroup H of G with $|H| = |D|$ is S-semipermutable in G, then there exists a maximal subgroup L of N which is normal in G.*

Lemma 2.7. *Suppose that a subgroup H of a group G is weakly S-semipermutable in G and N is a normal subgroup of G. Then the following hold:*

(1) *If $H \leq K \leq G$, then H is weakly S-semipermutable in K.*

(2) *If $(|H|, |N|) = 1$, then HN/N is weakly S-semipermutable in G/N.*

(3) *If $H \leq K \trianglelefteq G$, then G has a normal subgroup L contained in K such that HL is S-permutable in G and $H \cap L$ is S-semipermutable in G.*

(4) *If H is a p-subgroup of G, then HN/N is weakly S-semipermutable in G/N.*

(5) *If $N \leq U$ and U/N is weakly S-semipermutable in G/N, then U is weakly S-semipermutable in G.*

Proof. (1) Since H is weakly S-semipermutable in G, then there exists a normal subgroup T of G such that HT is S-permutable and $H \cap T$ is S-semipermutable in G. Let $T' = T \cap K$. Then $T' \lhd K$, $HT' = H(T \cap K) = HT \cap K$ is S-permutable in K by Lemma 2.1, and $H \cap T' = H \cap (T \cap K) = (H \cap T) \cap K = H \cap T$ is S-semipermutable in K by Lemma 2.2. Thus, H is weakly S-semipermutable in K.

(2) Since H is weakly S-semipermutable in G, there exists a normal subgroup T of G such that HT is S-permutable and $H \cap T$ is S-semipermutable in G. Let $T'/N = TN/N$. Then $T'/N \lhd G/N$, $(H/N)(T'/N) = HTN/N$ is S-permutable, and $(HN/N) \cap (T'/N) = (HN \cap TN)/N = (H \cap TN)N/N = (H \cap T)N/N$ is S-semipermutable in G/N by Lemma 2.2. Hence, HN/N is weakly S-semipermutable in G/N.

(3) Since H is weakly S-semipermutable in G, there exists a normal subgroup T of G such that HT is S-permutable and $H \cap T$ is S-semipermutable in G. Let $T' = T \cap K$. Then $T' \lhd G$, $HT' = H(T \cap K) = HT \cap K$ is S-permutable in G by Lemma 2.1, and $H \cap T' = H \cap (T \cap K) = H \cap T$ is S-semipermutable in G.

(4) Since H is weakly S-semipermutable in G, there exists a normal subgroup T of G such that HT is S-permutable and $H \cap T$ is S-semipermutable in G. Since $TN/N \lhd G/N$ and HT is S-permutable in G, we have that $(HN/N)(TN/N)$ is S-permutable in G/N by Lemma 2.1. Since $(H \cap T)N/N$ is S-semipermutable in G/N, we have that $(HN/N) \cap (TN/N) = (HN \cap TN)/N = (H \cap T)N/N$ is S-semipermutable in G/N.

(5) Apply the same arguments as in part (4). ■

LEMMA 2.8. *Let L be a minimal normal and elementary abelian subgroup of G. Then L has not a proper subgroup D with $1 < |D| < |L|$, such that every subgroup H of L with $|H| = |D|$ is weakly S-semipermutable in G.*

PROOF. Let L have a proper subgroup D with $1 < |D| < |L|$ such that every subgroup H of L with $|H| = |D|$ is weakly S-semipermutable in G. Then there exists a normal subgroup T of G with $T \leq L$ such that HT is S-permutable and $H \cap T$ is S-semipermutable in G by Lemma 2.7. If $T = L$, then $H = H \cap T$ is S-semipermutable in G. If $T = 1$, then $H = HT$ is S-permutable in G. Therefore L has a maximal subgroup which is a normal subgroup of G, a contradiction by Lemma 2.6 and the minimality of L. ∎

LEMMA 2.9 ([15, Theorem C]). *Let E be a normal subgroup of a group G. If $F^*(E)$ is hypercyclically embedded in G, then E is hypercyclically embedded in G.*

LEMMA 2.10 ([5, Lemma 2.2]). *Let N be a non-identity normal p-subgroup of a group G. Then the following hold:*

(1) *If N is elementary and every maximal subgroup of N is S-embedded in G, then some maximal subgroup of N is normal in G.*

