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A formula for the minimal perimeter of clusters with density

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Abstract – This paper deals with the isoperimetric problem for clusters in a Euclidean space with double density. In particular, we show that a limit of an isoperimetric minimizing sequence of clusters with volumes $\mathbf{V}$ is always isoperimetric for its own volumes (which may be smaller than $\mathbf{V}$). In particular, if it is strictly smaller, we provide an explicit formula.

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1. Introduction

In this paper, we examine some aspects of the isoperimetric problem with density for clusters; this arises as a fusion of two well-known problems which we are going to briefly recall, both readable as generalizations of the classical Euclidean isoperimetric problem.

The first one is the *minimal partitioning problem*. Given a positive integer N, we call N-*cluster* every family of N mutually disjoint (measure theoretically) sets of finite perimeter $\mathcal{E} = \{\mathcal{E}(h)\}_{h=1,\dots,N}$ and we look for a N-clusters satisfying the volume constraints $|\mathcal{E}(h)| = V(h)$ for every $h = 1, \dots, N$ which minimizes the perimeter

$$P(\mathcal{E}) = \mathcal{H}^{n-1}\Big(\bigcup_{h=1}^{N} \partial^* \mathcal{E}(h)\Big).$$

There is a huge literature on properties of minimal clusters, starting from the founding work of Almgren and Taylor [1, 23], where existence and regularity, among the many

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other results, have been proved. For what concerns the classification of minimal clusters, much is known about minimal 2-clusters [8, 12, 21], planar 3-clusters [24], and planar 4-clusters with chambers with equal area [17, 18], while there are still open problems regarding the structure of minima for more than three chambers, though symmetry properties are known, under restrictions on dimension and number of chambers.

The other well studied generalization of the Euclidean isoperimetric problem is the so called *isoperimetric problem with (double) density*: given two lower semi-continuous and locally summable functions $f\colon \mathbb{R}^n \to \mathbb{R}_+$ and $g\colon \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}_+$, which we will call the *density* functions, we measure the f-volume and the g-perimeter of a Borel subset $E \subseteq \mathbb{R}^n$ as

$$|E|_f = \int_E f(x)\,dx, \quad P_g(E) = \int_{\partial^* E} g(x, \nu_E(x))\,d\mathcal{H}^{n-1}(x), \tag{1.1}$$

and we ask if there exists a set E which minimizes the g-perimeter among all sets of fixed f-volume V. In the previous definitions of perimeter, we consider $\partial^* E$ the reduced boundary of E and $\nu_E(x)$ the outer unit normal at $x \in \partial^* E$; for sufficient regular subsets of $\mathbb{R}^n$, the reduced boundary precisely corresponds to the usual topological boundary. Along with the problem of existence (or the *non*-existence) of isoperimetric sets, usual properties which are examined are boundedness and regularity of the boundary; in particular, information about boundedness of isoperimetric sets may be decisive in order to prove existence.

The isoperimetric problem with density may be seen as a generalization of the isoperimetric problem on Riemannian manifolds, since the density functions which weight volume and perimeter may be more general than the ones given by those related to the Riemannian metric [14, 15]. Moreover, we underline that the generalization is consistent as long as we allow the density for the perimeter to be different from the one on the volume and to depend on the normal on $\partial^* E$. As one expects, the existence of isoperimetric sets and their geometric properties are intimately related to the densities f and g; a partial list of results is [3, 5, 6, 11, 16, 22] in case $f = g$ (single density), and [4, 9, 10, 19, 20] for the general case (double density).

As anticipated, the isoperimetric problem with density for clusters is a combination of the two: we look for a N-cluster which minimizes the g-perimeter among those having chambers of fixed f-volume $(V(h))_{h=1,\dots,N}$. More precisely, if we define the g-perimeter of a cluster $\mathcal{E}$ by

$$P_g(\mathcal{E}) := \frac{1}{2}\Big(\sum_{h=1}^{N} P_g(\mathcal{E}(h)) + P_g\Big(\bigcup_{h=1}^{N} \mathcal{E}(h)\Big)\Big), \tag{1.2}$$

its f-volume as the vector

$$|\mathcal{E}|_f := (|\mathcal{E}(h)|_f)_{h=1,\dots,N}, \tag{1.3}$$

and the (f, g)-*isoperimetric profile* as the function which assigns to each volume $\mathbf{V} = (V(h))_{h=1,\dots,N} \in \mathbb{R}^N_+$ the quantity

$$\mathcal{I}_{(f,g)}(\mathbf{V}) := \inf\{P_g(\mathcal{E}) : \mathcal{E} \text{ an } N\text{-cluster}, |\mathcal{E}(h)|_f = V(h),\ h = 1, \dots, N\}, \tag{1.4}$$

we ask if the infimum is reached.

