Rend. Sem. Mat. Univ. Padova 150 (2023), 181–226
DOI 10.4171/RSMUP/124

Central algebraic geometry and seminormality

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Abstract – We develop the theory of central ideals on commutative domains. We introduce and study the central seminormalization of a ring in another one. This seminormalization is related to the theory of regulous functions on real algebraic varieties. We provide a construction of the central seminormalization by a decomposition theorem in elementary central gluings. The existence of a central seminormalization is established in the affine case and for real schemes.

Mathematics Subject Classification (2020) – Primary 14P99; Secondary 13B22, 26C15.

Keywords – Real algebraic geometry, normalization, continuous rational functions, regulous functions, seminormalization.

1. Introduction

The present paper is devoted to the study of the seminormalization in the real setting. The operation of seminormalization was formally introduced around fifty years ago first in the case of analytic spaces by Andreotti and Norguet [2] and later in the abstract scheme setting by Andreotti and Bombieri [1]. The notion arose from a classification problem. For algebraic varieties, the seminormalization of X in Y is basically the biggest intermediate variety which is bijective with X. Recently, the concept of seminormalization appears in the study of singularities of algebraic varieties, in particular in the minimal model program of Kollár and Kovács (see [14, 15]).

Around 1970 Traverso [26] introduced the closely related notion of the seminormalization A_B^* of a commutative ring A in an integral extension B. The idea is to glue together the prime ideals of B lying over the same prime ideal of A. The seminor-

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malization A_B^* has the property that it is the biggest extension of A in a subring C of B which is subintegral, i.e., such that the map $\operatorname{Spec} C \to \operatorname{Spec} A$ is bijective and equiresidual (it gives isomorphisms between the residue fields). For geometric rings all these notions of seminormalizations are equivalent and are strongly related with the Grothendieck notion of universal homeomorphism [13, I 3.8]. We refer to Vitulli [28] for a survey on seminormality for commutative rings and algebraic varieties. See also [11, 19, 25, 27] for more detailed information on seminormalization.

For an integral extension B of a commutative ring A, using the classical notion of real ideal [4], we may try to copy Traverso's construction by gluing together all the real prime ideals of B lying over the same real prime ideal of A. Unfortunately it does not give an acceptable notion of real seminormalization since real prime ideals do not satisfy a lying-over property for integral extensions. Normalization in the real setting is deeply studied in [9], the aim of the paper is to develop the theory of central seminormalization introduced in [10].

The paper is organized as follows. In Section 2 we recall some classical results on real algebra and more precisely about the theory of real ideals as it is developed in [4]. In Section 3 we introduce the notion of central ideal: If I is an ideal of an integral domain A with fraction field $\mathcal{K}(A)$, we say that I is a central ideal if for all $a \in A$ and $b \in \sum \mathcal{K}(A)^2$, where $\sum \mathcal{K}(A)^2$ is the set of sums of squares of elements in $\mathcal{K}(A)$, we have

$$a + b \in I \;\Rightarrow\; a \in I.$$

The origin of the adjective "central", which can be considered the key word in this paper, comes from [5], where it was proved that given an algebraic variety X defined over a subfield K of $\mathbb{R}$, if a point $x \in X$ belongs to the closure, in the euclidean topology, of the set of smooth points of local dimension $d := \dim(X)$ (the set of central points of X, see below), then there exists a real place centered at x. We develop the theory of central ideals similarly to the theory of real ideals done in [4] proving in particular that an ideal is central if and only if it is equal to its central radical (the intersection of the central prime ideals containing it). We prove that the notion of central ideal developed here is compatible for geometric rings (coordinate rings of affine variety over $\mathbb{R}$) with the central Nullstellensatz [4, Cor. 7.6.6] and also coincides for prime ideals with that of [10]. For a domain A, the central spectrum of A (the set of central prime ideals of A) is denoted by $\operatorname{C-Spec} A$. For an extension $A \to B$ of domains we show that we have a well-defined associated map $\operatorname{C-Spec} B \to \operatorname{C-Spec} A$.

In Section 4 we show that central ideals (that are real ideals) behave much better than real ideals when we consider integral extensions of rings. This is the principal reason we prefer working with central ideals in this paper. Especially, we have the following lying-over property: Let $A \to B$ be an integral and birational extension of domains (birational means $\mathcal{K}(A) \simeq \mathcal{K}(B)$), then $\operatorname{C-Spec} B \to \operatorname{C-Spec} A$ is surjective.

Regarding the classical case, we say that an extension $A \to B$ of domains is centrally subintegral if it is an integral extension such that the associated map C-Spec $B \to$ C-Spec A is bijective and equiresidual. Surprisingly, centrally subintegral extensions of geometric rings are strongly linked with the recent theory of rational continuous and regulous functions on real algebraic varieties introduced by Fichou, Huisman, Mangolte and the author [7] and by Kollár and Nowak [17]. Let X be an irreducible affine algebraic variety over $\mathbb{R}$ with coordinate ring $\mathbb{R}[X]$. The central locus Cent X is the subset of the set of real closed points $X(\mathbb{R})$ such that the associated ideal is central. By the central Nullstellensatz, Cent X coincides with the euclidean closure of the set of smooth real closed points. Following [10], we denote by $\mathcal{K}^0(\operatorname{Cent} X)$, called the ring of rational continuous functions on Cent X, the ring of continuous functions on Cent X that are rational on X. We denote by $\mathcal{R}^0(\operatorname{Cent} X)$, called the ring of regulous functions on Cent X, the subring of $\mathcal{K}^0(\operatorname{Cent} X)$ given by rational continuous functions that satisfies the additional property that they are still rational by restriction to any subvariety intersecting Cent X in maximal dimension. The link between centrally subintegral extensions and regulous functions is given by the following result: Given a finite morphism $\pi : Y \to X$ between two irreducible affine algebraic varieties over $\mathbb{R}$, $\pi^* : \mathbb{R}[X] \to \mathbb{R}[Y]$ is centrally subintegral if and only if the map $\pi_{|\operatorname{Cent} Y}$ Cent $Y \to$ Cent X is biregulous if and only if X and Y have the same regulous functions, i.e., the map $\mathcal{R}^0(\operatorname{Cent} X) \to \mathcal{R}^0(\operatorname{Cent} Y)$, $f \mapsto f \circ \pi_{|\operatorname{Cent} Y}$ is an isomorphism. The rational continuous and regulous functions are now extensively studied in real geometry, we refer for example to [8, 16, 18, 22] for further readings related to the subject.

Similarly to the standard case we prove in Section 5 that given an extension $A \to B$ of domains there is a biggest extension of A in a subring of B which is centrally subintegral. The target of this biggest extension is denoted by $A_B^{s_C,*}$ and is called the central seminormalization of A in B. This result is a deep generalization of [10, Prop. 2.23]. To get the existence of such seminormalization we have introduced and studied several concepts: the central gluing of an integral extension, the birational and birational-integral closure of a ring in another one.

In Section 6 we obtain the principal result of the paper. We have proved the existence of a central seminormalization of a ring in another one; but if we take an explicit geometric example, i.e., a finite extension of coordinate rings of two irreducible affine algebraic varieties over $\mathbb{R}$, due to the fact that when we do the central gluing, we glue together infinitely many ideals, then it is in general not easy to compute the central seminormalization. In the main result of the paper, we prove that, under reasonable hypotheses on the extension $A \to B$, we can obtain the central seminormalization $A_B^{s_C,*}$ from B by a birational gluing followed by a finite number of successive elementary central gluings almost like Traverso's decomposition theorem for classical seminormal extensions [26]. We use this construction to compute the central seminormalization in several examples. This decomposition result allows us to prove in Section 7 that the processes of central seminormalization and localization commute together. The proof of the decomposition theorem makes strong use of the results on central ideals developed in Section 3.

Section 8 is devoted to the existence, given a finite type morphism $\pi : Y \to X$ of irreducible affine algebraic varieties over $\mathbb{R}$ or integral schemes of finite type over $\mathbb{R}$, of a central seminormalization of X in Y denoted by $X_Y^{s_c,*}$. It can be seen as a real or central version of Andreotti and Bombieri's construction of the classical seminormalization of a scheme in another one [1]. We show that the ring $\mathbb{R}[X]^{s_c,*}_{\mathbb{R}[Y]}$ is a finitely generated algebra over $\mathbb{R}$ in the affine case and the $\mathcal{O}_X$-algebra $(\mathcal{O}_X)^{s_c,*}_{\pi_*\mathcal{O}_Y}$ is a coherent sheaf when we work with schemes. In the affine case, we prove that the coordinate ring of $X_Y^{s_c,*}$ is the integral closure of the coordinate ring of X in a certain ring of regulous functions generalizing one of the main results in [10].

2. Real algebra

Let A be ring. We assume in the paper that all the rings are commutative and contain $\mathbb{Q}$.

Recall that an ideal I of A is called real if, for every sequence $a_1, \ldots, a_k$ of elements of A, $a_1^2 + \cdots + a_k^2 \in I$ implies $a_i \in I$ for $i = 1, \ldots, k$. We denote by Spec A (resp. R-Spec A) the Zariski spectrum (resp. real Zariski spectrum) of A, i.e., the set of all prime (resp. real prime) ideals of A. The set of maximal (resp. maximal and real) ideals is denoted by Max A (resp. R-Max A). We endow Spec A with the Zariski topology whose closed subsets are given by the sets $\mathcal{V}(I) = \{\mathfrak{p} \in \operatorname{Spec} A \mid I \subset \mathfrak{p}\}$ where I is an ideal of A. If $f \in A$, we simply denote $\mathcal{V}((f))$ by $\mathcal{V}(f)$. The subsets R-Spec A, Max A and R-Max A of Spec A are endowed with the induced Zariski topology. The radical of I, denoted by $\sqrt{I}$, is defined as

$$\sqrt{I} = \{a \in A \mid \exists m \in \mathbb{N} \text{ such that } a^m \in I\},$$

which is also the intersection of the prime ideals of A that contain I. If B is a ring, we denote in the sequel by $\sum B^2$ the set of (finite) sums of squares of elements of B. The real radical of I, denoted by $\sqrt[R]{I}$, is defined as

$$\sqrt[R]{I} = \Big\{a \in A \mid \exists m \in \mathbb{N}\ \exists b \in \sum A^2 \text{ such that } a^{2m} + b \in I\Big\}.$$

Proposition 2.1 ([4, Prop. 4.1.7]). *We have:*

(1) $\sqrt[R]{I}$ *is the smallest real ideal of A containing I.*

(2) $\sqrt[R]{I} = \bigcap_{\mathfrak{p} \in \text{R-Spec}\, A,\, I \subset \mathfrak{p}} \mathfrak{p}$.

It follows that I is a real ideal if and only if $I = \sqrt[R]{I}$ and that a real ideal is radical.

An order α in A is given by a real prime ideal $\mathfrak{p}$ of A (called the support of α and denoted by $\operatorname{supp}(\alpha)$) and an ordering on the residue field $k(\mathfrak{p})$ at $\mathfrak{p}$. An order can equivalently be given by a morphism ϕ from A to a real closed field (the kernel is then the support). The set of orders of A is called the real spectrum of A and we denote it by $\operatorname{Spec}_r A$. One endows $\operatorname{Spec}_r A$ with a natural topology whose open subsets are generated by the sets $\{\alpha \in \operatorname{Spec}_r A \mid \alpha(a) > 0\}$. Let $\phi : A \to B$ be a ring morphism. It canonically induces continuous maps $\operatorname{Spec} B \to \operatorname{Spec} A$, R-Spec $B \to$ R-Spec A and $\operatorname{Spec}_r B \to \operatorname{Spec}_r A$.

Assume $X = \operatorname{Spec} \mathbb{R}[X]$ is an affine algebraic variety over $\mathbb{R}$ with coordinate ring $\mathbb{R}[X]$ (see [20] for a description of the different notions of real algebraic varieties), we denote by $X(\mathbb{R})$ the set of real closed points of X. We recall some classical notations. If $f \in \mathbb{R}[X]$, then $\mathcal{Z}(f) = \mathcal{V}(f) \cap X(\mathbb{R}) = \{x \in X(\mathbb{R}) \mid f(x) = 0\}$ is the real zero set of f. If A is a subset of $\mathbb{R}[X]$, then $\mathcal{Z}(A) = \bigcap_{f\in A} \mathcal{Z}(f)$ is the real zero set of A. If $W \subset X(\mathbb{R})$, then $\mathcal{I}(W) = \{f \in \mathbb{R}[X] \mid W \subset \mathcal{Z}(f)\}$ is an ideal, called the ideal of functions vanishing on W. We recall the real Nullstellensatz [4, Thm. 4.1.4]:

Theorem 2.2 (Real Nullstellensatz). *Let X be an affine algebraic variety over $\mathbb{R}$. Then*

$$I \subset \mathbb{R}[X] \text{ is a real ideal} \Leftrightarrow I = \mathcal{I}(\mathcal{Z}(I)).$$

Corollary 2.3. *Let X be an affine algebraic variety over $\mathbb{R}$. The map*

$$\text{R-Max}\,\mathbb{R}[X] \to X(\mathbb{R}), \quad \mathfrak{m} \mapsto \mathcal{Z}(\mathfrak{m})$$

is bijective.

In the sequel, we will identify R-Max $\mathbb{R}[X]$ and $X(\mathbb{R})$ for an affine algebraic variety X over $\mathbb{R}$. For any ideal $I \subset \mathbb{R}[X]$ we have

$$\mathcal{Z}(I) = \mathcal{V}(I) \cap \text{R-Max}\,\mathbb{R}[X].$$

We can endow $X(\mathbb{R})$ with the induced Zariski topology, the closed subsets are of the form $\mathcal{Z}(I)$ for I an ideal of A.

3. Central algebra

The goal of this section is to develop the theory of central ideals similarly to the theory of real ideals done in [4] or in the previous section. We also prove that the notion of central ideal developed here coincides for prime ideals with that of [10]. This section will serve as a basis for developing the theory of central seminormalization and especially to prove a central version of Traverso's decomposition theorem.

In this section, A is a domain containing $\mathbb{Q}$. We denote by $\mathcal{K}(A)$ its fraction field.

PROPOSITION 3.1. *The following properties are equivalent:*

(1) $-1 \notin \sum \mathcal{K}(A)^2$.

(2) $\operatorname{Spec}_r \mathcal{K}(A) \neq \emptyset$.

(3) (0) *is a real ideal of* $\mathcal{K}(A)$.

(4) (0) *is a real ideal of* A.

PROOF. See the first chapter of [4] to get the equivalence between the first three properties. Since the contraction of a real ideal is a real ideal, property (3) implies (4). Assume (0) is a real ideal of A and $-1 \in \sum \mathcal{K}(A)^2$. We have $-1 = \sum_{i=1}^n (\frac{a_i}{b_i})^2$ with the a_i and b_i in $A \setminus \{0\}$ and consequently

$$\left(\prod_{i=1}^n b_i\right)^2 + \sum_{i=1}^n \left(a_i\left(\prod_{j=1, j\neq i}^n b_j\right)\right)^2 = 0$$

and since (0) is a real ideal of A, it follows that $a_i = 0$ for $i = 1, \ldots, n$, which is a contradiction. ∎

In the sequel, we say that A is a real domain if the equivalent properties of Proposition 3.1 are satisfied. In case A is the coordinate ring $\mathbb{R}[X]$ of an irreducible affine algebraic variety X over $\mathbb{R}$, we simply denote $\mathcal{K}(\mathbb{R}[X])$ by $\mathcal{K}(X)$ and it corresponds to the field of classes of rational functions on X and we call it the field of rational functions on X or the function field of X.

We modify a bit the definition of a real ideal.

DEFINITION 3.2. Let I be an ideal of A. We say that I is central if for every $a \in A$ and $b \in \sum \mathcal{K}(A)^2$ we have

$$a^2 + b \in I \;\Rightarrow\; a \in I.$$

REMARK 3.3. Clearly, I is central $\Rightarrow$ I is real $\Rightarrow$ I is radical.

We suggest the reader to look at Example 3.11 in order to get an example of a real but non-central ideal.

REMARK 3.4. An ideal $I \subset A$ is central if and only if I is $(\sum \mathcal{K}(A)^2 \cap A)$-radical in the sense of [4].

Definition 3.5. (1) We denote by C-Spec A the set of central prime ideals of A.

(2) We denote by C-Max A the set of central and maximal ideals of A.

(3) We say that A is a central ring if any real ideal is central.

(4) Assume X is an irreducible affine algebraic variety over $\mathbb{R}$ and let $\mathbb{R}[X]$ be the coordinate ring of X. We denote by Cent X the image of C-Max $\mathbb{R}[X]$ by the bijection R-Max $\mathbb{R}[X] \to X(\mathbb{R})$. We call Cent X the set of central real closed points of X. We say that X is central if $X(\mathbb{R}) = \text{Cent}\, X$.

Let X be an affine algebraic variety over $\mathbb{R}$. We recall some classical notations. The set of points of $X(\mathbb{R})$ which are smooth in X is denoted by $X_{\text{reg}}(\mathbb{R})$. If $W \subset X(\mathbb{R})$, we denote by $\overline{W}^Z$ (resp. $\overline{W}^E$) the closure of W for the Zariski (resp. euclidean) topology.

Our definition of central ideals is chosen in order to satisfy the central Nullstellensatz stated in [4, Cor. 7.6.6].

Theorem 3.6 (Central Nullstellensatz). *Let X be an irreducible affine algebraic variety over $\mathbb{R}$. Then*

$$I \subset \mathbb{R}[X] \text{ is a central ideal } \Leftrightarrow I = \mathcal{I}\big(\mathcal{Z}(I) \cap \overline{X_{\text{reg}}(\mathbb{R})}^E\big)$$
$$\Leftrightarrow I = \mathcal{I}\big(\mathcal{V}(I) \cap \overline{X_{\text{reg}}(\mathbb{R})}^E\big)$$

Proof. We assume $I = \mathcal{I}(\mathcal{Z}(I) \cap \overline{X_{\text{reg}}(\mathbb{R})}^E)$ and $a^2 + b \in I$ for $a \in \mathbb{R}[X]$ and $b \in \sum \mathcal{K}(X)^2$. Since $a^2 + b \in I$, we have $a \in \mathcal{I}(\mathcal{Z}(I) \cap \overline{X_{\text{reg}}(\mathbb{R})}^E)$ by [4, Cor. 7.6.6]. By hypothesis, we get $a \in I$ and thus I is central.