(2) *If N is a group of exponent p and every minimal subgroup of N is S-embedded in G, then $N \leq Z_{\mathfrak{U}}(G)$.*

LEMMA 2.11 ([15, Lemma 2.2]). *Let E be a normal p-subgroup of a group G. If $E \leq Z_{\mathfrak{U}}(G)$, then $(G/C_G(E))^{\mathcal{A}(p-1)} \leq O_p(G/C_G(E))$.*

LEMMA 2.12 ([14, Lemma 2.16]). *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G be a group with normal subgroup E such that $G/E \in \mathfrak{F}$. If E is cyclic, then $G \in \mathfrak{F}$.*

3. Main results

THEOREM 3.1. *Let G be a group and p be the smallest prime divisor of $|G|$. If G_p is a Sylow p-subgroup of G such that every maximal subgroup of G_p is weakly S-semipermutable in G, then G is soluble.*

THEOREM 3.2. *Let G be a group, $p \in \pi(G)$ with $(|G|, p - 1) = 1$, and G_p be a Sylow p-subgroup of G. If every maximal subgroup of G_p is weakly S-semipermutable in G, then G is p-nilpotent.*

Theorem 3.3. *Let G be a group, $p \in \pi(G)$ with $(|G|, p-1) = 1$, and G_p be a Sylow p-subgroup of G. If every cyclic subgroup of G_p which does not have any p-nilpotent supplement in G with order p or 4 (G_2 is a non-abelian 2-group) is weakly S-semipermutable in G, then G is p-nilpotent.*

Theorem 3.4. *Let G be a group and N be a normal subgroup of G. If there exists a normal subgroup L of G such that $F^*(N) \leq L \leq N$ and for any non-cyclic Sylow p-subgroup L_p of L, either every maximal subgroup of L_p or every cyclic subgroup of L_p with order p or 4 (L_2 is a non-abelian 2-group) is weakly S-semipermutable in G, then $N \leq Z_{\mathfrak{U}}(G)$ (i.e., each G-chief factor below N is cyclic).*

Theorem 3.5. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. If G has a normal subgroup N such that $G/N \in \mathfrak{F}$ and there exists a normal subgroup L of G such that $F^*(N) \leq L \leq N$ and for any non-cyclic Sylow p-subgroup L_p of L, either every maximal subgroup of L_p or every cyclic subgroup of L_p with order p or 4 (L_2 is a non-abelian 2-group) is weakly S-semipermutable in G, then $G \in \mathfrak{F}$.*

Proof of Theorem 3.1. If $p \neq 2$, then G is soluble by Feit–Thompson's theorem. So let $p = 2$, suppose that the theorem is false and consider a counterexample G of minimal order.

Step 1. $O_p(G) = 1$ and $O_{p'}(G) = 1$.

Suppose that $O_p(G) \neq 1$. It is clear that $|G_p/O_p(G)| \geq 2$. Let $P_1/O_p(G)$ be a maximal subgroup of $G_p/O_p(G)$. Then P_1 is a maximal subgroup of G_p, therefore P_1 is weakly S-semipermutable in G. Then $G/O_p(G)$ is soluble by Lemma 2.7 and the choice of G, hence G is soluble, a contradiction. Suppose that $O_{p'}(G) \neq 1$. Let $M/O_{p'}(G)$ be a maximal subgroup of $G_p O_{p'}(G)/O_{p'}(G)$, $M = P_1 O_{p'}(G)$ for some maximal subgroup P_1 of G_p. So, by the hypothesis of the theorem and Lemma 2.7 (2), $M/O_{p'}(G)$ is weakly S-semipermutable in $G/O_{p'}(G)$. Thus, $G/O_{p'}(G)$ is soluble by our choice of G and hence G is soluble, a contradiction.

Step 2. Final contradiction.

