Degeneration of K3 surfaces with non-symplectic automorphisms

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Abstract – We prove that a K3 surface with an automorphism acting on the global 2-forms by a primitive m-th root of unity, $m \neq 1, 2, 3, 4, 6$, does not degenerate (assuming the existence of the so-called Kulikov models). A key result used to prove this is the rationality of the actions of automorphisms on the graded quotients of the weight filtration of the l-adic cohomology groups of the surface.

1. Introduction

Let $\mathcal{O}_K$ be a Henselian discrete valuation ring (DVR) with fraction field K and residue field k. We consider the problem of degeneration of K3 surfaces: Given a K3 surface X over K, we investigate its possible extensions $\mathcal{X}$ over $\mathcal{O}_K$ and their reductions X_0 over k.

The so-called Kulikov models, which are semistable models of X over $\mathcal{O}_K$ with nice properties, is a standard tool to discuss degeneration of K3 surfaces. The special fibers of Kulikov models are classified into three types: Type I, smooth K3 surfaces, and Types II and III, which are reducible surfaces satisfying certain conditions. It is

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conjectured that any K3 surface over K admits a Kulikov model after replacing K by a finite extension, but not yet proved in general. See Section 4 for details.

In this paper we relate the properties of Kulikov models with (non-symplectic) automorphisms of X:

THEOREM 1.1. *Assume* char $K \neq 2$. *Let* $\mathcal{X}$ *be a Kulikov model over* $\mathcal{O}_K$ *of a K3 surface* X *over* K *and* X_0 *its special fiber over* k. *Denote by* $m = m(X)$ *the order of the image of* $\rho \colon \mathrm{Aut}_K(X) \to \mathrm{GL}(H^0(X, \Omega^2_{X/K}))$ *(which is finite).*

(1) *Assume* $m \neq 1, 2, 3, 4, 6$. *Then* X_0 *is of Type I, i.e. a smooth K3 surface.*

(2) *Assume* $m \neq 1, 2$. *Then* X_0 *is either of Type I or II.*

The key idea of the proof is describing the action of $\mathrm{Aut}(X)$ on the n-th graded quotients gr_n^W of the weight filtration of $H^2_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_l)$ in terms of ρ (see Lemma 5.1). We also show the rationality (and l-independence) of such action (see Theorem 3.3), and using this we derive some restrictions on $\dim \mathrm{gr}_n^W$. By calculating $\dim \mathrm{gr}_n^W$ (using the classification of the special fibers of Kulikov models, see Section 4) we can exclude certain types of degeneration, and then the remaining possibilities are as stated in the theorem.

The assumption on m in Theorem 1.1 is optimal: see Examples 7.1–7.2.

In Section 6 we give an application on K3 surfaces with non-symplectic automorphisms of prime order $p \geq 5$: we can show that the moduli space of such K3 surfaces (in characteristic 0) is compact and that any such surface defined over a number field has everywhere potential good reduction. Since these moduli spaces for $p = 5, 7, 11$ have positive dimension 4, 2, 1 respectively, there are plenty of such surfaces.

We also have a conjectural generalization of Theorem 1.1:

CONJECTURE 1.2. *Assume* char $K = 0$. *Let* X *and* $\mathcal{X}$ *be as in the previous theorem. Let* E *be the Hodge endomorphism field of* X.

(1) *Assume* E *is neither* $\mathbb{Q}$ *nor an imaginary quadratic field. Then* X_0 *is of Type I.*

(2) *Assume* $E \neq \mathbb{Q}$. *Then* X_0 *is either of Type I or II.*

THEOREM 1.3. *If the Hodge conjecture for* $X \times X$ *is true, then Conjecture 1.2 for* X *is true.*

It is known that the Hodge conjecture for $X \times X$ is true if E is a CM field (see Section 5 for the definition of the Hodge endomorphism field). It is either a totally real field or a CM field.

REMARK 1.4. The Hodge endomorphism field of X clearly contains $\mathbb{Q}(\zeta_m)$, where $m = m(X)$. The cyclotomic field $\mathbb{Q}(\zeta_m)$ is a CM field for $m \neq 1, 2$ and is imaginary quadratic only if $m = 3, 4, 6$. Hence, Theorem 1.3 generalizes Theorem 1.1.

2. Transcendental lattices and 2-forms of K3 surfaces

In this section X is a K3 surface over a field k and l is an arbitrary prime different from char k.

Let $\rho\colon \mathrm{Aut}_k(X) \to \mathrm{GL}(H^0(X, \Omega^2_{X/k})) = k^*$ be the natural action (notice that we have dim $H^0(X, \Omega^2_{X/k}) = 1$ since X is a K3 surface). An element of $\mathrm{Aut}_k(X)$ is called *symplectic* if it belongs to Ker ρ.

LEMMA 2.1. *Im ρ is a finite (cyclic) group.*

We denote by $m(X)$ (resp. $m(g)$) the order of the group Im ρ (resp. of the element $\rho(g)$). We denote by μ_m the group of m-th roots of 1.

PROOF. *Characteristic* 0: Finiteness follows from a general result [28, Theorem 14.10] by Ueno. Nikulin also showed in [19, Theorem 10.1.2] finiteness and moreover showed $\phi(m(X)) \le 20$, where $\phi(m) = \#(\mathbb{Z}/m\mathbb{Z})^*$ is the number of invertible classes modulo m. (In particular, we have $m(X) \le 66$.)