Assume I is a central ideal of $\mathbb{R}[X]$. Let $a \in \mathcal{I}(\mathcal{Z}(I) \cap \overline{X_{\text{reg}}(\mathbb{R})}^E)$. By [4, Cor. 7.6.6], there exist $m \in \mathbb{N}$ and $b \in \sum \mathcal{K}(X)^2$ such that $a^{2m} + b \in I$. Since I central, $a^m \in I$. Since I is radical, it follows that $a \in I$.

To end the proof, we remark that $\mathcal{Z}(I) \cap \overline{X_{\text{reg}}(\mathbb{R})}^E = \mathcal{V}(I) \cap \overline{X_{\text{reg}}(\mathbb{R})}^E$. ■

From the previous theorem, it follows that our notion of central locus coincides with that of [4, Def. 7.6.3].

Corollary 3.7. *Let X be an irreducible affine algebraic variety over $\mathbb{R}$. Then*

$$\text{Cent}\, X = \overline{X_{\text{reg}}(\mathbb{R})}^E.$$

Corollary 3.8. *Let X be an irreducible affine algebraic variety over $\mathbb{R}$. The ring $\mathbb{R}[X]$ is central if and only if X is central.*

Proof. It follows from Theorems 2.2 and 3.6, that any real ideal of $\mathbb{R}[X]$ is central if and only if Cent $X = \overline{X_{\text{reg}}(\mathbb{R})}^E = X(\mathbb{R})$. ■

We prove that we recover the definition of central prime ideal given in [10].

Proposition 3.9. *Let X be an irreducible affine algebraic variety over $\mathbb{R}$. Let $\mathfrak{p} \in \operatorname{Spec} \mathbb{R}[X]$. The following properties are equivalent:*

(1) $\mathfrak{p} \in \text{C-Spec}\, \mathbb{R}[X]$.

(2) $\overline{\mathcal{Z}(\mathfrak{p}) \cap \operatorname{Cent} X}^{Z} = \mathcal{Z}(\mathfrak{p})$.

(3) $\overline{\mathcal{V}(\mathfrak{p}) \cap \operatorname{Cent} X}^{Z} = \mathcal{V}(\mathfrak{p})$.

(4) $\mathfrak{p} = \mathcal{I}(\mathcal{Z}(\mathfrak{p}) \cap \operatorname{Cent} X) = \mathcal{I}(\mathcal{V}(\mathfrak{p}) \cap \operatorname{Cent} X)$.

(5) $\mathfrak{p} = \{f \in \mathbb{R}[X] \mid \exists m \in \mathbb{N}\ \exists g \in \sum \mathcal{K}(X)^2$ *such that* $f^{2m} + g \in \mathfrak{p}\}$.

(6) *There exists an* $\alpha \in \operatorname{Spec}_r(\mathcal{K}(X) \cap \mathbb{R}[X])$ *which specializes in an order* β *with support* $\mathfrak{p}$*, i.e.,* β *is in the closure of the singleton* $\{\alpha\}$ *for the topology of* $\operatorname{Spec}_r \mathbb{R}[X]$.

Proof. The equivalence between (1) and (4) is given by Theorem 3.6.

The equivalence between (2) and (6) is [9, Lem. 2.9].

Let us prove that (4), (3) and (2) are equivalent. We remark that we always have $\mathfrak{p} \subset \mathcal{I}(\mathcal{Z}(\mathfrak{p})) \subset \mathcal{I}(\mathcal{Z}(\mathfrak{p}) \cap \operatorname{Cent} X)$ and $\mathfrak{p} \subset \mathcal{I}(\mathcal{V}(\mathfrak{p})) \subset \mathcal{I}(\mathcal{V}(\mathfrak{p}) \cap \operatorname{Cent} X)$. Thus if we assume that $\mathfrak{p} = \mathcal{I}(\mathcal{Z}(\mathfrak{p}) \cap \operatorname{Cent} X)$ (resp. $\mathfrak{p} = \mathcal{I}(\mathcal{V}(\mathfrak{p}) \cap \operatorname{Cent} X)$), then $\mathcal{I}(\mathcal{Z}(\mathfrak{p})) = \mathfrak{p}$ (resp. $\mathcal{I}(\mathcal{V}(\mathfrak{p})) = \mathfrak{p}$) and it follows that $\mathfrak{p}$ is a real ideal by the real Nullstellensatz (resp. $\mathfrak{p}$ is a radical ideal by the classical Nullstellensatz), in addition,

$$\mathcal{I}(\mathcal{Z}(\mathfrak{p})) = \mathcal{I}(\mathcal{Z}(\mathfrak{p}) \cap \operatorname{Cent} X) \quad (\text{resp. } \mathcal{I}(\mathcal{V}(\mathfrak{p})) = \mathcal{I}(\mathcal{V}(\mathfrak{p}) \cap \operatorname{Cent} X))$$

and it says that $\overline{\mathcal{Z}(\mathfrak{p}) \cap \operatorname{Cent} X}^{Z} = \mathcal{Z}(\mathfrak{p})$ (resp. $\overline{\mathcal{V}(\mathfrak{p}) \cap \operatorname{Cent} X}^{Z} = \mathcal{V}(\mathfrak{p})$). We have proved that (4) implies (2) and (3).

Assume (2) (resp. (3)) holds and let $f \in \mathbb{R}[X]$. It follows that $\mathcal{Z}(\mathfrak{p}) \subset \mathcal{Z}(f)$ (resp. $\mathcal{V}(\mathfrak{p}) \subset \mathcal{V}(f)$) if and only if $(\mathcal{Z}(\mathfrak{p}) \cap \operatorname{Cent} X) \subset \mathcal{Z}(f)$ (resp. $(\mathcal{V}(\mathfrak{p}) \cap \operatorname{Cent} X) \subset \mathcal{V}(f)$) and thus we get (4).

Clearly (5) implies (1). Assume $\mathfrak{p}$ is central. Let $f \in \mathbb{R}[X]$ such that there exist $m \in \mathbb{N}$ and $g \in \sum \mathcal{K}(X)^2$ satisfying $f^{2m} + g \in \mathfrak{p}$. Then $f^m \in \mathfrak{p}$ and thus $f \in \mathfrak{p}$ since $\mathfrak{p}$ is radical, it proves that (1) implies (5). ∎

Example 3.10. Let X be the Whitney umbrella, i.e., the real algebraic surface with equation $y^2 = zx^2$. Then $\mathfrak{p} = (x, y) \subset \mathbb{R}[X]$ is a central prime ideal since the stick of the umbrella, which is exactly the "z"-axis $= \mathcal{Z}(\mathfrak{p})$, meets $\operatorname{Cent} X$ in dimension one (the intersection is half of the stick).

Example 3.11. Let X be the Cartan umbrella, i.e., the real algebraic surface with equation $x^3 = z(x^2 + y^2)$. Then $\mathfrak{p} = (x, y) \subset \mathbb{R}[X]$ is a real prime ideal but not a central ideal by Proposition 3.9 (2) since the stick of the umbrella, which is exactly the

"z"-axis $= \mathcal{Z}(\mathfrak{p})$, meets Cent X in a single point. We prove now directly that $\mathfrak{p}$ is not central: We have

$$\begin{aligned} b = x^2 + y^2 - z^2 &= x^2 + y^2 - \frac{x^6}{(x^2+y^2)^2} \\ &= \frac{3x^4y^2 + 3x^2y^4 + y^6}{(x^2+y^2)^2} \in \left(\sum \mathcal{K}(X)^2\right) \cap \mathbb{R}[X], \end{aligned}$$

thus $z^2 + b = x^2 + y^2 \in \mathfrak{p}$ but $z \notin \mathfrak{p}$. This example shows that even the "trivial" relationship between positivity and sums of squares is not so trivial: b is a sum of squares in $\mathcal{K}(X)$ but it is negative on the stick outside the origin.

We give a central version of [4, Lem. 4.1.5].

Proposition 3.12. *Assume A is noetherian. If $I \subset A$ is a central ideal, then the minimal prime ideals containing I are central ideals.*

Proof. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_l$ be the minimal prime ideals containing I. If $l = 1$, then $I = p_1$ since I is radical and thus the proof is done in this case. So we assume $l > 1$ and $\mathfrak{p}_1$ is not central. There exist $a \in A \setminus \mathfrak{p}_1$, $b_1, \dots, b_k \in \mathcal{K}(A)$ such that $a^2 + b_1^2 + \cdots + b_k^2 \in \mathfrak{p}_1$. We choose $a_2, \dots, a_l$ such that $a_i \in \mathfrak{p}_i \setminus \mathfrak{p}_1$ and we set $c = \prod_{i=2}^{l} a_i$. Then $(ac)^2 + (b_1c)^2 + \cdots + (b_kc)^2 \in \bigcap_{i=1,\dots,l} \mathfrak{p}_i = I$ (I is radical). Thus $ac \in \mathfrak{p}_1$, a contradiction. ∎

Definition 3.13. Let $I \subset A$ be an ideal. We define the central radical of I, denoted by $\sqrt[C]{I}$, as follows:

$$\sqrt[C]{I} = \left\{a \in A \mid \exists m \in \mathbb{N}\ \exists b \in \sum \mathcal{K}(A)^2 \text{ such that } a^{2m} + b \in I\right\}.$$

We give a central version of Proposition 2.1.

Proposition 3.14. *Let $I \subset A$ be an ideal. Then:*

(1) *$\sqrt[C]{I}$ is the smallest central ideal of A containing I.*

(2) $\sqrt[C]{I} = \bigcap_{\mathfrak{p} \in \text{C-Spec}\, A,\, I \subset \mathfrak{p}} \mathfrak{p}$.

Proof. We show that $\sqrt[C]{I}$ is an ideal. It is clear that $0 \in \sqrt[C]{I}$. Let $a \in \sqrt[C]{I}$. There exist $m \in \mathbb{N}, b_1, \dots, b_k \in \mathcal{K}(A)$ such that $a^{2m} + b_1^2 + \cdots + b_k^2 \in I$. Let $a' \in A$. Since $(aa')^{2m} + (b_1(a')^m)^2 + \cdots + (b_k(a')^m)^2 \in I$ and since $b_i(a')^m \in \mathcal{K}(A)$, we have $aa' \in \sqrt[C]{I}$. To show that $\sqrt[C]{I}$ is closed under addition, copy the proof of [4, Prop. 4.1.7] with the conditions that the b_i and b'_j are only in $\mathcal{K}(A)$ rather than in A.

We show that $\sqrt[C]{I}$ is a central ideal. Let $a \in A$ and $b_1, \dots, b_k \in \mathcal{K}(A)$ such that $a^2 + b_1^2 + \cdots + b_k^2 \in \sqrt[C]{I}$. Thus there exist $m \in \mathbb{N}$ and $c_1, \dots, c_l \in \mathcal{K}(A)$

such that $(a^2 + b_1^2 + \cdots + b_k^2)^{2m} + c_1^2 + \cdots + c_l^2 \in I$. It follows that there exist $d_1, \ldots, d_t \in \mathcal{K}(A)$ such that $a^{4m} + d_1^2 + \cdots + d_t^2 \in I$ and thus $a \in \sqrt[C]{I}$.

Let J be a central ideal of A containing I. Let $a \in \sqrt[C]{I}$. There exist $m \in \mathbb{N}$, $b \in \sum \mathcal{K}(A)^2$ such that $a^{2m} + b \in J$. Thus $a^m \in J$ by centrality of J and finally $a \in J$ by radicality of J. The proof of (1) is done.

We denote by I' the ideal

$$\bigcap_{\mathfrak{p} \in \text{C-Spec}\, A,\, I \subset \mathfrak{p}} \mathfrak{p}.$$

From (1) we get $\sqrt[C]{I} \subset I'$. Let us show the converse inclusion. Let $a \in A \setminus \sqrt[C]{I}$. Let J be maximal among the central ideals containing I but not a. If J is not prime, then, following the proof of [4, Prop. 4.1.7], we can find $m \in \mathbb{N}$, $b \in \sum \mathcal{K}(A)^2$ such that $a^{2m} + b \in I$, which gives a contradiction. Hence J is a prime ideal and thus $a \notin I'$. ∎

Corollary 3.15. *Let $I \subset A$ be an ideal. Then, I is a central ideal if and only if $I = \sqrt[C]{I}$.*

To end this section, we study the existence of a central ideal.

Proposition 3.16. *The following properties are equivalent:*

(1) *A is a real domain.*

(2) C-Spec $A \neq \emptyset$.

(3) *A has a proper central ideal.*

(4) (0) *is a central ideal of A.*

Assume A is the coordinate ring of an irreducible affine algebraic variety X over $\mathbb{R}$. Then the previous properties are equivalent to the two following ones:

(5) $X_{\mathrm{reg}}(\mathbb{R}) \neq \emptyset$.

(6) *$X(\mathbb{R})$ is Zariski dense in $X(\mathbb{C})$ and* Spec A.

Proof. It is clear that (4) implies (2) and (3). By Proposition 3.1, we see that (4) implies (1). Assume A is a real domain. By Proposition 3.1, we know that (0) is a real ideal and we will prove that, in addition, it is a central ideal. Assume $a^2 + b = 0$ with $a \in A$ and $b \in \sum \mathcal{K}(A)^2$. It gives an identity $c^2a^2 + s = 0$ with $c \in A \setminus \{0\}$ and $s \in \sum A^2$. Since (0) is a real ideal, it follows that $a = 0$. We get (1) implies (4). Since a prime ideal is proper, (2) implies (3). Assume $I \subset A$ is a proper central ideal. By Corollary 3.15, we have $I = \sqrt[C]{I}$. By Proposition 3.14, I is the intersection of the central prime ideals of A containing I, it follows that the set of central prime ideals of A containing I is non-empty and (3) implies (2). Let $I \subset A$ be a proper and central ideal of A. Assume A is not a real domain. By Proposition 3.1, we get that $-1 \in \sum \mathcal{K}(A)^2$

and since $1^2 + (-1) = 0 \in I$ and I is central, it follows that $1 \in I$, which is impossible. Thus (3) implies (1).

Assume A is the coordinate ring of an irreducible affine algebraic variety X over $\mathbb{R}$. Assume A is a real domain. We have proved that it implies that (0) is a central ideal. By Proposition 3.9 (3), it follows that $\operatorname{Cent} X$ is Zariski dense in $\operatorname{Spec} A$. Hence $X(\mathbb{R})$ is also Zariski dense in $\operatorname{Spec} A$ (and in $X(\mathbb{C})$). It proves that (1) implies (6). If $X_{\mathrm{reg}}(\mathbb{R}) \neq \varnothing$, then $\operatorname{Cent} X \neq \varnothing$ and thus (5) implies (2). Assume $X(\mathbb{R})$ is Zariski dense in $\operatorname{Spec} A$, then it intersects the set of regular prime ideals of A which is a non-empty Zariski open subset of $\operatorname{Spec} A$, and thus (6) implies (5). ∎

4. Integral extensions and lying-over properties

In the sequel, we consider rings up to isomorphisms and affine algebraic varieties up to isomorphisms. In particular, when we write an equality of rings it means they are isomorphic, the reader should remember this especially when speaking about uniqueness.

4.1 – *Integral extensions and normalization*

Let $A \to B$ be an extension of domains. The extension is said to be of finite type (resp. finite) if it makes B a finitely generated A-algebra (resp. A-module). We say that $A \to B$ is birational if it induces an isomorphism between the fraction fields $\mathcal{K}(A)$ and $\mathcal{K}(B)$. We say that an element $b \in B$ is integral over A if b is the root of a monic polynomial with coefficients in A. By [3, Prop. 5.1], b is integral over A if and only if $A[b]$ is a finite A-module. This equivalence allows us to prove that $A'_B = \{b \in B \mid b \text{ is integral over } A\}$ is a ring called the integral closure of A in B. The extension $A \to B$ is said to be integral if $A'_B = B$. In case $B = \mathcal{K}(A)$, the ring $A'_{\mathcal{K}(A)}$ is denoted by A' and is simply called the integral closure of A. The ring A is called integrally closed (resp. in B) if $A = A'$ (resp. $A = A'_B$). If A is the coordinate ring of an irreducible affine algebraic variety X over a field k, then A' is a finite A-module (a theorem of Emmy Noether [6, Thm. 4.14]) and thus it is a finitely generated k-algebra and so A' is the coordinate ring of an irreducible affine algebraic variety over k, denoted by X', called the normalization of X. For a morphism $\pi : X \to Y$ between two affine algebraic varieties over a field k, we denote by $\pi^* : k[Y] \to k[X]$, $f \mapsto f \circ \pi$ the associated ring morphism. We recall that a morphism $X \to Y$ between two irreducible affine algebraic varieties over a field k is said to be of finite type (resp. finite) (resp. birational) if the ring morphism $k[Y] \to k[X]$ is of finite type (resp. finite) (resp. birational). The inclusion $k[X] \subset k[X'] = k[X]'$ induces a finite and birational morphism which we denote by $\pi' : X' \to X$, called the normalization morphism. We say that an irreducible affine algebraic variety X over a field k is normal if its coordinate ring is integrally closed.

4.2 – *Contraction and lying-over properties*

For an extension of rings, it is clear that the contraction of a real ideal is a real ideal. We prove that, for an extension of domains, the contraction of a central ideal remains central.

Proposition 4.1. *Let $A \to B$ be an extension of domains. If I is a central ideal of B, then $I \cap A$ is a central ideal of A. In particular, the map*

$$\text{C-Spec}\, B \to \text{C-Spec}\, A, \quad \mathfrak{q} \mapsto \mathfrak{q} \cap A$$

is well defined.