Let N be a minimal normal subgroup of G. If $N \cap G_p \leq \Phi(G_p)$, then N is p-nilpotent by [7, IV, Theorem 4.7]. Let $N_{p'}$ be a normal p-complement of N. Then $N_{p'} \leq O_{p'}(G) = 1$, N is a p-subgroup of G. This is a contradiction as $O_p(G) = 1$ by Step 1. Thus $N \cap G_p \nleq \Phi(G_p)$. Then there exists a maximal subgroup M of G_p such that $G_p = M(G_p \cap N)$. There exists a normal subgroup T of G such that MT is S-permutable in G and $M \cap T$ is S-semipermutable in G. If $T = 1$, then M is S-permutable in G. Then by Lemma 2.1 (4), $M \leq O_p(G) = 1$, thus $|G_p| = 2$. By [13, Theorem 10.1.9], G is 2-nilpotent. Then G is soluble, a contradiction.

Hence $T \neq 1$ and $N \leq T$, $N \cap M = N \cap M \cap T$ is S-semipermutable in G, thus for every Sylow q-subgroup N_q of N, $N_q(N \cap M) = (N \cap M)N_q$ and therefore $(N \cap M)N_q$ is a subgroup of G. Then there exists a proper normal subgroup N' of N such that either $N \cap M \leq N'$ or $N_q < N'$ by Lemma 2.1. If $N_q \leq N'$, then $N = N'$ by [7, I, Theorem 9.12], a contradiction. Thus $N \cap M \leq N' \cap M$ and $|N/N'|_2 = |N|_2/|N'|_2 = |G_2 \cap N|/|G_2 \cap N'| \leq |G_2 \cap N|/|M \cap N| \leq 2$. Hence N/N' is p-nilpotent by [13, Theorem 10.1.9]. Then N/N' is soluble, so N/N' is an elementary p-group, therefore N is a p-group, a contradiction. ■

Proof of Theorem 3.2. Suppose that the theorem is false and consider a counterexample G of minimal order.

Step 1. G is soluble.

G is soluble by Feit–Thompson's theorem and Theorem 3.1.

Step 2. G_p is not cyclic and G is not a non-abelian simple group.

G has a non-cyclic Sylow p-subgroup G_p by Lemma 2.4, thus G_p is not cyclic. Let G be a non-abelian simple group and G_p^* be a maximal subgroup of G_p. Then there exists a normal subgroup T of G such that G_p^*T is S-permutable in G and $G_p^* \cap T$ is S-semipermutable in G. If $T = 1$, then G_p^* is a proper subnormal subgroup of G, a contradiction. If $T = G$, we also get a contradiction.

Step 3. There exists a unique minimal normal subgroup N of G such that G/N is p-nilpotent and $G = N \rtimes T$ where T is a maximal subgroup of G.

Assume that N is a minimal normal subgroup of G and $G_p \in \mathrm{Syl}_p(G)$. Hence G_pN/N is a Sylow p-subgroup of G/N. Let M/N be a maximal subgroup of G_pN/N. Then there exists a maximal subgroup P_1 of G_p such that $M = P_1N$ and $G_p \cap N = P_1 \cap N$ is a Sylow p-subgroup of N. Since P_1 is weakly S-semipermutable in G, there exists a normal subgroup K of G such that P_1K is S-permutable and $P_1 \cap K$ is S-semipermutable in G. KN/N is a normal subgroup of G/N. P_1KN/N is S-permutable in G/N by Lemma 2.1, so $(M/N)(KN/N) = (P_1N/N)(KN/N) = P_1KN/N$ is S-permutable in G/N. $(P_1 \cap K)N/N$ is S-semipermutable in G/N by Lemma 2.2. $(P_1 \cap N)(K \cap N) = P_1K \cap N$, hence $P_1N \cap KN = (P_1 \cap K)N$ by Lemma 2.5, thus $M/N \cap K/N = P_1N/N \cap KN/N = (P_1N \cap KM)/N$ is S-semipermutable in G/N. Then M/N is weakly S-semipermutable in G/N, therefore G/N is p-nilpotent by the choice of G. Since the class of all p-nilpotent groups is a saturated formation, N is a unique minimal normal subgroup of G and $N \not\leq \Phi(G)$. Then there exists a maximal subgroup T of G such that $G = N \rtimes T$ and $C_G(N) \cap T = 1$, thus $N = C_G(N)$.

Step 4. $O_{p'}(G) = 1$.

Let $O_{p'}(G) \neq 1$. Then $G/O_{p'}(G)$ is p-nilpotent by the choice of G, hence G is p-nilpotent, which is a contradiction.