This question inherits the difficulties of both problems it generalizes; in particular, the existence of isoperimetric clusters is strictly related to the density. Nevertheless, we can take advantage of strategies already working for the case of single sets. The basic idea, as customary in the Calculus of Variations, is to consider a minimizing sequence $\{\mathcal{E}_j\}_{j\in\mathbb{N}}$, that is, $|\mathcal{E}_j(h)|_f = V(h)$ for each $h \in \{1, \dots, N\}$, and $P_g(\mathcal{E}_j) \to \mathcal{I}_{(f,g)}(\mathbf{V})$, in order to apply a standard compactness-semi-continuity argument: by compactness properties of BV functions, up to subsequences we can assume $\mathcal{E}_j \to \mathcal{E}$ as $j \to \infty$, and by semi-continuity of the perimeter we have $P_g(\mathcal{E}) \le \liminf_{j\to\infty} P_g(\mathcal{E}_j)$. Actually, the limit cluster may not have the right f-volume, since there may be loss of mass at infinity for one or more than one of the chambers. This cannot happen if $f \in L^1(\mathbb{R}^n)$, since obviously $|\mathcal{E}|_f = \lim_{j\to\infty}|\mathcal{E}_j|_f$; this means that $\mathcal{E}$ is a competitor for the isoperimetric problem, and by lower semi-continuity $P_g(\mathcal{E}) \le \liminf_{j\to\infty} P_g(\mathcal{E}_j) = \mathcal{I}_{(f,g)}(\mathbf{V})$, thus we have that isoperimetric clusters exist for every volume $\mathbf{V}$. For general $f \in L^1_{\mathrm{loc}} \setminus L^1$, we only have the inequality $|\mathcal{E}(h)|_f \le V(h)$.

Let us focus for a moment on the single-set case (i.e. $N = 1$). As already shown in [6, 19], even in the case of loss of volume at infinity, limits of minimizing sequences are *isoperimetric sets for their own volumes*. Moreover, if the density f and g converge at infinity both to a finite positive value a, the following formula holds:

$$\begin{aligned}\mathcal{I}_{(f,g)}(V) &:= \inf\{P_g(F) : |F|_f = V\}\\ &= P_g(E) + n(a\omega_n)^{\frac{1}{n}}(V - |E|_f)^{\frac{n-1}{n}},\end{aligned}$$

ω_n being the Lebesgue measure of the unit ball in $\mathbb{R}^n$ and E being a limit set of any minimizing sequence for the problem; that is, the optimal profile is obtained as the union of E and a *ball at infinity* of volume $V - |E|_f$, where the density is constantly equal to a.

As we are going to prove in the article, limit points of minimizing sequences of clusters behave in a similar fashion; moreover, we can notice some extra structure if the densities f and g are converging to positive limits at infinity.

THEOREM 1.1. *Let f and g be L^1_{loc} and lower semi-continuous functions and assume them to be bounded from above and below (away from* 0*) away from the origin. Define*

$$g^+(x) := \sup_{\nu \in \mathbb{S}^{n-1}} g(x, \nu)$$

and assume it is locally integrable in $\mathbb{R}^n$. Let also $\{\mathcal{E}_j\}_{j\in\mathbb{N}}$ be an isoperimetric sequence of clusters with volume $\mathbf{V}$ which converges to a cluster $\mathcal{E}$ in the L^1_{loc} sense. Then:

(i) *$\mathcal{E}$ is a cluster of minimal g-perimeter for its own volume.*

(ii) *If in addition $\lim_{|x|\to\infty} f(x) = a \in (0,\infty)$ and $\lim_{|x|\to\infty} g(x,\nu) = b \in (0,\infty)$ uniformly in ν, then*

$$\mathcal{I}_{(f,g)}(\mathbf{V}) = P_g(\mathcal{E}) + b a^{-\frac{n-1}{n}} \mathcal{I}_{\mathrm{eucl}}(\mathbf{V} - |\mathcal{E}|_f). \tag{1.5}$$

For converging densities, as in [6, 19] for single sets, formula (1.5) suggests that an isoperimetric cluster for volume $\mathbf{V}$ is given by the union of the limit $\mathcal{E}$ and an "Euclidean cluster at infinity" which has precisely the missing f-volume.

For $N = 1$, in [6, 19] this heuristic is actually made rigorous, under some extra hypothesis on the densities f and g. Indeed, for a fixed $V > 0$, if we call E a limit point of a minimizing sequence $\{E_j\}_{j\in\mathbb{N}}$ for $\mathcal{I}_{(f,g)}(V)$, it is possible to find a set $\widetilde{B}$ far from the origin, not intersecting E, having exactly the missing f-volume $V - |E|_f$, and g-perimeter smaller than or equal to

$$b a^{-\frac{n-1}{n}} n \omega_n^{\frac{1}{n}} (\mathbf{V} - |\mathcal{E}|_f)^{\frac{n-1}{n}};$$

by (1.5), we can conclude that $E \cup \widetilde{B}$ is an isoperimetric set of volume V, thus obtaining an existence result.

For $N \geq 2$, we expect that Theorem 1.1 may be the starting point for deducing an existence result for isoperimetric clusters, in analogy with the single set case. The construction of a candidate cluster $\widetilde{\mathcal{B}}$ which recovers the missing f-volume with a controlled increase of the g-perimeter is much more delicate, due to the fact that the structure of Euclidean isoperimetric clusters in general is unknown (with the exceptions already discussed). The existence problem, in particular in the case $N = 2$, will be addressed in a forthcoming work.

The article is structured in the following way. In Section 2 we introduce the main definitions and we recall the basic properties of sets of finite perimeter we will need. Section 3 covers the proof of Theorem 1.1.