Proof. The proof is clear since $\sum \mathcal{K}(A)^2 \subset \sum \mathcal{K}(B)^2$. ∎

Remark 4.2. As noticed in [10] the result of the previous proposition cannot be generalized in the reducible case (even for extensions of reduced rings with a finite number of minimal prime ideals that are real) and it is the reason we restrict ourself to extension of domains in this paper. There are some problems if for example the contraction of a minimal prime ideal of B is not a minimal prime ideal of A. Consider the extension

$$A = \mathbb{R}[C] = \mathbb{R}[x,y]/(y^2 - x^2(x-1)) \to B = \mathbb{R}[C] \times (\mathbb{R}[C]/(x,y)),$$
$$f \mapsto (f, f(0,0)).$$

The extension $A \to B$ is associated to the morphism of affine algebraic varieties $C' \to C$ where C is the plane cubic with a real isolated point, C' is the disjoint union of C and a real point, the morphism is the identity on C and maps the point onto the origin. The contraction to A of the minimal and central prime ideal $\mathbb{R}[C] \times (0)$ (central here means central in its irreducible component) of B is the real maximal ideal corresponding to the isolated real point and thus the contracted ideal is not central (Proposition 3.9).

We have the following lying-over properties:

Proposition 4.3. *Let $A \to B$ be an integral extension of domains. Then:*

(1) $\operatorname{Spec} B \to \operatorname{Spec} A$, $\mathfrak{q} \mapsto \mathfrak{q} \cap A$ *is surjective.*

(2) $\operatorname{Max} B \to \operatorname{Max} A$ *is well defined and surjective.*

(3) *If* $A \to B$ *is birational, then the map* $\text{C-Spec}\, B \to \text{C-Spec}\, A$ *is surjective.*

(4) *If* $A \to B$ *is birational, then the map* $\text{C-Max}\, B \to \text{C-Max}\, A$ *is well defined and surjective.*

Proof. See [21, Thm. 9.3] or [3, Thm. 5.10, Cor. 5.8] for statements (1) and (2). From Proposition 4.1 and [10, Prop. 2.8] we get (3) in the case A is a real domain. Assume A is a domain but $\mathcal{K}(A)$ is not real and $A \to B$ is integral and birational. Then $\mathcal{K}(B)$ is not real and, by Proposition 3.16, $\text{C-Spec}\, A = \text{C-Spec}\, B = \varnothing$ and we get (3) in this case. Statement (4) is a consequence of (2) and (3). ∎

Remark 4.4. We do not have a lying-over property for real prime ideals even for birational extensions. Consider for example the integral and birational extension $A = \mathbb{R}[x, y]/(y^2 - x^2(x-1)) \to \mathbb{R}[x, Y]/(Y^2 - (x-1)) = B$ given by $x \mapsto x$ and $y \mapsto Yx$. The extension is integral and birational since it corresponds to the normalization of the plane cubic curve with a real isolated node given by the equation $y^2 - x^2(x-1) = 0$ and thus $B = A'$. Over the real but not central ideal (x, y) in A there is a unique ideal of B given by $(x, Y^2 + 1)$ and this ideal is not real. This is the principal reason we work here with the central spectrum rather than the real spectrum.

Remark 4.5. Consider for example the integral extension

$$A = \mathbb{R}[x] \to \mathbb{R}[x, y]/(y^2 - x) = B.$$

We do not have any real prime ideal of B lying over the real and central prime ideal $(x+1)$ of A. This example shows that we do not have a central lying-over property for integral extensions of domains which are not birational. From [10], the central lying-over property exists more generally for an integral extension $A \to B$ of domains such that $\operatorname{Spec}_r \mathcal{K}(B) \to \operatorname{Spec}_r \mathcal{K}(A)$ is surjective.

5. Central seminormalization for rings

5.1 – *Centrally subintegral extension*

Recall from [28] that an extension $A \to B$ is called subintegral if it is an integral extension, for any prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ there exists a unique prime ideal $\mathfrak{q} \in \operatorname{Spec} B$ lying over $\mathfrak{p}$ (it means $\operatorname{Spec} B \to \operatorname{Spec} A$ is bijective), and furthermore for any such pair $\mathfrak{p}$, $\mathfrak{q}$ the induced injective map $k(\mathfrak{p}) \to k(\mathfrak{q})$ on the residue fields is an isomorphism. To characterize the last property, we say that $\operatorname{Spec} B \to \operatorname{Spec} A$ is equiresidual. In summary an integral extension $A \to B$ is subintegral if and only if Spec $B \to$ Spec A is bijective and equiresidual. Such a concept is related to the notion of “radiciel” morphism of schemes introduced by Grothendieck [13, I Def. 3.7.2]. Swan gave another nice characterization of a subintegral extension [25, Lem. 2.1]: an extension $A \to B$ is subintegral if B is integral over A and for all morphisms $A \to K$ into a field K, there exists a unique extension $B \to K$.

In the same spirit, we can give a natural definition of a central subintegral extension.

Definition 5.1. Let $A \to B$ be an extension of domains. We say that $A \to B$ is centrally subintegral or s_c-subintegral for short (we follow the notation used in [10]) if it is an integral extension, and if the map C-Spec $B \to$ C-Spec A is bijective and equiresidual.

Remark 5.2. From Propositions 3.16 and 4.1, any integral extension $A \to B$ of a non-real domain A is trivially centrally subintegral since C-Spec $A =$ C-Spec $B = \emptyset$.

Remark 5.3. Let $A \to B$ be a centrally subintegral extension of domains and assume A is real. By property (4) of Proposition 3.16, (0) is a central ideal of A. Since the null ideal of B is the unique prime ideal of B lying over the null ideal of A, bijectivity of the central spectra implies that (0) is also a central ideal of B. By equiresiduality, the extension $A \to B$ is birational.

Example 5.4. The finite extension $A = \mathbb{R}[x] \to \mathbb{R}[x, y]/(y^2 - x^3) = B$ satisfies the property that C-Max $B \to$ C-Max A is bijective and equiresidual but $A \to B$ is not birational and so $A \to B$ is not centrally subintegral.

Example 5.5. The finite extension $\mathbb{R}[x, y]/(y^2 - x^3) \to \mathbb{R}[t]$ given by $x \mapsto t^2$ and $y \mapsto t^3$ (corresponding to the normalization of the cuspidal curve) is centrally subintegral.

From [25, Lem. 2.1] we derive another characterization of a centrally subintegral extension.

Proposition 5.6. *An extension $A \to B$ is centrally subintegral if B is integral over A and for all morphisms $\varphi : A \to K$ into a field K with* $\ker \varphi \in$ C-Spec A*, there exists a unique extension $\psi : B \to K$ such that* $\ker \psi \in$ C-Spec B.

We want now to characterize differently these centrally subintegral extensions in the case we work with geometric rings.

Let X be an irreducible affine algebraic variety over $\mathbb{R}$. A real (resp. irreducible real) algebraic subvariety V of X is a closed Zariski subset of Spec $\mathbb{R}[X]$ of the form $V = \mathcal{V}(I) = \{\mathfrak{p} \in \operatorname{Spec} \mathbb{R}[X] \mid I \subset \mathfrak{p}\} \simeq \operatorname{Spec}(\mathbb{R}[X]/I)$ for I an ideal (resp. prime ideal) of $\mathbb{R}[X]$. In this case, the real part of V, denoted by $V(\mathbb{R})$, is the closed Zariski subset of $X(\mathbb{R})$ given by $\mathcal{Z}(I)$. An algebraic subvariety V of X is called central in X if $V = \mathcal{V}(I) \simeq \operatorname{Spec}(\mathbb{R}[X]/I)$ for I a central ideal in $\mathbb{R}[X]$. By Theorem 3.6, an irreducible real algebraic subvariety V of X is central in X if and only if $\overline{V(\mathbb{R}) \cap \operatorname{Cent} X}^{Z} = V(\mathbb{R})$.

Remark 5.7. The stick is central in the Whitney umbrella but it is not the case in the Cartan umbrella.

Remark 5.8. For an irreducible real algebraic subvariety V of X, the properties "V is central" and "V is central in X" are distinct. As example, take the stick of the Cartan umbrella.

Definition 5.9. Let $\pi : Y \to X$ be a dominant morphism between irreducible affine algebraic varieties over $\mathbb{R}$. We say that π is centrally subintegral or s_c-subintegral if the extension $\pi^* : \mathbb{R}[X] \to \mathbb{R}[Y]$ is s_c-subintegral.

Let $\pi : Y \to X$ be a dominant morphism between irreducible affine algebraic varieties over $\mathbb{R}$. By Proposition 4.1, we have an associated map C-Spec $\mathbb{R}[Y] \to$ C-Spec $\mathbb{R}[X]$. We say that $\pi : Y \to X$ is centrally hereditarily birational if for any irreducible real algebraic subvariety $V = \mathcal{V}(\mathfrak{p}) \simeq \operatorname{Spec}(\mathbb{R}[Y]/\mathfrak{p})$ central in Y, the morphism $\pi_{|V} : V \to W = \mathcal{V}(\mathfrak{p} \cap \mathbb{R}[X]) \simeq \operatorname{Spec}(\mathbb{R}[X]/(\mathfrak{p} \cap \mathbb{R}[X]))$ is birational, i.e., the extension $k(\mathfrak{p} \cap \mathbb{R}[X]) = \mathcal{K}(W) \to k(\mathfrak{p}) = \mathcal{K}(V)$ is an isomorphism. By Proposition 3.16, a centrally hereditarily birational morphism $Y \to X$ is birational if $X_{\mathrm{reg}}(\mathbb{R}) \neq \varnothing$. From the above remarks we easily get:

Proposition 5.10. *Let $\pi : Y \to X$ be a dominant morphism between irreducible affine algebraic varieties over $\mathbb{R}$. The following properties are equivalent:*

(1) *The morphism $\pi : Y \to X$ is centrally hereditarily birational.*

(2) *The map* C-Spec $\mathbb{R}[Y] \to$ C-Spec $\mathbb{R}[X]$ *is equiresidual.*

From Proposition 4.3, with an additional finiteness hypothesis we get:

Corollary 5.11. *Let $\pi : Y \to X$ be a finite morphism between irreducible affine algebraic varieties over $\mathbb{R}$. The following properties are equivalent:*

(1) *The morphism $\pi : Y \to X$ is centrally hereditarily birational and the map* C-Spec $\mathbb{R}[Y] \to$ C-Spec $\mathbb{R}[X]$ *is bijective.*

(2) *π is s_c-subintegral.*

Let X be an irreducible affine algebraic variety over $\mathbb{R}$ such that $X_{\mathrm{reg}}(\mathbb{R}) \neq \varnothing$. Following [10], we denote by $\mathcal{K}^0(\operatorname{Cent} X)$, called the ring of rational continuous functions on Cent X, the ring of continuous functions on Cent X that are rational on X, i.e., coincide with a regular function on a non-empty Zariski open subset of $X(\mathbb{R})$ intersected with $\mathrm{Cent}\, X$. We denote by $\mathcal{R}^0(\mathrm{Cent}\, X)$, called the ring of regulous functions on $\mathrm{Cent}\, X$, the subring of $\mathcal{K}^0(\mathrm{Cent}\, X)$ given by rational continuous functions $f \in \mathcal{K}^0(\mathrm{Cent}\, X)$ that satisfies the additional property that for any irreducible real algebraic subvariety $V = \mathcal{V}(\mathfrak{p})$ of X for $\mathfrak{p} \in \text{C-Spec}\, \mathbb{R}[X]$, the restriction of f to $V(\mathbb{R}) \cap \mathrm{Cent}\, X$ is rational on V, i.e., lies in $k(\mathfrak{p})$. Let us remark that for a variety with at least a smooth real closed point, being rational, rational on the real closed points or rational on the central closed points is the same (Proposition 3.16).

Let $\pi : Y \to X$ be a finite and birational morphism between irreducible affine algebraic varieties over $\mathbb{R}$. By Proposition 4.3, we have associated surjective maps $\text{C-Spec}\, \mathbb{R}[Y] \to \text{C-Spec}\, \mathbb{R}[X]$ and $\mathrm{Cent}\, Y \to \mathrm{Cent}\, X$. The composition by π induces natural morphisms $\mathcal{K}^0(\mathrm{Cent}\, X) \to \mathcal{K}^0(\mathrm{Cent}\, Y)$ and $\mathcal{R}^0(\mathrm{Cent}\, X) \to \mathcal{R}^0(\mathrm{Cent}\, Y)$. We say that the map $\mathrm{Cent}\, Y \to \mathrm{Cent}\, X$ is biregulous if it is bijective and the inverse bijection is a regulous map, i.e., its components are in $\mathcal{R}^0(\mathrm{Cent}\, X)$. Such a concept is related to Grothendieck's notion of universal homeomorphism between schemes [13, I 3.8].

The following result from [10] explains how s_c-subintegral extensions and regulous functions are related.

Theorem 5.12 ([10, Lem. 3.13, Thm. 3.16]). *Let $\pi : Y \to X$ be a finite and birational morphism between irreducible affine algebraic varieties over $\mathbb{R}$. The following properties are equivalent:*

(1) *π is s_c-subintegral.*

(2) *The morphism $\pi : Y \to X$ is centrally hereditarily birational and the map $\text{C-Spec}\, \mathbb{R}[Y] \to \text{C-Spec}\, \mathbb{R}[X]$ is bijective.*

(3) *$\mathcal{R}^0(\mathrm{Cent}\, X) \to \mathcal{R}^0(\mathrm{Cent}\, Y)$, $f \mapsto f \circ \pi_{|\mathrm{Cent}\, Y}$ is an isomorphism.*

(4) *The map $\pi_{|\mathrm{Cent}\, Y} : \mathrm{Cent}\, Y \to \mathrm{Cent}\, X$ is biregulous.*

(5) *For all $g \in \mathbb{R}[Y]$ there exists $f \in \mathcal{R}^0(\mathrm{Cent}\, X)$ such that $f \circ \pi_{|\mathrm{Cent}\, Y} = g$ on $\mathrm{Cent}\, Y$.*

Remark 5.13. All these equivalent properties are trivially satisfied if $\mathrm{Cent}\, X = \varnothing$ (Remark 5.2).

5.2 – *Classical algebraic seminormalization*

We recall in this section the principal result obtained by Traverso [26] concerning the seminormality of a ring in another one.

Definition 5.14. A ring C is called intermediate between the rings A and B if there exists a sequence of extensions $A \to C \to B$. In this case, we say that $A \to C$

and $C \to B$ are intermediate extensions of $A \to B$ and, in addition, that $A \to C$ is a subextension of $A \to B$.

Seminormal extensions are maximal subintegral extensions.

Definition 5.15. Let $A \to C \to B$ be a sequence of two extensions of rings. We say that C is seminormal between A and B if $A \to C$ is subintegral and, in addition, if for every intermediate domain D between C and B with $C \subsetneq D$, the extension $A \to D$ is not subintegral. We say that A is seminormal in B if A is seminormal between A and B.

Definition 5.16. Let A be a ring and let I be an ideal of A. The Jacobson radical of A, denoted by $\operatorname{Rad}(A)$, is the intersection of the maximal ideals of A.

For a given extension of rings $A \to B$, Traverso (see [26] or [28]) proved that there exists a unique intermediate ring which is seminormal between A and B.

Theorem 5.17. *Let $A \to B$ be an extension of rings. There exists a unique ring A_B^* between A and B, which is seminormal between A and B and satisfies*

$$A_B^* = \{b \in A'_B \mid \forall \mathfrak{p} \in \operatorname{Spec} A,\ b_{\mathfrak{p}} \in A_{\mathfrak{p}} + \operatorname{Rad}((A'_B)_{\mathfrak{p}})\}.$$

This ring is called the seminormalization of A in B.

We remark that to build A_B^*, for all $\mathfrak{p} \in \operatorname{Spec} A$, we glue together all the prime ideals of A'_B lying over $\mathfrak{p}$.

5.3 – *Introduction to the central algebraic seminormalization existence problem*

Intermediate extensions of a centrally subintegral extension are still centrally subintegral extensions:

Lemma 5.18. *Let $A \to C \to B$ be a sequence of extensions of domains. Then $A \to B$ is s_c-subintegral if and only if $A \to C$ and $C \to B$ are both s_c-subintegral.*

Proof. Assume $A \to B$ is s_c-subintegral. Clearly, $A \to C$ and $B \to C$ are both integral extensions. It follows that C-Spec $B \to$ C-Spec A is bijective and equiresidual. If A is not a real domain, then it follows from Remark 5.2 that $A \to C$ and $C \to B$ are trivially s_c-subintegral. Assume now A is a real domain. By equiresiduality (Remark 5.3), $A \to B$ is birational and thus $A \to C$ and $C \to B$ are also both birational. By Proposition 4.3, the maps C-Spec $B \to$ C-Spec C and C-Spec $C \to$ C-Spec A are surjective, and since the composition is bijective, they are both bijective. Let

$\mathfrak{q} \in \text{C-Spec}\, B$; then we have the sequence $k(\mathfrak{q} \cap A) \to k(\mathfrak{q} \cap C) \to k(\mathfrak{q})$ of extensions of residue fields, showing that $\text{C-Spec}\, B \to \text{C-Spec}\, C$ and $\text{C-Spec}\, C \to \text{C-Spec}\, A$ are both equiresidual.

The converse implication is clear. ∎

By Lemma 5.18, a subextension of a centrally subintegral extension is still centrally subintegral, so we may consider maximal centrally subintegral subextensions.

Definition 5.19. Let $A \to C \to B$ be a sequence of two extensions of domains. We say that C is centrally seminormal (or s_c-normal for short) between A and B if $A \to C$ is s_c-subintegral and, in addition, if for every intermediate domain C' between C and B with $C \neq C'$, the extension $A \to C'$ is not s_c-subintegral. We say that A is s_c-normal in B if A is s_c-normal between A and B.