Step 5. $O_p(G) \neq 1$ and $O_p(G) = N$.

Let $O_p(G) = 1$. Then $N \leq O_{p'}(G) = 1$, thus $G = G/N$ is p-nilpotent by Step 3, a contradiction. There exists a normal subgroup L of G such that $G = LN$ and $L \cap N = 1$. Since $O_p(G) \cap L$ is normalized by N and L, thus $O_p(G) \cap L$ is normalized by $G = LN$. This implies $O_p(G) = N$.

Step 6. Final contradiction.

We have $G_p = N \rtimes (G_p \cap T)$ by Steps 3 and 4. Then there exists a maximal subgroup P' of G_p such that $G_p \cap T \leq P'$, $G_p = NP'$. There is a normal subgroup L of G such that $P'L$ is S-permutable and $P' \cap L$ is S-semipermutable in G. Since N is minimal normal in G, we have $N \leq (P')^G = (P')^{O^p(G)G_p}$ by $G = O^p(G)G_p$. If $L = 1$, then P' is S-permutable in G and $N \leq (P')^{G_p} = P'$ by Lemma 2.1. Since $G_p = NP'$, we have $G_p = P'$, a contradiction. Hence $L \neq 1$ and $N \leq L$. Then $N \cap P' = N \cap (P' \cap L)$ is S-semipermutable in G by Lemma 2.2. If $N \cap P' \neq 1$, then $N \leq (N \cap P')^G = (N \cap P')^{O^p(G)G_p} \leq (P')^{G_p} = P'$, a contradiction. Thus $N \cap P' = 1$ and $|N| = p$. Then $G/N = G/C_G(N) \cong A \leq \mathrm{Aut}(N)$ and $(|G|, p-1) = 1$ lead to $G = N$, a contradiction. ∎

Proof of Theorem 3.3. Suppose that the theorem is false and consider a counterexample G of minimal order.

Step 1. G is a minimal non-nilpotent group and G_p is non-cyclic.

Clearly G is non-nilpotent. If G_p is cyclic, then G is p-nilpotent by Lemma 2.4, a contradiction. Hence G_p is non-cyclic. Let $M \lneqq G$ and L be a cyclic subgroup of $G_p \cap M$ with order p or 4 (G_2 is a non-abelian 2-group) which does not have any p-nilpotent supplement in M, hence L has no p-nilpotent supplement in G. Therefore L is weakly S-semipermutable in G by hypothesis. Thus L is weakly S-semipermutable in M by Lemma 2.7. Hence M is p-nilpotent, by the choice of G. Then G is minimal non-nilpotent and G satisfies the results of Lemma 2.3.

Step 2. There exists a minimal normal subgroup $L/\Phi(G_p)$ of $G_p/\Phi(G_p)$ such that $L/\Phi(G_p)$ is not S-permutable in $G/\Phi(G_p)$.

If all minimal normal subgroups of $G_p/\Phi(G_p)$ are S-permutable in $G/\Phi(G_p)$, we get a contradiction by Lemma 2.8. Hence $G_p/\Phi(G_p)$ has a minimal normal subgroup $L/\Phi(G_p)$ such that $L/\Phi(G_p)$ is not S-permutable in $G/\Phi(G_p)$.

Step 3. Final contradiction.