Characteristic $p > 0$, *supersingular*: In [20, Theorem 2.1] (see Remark 2.2) Nygaard showed that $\rho(g) \in \mu_{p^{\sigma_0}+1}$ for every g, where σ_0 is the Artin invariant of X (which is a positive integer ≤ 10). Hence $m(g)$ and $m(X)$ divide $p^{\sigma_0} + 1$.

Characteristic $p > 0$, *finite height*: Let $W = W(\bar{k})$ be the ring of Witt vectors over $\bar{k}$ and let $K = \mathrm{Frac}\, W$. Lieblich–Maulik showed in [13, Corollary 4.2] that there exists a lifting $\widetilde{X}$ over W such that the specialization morphism $\mathrm{NS}(\widetilde{X}_{\bar{K}}) \to \mathrm{NS}(\widetilde{X}_{\bar{k}})$ is an isomorphism. They also showed in [13, Theorem 2.1] that for such $\widetilde{X}$ the restriction map $\mathrm{Aut}(\widetilde{X}) \to \mathrm{Aut}(X_K)$ is an isomorphism, and the same assertion holds for $\widetilde{X}_R := \widetilde{X} \otimes R$ for any finite extension R of W. Moreover, they showed [13, Section 6] that for such $\widetilde{X}$ the specialization map $\mathrm{Aut}(\widetilde{X}_{\bar{K}}) \to \mathrm{Aut}(X_{\bar{k}})$, defined as the limit of $\mathrm{Aut}(\widetilde{X}_{\mathrm{Frac}\,R}) \xrightarrow{\sim} \mathrm{Aut}(\widetilde{X}_R) \to \mathrm{Aut}(X_{\bar{k}})$, has finite cokernel. Comparing the actions on a 2-form on $\widetilde{X}_R$ and its mod p reduction, we observe that this specialization map is compatible with ρ. The assertion is reduced to the characteristic 0 case. ∎

REMARK 2.2. We cited a theorem of Nygaard, [20, Theorem 2.1], which is stated for $p \ne 2$. We show that this is still true if $p = 2$: the argument is due to Kazuhiro Ito. The only step where the assumption $p = 2$ is used is the inductive step of [21, Lemma 3.14]. If $p = 2$, we can argue as follows. If there exists $x \in \Gamma$ with $p \nmid \langle x, x \rangle$, then we argue as in [21]. Suppose there is no such x. There are still $x_1, x_2 \in \Gamma$ with $p \nmid \langle x_1, x_2 \rangle$, and then the matrix $(\langle x_i, x_j \rangle)^2_{i,j=1}$ is invertible (since $p \mid \langle x_i, x_i \rangle$), hence Γ decomposes as the sum of the subspace generated by (x_1, x_2) and its orthogonal complement. Apply the induction hypothesis to the complement.

We recall the transcendental lattices of K3 surfaces. Let $T_l(X)$ be the orthogonal complement of $\mathrm{NS}(X_{\bar{k}}) \otimes \mathbb{Z}_l(-1)$ in $H^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_l)$, and denote by

$$\chi_l \colon \mathrm{Aut}(X) \to \mathrm{GL}(T_l(X))$$

the natural action. If $k = \mathbb{C}$, we define $T(X) \subset H^2(X, \mathbb{Z})$ and $\chi \colon \mathrm{Aut}(X) \to \mathrm{GL}(T(X))$ similarly. If $\mathrm{char}\, k > 0$, we define

$$T_{\text{crys}}(X) \subset H^2_{\text{crys}}(X_{\bar{k}}/W(\bar{k})) \quad \text{and} \quad \chi_{\text{crys}} \colon \mathrm{Aut}(X) \to \mathrm{GL}(T_{\text{crys}}(X))$$

similarly, where $W(\bar{k})$ is the ring of Witt vectors over $\bar{k}$.

In characteristic 0 the following relationship between χ and ρ is well-known (see for example [26, Remark 3.4]). We include the proof for the reader's convenience.

LEMMA 2.3. *Assume* $\mathrm{char}\, k = 0$. *Then the characteristic polynomial of* $\chi_l(g)$ *belongs to* $\mathbb{Z}[x]$ *and is independent of* l. *It is a power of the* $m(g)$-*th cyclotomic polynomial* $\Phi_{m(g)}$. *If* $k = \mathbb{C}$, *then it is equal to that of* $\chi(g)$.

PROOF. We may assume $k = \mathbb{C}$. By the comparison of Betti and étale cohomology groups, $\chi(g)$ and $\chi_l(g)$ have the same characteristic polynomial $P \in \mathbb{Z}[x]$. Since $H^0(X, \Omega^2) \subset T(X) \otimes \mathbb{C}$ (by Hodge decomposition), P has $\rho(g)$ as a root, and hence is divisible by $\Phi_{m(g)}$. Note that $T(X)_{\mathbb{Q}}$ is irreducible as a rational Hodge structure (if it admits a decomposition $T(X)_{\mathbb{Q}} = T_1 \oplus T_2$, then since $\dim(T(X)_{\mathbb{C}})^{2,0} = h^{2,0}(X) = 1$ there exists $i \in \{1, 2\}$ for which $T_i \subset H^{1,1}$, but then by the Lefschetz $(1, 1)$ theorem we have $T_i \subset H^{1,1} \cap H^2(X, \mathbb{Q}) = \mathrm{NS}(X)_{\mathbb{Q}}$, whence $T_i = 0$). Therefore, P has no other irreducible factor in $\mathbb{Z}[x]$. ∎

We also need a positive characteristic version. We say that $g \in \mathrm{Aut}(X)$ in positive characteristic is *liftable to characteristic* 0 if there exist a pair $(\widetilde{X}, \widetilde{g})$ of a proper smooth scheme $\widetilde{X}$ over a DVR V that is finite over $W(\bar{k})$, and an automorphism $\widetilde{g} \in \mathrm{Aut}(\widetilde{X})$ satisfying $(\widetilde{X}, \widetilde{g}) \otimes_V \bar{k} = (X, g) \otimes_k \bar{k}$. (By the proof of [3, Corollaire 1.10], the generic fiber of $\widetilde{X}$ is then automatically a K3 surface.)