From Lemma 5.18, we get an equivalent definition of a centrally seminormal ring (between A and B):

Proposition 5.20. *Let $A \to C \to B$ be a sequence of two extensions of domains. Then, C is s_c-normal between A and B if and only if $A \to C$ is s_c-subintegral and C is s_c-normal in B.*

From Definition 5.19 we easily deduce the following property:

Proposition 5.21. *Let $A \to C \to B$ be a sequence of two extensions of domains. If A is s_c-normal in B, then A is s_c-normal in C.*

In view of the classical case (see the previous section), we state the following problem:

Given an extension $A \to B$ of domains, is there a unique intermediate domain C which is s_c-normal between A and B?

We define the central seminormalization (or s_c-normalization) of A in B as the ring which would give a solution to this problem. In the classical case, the problem is solved by Theorem 5.17.

Definition 5.22. Let $A \to B$ be an extension of domains. In case there exists a unique maximal element among the intermediate domains C between A and B such that $A \to C$ is s_c-subintegral, we denote it by $A_B^{s_c,*}$ and we call it the central seminormalization or s_c-normalization of A in B. In case $B = A'$, we omit B and we call $A^{s_c,*}$ the s_c-normalization of A.

The existence of a central seminormalization is already proved in [10] in the special case $B = A'$.

5.4 – *Central gluing over a ring*

Definition 5.23. Let A be a ring.

(1) The real Jacobson radical of A, denoted by $\mathrm{Rad}^{\mathrm{R}}(A)$, is the intersection of the maximal and real ideals of A.

(2) The central Jacobson radical of A, denoted by $\mathrm{Rad}^{\mathrm{C}}(A)$, is the intersection of the maximal and central ideals of A.

In view of the classical case (see Theorem 5.17), a candidate for the s_c-normalization of A in B when $A \to B$ is integral is the following ring.

Definition 5.24. Let $A \to B$ be an integral extension of domains. The ring

$$A_B^{s_c} = \{b \in B \mid \forall \mathfrak{p} \in \text{C-Spec}\, A,\ b_{\mathfrak{p}} \in A_{\mathfrak{p}} + \mathrm{Rad}^{\mathrm{C}}(B_{\mathfrak{p}})\}$$

is called the central gluing of B over A.

The central gluing is not the s_c-normalization.

Example 5.25. Consider the finite extension $A = \mathbb{R}[x] \to \mathbb{R}[x,y]/(y^2 + x^2 + 1) = B$. Then $A_B^{s_c} = B$ since $\text{C-Spec}\, B = \varnothing$ and $A \to B$ is not centrally subintegral.

In the following, if $\mathfrak{r}$ is a prime ideal of a ring C, we denote by $c(\mathfrak{r})$ the class of $c \in C$ in $k(\mathfrak{r})$.

The central gluing satisfies the following universal property again related to the notion of "radiciel" morphism of schemes introduced by Grothendieck [13, I Def. 3.7.2]:

Theorem 5.26. *Let $A \to B$ be an integral extension of domains. The central gluing $A_B^{s_c}$ of B over A is the biggest intermediate ring C between A and B satisfying the following properties:*

(1) *If $\mathfrak{q}_1, \mathfrak{q}_2 \in \text{C-Spec}\, B$ lie over $\mathfrak{p} \in \text{C-Spec}\, A$, then $\mathfrak{q}_1 \cap C = \mathfrak{q}_2 \cap C$.*

(2) *If $\mathfrak{q} \in \text{C-Spec}\, B$, then the residue fields extension $k(\mathfrak{q} \cap A) \to k(\mathfrak{q} \cap C)$ is an isomorphism.*

Proof. We first prove that $A_B^{s_c}$ satisfies (1) and (2). Let $\mathfrak{p} \in \text{C-Spec}\, A$ and let $\mathfrak{q}_1, \mathfrak{q}_2 \in \text{C-Spec}\, B$ lying over $\mathfrak{p}$. Since $\mathfrak{q}_1 B_{\mathfrak{p}}$ and $\mathfrak{q}_2 B_{\mathfrak{p}}$ are two maximal and central ideals of $B_{\mathfrak{p}}$, we get, by definition of $A_B^{s_c}$,

$$\mathfrak{q}_1 \cap A_B^{s_c} = \mathfrak{q}_2 \cap A_B^{s_c} = (\mathfrak{p}A_{\mathfrak{p}} + \mathrm{Rad}^{\mathrm{C}}(B_{\mathfrak{p}})) \cap A_B^{s_c}.$$

Since $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = (A_B^{s_c})_{\mathfrak{p}}/((\mathfrak{p}A_{\mathfrak{p}} + \mathrm{Rad}^{\mathrm{C}}(B_{\mathfrak{p}})) \cap A_B^{s_c})_{\mathfrak{p}}$, the first part of the proof is done.

To end the proof, it is sufficient to show that if C is intermediate between A and B and satisfies (1) and (2), then $C \subset A_B^{s_c}$. We have to show that if $\mathfrak{p} \in$ C-Spec A, then $C \subset (A_\mathfrak{p} + \mathrm{Rad}^\mathrm{C}(B_\mathfrak{p}))$. If there is no central prime ideal of B lying over $\mathfrak{p}$, then $C \subset A_\mathfrak{p} + \mathrm{Rad}^\mathrm{C}(B_\mathfrak{p}) = B_\mathfrak{p}$. Assume now there is at least one central prime ideal of B, say $\mathfrak{q}$, lying over $\mathfrak{p}$. Since C satisfies (1), $\mathfrak{q} \cap C$ is the unique central prime of C lying over $\mathfrak{p}$ that is the contraction of a central prime ideal of B. It follows that $(\mathfrak{q} \cap C)B_\mathfrak{p} \subset \mathrm{Rad}^\mathrm{C}(B_\mathfrak{p})$. We use the following commutative diagram:

$$\begin{tikzcd} k(\mathfrak{p}) \arrow[r, "\simeq"] & k(\mathfrak{q} \cap C) \\ A_\mathfrak{p} \arrow[u] \arrow[r] & C_\mathfrak{p}. \arrow[u] \end{tikzcd}$$

Let $c \in C$. By (2), there exist $a \in A$ and $s \in A \setminus \mathfrak{p}$ such that $(a/s)(\mathfrak{q} \cap C) = c(\mathfrak{q} \cap C)$. Hence $a - sc \in \mathfrak{q} \cap C$ and thus $c - a/s \in (\mathfrak{q} \cap C)C_\mathfrak{p} = \ker(C_\mathfrak{p} \to k(\mathfrak{q} \cap C))$. We get $c \in A_\mathfrak{p} + \mathrm{Rad}^\mathrm{C} B_\mathfrak{p}$. This concludes the proof. ∎

For integral extensions, the central gluing contains all centrally subintegral subextensions.

Corollary 5.27. *Let $A \to C \to B$ be a sequence of integral extensions of domains. If $A \to C$ is s_c-subintegral, then $C \subset A_B^{s_c}$.*

Proof. If $A \to C$ is s_c-subintegral, then it is easy to see that C satisfies the properties (1) and (2) of Theorem 5.26. We conclude by Theorem 5.26. ∎

5.5 – *Birational closure*

5.5.1. Definition.

Definition 5.28. Let $A \to B$ and $C \to B$ be two extensions of rings. The fibre product $A \times_B C$ is the ring defined by the following pull-back diagram:

$$\begin{tikzcd} A \times_B C \arrow[r] \arrow[d] & A \arrow[d] \\ C \arrow[r] & B. \end{tikzcd}$$

Definition 5.29. Let $A \to B$ be an extension of domains and let $\mathcal{K}(A) \to \mathcal{K}(B)$ be the associated extension of fields. We denote by $\widetilde{A}_B$ the fibre product $B \times_{\mathcal{K}(B)} \mathcal{K}(A)$ and we call it the birational closure of A in B.

The birational closure of A in B is the biggest intermediate ring between A and B which is birational with A.

PROPOSITION 5.30. *Let $A \to B$ be an extension of domains and let $\mathcal{K}(A) \to \mathcal{K}(B)$ be the associated extension of fields. Then, $\widetilde{A}_B$ is intermediate between A and B with $A \to \widetilde{A}_B$ birational and, in addition, if C is an intermediate ring between A and B and $A \to C$ is birational, then $C \to B$ factorizes uniquely through $\widetilde{A}_B$.*

PROOF. The commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ \mathcal{K}(A) \arrow[r] & \mathcal{K}(B) \end{tikzcd}$$

gives a factorization of $A \to B$ through $\widetilde{A}_B$ by the universal property of the fibre product. Since we have an extension $\widetilde{A}_B \to \mathcal{K}(A)$, we get $\mathcal{K}(\widetilde{A}_B) = \mathcal{K}(A)$ and thus $A \to \widetilde{A}_B$ is birational. Let C be an intermediate domain between A and B such that $A \to C$ is birational. We get the commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[rr] \arrow[d] & & B \arrow[d] \\ \mathcal{K}(A) \arrow[r, equal] & \mathcal{K}(C) \arrow[r, equal] & \mathcal{K}(A) \arrow[r] & \mathcal{K}(B). \end{tikzcd}$$

By the universal property of the fibre product, the extension $C \to B$ factorizes uniquely through $\widetilde{A}_B$. ■

5.5.2. Integral and birational closure. We prove that the operations "integral closure" and "birational closure" commute together.

PROPOSITION 5.31. *Let $A \to B$ be an extension of domains. Then*

$$A'_{\widetilde{A}_B} = \widetilde{A}_{A'_B}.$$

PROOF. The extension $A \to A'_{\widetilde{A}_B}$ is integral, so $A \to A'_B$ factorizes uniquely through $A'_{\widetilde{A}_B}$. Since $A \to A'_{\widetilde{A}_B}$ is also birational, $A \to \widetilde{A}_{A'_B}$ factorizes uniquely through $A'_{\widetilde{A}_B}$ by Proposition 5.30.

Conversely, the extension $A \to \widetilde{A}_{A'_B}$ is birational so $A \to \widetilde{A}_B$ factorizes uniquely through $\widetilde{A}_{A'_B}$. Since $A \to \widetilde{A}_{A'_B}$ is also integral, $A \to A'_{\widetilde{A}_B}$ factorizes uniquely through $\widetilde{A}_{A'_B}$. ■

Definition 5.32. Let $A \to B$ be an extension of domains. We simply denote by $\widetilde{A'_B}$ the ring $A'_{\widetilde{A}_B} = \widetilde{A}_{A'_B}$ and we call it the integral and birational closure of A in B.

From the above results we easily get a universal property for the integral and birational closure.

Proposition 5.33. *Let $A \to B$ be an extension of domains. Then, $\widetilde{A'_B}$ is intermediate between A and B with $A \to \widetilde{A'_B}$ integral and birational and, in addition, if C is an intermediate ring between A and B and $A \to C$ is integral and birational, then $C \to B$ factorizes uniquely through $\widetilde{A'_B}$.*

5.6 – *Resolution of the central seminormalization existence problem for rings*

The following theorem ensures the existence of the central seminormalization of a ring in another one.

Theorem 5.34. *Let $A \to B$ be an extension of domains. The central seminormalization $A_B^{s_c,*}$ of A in B exists. In addition, we have:*

(1) *If A is a real domain, then*

$$A_B^{s_c,*} = A^{s_c}_{\widetilde{A'_B}} = \big\{b \in \widetilde{A'_B} \mid \forall \mathfrak{p} \in \text{C-Spec}\, A,\ b_{\mathfrak{p}} \in A_{\mathfrak{p}} + \mathrm{Rad}^{\mathrm{C}}((\widetilde{A'_B})_{\mathfrak{p}})\big\}.$$

(2) *If A is not a real domain, then*

$$A_B^{s_c,*} = A'_B = A^{s_c}_{A'_B}.$$

Proof. Assume A is not a real domain. It follows from Remark 5.2 that $A \to A'_B$ is trivially s_c-subintegral. Since an s_c-subintegral extension is integral, we get (2).

Let us prove (1). We assume A is a real domain. Let C be an intermediate domain between A and B such that $A \to C$ is s_c-subintegral. By Remark 5.3, it follows that $A \to C$ is integral and birational and thus from Proposition 5.33 we get $C \subset \widetilde{A'_B}$. By Corollary 5.27, we get $C \subset A^{s_c}_{\widetilde{A'_B}}$.

To end the proof it is sufficient to prove that $A \to A^{s_c}_{\widetilde{A'_B}}$ is s_c-subintegral. We know that $A^{s_c}_{\widetilde{A'_B}}$ satisfies the properties (1) and (2) of Theorem 5.26 for the extension $A \to \widetilde{A'_B}$. It means that the map

$$\text{C-Spec}\, A^{s_c}_{\widetilde{A'_B}} \to \text{C-Spec}\, A$$

is injective and equiresidual by restriction to the image of $\text{C-Spec}(\widetilde{A'_B}) \to \text{C-Spec}\, A^{s_c}_{\widetilde{A'_B}}$. Since $A \to A^{s_c}_{\widetilde{A'_B}}$ and $A^{s_c}_{\widetilde{A'_B}} \to \widetilde{A'_B}$ are integral and birational, the maps

$$\text{C-Spec}\, A^{s_c}_{\widetilde{A'_B}} \to \text{C-Spec}\, A \quad \text{and} \quad \text{C-Spec}(\widetilde{A'_B}) \to \text{C-Spec}\, A^{s_c}_{\widetilde{A'_B}}$$

are surjective (Proposition 4.3), which gives the desired conclusion. ∎

For integral and birational extensions, we do not have to distinguish the empty case and we get:

Corollary 5.35. *Let $A \to B$ be an integral and birational extension of domains. The central seminormalization of A in B and the central gluing of B over A coincide, i.e.,*

$$A_B^{s_c,*} = A_B^{s_c}.$$

Remark 5.36. In the special case $B = A'$, Corollary 5.35 gives [10, Prop. 2.23].

Corollary 5.37. *Let $A \to B$ be an extension of domains. The following properties are equivalent:*

(1) *A is s_c-normal in B.*

(2) *$A = A^{s_c}_{\widetilde{A}'_B}$ if A is a real domain; A is integrally closed in B otherwise.*

Example 5.38. Consider the extension

$$\begin{aligned} A = \mathbb{R}[x,y]/(y^2 - x^3(x-1)^2(2-x)) \\ \to \mathbb{R}[x,z,u,v]/(z^2 - x(2-x), u^4 + z^2 + 1) = B \end{aligned}$$

such that $x \mapsto x$, $y \mapsto zx(x-1)$. We may decompose $A \to B$ in the following way:

(1) $A \to \mathbb{R}[x,Y]/(Y^2 - x(x-1)^2(2-x)) = C$ such that $x \mapsto x$ and $y \mapsto Yx$. This extension is clearly s_c-subintegral.

(2) $C \to \mathbb{R}[x,z]/(z^2 - x(2-x)) = D$ such that $x \mapsto x$ and $Y \mapsto z(x-1)$. This extension is integral and birational. We remark that D is the integral and birational closure of A and that C is s_c-normal in D.

(3) $D \to \mathbb{R}[x,z,u]/(z^2 - x(2-x), u^4 + z^2 + 1) = E$. This extension is integral but not birational.

(4) $E \to B = E[v]$.

We get here that $A'_B = E$, $\widetilde{A}'_B = D$ and $A_B^{s_c,*} = C$. Since C-Spec $E = \emptyset$, we have $A_E^{s_c} = E$ and thus

$$A_B^{s_c,*} = A^{s_c}_{\widetilde{A}'_B} \neq A^{s_c}_{A'_B} = A'_B$$

and it proves that in Theorem 5.34 (1) it is necessary to consider the integral and birational closure and not only the integral closure before doing the central gluing.

6. Traverso's type structural decomposition theorem

In this section, we establish the main result of the paper: the decomposition theorem. Motivations for obtaining this theorem are given in Section 1.

6.1 – *Central seminormality and conductor*

We prove in this section that the s_c-normality of an extension is strongly related to a property of the conductor.

Let $A \to B$ be an extension of rings. We recall that the conductor of A in B, denoted by $(A : B)$, is the set $\{a \in A \mid aB \subset A\}$. It is the biggest ideal in B contained in A.

Proposition 6.1. *Let $A \to B$ be an extension of domains with A a real domain. If A is s_c-normal in B, then $(A : \widetilde{A}'_B)$ is a central ideal in $\widetilde{A}'_B$.*

Proof. Assume A is s_c-normal in B. From Proposition 5.21, we know that A is s_c-normal in $\widetilde{A}'_B$. For the rest of the proof, we replace B by $\widetilde{A}'_B$. Let $I = (A : B)$. By Corollary 3.15, we have to show that $\sqrt[C]{I} \subset A$ where $\sqrt[C]{I}$ is the central radical of I in B. Let $b \in B$ such that $b \in \sqrt[C]{I}$ and let $\mathfrak{p} \in \text{C-Spec}\, A$.

- Assume $I \subset \mathfrak{p}$. Since $b \in \sqrt[C]{I}$, we have $b \in \bigcap_{\mathfrak{q} \in \text{C-Spec}\, B,\, \mathfrak{q} \cap A = \mathfrak{p}} \mathfrak{q}$. Thus $b \in \operatorname{Rad}^{\mathrm{C}}(B_{\mathfrak{p}})$.
- Assume $I \not\subset \mathfrak{p}$. Then $B_{\mathfrak{p}} = A_{\mathfrak{p}}$.