2. Definitions and basic properties of sets of finite perimeter

In this section, we introduce the definitions, the notation and the basic results on sets of finite perimeter we will need in the proof of Theorem 1.1; for more information on definitions and results, the reader should refer to [2, 7, 13].

Let $E \subseteq \mathbb{R}^n$ be a set of (locally) finite measure; we say this is a *set of (locally) finite perimeter* if its characteristic function χ_E is a BV function (resp. BV_{loc} function), i.e. it is summable (resp. locally summable) and its distributional derivative $D\chi_E$ is a Radon measure, and we will put $\mu_E := -D\chi_E$. For any Borel subset $A \subseteq \mathbb{R}^n$, we define the *relative perimeter of E in A* by

$$P(E; A) := |\mu_E|(A),$$

and we define the *perimeter of E* by $P(E) := P(E; \mathbb{R}^n)$.

For a set of locally finite perimeter E, the *reduced boundary* is

$$\partial^* E := \left\{ x \in \mathrm{spt}(\mu_E) \;\middle|\; \exists \lim_{r \to 0^+} \frac{\mu_E(B(x,r))}{|\mu_E|(B(x,r))} =: \nu_E(x),\ |\nu_E(x)| = 1 \right\}$$

and we define $\nu_E(x)$ as the exterior normal to $\partial^* E$ at x.

We recall a fundamental result on sets of finite perimeter.

Theorem (Blow-up, structure). *Assume E is a set of locally finite perimeter. Then:*

- *For any $x \in \partial^* E$, define $E_{x,r} := (E - x)/r$; then, we have the L^1_{loc} convergence*

$$E_{x,r} \xrightarrow{r \to 0^+} H_{\nu_E(x)} = \{ y \in \mathbb{R}^n : y \cdot \nu_E(x) \leq 0 \},$$

  and if we put $\pi_{\nu_E(x)} = \partial H_{\nu_E(x)}$, we have

$$\mu_{E_{x,r}} \overset{*}{\rightharpoonup} \nu_E(x) \mathcal{H}^{n-1} \llcorner \pi_{\nu_E(x)}, \qquad |\mu_{E_{x,r}}| \overset{*}{\rightharpoonup} \mathcal{H}^{n-1} \llcorner \pi_{\nu_E(x)}.$$

- *The reduced boundary $\partial^* E$ is a $(n-1)$-dimensional rectifiable set, and the measure μ_E satisfies*

$$\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial^* E, \qquad |\mu_E| = \mathcal{H}^{n-1} \llcorner \partial^* E.$$

In particular, this allows to rewrite the perimeter of E in the equivalent form $P(E) = \mathcal{H}^{n-1}(\partial^* E)$.

We say that a point $x \in \mathbb{R}^n$ is *of density $d \in [0, 1]$ for the set E* if

$$\lim_{r \to 0^+} \frac{|E \cap B(x,r)|}{\omega_n r^n} = d,$$

where $|\cdot| := \mathcal{L}^n$ is the Lebesgue measure on $\mathbb{R}^n$ and $\omega_n = |B(0,1)|$.

We define the *essential boundary* of E

$$\partial^e E := \mathbb{R}^n \setminus (E^{(0)} \cup E^{(1)}).$$

By Federer's theorem, we have that

$$\partial^* E = E^{(1/2)} = \partial^e E,$$

up to $\mathcal{H}^{n-1}$-negligible sets.

Given a positive integer N, a N*-cluster* is a family of sets of finite perimeter $\{\mathcal{E}(h)\}_{h=1,\dots,N}$, called *chambers*, such that

$$\begin{aligned} &|\mathcal{E}(h)| \in (0,\infty), && h = 1,\dots,N, \\ &|\mathcal{E}(h) \cap \mathcal{E}(k)| = 0, && h,k = 1,\dots,N, \quad h \neq k. \end{aligned}$$

If we put $\partial^*\mathcal{E} := \bigcup_{h=1}^N \partial^*\mathcal{E}(h)$, the Euclidean perimeter of the cluster is defined as

$$P(\mathcal{E}) := \mathcal{H}^{n-1}(\partial^*\mathcal{E}).$$

We can think to the perimeter of a cluster as given by the sum of the perimeter of each chamber, counting only once each interface, meaning the non-empty intersection of two chambers. If we define $\mathcal{E}(0) := \mathbb{R}^n \setminus \bigcup_{h=1}^N \mathcal{E}(h)$ the *exterior chamber* of $\mathcal{E}$, we can equivalently define the perimeter of the cluster as

$$P(\mathcal{E}) := \sum_{0 \le h < k \le N} \mathcal{H}^{n-1}(\partial^*\mathcal{E}(h) \cap \partial^*\mathcal{E}(k)) = \frac{1}{2}\sum_{h=0}^N P(\mathcal{E}(h)),$$

where the second equality is a consequence of Federer's theorem.

We define the g-perimeter and the f-volume of a cluster respectively as in (1.2) and (1.3). In the following, we will use $|\cdot|_{\mathrm{eucl}}$ and P_{eucl} to define the Euclidean volume and perimeter, while we will use $|\cdot|_f$ and P_g for the weighted volume and perimeter.

