

## A short proof of a non-vanishing result by Conca, Krattenthaler and Watanabe

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ABSTRACT – In this note, we propose a short and elementary proof of a non-vanishing result by Conca, Krattenthaler and Watanabe (2009).

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In their paper *Regular sequences of symmetric polynomials* [1], Aldo Conca, Christian Krattenthaler and Junzo Watanabe needed to prove, as an intermediate result, the fact that for any  $h \geq 1$ , the rational number

$$\sum_{b=0}^{\lfloor h/3 \rfloor} \frac{(-1)^{h-b}}{h-b} \binom{h-b}{2b} \left(\frac{2}{3}\right)^b$$

is non-zero, except for  $h = 3$ . The proof in [1, Appendix, pp. 190–199] performs a (quite intricate) 3-adic analysis. In this note, we propose a shorter and elementary proof, based on the following observation.

THEOREM 1. *For any  $h \geq 1$ , consider the polynomials*

$$a_h := \sum_{b=0}^{\lfloor h/3 \rfloor} \frac{(-1)^{h-b}}{h-b} \binom{h-b}{2b} U^b \in \mathbb{Q}[U]$$

and  $s_h := h \cdot a_h$ . Then, the sequence  $(s_h)_{h \geq 1}$  satisfies the linear recurrence

$$(1) \quad s_{h+3} + 2s_{h+2} + s_{h+1} = U \cdot s_h \quad \text{for all } h \geq 1.$$

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PROOF. Using  $h/(h-b) = 1 + b/(h-b)$  and  $2b \cdot \binom{h-b}{2b} = (h-b) \cdot \binom{h-b-1}{2b-1}$  yields the additive decomposition  $s_h = p_h + q_h$ , where

$$p_h := \sum_{b=0}^{\lfloor h/3 \rfloor} (-1)^{h-b} \binom{h-b}{2b} U^b, \quad q_h := \frac{1}{2} \cdot \sum_{b=0}^{\lfloor h/3 \rfloor} (-1)^{h-b} \binom{h-b-1}{2b-1} U^b.$$

It is thus enough to prove that both  $(p_h)_{h \geq 1}$  and  $(q_h)_{h \geq 1}$  satisfy recurrence (1). We prove this for  $(p_h)_{h \geq 1}$ , the proof for  $(q_h)_{h \geq 1}$  being similar. Extracting the coefficient of  $U^n$  on both sides of (1) with  $(s_h)$  replaced by  $(p_h)$  is equivalent to

$$\binom{h+3-n}{2n} - 2 \binom{h+2-n}{2n} + \binom{h+1-n}{2n} = \binom{h+1-n}{2n-2}$$

and this identity is an immediate consequence of the Pascal triangle rule.  $\blacksquare$

COROLLARY 2. *For any  $h \geq 1$ , the rational number*

$$\sum_{b=0}^{\lfloor h/3 \rfloor} \frac{(-1)^{h-b}}{h-b} \binom{h-b}{2b} \left(\frac{2}{3}\right)^b$$

*is non-zero, except for  $h = 3$ .*

PROOF. With previous notation, we need to prove that  $a_h(2/3) = 0$  if and only if  $h = 3$ . By Theorem 1, the sequence

$$(u_h)_{h \geq 1} := (3^{h-1} \cdot h \cdot a_h(2/3))_{h \geq 1} = (-1, 3, 0, -45, 324, \dots)$$

satisfies the linear recurrence relation

$$(2) \quad u_{h+3} + 6u_{h+2} + 9u_{h+1} = 18u_h \quad \text{for all } h \geq 1.$$

It is clearly enough to prove that  $u_h = 0$  if and only if  $h = 3$ . First, the terms  $u_h$  are all integers, by induction. Recurrence (2) shows that  $u_{h+3}$  and  $u_{h+1}$  have the same parity for all  $h \geq 1$ ; since  $u_2 = 3$ , this implies that  $u_{2h}$  is an odd integer, and in particular it is non-zero, for all  $h \geq 1$ . It remains to consider the odd subsequence  $(v_h)_{h \geq 1} := (u_{2h-1})_{h \geq 1} = (-1, 0, 324, 5508, 2916, \dots)$ . From (2) it follows that the sequence  $(v_h)_{h \geq 1}$  satisfies the recurrence relation

$$v_{h+3} - 18v_{h+2} + 297v_{h+1} = 324v_h \quad \text{for all } h \geq 1.$$

The same recurrence is also satisfied with  $(v_h)_{h \geq 1}$  replaced by the sequence  $(w_h)_{h \geq 1} := (v_h/4)_{h \geq 3} = (81, 1377, 729, -369603, \dots)$ . In particular,  $w_{h+3}$  and  $w_{h+1}$  have the same parity for all  $h \geq 1$ , hence  $w_h$  is odd for any  $h \geq 1$ . It follows that  $v_h$  is non-zero for all  $h \geq 3$ , which concludes the proof.  $\blacksquare$

REMARK 3. An equivalent, equally simple, but slightly more “conceptual” proof of Theorem 1 is expressed in terms of generating functions. One starts with the Pascal triangle rule in its “generating function” form  $\sum_{a,b} \binom{a}{b} U^b z^a = 1/(1 - (1 + U)z)$ , then extracts odd and even parts (with respect to  $U$ ) from it,

$$\sum_{a,b} \binom{a}{2b} U^b z^a = \frac{1-z}{(1-z)^2 - Uz^2},$$

$$\sum_{a,b} \binom{a-1}{2b-1} U^b z^a = \frac{Uz^2}{(1-z)^2 - Uz^2},$$

and finally substitutes successively  $a \leftarrow h - b$ ,  $z \leftarrow -z$ ,  $U \leftarrow Uz$ ; this yields

$$\begin{aligned} \sum_{h \geq 1} s_h z^h &= \left( \frac{z+1}{(1+z)^2 - Uz^3} - 1 \right) + \frac{1}{2} \cdot \frac{Uz^3}{(1+z)^2 - Uz^3} \\ &= \frac{z+1 + Uz^3/2}{(1+z)^2 - Uz^3} - 1. \end{aligned}$$

Recurrence (1) is now read off the denominator of the last rational function.

REMARK 4. We leave it as an open problem to prove that the polynomials  $a_h(U)$  and  $s_h(U)$  are irreducible in  $\mathbb{Q}[U]$  for all  $h \geq 3$ . (Computer calculations show that this holds for  $3 \leq h \leq 10000$ .) If true, this would imply a generalization of Corollary 2.

#### REFERENCES

- [1] A. CONCA – C. KRATTENTHALER – J. WATANABE, Regular sequences of symmetric polynomials. *Rend. Semin. Mat. Univ. Padova* **121** (2009), 179–199. Zbl [1167.05051](#)  
MR [2542141](#)

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