We have proved that $b \in A_B^{s_c}$. The proof is done since $A = A_B^{s_c}$ (Corollary 5.35). ∎

Corollary 6.2. *Let $A \to B$ be an extension of domains with A a real domain. If A is s_c-normal in B, then $(A : \widetilde{A}'_B)$ is a central ideal in A.*

Proof. Since the contraction of a central ideal remains central (Proposition 4.1), the proof follows from Proposition 6.1. ∎

Proposition 6.3. *Let $A \to B \to C$ be a sequence of integral and birational extensions of domains. If A is s_c-normal in B and B is s_c-normal in C, then A is s_c-normal in C.*

Proof. Let $c \in A_C^{s_c}$ and let $\mathfrak{q} \in \text{C-Spec}\, B$. Let $\mathfrak{p} = \mathfrak{q} \cap A \in \text{C-Spec}\, A$. We have $c = \alpha + \beta$ with $\alpha \in A_{\mathfrak{p}}$ and $\beta \in \operatorname{Rad}^{\mathrm{C}}(C_{\mathfrak{p}})$. Since the central prime ideals of C lying over $\mathfrak{q}$ lie over $\mathfrak{p}$, we get $\operatorname{Rad}^{\mathrm{C}}(C_{\mathfrak{p}}) \subset \operatorname{Rad}^{\mathrm{C}}(C_{\mathfrak{q}})$ (the inclusion is seen in $\mathcal{K}(A) = \mathcal{K}(B) = \mathcal{K}(C)$). Since $A_{\mathfrak{p}} \subset B_{\mathfrak{q}} \subset \mathcal{K}(A)$, we have $c \in B_{\mathfrak{q}} + \operatorname{Rad}^{\mathrm{C}}(C_{\mathfrak{q}})$. It follows that $b \in B_C^{s_c} = B$ (Corollary 5.35) and thus $A_C^{s_c} \subset B$. Since $A \to A_C^{s_c}$ is s_c-subintegral, we get $A_C^{s_c} \subset A_B^{s_c}$. Since A is s_c-normal in B, we get $A = A_C^{s_c}$, i.e., A is s_c-normal in C. ∎

Proposition 6.4. *Let $A \to B$ be an extension of domains. Let C be an intermediate ring between A and $A_B^{s_c,*}$. If $C \neq A_B^{s_c,*}$, then C is not s_c-normal in B.*

Proof. Assume $C \neq A_B^{s_c,*}$ and C is s_c-normal in B. By Proposition 5.18, we get that $C \to A_B^{s_c,*}$ is s_c-subintegral, contradicting the s_c-normality of C in B. ∎

6.2 – *Elementary central gluings*

We will adapt and revisit the definition of elementary gluing developed by Traverso [26] in the central case.

6.2.1. Non-trivial elementary central gluings. Throughout this section we consider the following situation: Let $A \to B$ be an integral extension of domains with A a real domain and let $\mathfrak{p} \in \text{C-Spec}\, A$. We assume that the set of central prime ideals of B lying over $\mathfrak{p}$ is finite and non-empty (it follows from Propositions 3.16 and 4.1 that B is also a real domain), and we denote these prime ideals by $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$. Let us remark that by Proposition 4.3 the previous condition is met if $A \to B$ is finite and birational. We denote by $\gamma_i : k(\mathfrak{p}) \to k(\mathfrak{q}_i)$, $i = 1, \ldots, t$ the canonical extensions of the residue fields. Let $Q = \prod_{i=1}^t k(\mathfrak{q}_i)$ and $\gamma = \prod_{i=1}^t \gamma_i$ be the injection $k(\mathfrak{p}) \to Q$. We remark that γ identifies $k(\mathfrak{p})$ with a subset of the diagonal of Q.

Definition 6.5. We define the “central gluing of B over $\mathfrak{p}$”, denoted by $A_B^{s_c,\mathfrak{p}}$, as the fibre product $B \times_Q k(\mathfrak{p})$, i.e., the domain defined by the following pull-back diagram of commutative rings:

$$\begin{array}{ccc} A_B^{s_c,\mathfrak{p}} & \xrightarrow{i} & B \\ \Big\downarrow h & & \Big\downarrow g \\ k(\mathfrak{p}) & \xrightarrow{\gamma} & Q \end{array}$$

where g is the composite $B \to \prod_{i=1}^t B/\mathfrak{q}_i \to Q$. A central gluing over a central prime ideal of this type is called a “non-trivial elementary central gluing”.

Remark 6.6. The term “non-trivial” in the previous definition corresponds to the fact that the set of central primes of B lying over $\mathfrak{p}$ is non-empty and thus we really glue something.

Remark 6.7. Back to the diagram of Definition 6.5. The map i is an injection and $i(A_B^{s_c,\mathfrak{p}})$ is the set of elements b in B such that $g(b) \in \gamma(k(\mathfrak{p}))$. In particular, we have that $i(A_B^{s_c,\mathfrak{p}})$ contains A. We identify $A_B^{s_c,\mathfrak{p}}$ with $i(A_B^{s_c,\mathfrak{p}})$. From the above commutative diagram, the ring $A_B^{s_c,\mathfrak{p}}$ is determined by

$$\text{(6.1)} \qquad A_B^{s_c,\mathfrak{p}} = \Big\{ b \in B \mid \forall i \in \{1, \ldots, t\}\ b(\mathfrak{q}_i) \in k(\mathfrak{p})\ (*) \text{ and } \\ \forall (i,j) \in \{1, \ldots, t\}^2\ b(\mathfrak{q}_i) = b(\mathfrak{q}_j)\ (**) \Big\}.$$

Proposition 6.8. *We set* $\mathfrak{q} = \bigcap_{i=1}^t \mathfrak{q}_i \cap A_B^{s_c,\mathfrak{p}}$. *Then:*

(1) $\bigcap_{i=1}^t \mathfrak{q}_i \subset A_B^{s_c,\mathfrak{p}}$ *so* $\mathfrak{q} = \bigcap_{i=1}^t \mathfrak{q}_i$ *and* $\mathfrak{q} \subset (A_B^{s_c,\mathfrak{p}} : B)$.

(2) $\mathfrak{q}_i \cap A_B^{sc,\mathfrak{p}} = \mathfrak{q}$ *for all* $i \in \{1,\dots,t\}$, *and thus* $\mathfrak{q}$ *is a central prime ideal of* $A_B^{sc,\mathfrak{p}}$ *lying over* $\mathfrak{p}$.

(3) *The extension* $k(\mathfrak{p}) \to k(\mathfrak{q})$ *is an isomorphism.*

Proof. Let $b \in \bigcap_{i=1}^t \mathfrak{q}_i$. We have $b(\mathfrak{q}_i) = 0 \in k(\mathfrak{p})$ for all $i \in \{1,\dots,t\}$, so $b \in A_B^{sc,\mathfrak{p}}$ and we get (1).

Let $b \in \mathfrak{q}_1 \cap A_B^{sc,\mathfrak{p}}$. We have $b(\mathfrak{q}_1) = 0$ and $h(b) \in k(\mathfrak{p})$. Thus we get

$$(\gamma_i \circ h)(b) = (\gamma_1 \circ h)(b) = b(\mathfrak{q}_i) = 0$$

for all $i \in \{1,\dots,t\}$. It follows that $\gamma \circ h(b) = (0,\dots,0) \in Q$. It means $g(b) = (0,\dots,0)$ and thus $b \in \mathfrak{q} = \ker g$. Hence $\mathfrak{q}_1 \cap A_B^{sc,\mathfrak{p}} \subset \mathfrak{q}$ and thus $\mathfrak{q}_1 \cap A_B^{sc,\mathfrak{p}} = \mathfrak{q}$. Since $\mathfrak{q} = \mathfrak{q}_1 \cap A_B^{sc,\mathfrak{p}}$ and $\mathfrak{q}_1$ is central, $\mathfrak{q}$ is also central (Proposition 4.1). It is clear that $\mathfrak{q}$ is an ideal in $A_B^{sc,\mathfrak{p}}$ and B and thus $\mathfrak{q} \subset (A_B^{sc,\mathfrak{p}} : B)$. Therefore assertion (2) is true.

It follows from $\mathfrak{q} = \ker(g \circ i)$ and the above commutative diagram that $(A_B^{sc,\mathfrak{p}}/\mathfrak{q}) \subset k(\mathfrak{p})$ and thus $k(\mathfrak{q}) \subset k(\mathfrak{p})$. By (2), $\mathfrak{q}$ is a prime ideal of $A_B^{sc,\mathfrak{p}}$ lying over $\mathfrak{p}$. Thus we have $k(\mathfrak{p}) \subset k(\mathfrak{q})$, which gives (3). ∎

A non-trivial elementary central gluing satisfies the following universal property.

Proposition 6.9. *The ring* $A_B^{sc,\mathfrak{p}}$ *is the biggest intermediate ring* C *between* A *and* B *satisfying:*

(1) $\mathfrak{q}_i \cap C = \mathfrak{q}_j \cap C$ *for all* $(i,j) \in \{1,\dots,t\}^2$.

(2) $k(\mathfrak{p}) \to k(\mathfrak{q}_i \cap C)$ *is an isomorphism for all* $i \in \{1,\dots,t\}$.

Proof. The ring $A_B^{sc,\mathfrak{p}}$ satisfies (1) and (2) by Proposition 6.8. It is clear that a subring of B containing A satisfying (1) and (2) satisfies the properties $(*)$ and $(**)$ given in (6.1) and thus is contained in $A_B^{sc,\mathfrak{p}}$. ∎

Proposition 6.10. *We have*

$$A_B^{sc,\mathfrak{p}} = \big\{b \in B \mid b \in A_{\mathfrak{p}} + \mathrm{Rad}^{\mathrm{C}}(B_{\mathfrak{p}})\big\}.$$

Proof. Let D denote the ring $\{b \in B \mid b \in A_{\mathfrak{p}} + \mathrm{Rad}^{\mathrm{C}}(B_{\mathfrak{p}})\}$. From the proof of Theorem 5.26 we see that D is the biggest subring of B satisfying properties (1) and (2) of Proposition 6.9, therefore we get the proof. ∎

In case $A \to B$ is birational with A a real domain, any elementary central gluing is non-trivial and we can strengthen the universal property of (non-trivial) elementary central gluings.

Proposition 6.11. *Under the above notation and hypotheses, assume in addition that $A \to B$ is birational. The ring $A_B^{s_c,\mathfrak{p}}$ defined above is the biggest element among the subrings C of B containing A satisfying the following properties:*

(a) *There exists a unique ideal $\mathfrak{q}'$ in* C-Spec C *lying over $\mathfrak{p}$.*

(b) *$k(\mathfrak{p}) \to k(\mathfrak{q}')$ is an isomorphism.*

Proof. We first prove that $A_B^{s_c,\mathfrak{p}}$ satisfies the properties (a) and (b) of the proposition. Let $\mathfrak{q}' \in \text{C-Spec}\, A_B^{s_c,\mathfrak{p}}$ lying over $\mathfrak{p}$. By Proposition 4.3, there exists a central prime ideal $\mathfrak{q}''$ of B lying over $\mathfrak{q}'$. Since $\mathfrak{q}'$ is lying over $\mathfrak{p}$, we deduce that $\mathfrak{q}''$ must be one of the $\mathfrak{q}_i$ and thus $\mathfrak{q}' = \mathfrak{q}_i \cap A_B^{s_c,\mathfrak{p}}$. The properties (a) and (b) come from (1) and (2) of Proposition 6.9.

Let C be an intermediate ring between A and B satisfying (a) and (b). Let $\mathfrak{q}'$ be the unique central prime ideal of C lying over $\mathfrak{p}$. By arguments similar to those in the first paragraph of the proof, there exists one of the $\mathfrak{q}_i$ lying over $\mathfrak{q}'$. By unicity, we must have $\mathfrak{q}_i \cap C = \mathfrak{q}_j \cap C = \mathfrak{q}'$ for all $(i, j) \in \{1, \dots, t\}^2$, and thus C satisfies Proposition 6.9 (1). By (b), we get that C satisfies Proposition 6.9 (2), and thus $C \subset A_B^{s_c,\mathfrak{p}}$. ■

We give some properties of non-trivial elementary central gluings. We start with an s_c-normality property.

Proposition 6.12. *We have $A_B^{s_c,\mathfrak{p}} = (A_B^{s_c,\mathfrak{p}})_B^{s_c}$ and thus $A_B^{s_c,\mathfrak{p}}$ is s_c-normal in B.*

Proof. By Theorem 5.26, it is clear that $(A_B^{s_c,\mathfrak{p}})_B^{s_c}$ satisfies the properties (1) and (2) of Proposition 6.9 and thus $A_B^{s_c,\mathfrak{p}} = (A_B^{s_c,\mathfrak{p}})_B^{s_c}$. By Corollary 5.27, it follows that $A_B^{s_c,\mathfrak{p}}$ is s_c-normal in B. ■

We prove that the operations of localization and non-trivial elementary central gluings commute together.

Proposition 6.13. *We assume S is a multiplicative closed subset of A. Then:*

(1) *If $S \cap \mathfrak{p} \neq \varnothing$, then $S^{-1}(A_B^{s_c,\mathfrak{p}}) = S^{-1}B$.*

(2) *If $S \cap \mathfrak{p} = \varnothing$, then $S^{-1}(A_B^{s_c,\mathfrak{p}})$ is the non-trivial elementary central gluing of $S^{-1}B$ over $S^{-1}\mathfrak{p}$, i.e.,*

$$S^{-1}A_B^{s_c,\mathfrak{p}} = (S^{-1}A)_{S^{-1}B}^{s_c,S^{-1}\mathfrak{p}}.$$

In particular,

$$S^{-1}(A_B^{s_c,\mathfrak{p}}) = (S^{-1}(A_B^{s_c,\mathfrak{p}}))_{S^{-1}B}^{s_c}$$

and thus $S^{-1}(A_B^{s_c,\mathfrak{p}})$ is s_c-normal in $S^{-1}B$.

Proof. Let $\mathfrak{q} = \bigcap_{i=1}^{t} \mathfrak{q}_i$. If $S \cap \mathfrak{p} \neq \varnothing$, then $S^{-1}\mathfrak{q} = S^{-1}(A_B^{s_c,\mathfrak{p}})$. The conductor commutes with localization so $S^{-1}(A_B^{s_c,\mathfrak{p}} : B) = (S^{-1}(A_B^{s_c,\mathfrak{p}}) : S^{-1}B)$ contains $S^{-1}\mathfrak{q} = S^{-1}(A_B^{s_c,\mathfrak{p}})$ (Proposition 6.8) and thus $S^{-1}(A_B^{s_c,\mathfrak{p}}) = S^{-1}B$.

Assume $S \cap \mathfrak{p} = \varnothing$. Since $S^{-1}\mathfrak{p}$ and $S^{-1}\mathfrak{q}$ are central prime ideals, since the $S^{-1}\mathfrak{q}_i$, $i = 1, \ldots, t$, are the central prime ideals of $S^{-1}B$ lying over $S^{-1}\mathfrak{p}$, since $S^{-1}\mathfrak{q} = \bigcap_{i=1}^{t}(S^{-1}\mathfrak{q}_i)$, localization commutes with quotient, and since $k(\mathfrak{p}) = k(S^{-1}\mathfrak{p})$, $k(\mathfrak{q}) = k(S^{-1}\mathfrak{q})$, and $k(\mathfrak{q}_i) = k(S^{-1}\mathfrak{q}_i)$ for $i = 1, \ldots, t$, we have

$$\begin{aligned}(S^{-1}A)_{S^{-1}B}^{s_c,S^{-1}\mathfrak{p}} &= S^{-1}B \times_{S^{-1}Q} k(S^{-1}\mathfrak{p}) = S^{-1}B \times_Q k(\mathfrak{p}) \\ &= S^{-1}(B \times_Q k(\mathfrak{p})) = S^{-1}A_B^{s_c,\mathfrak{p}}.\end{aligned}$$

It shows that the diagram

$$\begin{tikzcd} A_B^{s_c,\mathfrak{p}} \arrow[r, "i"] \arrow[d, "h"] & B \arrow[d, "g"] \\ k(\mathfrak{p}) \arrow[r, "\gamma"] & Q \end{tikzcd}$$

commutes with localization by S. Proposition 6.12 completes the proof. ∎

Proposition 6.14. *Let $\mathfrak{p}' \in \operatorname{Spec} A$ such that $\mathfrak{p} \not\subset \mathfrak{p}'$. The prime ideals of $A_B^{s_c,\mathfrak{p}}$ lying over $\mathfrak{p}'$ are in bijection with the prime ideals of B lying over $\mathfrak{p}'$. Moreover, they have the same nature: real, non-real, central, non-central.*

Proof. Let $\mathfrak{q} = \bigcap_{i=1}^{t} \mathfrak{q}_i$. By Proposition 6.8 (1), we have $\mathfrak{q} \subset (A_B^{s_c,\mathfrak{p}} : B)$, thus $\mathfrak{p} \subset (A_B^{s_c,\mathfrak{p}} : B) \cap A$ and therefore $(A_B^{s_c,\mathfrak{p}} : B) \cap A \not\subset \mathfrak{p}'$. Thus $(A_B^{s_c,\mathfrak{p}})_{\mathfrak{p}'} = B_{\mathfrak{p}'}$ and the proof is complete. ∎

6.2.2. Generalized elementary central gluings, the birational gluing and examples. We generalize the concept of elementary central gluing.

Definition 6.15. Let $A \to B$ be a finite extension of domains with A a real domain. Let $\mathfrak{p} \in \text{C-Spec}\, A$. The “central gluing of B over $\mathfrak{p}$”, denoted by $A_B^{s_c,\mathfrak{p}}$, is defined as follows:

- If the set of central prime ideals of B lying over $\mathfrak{p}$ is non-empty, then $A_B^{s_c,\mathfrak{p}}$ is the ring defined in Definition 6.5 and we say that it is a non-trivial elementary central gluing.
- If not, then $A_B^{s_c,\mathfrak{p}} := B$ and we say that it is a trivial elementary central gluing.

We can easily generalize Proposition 6.10 and show that central gluings over rings can be written in terms of elementary central gluings.

Proposition 6.16. *Let $A \to B$ be a finite extension of domains with A a real domain.*

(1) *Let $\mathfrak{p} \in \text{C-Spec}\, A$. We have*

$$A_B^{s_c,\mathfrak{p}} = \{b \in B \mid b \in A_{\mathfrak{p}} + \mathrm{Rad}^{\mathrm{C}}(B_{\mathfrak{p}})\}.$$

(2) *The central gluing $A_B^{s_c}$ of B over A can be seen as simultaneous elementary central gluings of B over all the central prime ideals of A. Namely, we have*

$$A_B^{s_c} = \bigcap_{\mathfrak{p} \in \text{C-Spec}\, A} A_B^{s_c,\mathfrak{p}}.$$

The following property will be useful in the next section.