LEMMA 2.4. *Assume* $\mathrm{char}\, k = p > 0$.

(1) *The characteristic polynomial* P *of* $\chi_l(g)$ *belongs to* $\mathbb{Z}[x]$, *it is independent of* l *and equal to that of* $\chi_{\text{crys}}(g)$.

(2) *If* g *is liftable to characteristic* 0, *then* P *is a power of* $\Phi_{m(g)p^e}$, *for some integer* $e \geq 0$.

(3) *If* $p > 2$ *and* X *is of finite height (we no longer assume liftability), then* P *is a product of cyclotomic polynomials of the form* $\Phi_{m(g)p^{e_i}}$, *for some integers* $e_i \geq 0$.

PROOF. We may assume k is algebraically closed.

(1) This follows from the corresponding assertions for the actions on $H^2_{\text{ét}}$ and H^2_{crys} (showed in [5, (3.7.3)]) and on their subspaces generated by NS.

(2) Let $(\widetilde{X}, \widetilde{g})$ be a lifting over V and let $K = \text{Frac } V$. Comparing the actions of $\widetilde{g}$ on a 2-form on $\widetilde{X}$ and its mod p reduction, we observe that $\rho(\widetilde{g}|_{\widetilde{X}_K})$ maps to $\rho(g)$ under the map $\mu_m(K) \to \mu_m(\bar{k})$, where $m = m(\widetilde{g}|_{\widetilde{X}_K})$. Since the kernel of this map is formed precisely by the elements whose order is a power of p, we obtain $m(\widetilde{g}|_{\widetilde{X}_K}) = m(g)\,p^e$, for some $e \geq 0$. Since $\text{NS}(\widetilde{X}_{\bar{K}}) \hookrightarrow \text{NS}(X_{\bar{k}})$, we have an equivariant injection $T_l(X) \hookrightarrow T_l(\widetilde{X}_K)$, and the assertion follows from Lemma 2.3.

(3) The images of χ_l and χ_{crys} are finite (this can be shown by reducing to characteristic 0 as in the proof of Lemma 2.1). By replacing g with its p^N-th power for some N, we may assume that the order of $\chi_{\text{crys}}(g)$ is prime to p; this does not change $m(g)$ because $p \nmid m(g)$ (since there are no primitive p-th roots of 1 in k). By [7, Theorem 3.2], an automorphism of a K3 surface of finite height in characteristic $p > 2$ with this property is liftable to characteristic 0. Hence, the assertion is reduced to point (2). ∎

In general, e and e_i in (2) and (3) may be nonzero: see Example 7.3. We do not know whether P in (3) can have more than one different factors.

3. Action of correspondences on the weight spectral sequence

Let l be an arbitrary prime different from char k. In this section we study the actions of automorphisms, and more generally of algebraic correspondences, on the l-adic cohomology groups of varieties X over the fraction field K of a Henselian DVR $\mathcal{O}_K$. We show that they act on the graded quotients gr_n^W of the weight filtration and that for certain n these actions are rational, i.e. their characteristic polynomials have coefficients in $\mathbb{Q}$.

In this paper, an algebraic space $\mathcal{X}$ over $\mathcal{O}_K$ is said to be *strictly semistable model* of its generic fiber X if it is regular and flat over $\mathcal{O}_K$, its generic fiber X over K is a smooth *scheme*, and its special fiber X_0 over k is a simple normal crossing divisor that is a *scheme*. ($\mathcal{X}$ itself is not assumed to be a scheme.)

We review the following results on the *weight spectral sequence*.

THEOREM 3.1. *Let $\mathcal{X}$ be a strictly semistable model over $\mathcal{O}_K$ of a variety X. Then one can attach to $\mathcal{X}$ a spectral sequence*

$$E_1^{p,q} = \bigoplus_{i \geq \max\{0,-p\}} H_{\text{ét}}^{q-2i}(X_0^{(p+2i)}, \mathbb{Q}_l(-i)) \Rightarrow H_{\text{ét}}^{p+q}(X_{\bar{K}}, \mathbb{Q}_l),$$

where $X_{\overline{0}}^{(p)}$ are the disjoint unions of $(p + 1)$-fold intersections of the irreducible components of $X_{\overline{0}} := X_0 \otimes_k \overline{k}$. The spectral sequence is compatible with automorphisms of $\mathcal{X}$. The spectral sequence degenerates at E_2.

Under the assumption that $\mathcal{X}$ is a scheme, the first assertion is proved by Rapoport–Zink (see [23, Satz 2.10]) and also by Saito (see [25, Corollary 2.8]), whereas E_2-degeneration is proved by Nakayama (see [17, Proposition 1.9]). Compatibility with automorphisms follows from the proof of Saito. In [16, Proposition 2.3] we removed the assumption that $\mathcal{X}$ is a scheme.