For every cluster $\mathcal{E}$ and every Borel set B, we define the relative g-perimeter of $\mathcal{E}$ in B by

$$\begin{aligned} P_g(\mathcal{E}; B) := \frac{1}{2}\Bigg(\sum_{h=1}^N \int_{\partial^*\mathcal{E}(h)\cap B} & g(x, \nu_{\mathcal{E}(h)}(x))\, d\mathcal{H}^{n-1}(x) \\ & + \int_{\partial^*(\cup\mathcal{E})\cap B} g(x, \nu_{\cup\mathcal{E}}(x))\, d\mathcal{H}^{n-1}(x)\Bigg), \end{aligned}$$

where we put $\cup\mathcal{E} := \bigcup_{h=1}^N \mathcal{E}(h)$.

3. Proof of Theorem 1.1

For a fixed volume $\mathbf{V} = (V(h))_{h=1,\dots,N} \in \mathbb{R}^N_+$, let us consider a minimizing sequence of clusters $\{\mathcal{E}_j\}_{j\in\mathbb{N}}$ for $\mathcal{I}_{(f,g)}(\mathbf{V})$, that is, $|\mathcal{E}_j|_f = \mathbf{V}$ for any $j \in \mathbb{N}$ and

$$\mathcal{I}_{(f,g)}(\mathbf{V}) = \lim_{j\to\infty} P_g(\mathcal{E}_j).$$

By assumption, we consider $\mathcal{E}$ such that $\mathcal{E}_j \xrightarrow{L^1_{\mathrm{loc}}} \mathcal{E}$.

If $|\mathcal{E}|_f = \mathbf{V}$, there is nothing to prove; the cluster is isoperimetric for its own volume, thanks to the lower semi-continuity of the perimeter.

Therefore, henceforth we assume $|\mathcal{E}|_f < \mathbf{V}$, meaning that there exists $h \in \{1,\dots,N\}$ such that $|\mathcal{E}(h)|_f < V(h)$. Without loss of generality, we may assume that $|\mathcal{E}(h)|_f > 0$ for every $h = 1,\dots,N$; if $\mathcal{E}$ does not verify this condition, we simply consider it as a cluster with a smaller number of chambers.

We assume by contradiction that there exists a cluster $\mathcal{F}$ such that

$$|\mathcal{F}|_f = |\mathcal{E}|_f, \qquad \frac{P_g(\mathcal{E}) - P_g(\mathcal{F})}{6} =: \eta > 0. \tag{3.1}$$

We can find points $x_1,\dots,x_N$ of density 1 respectively for $\mathcal{F}(1),\dots,\mathcal{F}(N)$ and which are Lebesgue points for f and g^+ so that $f(x_h) > 0$ for every $h = 1,\dots,N$; hence, there exists $\overline{r} > 0$ such that for every $h = 1,\dots,N$:

$$\frac{1}{2}\omega_n f(x_h) r^n \le |B(x_h,r) \cap \mathcal{F}(h)|_f \le |B(x_h,r)|_f \le 2\omega_n f(x_h) r^n, \tag{3.2}$$

$$P_g(\mathbb{R}^n \setminus B(x_h,r)) \le 2n\omega_n g^+(x_h) r^{n-1}, \tag{3.3}$$

where (3.2) holds true for every $0 < r < \overline{r}$, (3.3) holds true for arbitrarily many r smaller than $\overline{r}$ (to prove (3.3), one can consider the analogue of (3.2) for g^+ and work by contradiction).

Since $f \notin L^1$ (otherwise we would have had $|\mathcal{E}|_f = \mathbf{V}$, a contradiction, we can find points $y_1,\dots,y_N$ of density 0 for $\cup\mathcal{F} := \bigcup_{h=1}^N \mathcal{F}(h)$ which are Lebesgue points for f and g^+ and verifying $f(y_h) > 0$ for every $h = 1,\dots,N$, far enough from the origin to assume $\frac{1}{M} \le f, g \le M$ for some $M > 0$, by the assumptions on f and g; hence, we obtain the estimates

$$|B(y_h,\rho) \setminus (\cup\mathcal{F})|_f \ge \frac{f(y_h)}{2}\omega_n \rho^n, \tag{3.4}$$

$$P_g(B(y_h,\rho)) \le M n \omega_n \rho^{n-1}, \tag{3.5}$$

both inequalities being true for every $\rho \in (0,\overline{\rho})$, for some $\overline{\rho} > 0$ small enough.

Up to consider a smaller $\overline{\rho}$, we define a constant $\delta > 0$ so that

$$M^2\delta < \eta, \quad Mn\omega_n\overline{\rho}^{n-1} < \frac{\eta}{N}, \quad \frac{f(y_h)}{2}\omega_n\overline{\rho}^n > \delta \quad \forall h = 1, \dots, N. \tag{3.6}$$

We claim that there exists a N-cluster $\mathcal{F}'$ and $R > 0$ big enough such that $\mathcal{F}' \subseteq B_R$ and

$$P_g(\mathcal{F}') < P_g(\mathcal{E}) - 5\eta, \tag{3.7}$$

$$0 < \delta'_h := |\mathcal{E}(h)|_f - |\mathcal{F}'(h)|_f < \frac{\delta}{2}, \tag{3.8}$$

for every $h = 1, \dots, N$.

Case 1: The cluster $\mathcal{F}$ is bounded.