Let $x \in L \setminus \Phi(G_p)$. Then $\langle x \rangle$ is a cyclic group of order p or 4. Suppose that T is an arbitrary supplement of $\langle x \rangle$ in G. Then $G_p = G_p \cap \langle x \rangle T = \langle x \rangle (G_p \cap T)$. Since $G_p/\Phi(G_p)$ is a normal subgroup of $G/\Phi(G_p)$, we have $((G_p \cap T)\Phi(G_p)/\Phi(G_p)) \lhd G/\Phi(G_p)$, then $(G_p \cap T)\Phi(G_p) \lhd G$. The group $G_p/\Phi(G_p)$ is a minimal normal subgroup of $G/\Phi(G_p)$ by Lemma 2.3, hence $(G_p \cap T) \leq \Phi(G_p)$ or $(G_p \cap T) = G_p$, as $G_p/\Phi(G_p)$ is a chief factor of G. If $(G_p \cap T) \leq \Phi(G_p)$, then $G_p = \langle x \rangle$, a contradiction. If $(G_p \cap T) = G_p$, then $T = G$ and T is a unique supplement of $\langle x \rangle$ in G. A cyclic subgroup $\langle x \rangle$ is weakly S-semipermutable in G by Step 2. Then there exists a normal subgroup K of G with $K \leq G_p$ such that $K\langle x \rangle$ is S-permutable and $\langle x \rangle \cap K$ is S-semipermutable in G by Lemma 2.7. Then $\langle x \rangle \cap K$ is S-permutable in G by Lemma 2.2. Since $G_p/\Phi(G_p)$ is a chief factor of G, we have $K \leq \Phi(G_p)$ or $K = G_p$. If $K \leq \Phi(G_p)$, then $L/\Phi(G_p) = (\langle x \rangle K\Phi(G_p))/\Phi(G_p)$ is S-permutable in $G/\Phi(G_p)$, a contradiction. If $K = G_p$, then $\langle x \rangle = \langle x \rangle \cap K$ is S-permutable in G, and $L/\Phi(G_p) = (\langle x \rangle \Phi(G_p))/\Phi(G_p)$ is S-permutable in $G/\Phi(G_p)$, a contradiction. ■

Proof of Theorem 3.4. Since $F^*(N) \leq L$, it is sufficient to show that $L \leq Z_{\mathfrak{U}}(G)$ by Lemma 2.9. Suppose that the theorem is false and consider a counterexample (G, L) for which $|G||L|$ is minimal. Let $L_p \in \mathrm{Syl}_p(L)$ where p is the smallest prime number dividing $|L|$. By [7, IV, Theorem 2.8], L_p is not cyclic.

Step 1. L is p-nilpotent.

Since L_p is not cyclic, either every maximal subgroup of L_p or any cyclic subgroup of L_p, with order p or 4 (L_2 is a non-abelian 2-group) is weakly S-semipermutable in L by Lemma 2.7. Then L is p-nilpotent by Theorems 3.2 and 3.3.

Step 2. $L = L_p$.

L is p-nilpotent by Step 1. Therefore there exists a normal p-complement subgroup K of L, thus $K \lhd G$. If $K \neq 1$, then $L/K \leq Z_{\mathfrak{U}}(G/K)$. Since $K \leq Z_{\mathfrak{U}}(G)$, we have $L \leq Z_{\mathfrak{U}}(G)$ by the choice of (G, L), thus $N \leq Z_{\mathfrak{U}}(G)$, which is a contradiction. So $K = 1$ and then $L = L_p$.

Step 3. L_p is not a minimal normal subgroup of G and every cyclic subgroup of L_p with order p or 4 (L_2 is a non-abelian 2-group) is S-embedded in G.

Assume that L_p is a minimal normal subgroup of G. Thus L_p is a normal elementary abelian p-subgroup of G by Step 2. Let M be a maximal subgroup of L_p. Thus ML_p is S-permutable in G and $M \cap L_p = M \leq L_p = (L_p)_{sG}$. Then M is S-embedded in G. Therefore L_p has a maximal subgroup such that it is normal in G by Lemma 2.10, a contradiction. This contradiction shows that L_p is not a minimal normal subgroup of G. Assume that any maximal subgroup M of L_p is weakly S-semipermutable in G.