The filtration induced by this spectral sequence is called the *weight filtration* and denoted $W_n H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Q}_l)$. This filtration is independent of the choice of a proper strictly semistable model $\mathcal{X}$ (use the argument in [6, Section 2.3]). We let

$$\text{gr}_n^W := \text{gr}_n^W H_{\text{ét}}^i = W_n H_{\text{ét}}^i / W_{n-1} H_{\text{ét}}^i.$$

More generally, we can define the weight filtration on $H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Q}_l)$ in the following way, without assuming the existence of a proper strictly semistable model of X. It is known by [2, Theorem 6.5] that, after replacing $\mathcal{O}_K$ by a finite extension of its completion, there exists an alteration Y of X (i.e. a proper surjective generically-finite morphism $f: Y \to X$ from a variety Y) that admits a proper strictly semistable model that is a scheme. The composite

$$f_* \circ f^*: H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Q}_l) \to H_{\text{ét}}^i(Y_{\overline{K}}, \mathbb{Q}_l) \to H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Q}_l)$$

is equal to the multiplication by $\deg(f)$, and we regard $H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Q}_l)$ to be a direct summand of $H_{\text{ét}}^i(Y_{\overline{K}}, \mathbb{Q}_l)$. We define $W_n H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Q}_l)$ as the restriction of $W_n H_{\text{ét}}^i(Y_{\overline{K}}, \mathbb{Q}_l)$ (the weight filtration of $H_{\text{ét}}^i(Y_{\overline{K}}, \mathbb{Q}_l)$, defined using a proper strictly semistable model of Y). Arguing as in [6, Section 2.3] again, we can show that this filtration is independent of the choice of Y and a strictly semistable model of Y. In particular, if X itself admits a strictly semistable model, then this filtration coincides with the one defined in the previous paragraph.

We also recall the *monodromy filtration* $M_r H_{\text{ét}}^i$ on $H_{\text{ét}}^i = H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Q}_l)$ (see [25, Section 2.1] for details). Let $t_l: I_K \to \mathbb{Z}_l(1)$ be the homomorphism defined by $t_l(\sigma) = (\sigma(\pi^{1/l^n})/\pi^{1/l^n})_{n \in \mathbb{Z}_{\geq 0}}$, where π is a uniformizer of $\mathcal{O}_K$, $(\pi^{1/l^n})_{n \in \mathbb{Z}_{\geq 0}}$ is a system of l^n-th roots of π, and $I_K = \text{Ker}(\text{Gal}(\overline{K}/K) \to \text{Gal}(\overline{k}/k))$ is the inertia subgroup. The monodromy operator N is the unique nilpotent map $N: H_{\text{ét}}^i(1) \to H_{\text{ét}}^i$ such that there exists an open subgroup $J \subset I_K$ for which any element $\sigma \in J$ acts on $H_{\text{ét}}^i$ by $\exp(t_l(\sigma)N)$. Then the monodromy filtration is defined as the unique increasing filtration satisfying $M_r = 0$ for $r \ll 0$, $M_r = H_{\text{ét}}^i$ for $r \gg 0$, $N(M_r) \subset M_{r-2}$, and

$N^r \colon \mathrm{gr}_r^M \xrightarrow{\sim} \mathrm{gr}_{-r}^M$. An equivalent definition is

$$M_r H_{\mathrm{\acute{e}t}}^i = \sum_{\substack{p,q\in\mathbb{Z}_{\geq 0} \\ p-q=r}} (\mathrm{Ker}\, N^{p+1} \cap \mathrm{Im}\, N^q).$$

In particular, since N acts by zero on the classes of algebraic cycles, we have $\mathrm{NS}(X) \subset \mathrm{Ker}\, N \subset M_0 H_{\mathrm{\acute{e}t}}^2(1)$.

The *weight-monodromy conjecture* states that $M_r H_{\mathrm{\acute{e}t}}^i = W_{r+i} H_{\mathrm{\acute{e}t}}^i$ for any X and any i.

THEOREM 3.2. *If* $\dim X = 2$, *then the weight-monodromy conjecture for* $H_{\mathrm{\acute{e}t}}^2$ *is true, i.e. we have* $M_r H_{\mathrm{\acute{e}t}}^2 = W_{r+2} H_{\mathrm{\acute{e}t}}^2$.

This is proved by Rapoport–Zink in [23, Satz 2.13] in the case X admits a strictly semistable model that is a scheme. The general case is reduced to this case by de Jong's alteration (see [25, Lemma 3.9]).

Now we consider the actions of algebraic correspondences on the l-adic cohomology groups.

THEOREM 3.3. *Let* X *be a proper smooth variety over* K *of dimension* d, *and* $\Gamma \in \mathrm{CH}^d(X \times X)$ *an algebraic correspondence (i.e.* Γ *is a* $\mathbb{Z}$-*linear combination of codimension* d *subvarieties of* $X \times X$). *Then:*

(1) *For each integer* $0 \leq i \leq 2d$, *the action of* Γ^* *on* $H_{\mathrm{\acute{e}t}}^i(X_{\overline{K}}, \mathbb{Q}_l)$ *preserves the weight filtration. Hence it acts on* $\mathrm{gr}_n^W = \mathrm{gr}_n^W H_{\mathrm{\acute{e}t}}^i(X_{\overline{K}}, \mathbb{Q}_l)$.

(2) *For each integer* $0 \leq i \leq 2d$ *and each* $n \in \{0, 1, 2d-1, 2d\}$, *the characteristic polynomial of* $\Gamma^*|_{\mathrm{gr}_n^W}$ *is in* $\mathbb{Z}[x]$ *and independent of* l. *If* $d \leq 2$, *then this holds for all* $0 \leq n \leq 2d$.