Proposition 6.17. *Let $A \to C \to B$ be a sequence of two finite extensions of domains such that A is a real domain and $A \to C$ is s_c-subintegral. Let $\mathfrak{p} \in \text{C-Spec}\, A$ and let $\mathfrak{p}' \in \text{C-Spec}\, C$ be the unique central prime ideal lying over $\mathfrak{p}$. We have*

$$A_B^{s_c,\mathfrak{p}} = C_B^{s_c,\mathfrak{p}'}.$$

Proof. If the set of central prime ideals of B lying over $\mathfrak{p}$ is empty, then the set of central prime ideals of B lying over $\mathfrak{p}'$ is empty and $A_B^{s_c,\mathfrak{p}} = C_B^{s_c,\mathfrak{p}'} = B$.

Assume the set of central prime ideals of B lying over $\mathfrak{p}$ is non-empty. Let $\mathfrak{q}$ be one of these ideals. Since $\mathfrak{p}'$ is the unique central ideal of C lying over $\mathfrak{p}$ and since $\mathfrak{q} \cap C$ is central, $\mathfrak{q}$ lies over $\mathfrak{p}'$. We have proved that the central prime ideals of B lying over $\mathfrak{p}$ or $\mathfrak{p}'$ are the same, since $k(\mathfrak{p}) = k(\mathfrak{p}')$. It follows from Definition 6.5 that $C_B^{s_c,\mathfrak{p}'} = B \times_Q k(\mathfrak{p}') = B \times_Q k(\mathfrak{p}) = A_B^{s_c,\mathfrak{p}}$. ∎

Elementary central gluings are not sufficient to get a decomposition theorem due to the presence of the birational closure in Theorem 5.34. Let $A \to B$ be an integral extension of domains and let $\mathfrak{p} \in \operatorname{Spec} A$. The elementary Traverso's gluing of B over $\mathfrak{p}$ can be defined similarly as in Definition 6.5, but here we consider all the prime ideals of B lying over $\mathfrak{p}$ and not only the central ones. Following the proof of Proposition 6.10, we see that this gluing is

$$\{b \in B \mid b \in A_{\mathfrak{p}} + \mathrm{Rad}(B_{\mathfrak{p}})\}.$$

From above and Definition 5.29, we see that the birational closure is an elementary Traverso's gluing:

Proposition 6.18. *Let $A \to B$ be an integral extension of domains. Then, the birational closure $\widetilde{A}_B$ of A in B is the elementary Traverso's gluing of B over the null ideal of A.*

In the sequel, the "birational closure" will be also called the "birational gluing". The birational gluing is sometimes an elementary central gluing:

PROPOSITION 6.19. *Let $A \to B$ be an integral extension of real domains. The central gluing $A_B^{s_c,(0)}$ of B over the null ideal of A is the birational closure $\widetilde{A}_B$ of A in B.*

PROOF. By Proposition 3.1, the null ideal of B is the unique central prime ideal of B lying over the null ideal of A. The commutative diagram of Definition 6.5 becomes in this situation

$$\begin{tikzcd} A_B^{s_c,(0)} \arrow[r, "i"] \arrow[d, "h"] & B \arrow[d, "g"] \\ \mathcal{K}(A) \arrow[r, "\gamma"] & \mathcal{K}(B) \end{tikzcd}$$

and thus $A_B^{s_c,(0)} = B \times_{\mathcal{K}(B)} \mathcal{K}(A) = \widetilde{A}_B$. ∎

EXAMPLE 6.20. Consider the finite extension $A = \mathbb{R}[x] \to B = \mathbb{R}[x,y]/(y^2 - x)$ of real domains sending x to itself. Then A is the central gluing of B over the null ideal of A, i.e., $A = B \times_{\mathcal{K}(B)} \mathcal{K}(A)$ (Proposition 6.19). Indeed, if $f \in B$, then we may write $f = p + yq$ with $p, q \in \mathbb{R}[x]$ and if, in addition, $f \in \mathcal{K}(A)$, then $q = 0$.

EXAMPLE 6.21. Consider the finite extension $A = \mathbb{R}[x] \to B = \mathbb{R}[x,y]/(y^2 + x^2)$ of domains sending x to itself. Let us remark that the null ideal of B is not a central ideal since -1 is a square in $\mathcal{K}(B)$ (see Proposition 3.1). As in the previous example, we can prove that A is the birational gluing of B over A, but here the birational gluing is not an elementary central gluing.

EXAMPLE 6.22. Let V be the Whitney umbrella, i.e., the affine algebraic surface over $\mathbb{R}$ with coordinate ring $\mathbb{R}[V] = \mathbb{R}[x,y,z]/(y^2 - zx^2)$ and let V' be its normalization. The coordinate ring of V' is $\mathbb{R}[V'] = \mathbb{R}[x,Y,z]/(Y^2 - z)$ and consider the finite birational extension $\mathbb{R}[V] \to \mathbb{R}[V']$ given by $x \mapsto x$, $y \mapsto Yx$, $z \mapsto z$. We claim that $\mathbb{R}[V]$ is equal to the central gluing of $\mathbb{R}[V']$ over the central prime ideal $\mathfrak{p} = (x,y)$ of $\mathbb{R}[V]$. There is a unique prime ideal of $\mathbb{R}[V']$ lying over $\mathfrak{p}$, which we denote by $\mathfrak{q}$, and $\mathfrak{q} = (x, Y^2 - z)$ is also a central ideal. We have $k(\mathfrak{p}) = \mathbb{R}(z)$ and $k(\mathfrak{q}) = \mathbb{R}(z)[Y]/(Y^2 - z)$. Let $f \in \mathbb{R}[V]_{\mathbb{R}[V']}^{s_c,\mathfrak{p}} = \mathbb{R}[V'] \times_{k(\mathfrak{q})} k(\mathfrak{p})$, we may write $f = g + Yv$ with $g, v \in \mathbb{R}[x,z]$. From the commutative diagram

$$\begin{tikzcd} \mathbb{R}[V]_{\mathbb{R}[V']}^{s_c,\mathfrak{p}} \arrow[r, "i"] \arrow[d, "h"] & \mathbb{R}[V'] \arrow[d, "g"] \\ \mathbb{R}(z) \arrow[r, "\gamma"] & \mathbb{R}(z)[Y]/(Y^2 - z) \end{tikzcd}$$

we see that x must divide v. Thus we have $v = xs$ with $s \in \mathbb{R}[x, z]$. It follows that $f = g + Yxs = g + ys \in \mathbb{R}[V]$, which proves the claim.

6.3 – *Structural decomposition theorem*

As announced, we show that if a noetherian domain A is centrally seminormal in a domain B which is a finite A-module, then A can be obtained from B by the birational gluing followed by a finite number of successive non-trivial elementary central gluings.

Theorem 6.23. *Let $A \to B$ be a finite extension of domains and assume A is a noetherian ring. If A is s_c-normal in B, then there is a finite sequence $(B_i)_{i=0,\dots,n}$ of real domains such that:*

(1) $A = B_n \subset \cdots \subset B_1 \subset B_0 = B$.

(2) *B_1 is the birational gluing of B over A.*

(3) *For $i \geq 1$, B_{i+1} is the central gluing of B_i over a central prime ideal of A.*

Proof. We assume A is s_c-normal in B. If A is not a real domain, then $A = B$ (Corollary 5.37) and there is nothing to do. In the sequel of the proof, we assume A is a real domain.

First, remark that B is also a noetherian ring since it is a noetherian A-module. Indeed a finite module over a noetherian ring is a noetherian module.

Since A is s_c-normal in B, we have that A is s_c-normal in $\widetilde{A}_B$ (Proposition 5.21). Since $\widetilde{A}_B$ is the birational gluing of B over A (Proposition 6.18) and since $A \to \widetilde{A}_B$ is finite (every submodule of a noetherian module is finite), we may assume in the rest of the proof that $A \to B$ is birational.

Assume we have already built the sequence from B_0 to B_i and that $B_i \neq A$. We denote $(A : B_i)$ simply by I. By Proposition 5.21, A is s_c-normal in B_i. By Proposition 6.1 and Corollary 6.2, I is central in B_i and also in A. By Propositions 3.12 and 3.14, the minimal prime ideals of A containing I (in finite number by noetherianity) are all central ideals and their intersection is I. Let $\mathfrak{p}$ be one of these minimal prime ideals. Set $B_{i+1} = A_{B_i}^{s_c,\mathfrak{p}}$, the central gluing of B_i over $\mathfrak{p}$, and set $J = (A : B_{i+1})$.

We claim that $J \not\subset \mathfrak{p}$: Suppose $J \subset \mathfrak{p}$. We have $I \subset J$ and since $\mathfrak{p}$ is a minimal prime ideal of A containing I, we have that $\mathfrak{p}$ is also a minimal prime ideal of A containing J. Since A is s_c-normal in B, by Proposition 5.21 it is also s_c-normal in B_{i+1}, and thus J is a central ideal (Proposition 6.1 and Corollary 6.2). We denote by $\mathfrak{q}$ the unique central prime ideal of B_{i+1} lying over $\mathfrak{p}$ given by Proposition 6.11. After localization at $\mathfrak{p}$, we get $J_\mathfrak{p} = (A_\mathfrak{p} : (B_{i+1})_\mathfrak{p}) = \mathfrak{p}A_\mathfrak{p}$ since $\mathfrak{p}$ is a primary component of J. Since $J_\mathfrak{p}$ is a central ideal in $(B_{i+1})_\mathfrak{p}$, it is the intersection of the central prime ideals of $(B_{i+1})_{\mathfrak{p}}$ containing it (Proposition 3.14), so

$$J_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}} = \mathfrak{q}(B_{i+1})_{\mathfrak{p}}. \tag{6.2}$$

By Proposition 6.11, we have $k(\mathfrak{p}) = k(\mathfrak{q})$ and thus

$$(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}) = ((B_{i+1})_{\mathfrak{p}}/\mathfrak{q}(B_{i+1})_{\mathfrak{p}}). \tag{6.3}$$

Let $b \in (B_{i+1})_{\mathfrak{p}}$. By (6.3) (we may also use Proposition 6.16 and Proposition 6.8 (2)), we may write $b = \alpha + \beta$ with $\alpha \in A_{\mathfrak{p}}$ and $\beta \in \mathfrak{q}(B_{i+1})_{\mathfrak{p}}$. By (6.2), we get $\beta \in \mathfrak{p}A_{\mathfrak{p}}$ and thus $b \in A_{\mathfrak{p}}$. We have proved that $A_{\mathfrak{p}} = (B_{i+1})_{\mathfrak{p}}$. This is impossible (since $J \subset \mathfrak{p}$ by hypothesis) and we get the claim.

We have $I \subset J$, $I \subset \mathfrak{p}$ and $J \not\subset \mathfrak{p}$. Therefore $I \neq J$ and we may build a strictly ascending sequence of ideals as soon as $B_i \neq A$. By noetherianity of A, we get the proof of the theorem. ∎

Corollary 6.24. *Let $A \to B$ be a finite extension of real domains and assume A is a noetherian ring. If A is s_c-normal in B, then A can be obtained from B by a finite number of successive elementary central gluings over central prime ideals of A.*

Proof. Since here A and B are real domains, the birational gluing of B over A is an elementary central gluing (Proposition 6.19) and thus the proof follows from Theorem 6.23. ∎

From Theorem 6.23 we get a structural decomposition theorem for the central seminormalization of A in B with gluings over central prime ideals of $A_B^{s_c,*}$ and the birational gluing. We prove now a structural decomposition theorem for the central seminormalization of A in B using only gluings over central prime ideals of A and the birational gluing.

Theorem 6.25. *Let $A \to B$ be a finite extension of domains and assume A is a noetherian ring. The central seminormalization $A_B^{s_c,*}$ of A in B is B (if A is not a real domain) or can be obtained from B by the birational gluing over A followed by a finite number of successive elementary central gluings over central prime ideals of A.*

Proof. If A is not a real domain, then $A_B^{s_c,*} = B$ by Theorem 5.34 and there is nothing to do in this case. We assume A is a real domain in the sequel of the proof.

The extension $A \to A_B^{s_c,*}$ is finite since every submodule of a noetherian module is finite. It follows that $A_B^{s_c,*}$ is a noetherian ring and it is also a real domain (see Remark 5.3). By Theorem 5.34, $A_B^{s_c,*}$ is s_c-normal in B. By Theorem 6.23, $A_B^{s_c,*}$ can be obtained from B by the birational gluing of B over $A_B^{s_c,*}$ followed by a finite number of successive elementary central gluings over central prime ideals of $A_B^{s_c,*}$. Since $A \to A_B^{s_c,*}$ is birational, it follows from Proposition 5.30 that the birational gluings of B over A and $A_B^{s_c,*}$ are the same. Since $A \to A_B^{s_c,*}$ is s_c-subintegral, it follows from Proposition 6.17 that an elementary central gluing (of an intermediate ring between $A_B^{s_c,*}$ and B) over a central prime ideal of $A_B^{s_c,*}$ is an elementary central gluing over a central prime ideal of A. The proof is done. ∎

Corollary 6.26. *Let $A \to B$ be a finite extension of real domains and assume A is a noetherian ring. The central seminormalization $A_B^{s_c,*}$ of A in B can be obtained from B by a finite number of successive elementary central gluings over central prime ideals of A.*

From Corollaries 5.35 and 6.26 we get:

Corollary 6.27. *Let $A \to B$ be a finite and birational extension of real domains and assume A is a noetherian ring. The central gluing $A_B^{s_c}$ of B over A can be obtained from B by a finite number of successive elementary central gluings over central prime ideals of A.*

We want to replace the word “successive” by “simultaneous” in the statement of Corollary 6.27.

Lemma 6.28. *Let $A \to C \to B$ be a sequence of two integral and birational extensions of real domains. Let $\mathfrak{p} \in \text{C-Spec}\, A$. Then*

$$A_C^{s_c,\mathfrak{p}} = A_B^{s_c,\mathfrak{p}} \cap C = C \times_B A_B^{s_c,\mathfrak{p}}.$$

Proof. Since $C \to B$ is integral and birational, it follows from Proposition 4.3 (4) that $\operatorname{Rad}^{\mathrm{C}} B_{\mathfrak{p}} \cap C_{\mathfrak{p}} = \operatorname{Rad}^{\mathrm{C}} C_{\mathfrak{p}}$. From Proposition 6.10 it follows that

$$\begin{aligned} A_C^{s_c,\mathfrak{p}} &= \{c \in C \mid b \in A_{\mathfrak{p}} + \operatorname{Rad}^{\mathrm{C}}(C_{\mathfrak{p}})\} \\ &= \{c \in C \mid b \in A_{\mathfrak{p}} + (\operatorname{Rad}^{\mathrm{C}}(B_{\mathfrak{p}}) \cap C)\} \\ &= (\{b \in B \mid b \in A_{\mathfrak{p}} + \operatorname{Rad}^{\mathrm{C}}(B_{\mathfrak{p}})\}) \cap C \\ &= A_B^{s_c,\mathfrak{p}} \cap C = C \times_B A_B^{s_c,\mathfrak{p}}. \end{aligned}$$ ∎

Proposition 6.29. *Let $A \to B$ be a finite and birational extension of real domains and assume A is a noetherian ring. If A is s_c-normal in B and $A \neq B$, then there exists a finite number of central prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ of A such that A can be obtained by simultaneous elementary central gluings of B over $\mathfrak{p}_1, \dots, \mathfrak{p}_n$, namely*

$$A = \bigcap_{i=1}^{n} A_B^{s_c,\mathfrak{p}_i}.$$

Proof. By Corollary 6.24, there are a finite sequence $(B_i)_{i=0,\dots,n}$ ($n > 0$ since $A \neq B$) of real domains and a finite number of central prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ of A such that

(1) $A = B_n \subset \cdots \subset B_1 \subset B_0 = B$;

(2) B_{i+1} is the elementary central gluing of B_i over $\mathfrak{p}_{i+1}$ for $i = 0, \dots, n-1$.