Hence, for any minimal normal subgroup H of G with $H \leq L_p$, $L_p/H \leq Z_{\mathfrak{U}}(G/H)$ by Lemma 2.7 and the choice of (G, L). Therefore H is a unique minimal normal subgroup of G with $H \leq L_p$ and $|H| > p$. Now we consider two cases: $\Phi(L_p) = 1$, and $\Phi(L_p) \neq 1$. If $\Phi(L_p) = 1$, then L_p is an elementary abelian p-subgroup of G. Let H_1 be a maximal subgroup of H, let B be the complement of H in L_p, and let $L_1 = H_1B$. Then L_1 is a maximal subgroup of L_p, therefore L_1 is weakly S-semipermutable in G, thus there exists $T \lhd G$ with $T \leq L_p$ such that TL_1 is S-permutable and $T \cap L_p$ is S-semipermutable in G by Lemma 2.7. If $T = 1$, then $L_1 = TL_1$ is S-permutable in G, hence $H_1 = H_1(B \cap H) = H_1B \cap H = L_1 \cap H$ is S-permutable in G by Lemma 2.1. If $T = L_p$, then $L_1 = T \cap L_1$ is S-semipermutable in G, hence L_1 is S-permutable in G by Lemma 2.2, hence $H_1 = H_1(B \cap H) = H_1B \cap H = L_1 \cap H$ is S-permutable in G by Lemma 2.1. If $1 < T < L_p$, then $H \leq T$, hence $H_1 \leq T$, thus $H_1 \leq L_1 \cap T \leq (L_1)_{sG}$, since $H \cap L_1 = H_1$ and $H_1 = H \cap (L_1)_{sG}$ is S-permutable in G. Therefore every maximal subgroup of H is S-permutable in G. It is a contradiction by Lemma 2.8. Hence $\Phi(L_p) \neq 1$ and $H \leq \Phi(L_p)$. Thus $L_p/H \leq Z_{\mathfrak{U}}(G/H)$ by the choice of (G, L). Then $(G/C_G(L_p/\Phi(L_p)))^{\mathcal{A}(p-1)}$ is a p-group by Lemma 2.11. Hence $(G/C_G(H))^{\mathcal{A}(p-1)}$ is a p-group by [4, Theorem 5.1.4]. Therefore $O_p(G/C_G(H)) = 1$ by [17, Corollary 6.4] and $(G/C_G(H)) \in \mathcal{A}(p-1)$. Then $|H| = p$ by [17, I, Theorem 1.4]. This is a contradiction by $|H| > p$. Therefore every cyclic subgroup K of L_p with order p or 4 (L_2 is a non-abelian 2-group) is weakly S-semipermutable in G, thus K is S-embedded in G.

Step 4. There exists $R \lhd G$ with $R \leq L_p$, L_p/R is a non-cyclic chief factor of G, $R \leq Z_{\mathfrak{U}}(G)$ and $V \leq R$ for any normal subgroup $V \neq R$ of G contained in L_p.

Let L_p/R be a chief factor of G. Then $R \neq 1$ by Step 3, hence $R \leq Z_{\mathfrak{U}}(G)$ by the choice of (G, L). Therefore $L \leq Z_{\mathfrak{U}}(G)$ and we know that L_p/R is not cyclic. Let $V \lhd G$ with $V \lneqq L_p$. Then $V \leq Z_{\mathfrak{U}}(G)$ by the choice of (G, L). If $V \nleq R$, then $L_p \leq Z_{\mathfrak{U}}(G)$ by the G-isomorphism $L_p/R = VR/R \cong V/(V \cap R)$, a contradiction by the choice of (G, L), therefore $V \leq R$.

Let L_p be a non-abelian 2-subgroup of G. We use $\Omega := \Omega_2(L_p)$, otherwise, let $\Omega := \Omega_1(L_p)$.

Step 5. $C_G(\Omega)/C_G(H)$ is a p-group.

It is true by [2, A, Lemma 1.2].

Step 6. $\Omega \nleq Z_{\mathfrak{U}}(G)$.

Let $\Omega \leq Z_{\mathfrak{U}}(G)$. Then $(G/C_G(\Omega))^{\mathcal{A}(p-1)}$ is a p-group by Lemma 2.11. Hence $(G/C_G(L_p/R)) \in \mathcal{A}(p-1)$ and so $|L_p/R| = p$ by Step 5, a contradiction. Hence $\Omega \nleq Z_{\mathfrak{U}}(G)$.

Step 7. Final contradiction.

$\Omega = L_p$ by Steps 4 and 6. Let $W_1, W_2, \dots, W_m$ be the set of all cyclic subgroups of L_p with order p or 4 (L_2 is a non-abelian 2-group) and $W_i \not\leq R$. Then W_i is S-embedded in G for $i = 1, 2, \dots, m$. Then there exists $T_i \lhd G$ with $T_i \leq L_p$ such that $W_i T_i$ is S-permutable in G and $W_i \cap T_i \leq (W_i)_{sG}$ for $i = 1, 2, \dots, m$. Since $R \lhd G$, if W_i is S-permutable in G, then $W_i R/R$ is S-permutable in G/R. Suppose that W_i is not S-permutable in G, then $1 < T_i < L_p$. Thus $T_i \leq R$ and $W_i R = W_i T_i R$ is S-permutable in G. Therefore $W_i R$ is S-permutable in G in all of the cases, thus $W_i R/R$ is S-permutable in G/R in all of the cases. This is a contradiction by Lemma 2.8, since $(L_p/R) = (W_1 R/R)(W_2 R/R) \dots (W_m R/R)$ is an elementary abelian group. ∎