For every $h = 1, \dots, N$, choose $r_h < \overline{r}$ so small that all balls $B_h := B(x_h, r_h)$ are mutually disjoint and transversally intersect all the chambers of $\mathcal{F}$ (i.e., $\mathcal{H}^{n-1}(\partial^*\mathcal{F} \cap \partial B_h) = 0$). Define the new cluster

$$\mathcal{F}' := \mathcal{F} \setminus \Big(\bigcup_{h=1}^N B_h\Big) = \Big\{\mathcal{F}(h) \setminus \Big(\bigcup_{j=1}^N B_j\Big)\Big\}_{h=1,\dots,N},$$

which is obviously bounded.

We easily notice that, for a given open set of locally finite perimeter $B \subseteq \mathbb{R}^n$ transversal to each chamber:

$$P_g(\mathcal{F} \setminus B) = P_g(\mathcal{F}; \overline{B}^c) + \sum_{h=1}^N \int_{\partial^* B \cap \mathcal{F}(h)^{(1)}} g(x, -\nu_B(x))\, d\mathcal{H}^{n-1}(x). \tag{3.9}$$

By the previous relations (3.2) and (3.3), up to possibly decreasing the $r'_h s$, we have that

$$P_g(\mathcal{F}') \le P_g(\mathcal{F}) + \sum_{h=1}^N P_g(\mathbb{R}^n \setminus B_h) < P_g(\mathcal{E}) - 5\eta.$$

and (3.8) holds as well.

Case 2: The cluster $\mathcal{F}$ is unbounded.

Without loss of generality, let us assume that the chambers $\mathcal{F}(1), \dots, \mathcal{F}(L)$ are unbounded, for a certain $1 \le L \le N$.

We choose $R_0 > 0$ big enough so that $\mathcal{F}(h) \subset B_{R_0}$ for all $h = L+1, \ldots, N$, $\frac{1}{M} \leq f, g \leq M$ in $\mathbb{R}^n \setminus B_{R_0}$, and

$$|\mathcal{F}(h) \setminus B_{R_0}|_f < \frac{\delta}{2}$$

for all $h = 1, \ldots, L$.

Let us consider $R > R_0$ to be chosen later; we define the new cluster

$$\mathcal{F}' := \mathcal{F} \cap B_R = \{\mathcal{F}(h) \cap B_R\}_{h=1,\ldots,N}$$

and we notice that for every open set of locally finite perimeter B transversal to each chamber:

$$P_g(\mathcal{F} \cap B) = P_g(\mathcal{F}; B) + \sum_{h=1}^{N} \int_{\partial B \cap \mathcal{F}(h)^{(1)}} g(x, \nu_B(x))\, d\mathcal{H}^{n-1}(x). \tag{3.10}$$

We need to find a $R > R_0$ such that

$$P_g(\mathcal{F}') < P_g(\mathcal{E}) - \left(5 + \frac{1}{2}\right)\eta. \tag{3.11}$$

By contradiction, let us assume that for every $R > R_0$ the inequality (3.11) does not hold. By (3.1) and (3.10), we obtain

$$\sum_{h=1}^{L} \int_{\partial B_R \cap \mathcal{F}(h)^{(1)}} g(x, \nu_{B_R}(x))\, d\mathcal{H}^{n-1}(x) \geq \frac{\eta}{2}$$

and so

$$\begin{aligned}
+\infty > \sum_{h=1}^{L} |\mathcal{F}(h) \setminus B_R|_f &\geq \int_{R_0}^{+\infty} \sum_{h=1}^{L} \int_{\partial B_R \cap \mathcal{F}(h)^{(1)}} f(x)\, d\mathcal{H}^{n-1}(x)\, dR \\
&\geq \int_{R_0}^{+\infty} \frac{1}{M^2} \sum_{h=1}^{L} \int_{\partial B_R \cap \mathcal{F}(h)^{(1)}} g(x, \nu_{B_R}(x))\, d\mathcal{H}^{n-1}(x) \\
&\geq \int_{R_0}^{+\infty} \frac{\eta}{2M^2}\, dR = +\infty,
\end{aligned}$$

which is a contradiction. Thus, there must exist $R > R_0$ for which (3.11) holds.

Now, we want to reduce the volume of the bounded chambers and obtain the complete estimate (3.7). We apply the same strategy of *case 1* to $\mathcal{F}'$; we call $\mathcal{F}''$ the new cluster and we require that

$$P_g(\mathcal{F}'') < P_g(\mathcal{F}') + \frac{1}{2}\eta,$$
$$0 < \delta'_h := |\mathcal{E}(h)|_f - |\mathcal{F}''(h)|_f < \frac{\delta}{2}, \quad h = L+1, \dots, N.$$

Putting together the estimates and renaming $\mathcal{F}''$ in $\mathcal{F}'$, we obtain (3.7) and (3.8) also in this case.

The leading idea in the proof is to construct a new sequence of competitors $\{\mathcal{E}'_j\}_{j\in\mathbb{N}}$ with the right volume $\mathbf{V}$, but converging to the "wrong" perimeter, that is a little smaller than the minimum.