To the author's knowledge, this result has not previously been known.

We use the following lemma, which follows from [10, Lemma 2.8] (cf. also [25, Lemma 3.4]).

LEMMA 3.4. *Let* F *be a field of characteristic* 0 *and* f *an* F-*linear endomorphism of an* F-*vector space of finite dimension. If there exists a nonzero integer* N *such that any power of* f *has trace in* $N^{-1}\mathbb{Z}$, *then the characteristic polynomial of* f *has coefficients in* $\mathbb{Z}$.

PROOF OF THEOREM 3.3. First we prove the assertions (1) and (2) under the assumption that X admits a proper strictly semistable model $\mathcal{X}$ that is a scheme.

By [25, Proposition 2.20] there exists a collection of algebraic cycles $\overline{\Gamma}^{(p)} \in \mathrm{CH}^{d-p}(X_0^{(p)} \times X_0^{(p)})$ $(p \geq 0)$ such that there is an endomorphism of the weight

spectral sequence that acts on

$$E_1^{p,q} = \bigoplus_{i \geq \max\{0, -p\}} H_{\text{ét}}^{q-2i}(X_0^{(p+2i)}, \mathbb{Q}_l(-i))$$

by

$$(d!)^{-1} \cdot \bigoplus_{i \geq \max\{0, -p\}} \overline{\Gamma}^{(p+2i)^*}$$

and on $H_{\text{ét}}^{p+q}(X_{\bar{K}}, \mathbb{Q}_l)$ by Γ^*. Hence (1) follows.

Next we show (2). Assume $n = 0$. Then $\text{gr}_0^W H_{\text{ét}}^i = E_2^{i,0}$ is the i-th cohomology of the complex $E_1^{\bullet,0} = H_{\text{ét}}^0(X_0^{(\bullet)}, \mathbb{Q}_l)$. This complex is naturally isomorphic to the base change $V^{(\bullet)} \otimes_{\mathbb{Z}} \mathbb{Q}_l$ of the complex $V^{(\bullet)}$ of the $\mathbb{Z}$-modules freely generated by the components of $X_0^{(\bullet)}$. The algebraic correspondences $\overline{\Gamma}^{(p)}$ induce an endomorphism of this complex of $\mathbb{Z}$-modules, independent of l. Hence the coefficients of the characteristic polynomial of $\Gamma^*|_{\text{gr}_0^W}$ lie in $(d!)^{-1}\mathbb{Z}$, and the same assertion holds for any power of Γ. Hence, by Lemma 3.4, the coefficients actually lie in $\mathbb{Z}$.

Assume $n = 1$. Then $\text{gr}_1^W H_{\text{ét}}^i = E_2^{i-1,1}$ is the i-th cohomology of the complex $E_1^{\bullet-1,1} = H_{\text{ét}}^1(X_0^{(\bullet-1)}, \mathbb{Q}_l)$. By [25, Lemma 3.6], this complex is naturally isomorphic to $T_l(A^{(\bullet-1)}) \otimes \mathbb{Q}_l(-1)$ induced by the complex $A^{(\bullet-1)}$ of the Picard varieties of $X_0^{(\bullet-1)}$. As above, $\overline{\Gamma}^{(p)}$ induce an endomorphism of this complex of abelian varieties that is independent of l. Hence the action of Γ on $\text{gr}_1^W H_{\text{ét}}^i$ is $(d!)^{-1}$ times the l-adic realization of an endomorphism of an abelian variety, and therefore its characteristic polynomial has coefficients in $(d!)^{-1}\mathbb{Z}$ and is independent of l. By Lemma 3.4 again, the coefficients lie in $\mathbb{Z}$.

The cases $n = 2d - 1$ and $n = 2d$ are similar to the cases of $n = 1, 0$ respectively.

Consider the latter assertion of (2). Since gr_n^W is 0 outside the range $0 \leq n \leq 2d$, we are done if $d \leq 1$. If $d = 2$, then the only remaining case of $n = 2$ follows from the assertions for $H_{\text{ét}}^i$ (see [25, Corollary 0.2]) and for gr_n^W for the other n's ($n = 0, 1, 3, 4$).

Now we show that the assertions for the general case (i.e. not assuming the existence of a semistable model that is a scheme) can be reduced to this special case. Take an alteration $Y \to X$ as above (after replacing $\mathcal{O}_K$ if necessary). Assertion (1) applied to $f^* \circ f_* \in \text{CH}^d(Y \times Y)$ implies that $f_*(W_n H_{\text{ét}}^i(Y_{\bar{K}}, \mathbb{Q}_l))$ is contained in $W_n H_{\text{ét}}^i(X_{\bar{K}}, \mathbb{Q}_l)$. Hence $\Gamma^* = \deg(f)^{-2} \cdot f_* \circ (f^*\Gamma)^* \circ f^*$ preserves the filtration. Using Lemma 3.4 and the equality

$$\text{tr}(\Gamma^* \mid W_n H_{\text{ét}}^i(X_{\bar{K}}, \mathbb{Q}_l)) = \deg(f)^{-1} \cdot \text{tr}(f^* \circ \Gamma^* \circ f_* \mid W_n H_{\text{ét}}^i(Y_{\bar{K}}, \mathbb{Q}_l))$$

for Γ and its powers, we reduce assertion (2) for Γ to the case of Y. ∎

For $n = 1, 2d - 1$, we actually proved:

COROLLARY 3.5. *Let X and Γ be as in Theorem 3.3. Then for each integer $0 \leq i \leq 2d$ and each $n \in \{1, 2d-1\}$, $\Gamma^*|_{\operatorname{gr}_n^W} \in \operatorname{End}(\operatorname{gr}_n^W H_{\text{ét}}^i)$ lies in the image of the algebra* $\operatorname{End}(B_n^{(i)}) \otimes \mathbb{Q}$, *where $B_n^{(i)}$ is an abelian variety with* $\operatorname{gr}_n^W H_{\text{ét}}^i \cong H_{\text{ét}}^1((B_n^{(i)})_{\bar{k}}, \mathbb{Q}_l)$.