By successive applications of Lemma 6.28, we get for $i = 0, \dots, n-1$ that

$$B_{i+1} = \bigcap_{j=1}^{i+1} A_B^{s_c, \mathfrak{p}_j}. \qquad \blacksquare$$

From Corollary 6.27 and Proposition 6.29, we get:

Corollary 6.30. *Let $A \to B$ be a finite and birational extension of real domains and assume A is a noetherian ring. The central gluing $A_B^{s_c}$ of B over A is the intersection of a finite number of elementary central gluings of B over central prime ideals of A.*

Example 6.31. Let V be the Kollár surface, i.e., the affine algebraic surface over $\mathbb{R}$ with coordinate ring $\mathbb{R}[V] = \mathbb{R}[x, y, z]/(y^3 - (1+z^2)x^3)$ and let V' be its normalization. The coordinate ring of V' is $\mathbb{R}[V'] = \mathbb{R}[x, Y, z]/(Y^3 - (1+z^2))$. Consider the finite birational extension $\mathbb{R}[V] \to \mathbb{R}[V']$ given by $x \mapsto x$, $y \mapsto Yx$, $z \mapsto z$. We remark that V and V' are both central. Let $\mathfrak{p} = (x, y) \in \text{C-Spec}\,\mathbb{R}[V]$, we have $k(\mathfrak{p}) = \mathbb{R}(z)$. Let $\mathfrak{q}$ be the unique real (and central) ideal of $\mathbb{R}[V']$ lying over $\mathfrak{p}$. We have $k(\mathfrak{q}) = \mathbb{R}(z)(\sqrt[3]{1+z^2})$. Let W be the affine algebraic surface over $\mathbb{R}$ with coordinate ring $\mathbb{R}[V][y^2/x]$. Since $y^2/x \in \mathcal{R}^0(V(\mathbb{R}))$, it follows from Theorem 5.12 that $\mathbb{R}[V] \to \mathbb{R}[W]$ is s_c-subintegral. To illustrate Theorem 6.23, we claim that $\mathbb{R}[W]$ can be obtained from $\mathbb{R}[V']$ by a unique elementary central gluing, namely $\mathbb{R}[W] = \mathbb{R}[V]_{\mathbb{R}[V']}^{s_c, \mathfrak{p}}$. Since $\mathbb{R}[V] \to \mathbb{R}[W]$ is s_c-subintegral, we have

$$\mathbb{R}[W] \subset \mathbb{R}[V]^{s_c, *} = \mathbb{R}[V]_{\mathbb{R}[V']}^{s_c}.$$

By Proposition 6.12, we get

$$\mathbb{R}[V]^{s_c, *} \subset (\mathbb{R}[V]_{\mathbb{R}[V']}^{s_c, \mathfrak{p}})^{s_c, *} = \mathbb{R}[V]_{\mathbb{R}[V']}^{s_c, \mathfrak{p}}$$

and thus we have

$$\mathbb{R}[W] \subset \mathbb{R}[V]_{\mathbb{R}[V']}^{s_c, \mathfrak{p}}.$$

Let $f \in \mathbb{R}[V]_{\mathbb{R}[V']}^{s_c, \mathfrak{p}}$, we may write $f = g + Yh + Y^2 t$ with $g, h, t \in \mathbb{R}[x, z]$. From the commutative diagram

$$\begin{tikzcd}
\mathbb{R}[V]^{s_c,\mathfrak{p}}_{\mathbb{R}[V']} \arrow[r, "i"] \arrow[d, "h"] & \mathbb{R}[V'] \arrow[d, "g"] \\
\mathbb{R}(z) \arrow[r, "\gamma"] & \mathbb{R}(z)[Y]/(Y^3-(1+z^2))
\end{tikzcd}$$

we see that x must divide h and also t and thus $h = xs$ and $t = xr$ with $s, r \in \mathbb{R}[x, z]$. It follows that $f = g + Yxs + Y^2xr = g + ys + (y^2/x)r \in \mathbb{R}[W]$ and it proves the claim. Since $\mathbb{R}[V] \to \mathbb{R}[W]$ is s_c-subintegral, it follows from Proposition 6.12 that $\mathbb{R}[W]$ is the s_c-normalization of $\mathbb{R}[V]$, i.e.,

$$\mathbb{R}[W] = \mathbb{R}[V]^{s_c,*}.$$

Example 6.32. Let $n \in \mathbb{N} \setminus \{0\}$ and let C be the affine plane algebraic curve over $\mathbb{R}$ with coordinate ring $\mathbb{R}[C] = \mathbb{R}[x, y]/(y^2 - x\prod_{i=1}^n (x-i)^2)$. Let C' be the normalization of C, we have $\mathbb{R}[C'] = \mathbb{R}[x, Y]/(Y^2 - x)$ and the finite birational extension $\mathbb{R}[C] \to \mathbb{R}[C']$ is given by $x \mapsto x$ and $y \mapsto Y\prod_{i=1}^n (x-i)$. The curve C' is smooth and the curve C has only nodal and central singularities corresponding to the maximal ideals $\mathfrak{m}_i = (x-i, y)$ of $\mathbb{R}[C]$ for $i = 1, \ldots, n$. We denote by $\mathfrak{m}'_i$ and $\mathfrak{m}''_i$ the two distinct ideals of $\mathbb{R}[C']$ lying over $\mathfrak{m}_i$. We have $\mathfrak{m}'_i = (x-i, Y - \sqrt{i})$ and $\mathfrak{m}''_i = (x-i, Y + \sqrt{i})$. Since $k(\mathfrak{m}_i) = k(\mathfrak{m}'_i) = k(\mathfrak{m}''_i) = \mathbb{R}$, it is clear that $\mathbb{R}[C]$ is s_c-normal in $\mathbb{R}[C']$, i.e.,

$$\mathbb{R}[C] = \mathbb{R}[C]^{s_c}_{\mathbb{R}[C']} = \left\{f \in \mathbb{R}[C'] \mid f(\mathfrak{m}'_i) = f(\mathfrak{m}''_i) \text{ for } i = 1, \ldots, n\right\}.$$

We set $C_0 = C'$ and, for $i = 1, \ldots, n$, we set C_i to be the affine plane algebraic curve over $\mathbb{R}$ with coordinate ring $\mathbb{R}[C_i] = \mathbb{R}[x, Y_i]/(Y_i^2 - x\prod_{j=1}^i (x-j)^2)$. Note that $\mathbb{R}[C] = \mathbb{R}[C_n] \subset \cdots \subset \mathbb{R}[C_1] \subset \mathbb{R}[C_0] = \mathbb{R}[C']$. Since the extension $\mathbb{R}[C_{i+1}] \to \mathbb{R}[C_i]$ is given by $x \mapsto x$ and $Y_{i+1} \mapsto Y_i(x-(i+1))$, we have

$$\begin{aligned}
\mathbb{R}[C_{i+1}] &= \mathbb{R}[C]^{s_c,\mathfrak{m}_{i+1}}_{\mathbb{R}[C_i]} \\
&= \left\{f \in \mathbb{R}[C_i] \mid f(\mathfrak{m}'_{i+1} \cap \mathbb{R}[C_i]) = f(\mathfrak{m}''_{i+1} \cap \mathbb{R}[C_i]) \text{ for } i = 1, \ldots, n\right\}.
\end{aligned}$$

We have illustrated Theorem 6.23 by showing that $\mathbb{R}[C]$ can be obtained from $\mathbb{R}[C']$ by n successive non-trivial elementary central gluings. It is clear that the number n of elementary central gluings is the lowest we can obtain in this case.

7. Central seminormalization and localization

We may wonder if the processes of central seminormalization and localization commute together. It is known to be true for geometric rings in the special case we take the central seminormalization in the standard integral closure (i.e., $B = A'$) and, in addition, we only localize at a central prime ideal [10, Thm. 4.23]. The goal of this section is to show that it is true more generally.

An extension $A \to B$ of rings is called essentially of finite type if B is a localization of C with $A \to C$ an extension of finite type [23, Def. 53.1]. A domain A is called Japanese if for any finite extension $\mathcal{K}(A) \to L$ of fields the integral closure of A in L is a finitely generated A-module [23, Def. 159.1]. A ring A is a Nagata ring if A is noetherian and, for any prime ideal $\mathfrak{p}$ of A, the domain $A/\mathfrak{p}$ is Japanese [23, Def. 160.1]. As a representative example, a finitely generated algebra over a field is a Nagata ring [23, Prop. 160.3].

Proposition 7.1. *Let $A \to B$ be an essentially of finite type extension of domains and assume A is a Nagata ring. Let S be a multiplicative closed subset of A. If A is s_c-normal in B, then $S^{-1}A$ is s_c-normal in $S^{-1}B$.*

Proof. Assume A is s_c-normal in B. By Proposition 5.21, A is s_c-normal in the integral closure A'_B of A in B. By [23, Lem. 160.2], A'_B is a finitely generated A-module. By Theorem 6.23, there is a finite sequence $(B_i)_{i=0,\dots,n}$ of real domains such that:

(1) $A = B_n \subset \cdots \subset B_1 \subset B_0 = A'_B$.

(2) $B_1 = \widetilde{A'_B}$ is the birational gluing of A'_B over A.

(3) For $i \geq 1$, B_{i+1} is the central gluing of B_i over a central prime ideal of A.

By Proposition 6.13, $S^{-1}B_{i+1}$ is s_c-normal in $S^{-1}B_i$ for all $i \geq 1$. By Proposition 6.3, it follows that $S^{-1}A$ is s_c-normal in

$$S^{-1}B_1 = S^{-1}(A'_B \times_{\mathcal{K}(A'_B)} \mathcal{K}(A)) = (S^{-1}A'_B) \times_{\mathcal{K}(A'_B)} \mathcal{K}(A).$$

Let $S^{-1}A \to D \to S^{-1}B$ be a sequence of extensions such that $S^{-1}A \to D$ is s_c-subintegral (note that $D = S^{-1}C$ for C an intermediate domain between A and B). By [23, Lem. 35.1], $S^{-1}A'_B$ is the integral closure of $S^{-1}A$ in $S^{-1}B$. Since an s_c-subintegral extension is birational and integral, we have $D \subset (S^{-1}A'_B) \times_{\mathcal{K}(A'_B)} \mathcal{K}(A)$ and thus $D = S^{-1}A$. It proves that $S^{-1}A$ is s_c-normal in $S^{-1}B$. ∎

Let $A \to B$ be an extension of domains and let S be a multiplicative closed subset of A. Since $A \to A_B^{s_c,*}$ is s_c-subintegral, $S^{-1}A \to S^{-1}(A_B^{s_c,*})$ is also s_c-subintegral. By Definition 5.22, we get the following integral extension of domains:

$$S^{-1}(A_B^{s_c,*}) \to (S^{-1}A)_{S^{-1}B}^{s_c,*}. \tag{7.1}$$

It is an unsolved problem whether the extension (7.1) is an isomorphism.

Theorem 7.2. *Let $A \to B$ be an essentially of finite type extension of domains and assume A is a Nagata ring. Let S be a multiplicative closed subset of A. Then*

$$S^{-1}(A_B^{s_c,*}) = (S^{-1}A)_{S^{-1}B}^{s_c,*}.$$

Proof. We already know that $S^{-1}(A_B^{s_c,*}) \subset (S^{-1}A)_{S^{-1}B}^{s_c,*}$ by (7.1). Since $S^{-1}A \to (S^{-1}A)_{S^{-1}B}^{s_c,*}$ is s_c-subintegral, it follows from Lemma 5.18 that both

$$S^{-1}A \to S^{-1}(A_B^{s_c,*}) \quad \text{and} \quad S^{-1}(A_B^{s_c,*}) \to (S^{-1}A)_{S^{-1}B}^{s_c,*}$$

are s_c-subintegral. By Proposition 7.1, $S^{-1}(A_B^{s_c,*})$ is s_c-normal in $S^{-1}B$. The theorem follows by the arguments given previously in the proof. ∎

In particular, we get:

Corollary 7.3. *Let $A \to B$ be an essentially of finite type extension of domains and assume A is a Nagata ring. Let $\mathfrak{p} \in \operatorname{Spec} A$. Then*

$$(A_B^{s_c,*})_{\mathfrak{p}} = (A_{\mathfrak{p}})_{B_{\mathfrak{p}}}^{s_c,*}.$$

Using Corollary 5.35 and Theorem 7.2, we generalize [10, Thm. 4.23].

Corollary 7.4. *Let $A \to B$ be a finite and birational extension of domains and assume A is a Nagata ring. Let S be a multiplicative closed subset of A. Then*

$$S^{-1}(A_B^{s_c}) = (S^{-1}A)_{S^{-1}B}^{s_c}.$$

Corollary 7.5. *Let $A \to B$ be an essentially of finite type extension of domains and assume A is a Nagata ring. We have*

$$A_B^{s_c,*} = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} (A_{\mathfrak{p}})_{B_{\mathfrak{p}}}^{s_c,*}.$$

Proof. The proof follows from the equality $A_B^{s_c,*} = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} (A_B^{s_c,*})_{\mathfrak{p}}$ and Corollary 7.3. ∎

8. Central seminormalization of real algebraic varieties

8.1 – *Central seminormalization of an affine real algebraic variety*

In this section, we focus on the existence problem of a central seminormalization of an affine real algebraic variety in another one. Let us introduce the problem. Let $Y \to X$ be a dominant morphism of finite type between two irreducible affine algebraic varieties over $\mathbb{R}$. Does there exist a unique real algebraic variety Z such that $Y \to X$ factorizes through Z, satisfying the following property? For any irreducible affine algebraic variety V over $\mathbb{R}$ such that $Y \to X$ factorizes through V, the morphism $V \to X$ is centrally subintegral if and only if $Z \to X$ factorizes through V.

Definition 8.1. Let $Y \to X$ be a dominant morphism between two affine real algebraic varieties over $\mathbb{R}$. We say that an affine algebraic variety Z over $\mathbb{R}$ is intermediate between X and Y if $Y \to X$ factorizes through Z or equivalently if $\mathbb{R}[Z]$ is intermediate between $\mathbb{R}[X]$ and $\mathbb{R}[Y]$ (by considering the associated ring extension $\mathbb{R}[X] \to \mathbb{R}[Y]$).

We define the central seminormalization (or s_c-normalization) of X in Y as the variety that would give a solution to the problem posed here.

Definition 8.2. Let $Y \to X$ be a dominant morphism of finite type between two irreducible affine real algebraic varieties over $\mathbb{R}$. In case there exists a unique maximal element among the intermediate varieties V between X and Y such that $V \to X$ is centrally subintegral, we denote it by $X_Y^{s_c,*}$ and we call it the central seminormalization or s_c-normalization of X in Y. If Y equals the normalization X' of X, we omit Y and call $X^{s_c,*}$ the central seminormalization or s_c-normalization of X.

We need the following:

Lemma 8.3. *Let $\pi : Y \to X$ be a finite morphism between two irreducible affine algebraic varieties over $\mathbb{R}$. Let A be a ring such that $\mathbb{R}[X] \subset A \subset \mathbb{R}[Y]$. Then A is the coordinate ring of a unique irreducible affine algebraic variety over $\mathbb{R}$ and π factorizes through this variety.*

Proof. Since $\mathbb{R}[Y]$ is a finite module over the noetherian ring $\mathbb{R}[X]$, it is a noetherian $\mathbb{R}[X]$-module. Thus the ring A is a finite $\mathbb{R}[X]$-module as a submodule of a noetherian $\mathbb{R}[X]$-module. It follows that A is a finitely generated algebra over $\mathbb{R}$ and the proof is done. ∎

Some finiteness results:

Proposition 8.4. *Let $Y \to X$ be a dominant morphism of finite type between two irreducible affine real algebraic varieties over $\mathbb{R}$. The integral closure $\mathbb{R}[X]'_{\mathbb{R}[Y]}$ and the birational and integral closure $\widetilde{\mathbb{R}[X]}'_{\mathbb{R}[Y]}$ of $\mathbb{R}[X]$ in $\mathbb{R}[Y]$ are finite $\mathbb{R}[X]$-modules.*

Proof. By [23, Prop. 160.16], coordinate rings of irreducible affine real algebraic varieties over $\mathbb{R}$ are Nagata domains and thus $\mathbb{R}[X]'_{\mathbb{R}[Y]}$ is a finite $\mathbb{R}[X]$-module by

[23, Prop. 160.2]. The finiteness of $\widetilde{\mathbb{R}[X]}'_{\mathbb{R}[Y]}$ as a $\mathbb{R}[X]$-module is a consequence of Lemma 8.3. ■

Using the above proposition, we can define the normalization and the birational normalization of a variety in another one.

Definition 8.5. Let $Y \to X$ be a dominant morphism of finite type between two irreducible affine real algebraic varieties over $\mathbb{R}$.

(1) The variety with coordinate ring $\mathbb{R}[X]'_{\mathbb{R}[Y]}$ is called the normalization of X in Y and is denoted by X'_Y.

(2) The variety with coordinate ring $\widetilde{\mathbb{R}[X]}'_{\mathbb{R}[Y]}$ is called the birational normalization of X in Y and is denoted by $\widetilde{X}'_Y$.

We prove now the existence of a central seminormalization of an affine real algebraic variety in another one.

Theorem 8.6. *Let $Y \to X$ be a dominant morphism of finite type between two irreducible affine real algebraic varieties over $\mathbb{R}$. Then, the central seminormalization $X_Y^{sc,*}$ of X in Y exists and its coordinate ring is the central seminormalization $\mathbb{R}[X]^{sc,*}_{\mathbb{R}[Y]}$ of $\mathbb{R}[X]$ in $\mathbb{R}[Y]$, namely*

$$\mathbb{R}[X_Y^{sc,*}] = \mathbb{R}[X]^{sc,*}_{\mathbb{R}[Y]}.$$

Proof. Assume $X_{\mathrm{reg}}(\mathbb{R}) = \varnothing$. By Theorem 5.34, Propositions 3.16 and 8.4, the theorem is proved in this case and we get $X_Y^{sc,*} = X'_Y$.

Assume $X_{\mathrm{reg}}(\mathbb{R}) \neq \varnothing$, i.e., $\mathbb{R}[X]$ is a real domain (Proposition 3.16). By Theorems 5.12 and 5.34, we have to prove that $\mathbb{R}[X]^{sc,*}_{\mathbb{R}[Y]}$ is a finitely generated algebra over $\mathbb{R}$. By Theorem 5.34, we have

$$\mathbb{R}[X]^{sc,*}_{\mathbb{R}[Y]} = \mathbb{R}[X]^{sc}_{\widetilde{\mathbb{R}[X]}'_{\mathbb{R}[Y]}} = \mathbb{R}[X]^{sc}_{\mathbb{R}[\widetilde{X}'_Y]}.$$

By Lemma 8.4, the morphism $\widetilde{X}'_Y \to X$ is finite and thus, by Lemma 8.3, we get the proof. ■

Similarly to the classical normalization, the central normalization is a geometric process associated to an algebraic integral closure. It generalizes [10, Thm. 4.16].