For every $R' > R$ large enough (say, more than $\max|y_h| + \overline{\rho}$), we have that

$$|\mathcal{E}(h) \setminus B_{R'}|_f < \frac{\delta'_h}{2}, \quad h = 1, \dots, N, \tag{3.12}$$

$$P_g(\mathcal{E}; B_{R'}) > P_g(\mathcal{E}) - \eta. \tag{3.13}$$

By the L^1_{loc} convergence of $\mathcal{E}_j$ to $\mathcal{E}$, for j big enough and by lower semi-continuity of P_g:

$$|\mathcal{E}(h)|_f - \frac{\delta'_h}{N} < |\mathcal{E}_j(h) \cap B_{R'}|_f \le |\mathcal{E}_j(h) \cap B_{R'+1}|_f < |\mathcal{E}(h)|_f + \frac{\delta'_h}{N}, \tag{3.14}$$

$$P_g(\mathcal{E}; B_{R'}) \le P_g(\mathcal{E}_j; B_{R'}) + \eta. \tag{3.15}$$

Combining (3.14), (3.12) and (3.6), we notice that

$$\begin{aligned}
\int_{R'}^{R'+1} \sum_{h=1}^{N} \int_{\partial B \cap \mathcal{E}(h)^{(1)}} & g(x, -\nu_B(x))\, d\mathcal{H}^{n-1}(x)\, dR \\
&\le \int_{R'}^{R'+1} M^2 \sum_{h=1}^{N} \mathcal{H}^{n-1}_f(\mathcal{E}_j(h) \cap \partial B_R)\, dR \\
&= M^2 \sum_{h=1}^{N} |\mathcal{E}_j(h) \cap (B_{R'+1} \setminus B_{R'})|_f \\
&< 2M^2 \max_{h=1,\dots,N} \delta'_h < M^2 \delta < \eta,
\end{aligned}$$

and so for each j big enough there exists $R_j \in (R', R'+1)$ such that

$$\sum_{h=1}^{N} \int_{\partial B \cap \mathcal{E}(h)^{(1)}} g(x, -\nu_{B_{R_j}}(x))\, d\mathcal{H}^{n-1}(x) < \eta; \tag{3.16}$$

moreover, for each chamber we have the estimate on the volume

$$V(h) - |\mathcal{E}(h)|_f - \delta'_h < |\mathcal{E}_j(h) \setminus B_{R_j}|_f < V(h) - |\mathcal{E}(h)|_f + \delta'_h. \tag{3.17}$$

We define the new sequence of clusters $\mathcal{G}_j := \{\mathcal{F}'(h) \cup (\mathcal{E}_j(h) \setminus B_{R_j})\}_{h=1,\dots,N}$. By (3.17) and since $|\mathcal{E}_j(h)|_f = V(h)$ for every $h = 1, \dots, N$, we notice that

$$|\mathcal{G}_j(h)|_f \in (V(h) - \delta, V(h)), \tag{3.18}$$

and by (3.7), (3.16), (3.13), (3.15), we have the estimate on the perimeter

$$\begin{aligned}
P_g(\mathcal{G}_j) &\le P_g(\mathcal{F}') + P_g(\mathcal{E}_j \setminus B_{R_j}) \\
&< P_g(\mathcal{E}) - 5\eta + P_g(\mathcal{E}_j; \overline{B_{R_j}}^c) \\
&\qquad + \sum_{h=1}^{N} \int_{\partial B_{R_j} \cap \mathcal{E}(h)^{(1)}} g(x, -\nu_{B_{R_j}}(x))\, d\mathcal{H}^{n-1}(x) \\
&< P_g(\mathcal{E}; B_{R'}) - 3\eta + P_g(\mathcal{E}_j, \overline{B_{R_j}}^c) \\
&\le P_g(\mathcal{E}_j; B_{R'}) - 2\eta + P_g(\mathcal{E}_j, \overline{B_{R_j}}^c) \\
&\le P_g(\mathcal{E}_j) - 2\eta.
\end{aligned}$$

Finally, we define the new sequence

$$\widetilde{\mathcal{E}}_j := \{\mathcal{G}_j(h) \cup (B(y_h, \rho_h) \setminus \cup\mathcal{G})\}_{h=1,\dots,N},$$

where we choose each $\rho_h \in (0, \overline{\rho})$ so that $|\widetilde{\mathcal{E}}_j(h)|_f = V(h)$ for each $h = 1, \dots, N$ (this can occur, because of (3.4), (3.18) and the condition $R' > \max|y_h| + \overline{\rho}$ implies that $B(y_h, \rho) \setminus \cup\mathcal{G} = B(y_h, \rho) \setminus \cup\mathcal{F}$ for each $h = 1, \dots, N$). Each $\widetilde{\mathcal{E}}_j$ is a N-cluster of volume $\mathbf{V}$; putting together the preceding estimates and (3.5), we have

$$P_g(\widetilde{\mathcal{E}}_j) < P_g(\mathcal{E}_j) - \eta,$$

that is, we have built a sequence of competitors for the problem having perimeters strictly smaller than the infimum if j is large enough. This is a contradiction, and hence $\mathcal{E}$ is a minimal cluster for its own volume, concluding the proof of statement (i).

3.1 – *Proof of statement* (ii)

From now on, we assume that the densities f and g are converging to finite positive limits a and b at infinity. Our goal is to prove that

$$\mathcal{I}_{(f,g)}(\mathbf{V}) = P_g(\mathcal{E}) + ba^{-\frac{n-1}{n}} \mathcal{I}_{\mathrm{eucl}}(\mathbf{V} - |\mathcal{E}|_f), \tag{3.19}$$

$\mathcal{E}$ being limit of a minimizing sequence $\{\mathcal{E}_j\}_{j\in\mathbb{N}}$ with $|\mathcal{E}(h)|_f = V(h)$ for each $h = 1, \dots, N$.