Proof of Theorem 3.5. We have $N \leq Z_{\mathfrak{U}}(G)$ by Theorem 3.4. Since $G/N \in \mathfrak{F}$, Lemma 2.12 implies $G \in \mathfrak{F}$. ∎

Acknowledgments – The authors cordially thank the referees for their careful reading and helpful comments.

Funding – The authors thank Shahrekord University for the financial support.

References

[1] C. Chen, *Inner outer Σ-groups and minimal non Σ-groups* (in Chinese). Southwest Normal University Press, Chongqing, 1988.

[2] K. Doerk – T. Hawkes, *Finite soluble groups*. De Gruyter Exp. Math. 4, Walter de Gruyter, Berlin, 1992. Zbl 0753.20001 MR 1169099

[3] V. Go – I. Lu – V. Nyu, On S-embedded subgroups of finite groups. *Algebra Logic* **49** (2010), no. 4, 293–304. MR 2790171

[4] D. Gorenstein, *Finite groups*. Harper & Row, Publishers, New York-London, 1968. Zbl 0185.05701 MR 0231903

[5] W. Guo – K. P. Shum – A. N. Skiba, On solubility and supersolubility of some classes of finite groups. *Sci. China Ser. A* **52** (2009), no. 2, 272–286. Zbl 1189.20023 MR 2491727

[6] Z. Han, On s-semipermutable subgroups of finite groups and p-nilpotency. *Proc. Indian Acad. Sci. Math. Sci.* **120** (2010), no. 2, 141–148. Zbl 1214.20026 MR 2748585

[7] B. Huppert, *Endliche Gruppen. I*. Grundlehren Math. Wiss. 134, Springer, Berlin, 1967. Zbl 0217.07201 MR 0224703

[8] N. Itô – J. Szép, Über die Quasinormalteiler von endlichen Gruppen. *Acta Sci. Math. (Szeged)* **23** (1962), 168–170. Zbl 0112.02106 MR 138676

[9] O. H. Kegel, Sylow-Gruppen und Subnormalteiler endlicher Gruppen. *Math. Z.* **78** (1962), 205–221. Zbl 0102.26802 MR 147527

[10] Y. Li – S. Qiao – N. Su – Y. Wang, On weakly s-semipermutable subgroups of finite groups. *J. Algebra* **371** (2012), 250–261. Zbl 1269.20020 MR 2975395

[11] R. Maier – P. Schmid, The embedding of quasinormal subgroups in finite groups. *Math. Z.* **131** (1973), 269–272. Zbl 0259.20017 MR 364443

[12] O. Ore, Contributions to the theory of groups of finite order. *Duke Math. J.* **5** (1939), no. 2, 431–460. Zbl 0021.21104 MR 1546136

[13] D. J. S. Robinson, *A course in the theory of groups*. Second edn., Grad. Texts in Math. 80, Springer, New York, 1996. Zbl 0836.20001 MR 1357169

[14] A. N. Skiba, On weakly s-permutable subgroups of finite groups. *J. Algebra* **315** (2007), no. 1, 192–209. Zbl 1130.20019 MR 2344341

[15] A. N. Skiba, A characterization of the hypercyclically embedded subgroups of finite groups. *J. Pure Appl. Algebra* **215** (2011), no. 3, 257–261. Zbl 1206.20020 MR 2729221

[16] H. Wei – Y. Wang, The c-supplemented property of finite groups. *Proc. Edinb. Math. Soc. (2)* **50** (2007), no. 2, 493–508. Zbl 1132.20011 MR 2334959

[17] M. Weinstein (Ed.), *Between nilpotent and solvable*. Polygonal Publishing House, Washington, N.J., 1982. Zbl 0488.20001 MR 655785

Manoscritto pervenuto in redazione il 2 dicembre 2020.