The variety $B_1^{(i)}$ is obtained as the i-th "cohomology" of the complex $A^{(\bullet - 1)}$ above, while $B_{n-1}^{(i)}$ is the dual of $B_1^{(i)}$.

REMARK 3.6. We can also prove Theorem 3.3 without using alterations, if X admits a strictly semistable model that is not necessarily a scheme, which is the case in the setting of our main theorems. For this, we need to generalize [25, Proposition 2.20] to the case of algebraic spaces. Most of its proof is étale-local and hence can be reduced to the scheme case, and the only non-trivial part is the construction (see [25, Lemma 2.17]) of cycle classes $\Gamma^{(p)}$ on $X_{\bar{0}}^{(p)}$ for a cycle class Γ on the generic fiber X. Although we cannot mimic the construction of Saito (which uses locally-free resolution of coherent sheaves), we can take an algebraic cycle on $X_{\bar{0}}^{(p)}$ to be the intersection with the closure of (a cycle representing) Γ, and then check the required properties étale-locally. We omit the details.

4. Kulikov models

DEFINITION 4.1. Let X be a K3 surface or an abelian surface over K. An algebraic space $\mathcal{X}$ over $\mathcal{O}_K$ with generic fiber X is a *Kulikov model* of X if it is a proper strictly semistable model (in the sense of the previous section) and its relative canonical divisor $K_{\mathcal{X}/\mathcal{O}_K}$ is trivial.

REMARK 4.2. The standard, but conditional, recipe to construct a Kulikov model of a K3 (or an abelian) surface X is the following. Take a proper strictly semistable model of X after extending K, then apply a suitable MMP to get a log terminal model with nef canonical divisor (in this case, the canonical divisor is in fact trivial), and then apply Artin's simultaneous resolution to make it semistable. If the residue field k is of characteristic 0, this is unconditional. If char $k = p > 0$, then the existence of a semistable model is still conjectural and the MMP is proved only for $p \geq 5$ (see [8]).

A classification of the special fibers of Kulikov models is given by Kulikov in [12, Theorem II] when the characteristic is 0, and by Nakkajima in [18, Proposition 3.4] when the characteristic is > 0. Then we can compute $\dim \operatorname{gr}_n^W$ using the weight spectral sequence. We summarize:

PROPOSITION 4.3. *Let X be a K3 surface and $\mathcal{X}$ a Kulikov model of X.*

(1) *The geometric special fiber $X_{\bar{0}} = X_0 \otimes \bar{k}$ is one of the following types:*

(I) *A smooth K3 surface.*

(II) *$X_{\bar{0}} = Z_1 \cup \cdots \cup Z_N$, $N \geq 2$, Z_1 and Z_N are rational and the others are elliptic ruled (i.e. birational to a $\mathbb{P}^1$-bundle over an elliptic curve), $Z_i \cap Z_j$ is an elliptic curve if $|i - j| = 1$, and there are no other intersections. (The components form a "chain".)*

(III) *$X_{\bar{0}}$ is a union of rational surfaces, each double curve is rational, and the dual graph of the components forms a triangulation of S^2 (the 2-dimensional sphere).*

(2) *$\dim_{\mathbb{Q}_l} \operatorname{gr}_n^W H^2_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_l)$ ($n = 0, 1, 2, 3, 4$) of the generic fiber X depends only on the type, and are given by the following ones:*

(I) $0, 0, 22, 0, 0.$

(II) $0, 2, 18, 2, 0.$

(III) $1, 0, 20, 0, 1.$

We will simply say that X_0 is of Type I, II, or III if $X_{\bar{0}}$ is so.

5. Proof of the main theorems

LEMMA 5.1. *Let K, X, and $\mathcal{X}$ be as in Theorem 1.1 (so char $K \neq 2$). Assume either char $K = p > 2$ and $\operatorname{ht}(X) < \infty$, or char $K = 0$. Assume X_0 is of Type II or III, and let $n = 3$ or $n = 4$ respectively. Then any eigenvalue of the action of $g \in \operatorname{Aut}(X)$ on $\operatorname{gr}_n^W = \operatorname{gr}_n^W H^2_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_l)$ is a primitive m'-th root of 1, where $m' = m(g)p^e$ for some integer $e \geq 0$ (possibly depending on g and the eigenvalue) if char $K = p > 0$, and $m' = m(g)$ if char $K = 0$.*

PROOF. By Proposition 4.3, we have $\operatorname{gr}_n^W = W_n / W_2 \neq 0$.

As noted in Section 3, we have $\operatorname{NS}(X_{\bar{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}_l(-1) \subset \operatorname{Ker} N \subset M_0 H^2_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_l)$, and we have $M_0 H^2_{\text{ét}} = W_2 H^2_{\text{ét}}$ (Theorem 3.2). Hence, $\operatorname{gr}_n^W = W_n / W_2$ is a quotient of $T_l(X) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. The assertions follow from Lemmas 2.3–2.4. ∎

PROOF OF THEOREM 1.1. If char $K > 0$ and $\operatorname{ht}(X) \geq 3$, then it is proved in [24, Section 2, Corollary] (without the condition on automorphisms) that X_0 is of Type I: Since the height is upper semi-continuous, and since Type II (resp. III) surfaces have height ≤ 2 (resp. $= 1$), then X_0 cannot be of Type II nor III.