As already seen in statement (i), we can choose N points $y_1, \dots, y_N$ far away from the origin which are Lebesgue points for f and 0-density points of $\cup\mathcal{E}$. It follows that there exist $\overline{\rho} > 0$ such that for each $\rho \in (0, \overline{\rho})$ we have

$$\omega_n \frac{f(y_h)}{2}\rho^n \le |B(y_h,\rho) \setminus \mathcal{E}|_f \le |B(y_h,\rho)|_f \le \omega_n \rho^n 2a,$$
$$P_g(B(y_h,\rho)) \le n\omega_n \rho^{n-1} 2b. \tag{3.20}$$

We choose $\varepsilon > 0$ so small that for each $h = 1, \dots, N$

$$\omega_n \frac{f(y_h)}{2}(\overline{\rho})^n > \Big(\frac{2V(h)}{a} + 1\Big)\varepsilon; \tag{3.21}$$

we define $\mathcal{F} := \mathcal{E} \cap B_R$, with $R \gg 1$, $R > \max_{h=1,\dots,N}|y_h| + \overline{\rho}$ such that

$$|(\mathcal{E} \cap B_R)(h)|_f \ge |\mathcal{E}(h)|_f - \varepsilon,$$
$$P_g(\mathcal{E} \cap B_R) \le P_g(\mathcal{E}) + \varepsilon.$$

We choose an Euclidean minimal N-cluster $\mathcal{B}$, with *Euclidean* volume

$$\Big(\frac{V(h) - |\mathcal{E}(h)|_f}{a + \varepsilon}\Big)_{h=1,\dots,N},$$

so far from the origin that it does not intersect B_R and $a - \varepsilon < f < a + \varepsilon$, $b - \varepsilon < g < b + \varepsilon$. Clearly, we notice that

$$(a - \varepsilon)|\mathcal{B}(h)|_{\mathrm{eucl}} \le |\mathcal{B}(h)|_f \le V(h) - |\mathcal{E}(h)|_f$$

for each $h = 1, \dots, N$. We define the cluster $\mathcal{G} := \{\mathcal{F}(h) \cup \mathcal{B}(h)\}_{h=1,\dots,N}$ and we notice that

$$\begin{aligned}
P_g(\mathcal{G}) &\le P_g(\mathcal{E}) + \varepsilon + P_g(\mathcal{B}) \\
&\le P_g(\mathcal{E}) + \varepsilon + (b + \varepsilon)\mathcal{I}_{\mathrm{eucl}}\Big(\frac{\mathbf{V} - |\mathcal{E}|_f}{a + \varepsilon}\Big) \\
&= P_g(\mathcal{E}) + \varepsilon + (b + \varepsilon)(a + \varepsilon)^{\frac{1}{n}-1}\mathcal{I}_{\mathrm{eucl}}(\mathbf{V} - |\mathcal{E}|_f).
\end{aligned}$$

This is not yet a competitor for the minimization problem with volume $\mathbf{V}$; indeed, we have that

$$\begin{aligned}
V(h) - \Big(\frac{2V(h)}{a} + 1\Big)\varepsilon &\le |\mathcal{E}(h)|_f - \varepsilon + \frac{a - \varepsilon}{a + \varepsilon}(V(h) - |\mathcal{E}(h)|_f) \\
&\le |\mathcal{G}(h)|_f \le V(h).
\end{aligned} \tag{3.22}$$

We define the new cluster $\mathcal{E}'$ chamber by chamber by

$$\mathcal{E}'(h) := \mathcal{G}(h) \cup B(y_h, \rho_h),$$

with $0 < \rho_h < \overline{\rho}$ so that $|\mathcal{E}'(h)|_f = V(h)$ for each $h = 1, \ldots, N$, taking (3.20), (3.21) and (3.22) into account. Finally, we have that

$$\begin{aligned} \mathcal{I}_{(f,g)}(\mathbf{V}) \le P_g(\mathcal{E}') &= P_g(\mathcal{G}) + \sum_{h=1}^{N} P_g(B(y_h, \rho_h)) \\ &\le P_g(\mathcal{E}) + \varepsilon + (b+\varepsilon)(a+\varepsilon)^{\frac{1}{n}-1} \mathcal{I}_{\mathrm{eucl}}(\mathbf{V} - |\mathcal{E}|_f) + Nn\omega_n(\overline{\rho})^{n-1} 2b \end{aligned}$$

and since $\overline{\rho} \ll 1$ and $\varepsilon \ll \overline{\rho}$, by sending $\varepsilon, \rho \to 0$ we have one side of (3.19).

To get the reverse inequality, we need to act on a minimizing sequence converging to $\mathcal{E}$.