Theorem 8.7. *Let $Y \to X$ be a dominant morphism of finite type between two irreducible affine real algebraic varieties over $\mathbb{R}$.*

(1) *If $X_{\mathrm{reg}}(\mathbb{R}) \neq \varnothing$, then $\mathbb{R}[X_Y^{sc,*}]$ is the integral closure of $\mathbb{R}[X]$ in $\mathcal{R}^0(\operatorname{Cent} X) \times_{\mathcal{K}(Y)} \mathbb{R}[Y]$.*

(2) *If $X_{\mathrm{reg}}(\mathbb{R}) = \varnothing$, then $\mathbb{R}[X_Y^{sc,*}]$ is the integral closure of $\mathbb{R}[X]$ in $\mathbb{R}[Y]$.*

Proof. Assume $X_{\mathrm{reg}}(\mathbb{R}) = \varnothing$. Looking at the proof of Theorem 8.6, we get (2). Assume $X_{\mathrm{reg}}(\mathbb{R}) \neq \varnothing$. From the proof of Theorem 8.6, we get

$$\mathbb{R}[X_Y^{s_c,*}] = \mathbb{R}[X]^{s_c}_{\mathbb{R}[\widetilde{X}'_Y]}.$$

From the commutative diagram

$$\begin{tikzcd}
\mathbb{R}[X] \arrow[r] \arrow[d] & \mathbb{R}[\widetilde{X}'_Y] \arrow[r] \arrow[d] & \mathbb{R}[X'_Y] \arrow[r] \arrow[dd] & \mathbb{R}[Y] \arrow[dd] \\
\mathcal{R}^0(\operatorname{Cent} X) \arrow[r] \arrow[d] & \mathcal{R}^0(\operatorname{Cent} \widetilde{X}'_Y) \arrow[d] & & \\
\mathcal{K}(X) \arrow[r, dash, "\simeq"] & \mathcal{K}(\widetilde{X}'_Y) \arrow[r] & \mathcal{K}(X'_Y) \arrow[r] & \mathcal{K}(Y)
\end{tikzcd}$$

we see that the integral closure of $\mathbb{R}[X]$ in $\mathcal{R}^0(\operatorname{Cent} X) \times_{\mathcal{K}(Y)} \mathbb{R}[Y]$ is $\mathcal{R}^0(\operatorname{Cent} X) \times_{\mathcal{K}(X)} \mathbb{R}[\widetilde{X}'_Y]$, and we denote this latest domain by B. Let $g \in \mathbb{R}[X_Y^{s_c,*}]$. Let π denote the morphism $X_Y^{s_c,*} \to X$. We have $g \in \mathbb{R}[\widetilde{X}'_Y]$. By Theorem 5.12, there exists $f \in \mathcal{R}^0(\operatorname{Cent} X)$ such that $f \circ \pi = g$ on $\operatorname{Cent} X_Y^{s_c,*}$. It follows that $g \in B$.

By Lemma 8.3, we have $B = \mathbb{R}[Z]$ for an irreducible affine algebraic variety over $\mathbb{R}$ and we get a finite birational morphism $Z \to X$ factorizing $Y \to X$. Since $B \subset \mathcal{R}^0(\operatorname{Cent} X)$ and by Theorem 5.12 (5), we have that $\mathbb{R}[X] \to B$ is s_c-subintegral and thus $B \subset \mathbb{R}[X_Y^{s_c,*}]$. ∎

8.2 – *Central seminormalization of real schemes*

In this section, we prove the existence of a central seminormalization of a real scheme in another one. It can be seen as a real or central version of Andreotti and Bombieri's construction of the classical seminormalization of a scheme in another one [1].

8.2.1. Central locus of a scheme over $\mathbb{R}$. Let $X = (X, \mathcal{O}_X)$ be an integral scheme of finite type over $\mathbb{R}$ with field of rational functions $\mathcal{K}(X)$. We say that $x \in X$ is real if the residue field $k(x)$ is a real field. By [13, Prop. 6.4.2], the residue field at a closed point of X is $\mathbb{R}$ or $\mathbb{C}$, consequently the residue field at a real closed point is the field of real numbers. We denote by $X(\mathbb{R})$ (resp. $X_{\mathrm{reg}}(\mathbb{R})$) the set of real (resp. smooth real) closed points of X. We denote by η the generic point of X, we have $k(\eta) = \mathcal{K}(X)$. Note that if $U = \operatorname{Spec} A$ is a non-empty affine open subset of X, then U is (Zariski) dense in X and η is also the generic point of U, i.e., A is a domain. We say that $x \in X$ is central if there exists an affine neighborhood $U = \operatorname{Spec} A$ of x such that $x \in$ C-Spec A seeing x as a prime ideal of A. We denote by C-Spec $\mathcal{O}_X$ the set of central points of X. By Proposition 3.16, η is central if and only if $\mathcal{K}(X)$ is a real field if and only if $X_{\mathrm{reg}}(\mathbb{R}) \neq \emptyset$. We denote by Cent X the set of central closed points of X. By Theorem 3.6 and the definition, we get Cent $X = \overline{X_{\mathrm{reg}}(\mathbb{R})}^E$ with $\overline{X_{\mathrm{reg}}(\mathbb{R})}^E$ denoting the euclidean closure of the set of smooth real closed points.

From Propositions 4.1 and 4.3, we get:

Proposition 8.8. *Let $\pi : Y \to X$ be a dominant morphism between integral and finitely generated schemes over $\mathbb{R}$. Then π(C-Spec $\mathcal{O}_Y) \subset$ C-Spec $\mathcal{O}_X$. If π is finite and birational, then π(C-Spec $\mathcal{O}_Y) =$ C-Spec $\mathcal{O}_X$.*

8.2.2. Normalization and birational normalization of a scheme in another one. Let $\pi : Y \to X$ be a dominant morphism of finite type between integral schemes of finite type over $\mathbb{R}$. The integral closure $(\mathcal{O}_X)'_{\pi_*(\mathcal{O}_Y)}$ of $\mathcal{O}_X$ in $\pi_*(\mathcal{O}_Y)$ is a coherent sheaf (see [24, Lem. 52.15]) and by [12, II Prop. 1.3.1] it is the structural sheaf of a scheme denoted X'_Y and called the normalization of X in Y. We get a finite morphism $\pi' : X'_Y \to X$ factorizing π. If $U = \operatorname{Spec} A \subset X$, then $H^0(\pi'^{-1}(U), \mathcal{O}_{X'_Y})$ is the integral closure of $H^0(U, \mathcal{O}_X) = A$ in $H^0(\pi^{-1}(U), \mathcal{O}_Y)$. If $Y = \operatorname{Spec} \mathcal{K}(X)$, then we simply denote X'_Y by X' and we call it the normalization of X. We have the universal property that any finite morphism $Z \to X$, with Z an integral scheme over $\mathbb{R}$, factorizing π factorizes π'.

By definition, the birational and integral closure $(\widetilde{\mathcal{O}_X})'_{\pi_*(\mathcal{O}_Y)}$ of $\mathcal{O}_X$ in $\pi_*(\mathcal{O}_Y)$ is a quasi-coherent sheaf. Since $\mathcal{O}_{X'_Y}$ is coherent, it follows from Lemma 8.4 that $(\widetilde{\mathcal{O}_X})'_{\pi_*(\mathcal{O}_Y)}$ is also coherent. By [12, Prop. 1.3.1], it is the structural sheaf of a scheme denoted by $\widetilde{X}'_Y$ and called the birational normalization of X in Y. We have the universal property that any finite and birational morphism $Z \to X$, with Z an integral scheme over $\mathbb{R}$, factorizing π factorizes $\widetilde{X}'_Y \to X$.

8.2.3. Central gluing of a scheme over $\mathbb{R}$ over another one. Let X be an integral scheme of finite type over $\mathbb{R}$. For $x \in X$, we denote by $\mathfrak{m}_x$ the maximal ideal of the local ring $\mathcal{O}_{X,x}$. Since $\mathcal{K}(\mathcal{O}_{X,x}) = \mathcal{K}(X)$, the next proposition follows directly from Definition 3.2.

Proposition 8.9. *Let $x \in X$. Then $x \in$ C-Spec $\mathcal{O}_X$ if and only if $\mathfrak{m}_x \in$ C-Spec $\mathcal{O}_{X,x}$.*

We define an $\mathcal{O}_X$-algebra that corresponds to the central simultaneous gluings.

Definition 8.10. Let $\pi : Y \to X$ be a finite morphism with Y an integral scheme over $\mathbb{R}$. The central gluing of $\pi_*(\mathcal{O}_Y)$ over $\mathcal{O}_X$ is the $\mathcal{O}_X$-subalgebra of $\pi_*(\mathcal{O}_Y)$, denoted by $(\mathcal{O}_X)^{\mathcal{S}c}_Y$, whose sections $f \in H^0(U, (\mathcal{O}_X)^{\mathcal{S}c}_Y)$ on an open subset U of X are the $f \in H^0(U, \pi_*(\mathcal{O}_Y))$ such that

$$f_x \in \mathcal{O}_{X,x} + \mathrm{Rad}^{\mathrm{C}}(\pi_*(\mathcal{O}_Y))_x \quad \text{for any } x \text{ in } U.$$

Note that if $x \notin$ C-Spec $\mathcal{O}_X$, then $\mathcal{O}_{X,x} + \mathrm{Rad}^{\mathrm{C}}(\pi_*(\mathcal{O}_Y))_x = (\pi_*(\mathcal{O}_Y))_x$.

Proposition 8.11. *Let $\pi : Y \to X$ be a finite morphism with Y an integral scheme over $\mathbb{R}$. Then $(\mathcal{O}_X)_Y^{sc}$, the central gluing of $\pi_*(\mathcal{O}_Y)$ over $\mathcal{O}_X$, is a sheaf.*

Proof. From Definition 8.10, we easily see that $(\mathcal{O}_X)_Y^{sc}$ is a presheaf. To get now that it is a sheaf, use that $\pi_*(\mathcal{O}_Y)$ is a sheaf. ∎

Remark 8.12. Let $\pi : Y \to X$ be a finite morphism with Y an integral scheme over $\mathbb{R}$ and assume that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are affine. From above we get

$$(\mathcal{O}_X)_Y^{sc} = A_B^{sc}.$$

Lemma 8.13. *Let $\pi : Y \to X$ be a finite morphism with Y an integral scheme over $\mathbb{R}$. Let $x \in X$. We have*

$$\mathfrak{m}_x(\pi_*\mathcal{O}_Y) \subset (((\mathcal{O}_X)_Y^{sc})_x : (\pi_*\mathcal{O}_Y)_x).$$

Proof. By [21, Ch. 2, §9, Lem. 2] and Definition 8.10, we get

$$\mathfrak{m}_x(\pi_*\mathcal{O}_Y) \subset \mathrm{Rad}(\pi_*\mathcal{O}_Y)_x \subset \mathrm{Rad}^{\mathrm{C}}(\pi_*\mathcal{O}_Y)_x \subset ((\mathcal{O}_X)_Y^{sc})_x. \qquad ∎$$

Theorem 8.14. *Let $\pi : Y \to X$ be a finite morphism with Y an integral scheme over $\mathbb{R}$. Then $(\mathcal{O}_X)_Y^{sc}$ is a coherent sheaf on X.*

Proof. It is sufficient to prove that $(\mathcal{O}_X)_Y^{sc}$ is quasi-coherent since the finiteness property is given by Lemma 8.3. Since the property to be quasi-coherent can be verified locally, we assume $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ with A and B denoting respectively the coordinate rings of X and Y. We now have to check the two properties c 1) and c 2) given by Grothendieck in [13, I Thm. 1.4.1].

Let $f \in A$ and set $\mathcal{D}(f) = \{x \in X \mid f \notin \mathfrak{p}_x\}$ (here we identify $x \in X$ with the corresponding prime ideal $\mathfrak{p}_x \in \operatorname{Spec} A$). Let U be an open subset of X such that $\mathcal{D}(f) \subset U$ and let $s \in H^0(\mathcal{D}(f), (\mathcal{O}_X)_Y^{sc})$. We have to show that there exists $n \in \mathbb{N}$ such that $(f_{|\mathcal{D}(f)})^n s$ extends as a section in $H^0(U, (\mathcal{O}_X)_Y^{sc})$.

For $x \in \mathcal{D}(f)$, we have

$$s_x \in ((\mathcal{O}_X)_Y^{sc})_x = A_{\mathfrak{p}_x} + \mathrm{Rad}^{\mathrm{C}}\, B_{\mathfrak{p}_x} \subset (\pi_*\mathcal{O}_Y)_x = B_{\mathfrak{p}_x}.$$

Since $\pi_*\mathcal{O}_Y$ is quasi-coherent, by [13, I Thm. 1.4.1, d 1)], there exists $n \in \mathbb{N}^*$ such that $(f_{|\mathcal{D}(f)})^{n-1} s$ extends as a section $t \in H^0(U, \pi_*\mathcal{O}_Y)$. So,

$$t_x \in \begin{cases} ((\mathcal{O}_X)_Y^{sc})_x = A_{\mathfrak{p}_x} + \mathrm{Rad}^{\mathrm{C}}\, B_{\mathfrak{p}_x}, & \text{if } x \in \mathcal{D}(f), \\ (\pi_*\mathcal{O}_Y)_x = B_{\mathfrak{p}_x}, & \text{if } x \in U \setminus \mathcal{D}(f). \end{cases}$$

We claim that $f_{|U}t \in H^0(U, (\mathcal{O}_X)^{sc}_Y)$. If $x \in \mathcal{D}(f)$, then clearly $f_x t_x = f_x^n s_x \in ((\mathcal{O}_X)^{sc}_Y)_x$. Assume now $x \in U \setminus \mathcal{D}(f)$. Since $f_x \in \mathfrak{m}_x = \mathfrak{p}_x A_{\mathfrak{p}_x}$, it follows from Lemma 8.13 that $f_x t_x \in ((\mathcal{O}_X)^{sc}_Y)_x$ and we have proved the claim.

It follows that $(f_{|\mathcal{D}(f)})^n s$ extends as a section in $H^0(U, (\mathcal{O}_X)^{sc}_Y)$ and we have checked property c 1) of [13, I Thm. 1.4.1]. Since $\pi_* \mathcal{O}_Y$ is quasi-coherent, obviously $(\mathcal{O}_X)^{sc}_Y$ satisfies property c 2) of [13, I Thm. 1.4.1] and the proof is done. ∎

From [12, II Prop. 1.3.1] and Theorem 8.14, we get:

Corollary 8.15. *Let $\pi : Y \to X$ be a finite morphism with Y an integral scheme over $\mathbb{R}$. There exists an integral scheme X^{sc}_Y over $\mathbb{R}$, called the central gluing of Y over X, with a finite and birational morphism $\pi^{sc}_Y : X^{sc}_Y \to Y$ factorizing π such that $(\pi^{sc}_Y)_* \mathcal{O}_{X^{sc}_Y} = (\mathcal{O}_X)^{sc}_Y$, i.e.,*

$$X^{sc}_Y = \operatorname{Spec}(\mathcal{O}_X)^{sc}_Y.$$

8.2.4. Central seminormalization of a scheme over $\mathbb{R}$ in another one. Using Theorem 5.34, Proposition 3.16 and the above constructions we state the following definition:

Definition 8.16. Let $\pi : Y \to X$ be a dominant morphism of finite type with Y an integral scheme over $\mathbb{R}$. The central seminormalization of $\mathcal{O}_X$ in $\pi_*(\mathcal{O}_Y)$ is the $\mathcal{O}_X$-algebra denoted by $(\mathcal{O}_X)^{sc,*}_Y$ and defined as follows:

(1) $(\mathcal{O}_X)^{sc,*}_Y = (\mathcal{O}_X)^{sc}_{\widetilde{X}'_Y}$ if $X_{\mathrm{reg}}(\mathbb{R}) \neq \varnothing$.

(2) $(\mathcal{O}_X)^{sc,*}_Y = (\mathcal{O}_X)'_{\pi_*(\mathcal{O}_Y)}$ if $X_{\mathrm{reg}}(\mathbb{R}) = \varnothing$.

By the above constructions and results, we are able to prove the existence of the central seminormalization of a real scheme in another one:

Theorem 8.17. *Let $\pi : Y \to X$ be a dominant morphism of finite type with Y an integral scheme over $\mathbb{R}$. Then:*

(1) *The $\mathcal{O}_X$-algebra $(\mathcal{O}_X)^{sc,*}_Y$ is a coherent sheaf on X.*

(2) *There exists an integral scheme $X^{sc,*}_Y$ over $\mathbb{R}$, called the central seminormalization of X in Y, with a finite morphism $\pi^{sc,*}_Y : X^{sc,*}_Y \to Y$ factorizing π such that $(\pi^{sc,*}_Y)_* \mathcal{O}_{X^{sc,*}_Y} = \mathcal{O}^{sc,*}_Y$, i.e.,*

$$X^{sc,*}_Y = \operatorname{Spec} \mathcal{O}^{sc,*}_Y.$$

Proof. If $X_{\mathrm{reg}}(\mathbb{R}) = \varnothing$, then $(\mathcal{O}_X)^{sc,*}_Y$, is a coherent sheaf on X by [24, Lem. 52.15]. By Section 8.2.2 and Theorem 8.14, we have that $(\mathcal{O}_X)^{sc,*}_Y$ is a coherent sheaf on X in the case $X_{\mathrm{reg}}(\mathbb{R}) \neq \varnothing$. The rest of the proof follows from [12, II Prop. 1.3.1]. ∎

Remark 8.18. Let $\pi : Y \to X$ be a dominant morphism of finite type with Y an integral scheme over $\mathbb{R}$ and assume that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are affine. From Theorems 5.34 and 8.17 we get

$$\mathcal{O}_Y^{s_c,*} = A_B^{s_c,*}.$$

Definition 8.19. Let $\pi : Y \to X$ be a dominant morphism of finite type with Y an integral scheme over $\mathbb{R}$. We say that π is s_c-subintegral or centrally subintegral if π is finite and the induced map C-Spec $\mathcal{O}_Y \to$ C-Spec $\mathcal{O}_X$ is bijective and equiresidual (for all $y \in$ C-Spec Y we have $k(\pi(y)) \simeq k(y)$).

Remark 8.20. This notion of centrally subintegral morphism has similarities with the concept of universal homeomorphism introduced by Grothendieck [13, I 3.8].

A point $x \in X$ is central if and only if it is a central point in an affine neighborhood of x. We can thus derive the following result from Theorems 5.12 and 8.6:

Theorem 8.21. *Let $\pi : Y \to X$ be a dominant morphism of finite type with Y an integral scheme over $\mathbb{R}$. A morphism $Z \to X$ factorizing π, with Z an integral scheme over $\mathbb{R}$, is centrally subintegral if and only if it factorizes through $\pi_Y^{s_c,*} : X_Y^{s_c,*} \to X$.*

Acknowledgments – The author is deeply grateful to G. Fichou and R. Quarez for useful discussions.

Funding – The author benefited from the support of the Centre Henri Lebesgue ANR-11-LABX-0020-01 and from the project ENUMGEOM ANR-18-CE40-0009.

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Manoscritto pervenuto in redazione il 12 aprile 2021.