Now assume either char $K = p > 2$ and $\operatorname{ht}(X) < \infty$, or char $K = 0$. We can apply Lemma 5.1 to $\psi_l \colon \operatorname{Aut}(X) \to \operatorname{GL}(\operatorname{gr}_n^W)$.

Assume that X_0 is of Type II. Let $n = 3$. By Corollary 3.5, ψ_l factors through $(\mathrm{End}(C)_{\mathbb{Q}})^*$, where C is the elliptic curve appearing as the intersection of two components of $X_{\overline{0}}$. For any $g \in \mathrm{Aut}(X)$, $\psi_l(g)$ belongs to a (commutative) $\mathbb{Q}$-subalgebra of $\mathrm{End}(C)_{\mathbb{Q}}$ generated by a single element ($\psi_l(g)$ itself), and such a subalgebra is either $\mathbb{Q}$ or an imaginary quadratic field. Hence $m' \in \{1, 2, 3, 4, 6\}$ and so is $m(g)$.

Assume that X_0 is of Type III. Let $n = 4$. Similarly, by Theorem 3.3, we have $\psi_l(g) \in \mathrm{GL}_1(\mathbb{Q}) = \mathbb{Q}^*$. Hence $m' \in \{1, 2\}$ and so is $m(g)$. ∎

COROLLARY 5.2. *Assume that X_0 is of Type III. Let $g \in \mathrm{Aut}(X)$, and suppose that g extends to an automorphism of $\mathcal{X}$, so that g acts on X_0 and on the set of the irreducible components of $X_{\overline{0}}$. Then the induced action of g on the (2-element) set of orientations of S^2 (which the dual graph of $X_{\overline{0}}$ triangulates) coincides with the image $\rho(g) \in \{\pm 1\}$ of g by $\rho\colon \mathrm{Aut}(X) \to \mathrm{GL}(H^0(X, \Omega^2))$.*

PROOF. We have

$$\mathrm{gr}_4^W H_{\text{ét}}^2 = E_2^{-2,4} = \mathrm{Ker}\big(H_{\text{ét}}^0(X_0^{(2)}, \mathbb{Q}_l(-2)) \to H_{\text{ét}}^2(X_0^{(1)}, \mathbb{Q}_l(-1))\big)$$
$$\cong H_2(S^2, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_l(-2),$$

and the two generators of $H_2(S^2, \mathbb{Z})$ correspond to the two orientations. ∎

REMARK 5.3. Actually, it was the assertion of Corollary 5.2, proposed by Yuji Odaka, that led the author to the study of this paper. The author first looked for an example of a Type III degeneration with $m(X) = |\mathrm{Im}\,\rho| \geq 3$, which would be a counterexample to the assertion, but failed to find one. It turned out that such examples do not exist!

Next we prove Theorem 1.3. First we define the Hodge endomorphism field.

Let X be a K3 surface over a field F of characteristic 0. Let $X_{\mathbb{C}}$ be a K3 surface over $\mathbb{C}$ isomorphic to X over some field F' containing both F and $\mathbb{C}$ (such $X_{\mathbb{C}}$ always exists). We call $E = \mathrm{End}_{\mathrm{HS}}(T(X_{\mathbb{C}})_{\mathbb{Q}})$ the *Hodge endomorphism field* of X, where $\mathrm{End}_{\mathrm{HS}}$ denotes the endomorphisms of a rational Hodge structure. It is known that E is either a totally real field or a CM field ([29, Theorem 1.5.1]).

This definition of E depends a priori on the choice of $X_{\mathbb{C}}$. However, if the Hodge conjecture for the self-product of a K3 surface holds, then every element of E can be realized as the action of an algebraic cycle on $X_{\mathbb{C}} \times X_{\mathbb{C}}$, hence as the action of an algebraic cycle on $(X \times X)_{\overline{F}}$, and therefore E (up to isomorphism) depends only on X. The conjecture for $X_{\mathbb{C}} \times X_{\mathbb{C}}$ is proved by Ramón Marí [22, Theorem 5.4] under the assumption that E (of $X_{\mathbb{C}}$) is a CM field. Thus, for a CM field E, the statement "E is the Hodge endomorphism field of X" is well-defined, and the statement of Theorem 1.3 should be understood accordingly.

PROOF OF THEOREM 1.3. Assume that X_0 is of Type II or III, and let $n = 3$ or $n = 4$ respectively. We have two homomorphisms of $\mathbb{Q}$-algebras $\psi_l \colon \mathrm{CH}^2(X \times X)_\mathbb{Q} \to \mathrm{End}(\mathrm{gr}_n^W)$ and $\chi \colon \mathrm{CH}^2(X \times X)_\mathbb{Q} \to \mathrm{End}_{\mathrm{HS}} T(X_\mathbb{C})_\mathbb{Q} \cong E$. Since we are assuming that the Hodge conjecture for $X \times X$ is true, the E-action is realized as algebraic correspondences, i.e. χ is surjective. Since $T(X_\mathbb{C}) \otimes \mathbb{Q}_l \to \mathrm{gr}_n^W$ is surjective, we have a surjection $\mathrm{Im}\,\chi \twoheadrightarrow \mathrm{Im}\,\psi_l$. Since $\mathrm{Im}\,\chi \cong E$ is a field and $\mathrm{Im}\,\psi_l$ is nonzero, this surjection is an isomorphism.