By the continuity of the Euclidean isoperimetric function, for a fixed $\varepsilon' > 0$ there exists $\delta > 0$ such that

$$\left|\mathcal{I}_{\mathrm{eucl}}(\mathbf{V}') - \mathcal{I}_{\mathrm{eucl}}(\mathbf{V} - |\mathcal{E}|_f)\right| \le \varepsilon', \tag{3.23}$$

if $\left|\mathbf{V}' - (\mathbf{V} - |\mathcal{E}|_f)\right| \le \delta$. Choose ε such that

$$0 < \varepsilon\left(N + \frac{\left|V - |\mathcal{E}|_f\right| + \varepsilon}{a - \varepsilon}\right) < \delta.$$

By means of formulae (3.9) and (3.10), we find R big enough so that $a - \varepsilon < f < a + \varepsilon$, $b - \varepsilon < g < b + \varepsilon$ out of B_R, and for every $h = 1, \ldots, N$ we have

$$|\mathcal{E}(h)|_f - \varepsilon < |\mathcal{E}(h) \cap B_R|_f$$

and

$$P_g(\mathcal{E} \setminus B_R) \le \varepsilon. \tag{3.24}$$

We claim that, for every j big enough there exists $R_j \in (R, R+1)$ so that

$$|\mathcal{E}(h)|_f - \varepsilon \le |\mathcal{E}_j(h) \cap B_{R_j}|_f \le |\mathcal{E}(h)|_f + \varepsilon, \quad \text{for every } h = 1, \ldots, N, \tag{3.25}$$

$$\sum_{h=1}^{N} \int_{\partial B_{R_j} \cap \mathcal{E}(h)^{(1)}} g(x, \nu_{B_{R_j}}(x)) + g(x, -\nu_{B_{R_j}}(x))\, d\mathcal{H}^{n-1}(x) \le 2\varepsilon, \tag{3.26}$$

$$P_g(\mathcal{E}) \le P_g(\mathcal{E}_j \cap B_{R_j}) + 2\varepsilon. \tag{3.27}$$

Indeed, estimates (3.25) and (3.26) are perfectly analogous to what already seen in statement (i); by (3.10), (3.9), (3.24) and the lower semi-continuity of P_f, for $j \gg 1$

we get

$$
\begin{aligned}
P_g(\mathcal{E}) = P_g(\mathcal{E}; B_R) + P_g(\mathcal{E}; \overline{B_R}^c) &\leq P_g(\mathcal{E}; B_R) + P_g(\mathcal{E} \setminus B_R) \\
&< P_g(\mathcal{E}; B_R) + \varepsilon < P_g(\mathcal{E}_j; B_R) + 2\varepsilon \leq P_g(\mathcal{E}_j; B_{R_j}) + 2\varepsilon.
\end{aligned}
$$

By (3.25), we notice that

$$
\begin{aligned}
\big|\mathbf{V} - |\mathcal{E}|_f - a|\mathcal{E}_j \setminus B_{R_j}|_{\mathrm{eucl}}\big| &\leq \varepsilon\big(N + \big||\mathcal{E}_j \setminus B_{R_j}|_{\mathrm{eucl}}\big|\big) \\
&\leq \varepsilon\Big(N + \frac{\big||\mathcal{E}_j \setminus B_{R_j}|_f\big|}{a-\varepsilon}\Big) \\
&\leq \varepsilon\Big(N + \frac{\big|\mathbf{V} - |\mathcal{E}|_f\big| + \varepsilon}{a-\varepsilon}\Big) < \delta,
\end{aligned}
$$

by our choice of ε. Thanks to these estimates and by (3.23), we obtain

$$
\begin{aligned}
P_g(\mathcal{E}_j \setminus B_{R_j}) \geq (b-\varepsilon) P_{\mathrm{eucl}}(\mathcal{E}_j \setminus B_{R_j}) &\geq \frac{b-\varepsilon}{a^{\frac{n-1}{n}}} \mathcal{I}_{\mathrm{eucl}}(a|\mathcal{E}_j \setminus B_{R_j}|_{\mathrm{eucl}}) \\
&\geq \frac{b-\varepsilon}{a^{\frac{n-1}{n}}}[\mathcal{I}_{\mathrm{eucl}}(\mathbf{V} - |\mathcal{E}|_f) - \varepsilon'].
\end{aligned}
$$

Finally, by (3.26) and (3.27) we can conclude

$$
\begin{aligned}
P_g(\mathcal{E}_j) &= P_g(\mathcal{E}_j \cap B_{R_j}) + P_g(\mathcal{E}_j \setminus B_{R_j}) \\
&\qquad - \sum_{h=1}^{N} \int_{\partial B_{R_j} \cap \mathcal{E}_j(h)} g(x, \nu_{B_{R_j}}(x)) + g(x, -\nu_{B_{R_j}}(x))\, d\mathcal{H}^{n-1}(x) \\
&\geq P_g(\mathcal{E}) + \frac{b-\varepsilon}{a^{\frac{n-1}{n}}}[\mathcal{I}_{\mathrm{eucl}}(\mathbf{V} - |\mathcal{E}|_f) - \varepsilon'] - 6\varepsilon.
\end{aligned}
$$

Sending first $j \to \infty$ and then $\varepsilon' \to 0$ (hence $\varepsilon \to 0$ as well), we have that

$$
\mathcal{I}_{(f,g)}(\mathbf{V}) \geq P_g(\mathcal{E}) + ba^{-\frac{n-1}{n}} \mathcal{I}_{\mathrm{eucl}}(\mathbf{V} - |\mathcal{E}|_f),
$$

thus concluding formula (3.19).

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