Assume that X_0 is of Type II. Let $n = 3$. By Corollary 3.5, $\mathrm{Im}\,\psi_l$ is contained in $\mathrm{End}(C)_\mathbb{Q}$ for some elliptic curve C. A (commutative) subfield of $\mathrm{End}(C)_\mathbb{Q}$ is either $\mathbb{Q}$ or an imaginary quadratic field.

Assume that X_0 is of Type III. Let $n = 4$. Similarly by Theorem 3.3, $\mathrm{Im}\,\psi_l$ is contained in $M_1(\mathbb{Q}) = \mathbb{Q}$. ∎

6. Application: moduli spaces of K3 surfaces with non-symplectic automorphisms of prime order

We apply Theorem 1.1 to obtain a compactification of the moduli spaces of K3 surfaces with non-symplectic automorphisms of fixed prime order ≥ 5.

For a moment we work over $\mathbb{C}$. In this section, a *lattice* is a free $\mathbb{Z}$-module of finite rank equipped with a $\mathbb{Z}$-valued symmetric bilinear form. Let U be the hyperbolic plane (i.e. the rank 2 lattice with Gram matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$) and E_8 the (negative definite) root lattice of type E_8. Then $L_{\mathrm{K3}} := U^{\oplus 3} \oplus E_8^{\oplus 2}$ is isometric to $H^2(X, \mathbb{Z})$, for any K3 surface X over $\mathbb{C}$, and is called the K3 lattice.

We recall the notation of [1]. Fix a prime $p \leq 19$ and a primitive p-th root ζ_p of 1. Fix an isometry $\sigma \in \mathrm{O}(L_{\mathrm{K3}})$ of order p, and denote by $[\sigma]$ its conjugacy class. We write $S(\sigma) = (L_{\mathrm{K3}})^{\sigma=1}$. Let $\mathcal{M}^\sigma$ be the moduli space of a pair (X, g) consisting of a complex K3 surface X and an automorphism g of X of order p with $\rho(g) = \zeta_p$ and acting on $H^2(X, \mathbb{Z})$ by σ via some marking (i.e. isometry) $L_{\mathrm{K3}} \xrightarrow{\sim} H^2(X, \mathbb{Z})$. We call such (X, g) a $[\sigma]$-*polarized* K3 *surface*. Such X is automatically algebraic and has an ample class in $S(g) = H^2(X, \mathbb{Z})^{g=1}$. The marking induces an isometry $S(\sigma) \cong S(g)$.

Let $D^\sigma = \{w \in \mathbb{P}((L_{\mathrm{K3}} \otimes \mathbb{C})^{\sigma=\zeta_p}) : (w, w) = 0, (w, \overline{w}) > 0\}$: this is a type IV Hermitian symmetric space if $p = 2$, and a complex ball if $p > 2$. Define the divisor

$$\Delta^\sigma = \bigcup_{\substack{\delta \in (S(\sigma))^\perp \\ \delta^2 = -2}} (D^\sigma \cap \delta^\perp)$$

and the discrete group

$$\Gamma^\sigma = \{\gamma \in \mathrm{O}(L_{\mathrm{K3}}) : \gamma\sigma = \sigma\gamma\}.$$

Then the space $\Gamma^\sigma \backslash (D^\sigma \setminus \Delta^\sigma)$ is naturally isomorphic to the space $\mathcal{M}^\sigma(\mathbb{C})$ of $\mathbb{C}$-valued points of $\mathcal{M}^\sigma$ (see [1, Theorem 9.1]).

Hereafter, we consider only isometries σ for which $\mathcal{M}^\sigma$ is nonempty.

In fact, the moduli space $\mathcal{M}^\sigma$ can be defined algebraically over $\mathbb{Q}(\zeta_p)$. Indeed, by [1, Proposition 9.3], if $p \geq 3$, then $[\sigma]$ is uniquely determined by the topology of $\mathrm{Fix}(g) \subset X$, and if $p = 2$, then $[\sigma]$ is uniquely determined by the parameters r, a, δ of the lattice $S(\sigma) \cong S(g)$, which can be read off from the action of g on the 2-adic étale cohomology group $H^2_{\text{ét}}(X, \mathbb{Z}_2)$.

Now we consider the compactification of $\mathcal{M}^\sigma$. Theorem 1.1 does not imply that $\mathcal{M}^\sigma$ is compact, since the action on the K3 lattice may change by specialization. So we need to attach some $\mathcal{M}^\tau$'s on the boundary.

Consider the space $\Gamma^\sigma \backslash D^\sigma$. The points on the boundary correspond to pairs (X, g) with non-symplectic automorphism of order p but acting on H^2 by some $\tau \neq \sigma$. We can translate this into an algebraic construction of $\overline{\mathcal{M}^\sigma}$ by attaching $\mathcal{M}^\tau$'s to $\mathcal{M}^\sigma$, and we have $\overline{\mathcal{M}^\sigma}(\mathbb{C}) = \Gamma^\sigma \backslash D^\sigma$. Then we have the following:

PROPOSITION 6.1. *Let σ be an isometry of L_{K3} of order $p \geq 5$. Then $\overline{\mathcal{M}^\sigma}$ is proper.*

PROOF. It suffices to show that any K-rational point, $K = \mathbb{C}((t))$, extends to an $\mathcal{O}_K$-rational point, $\mathcal{O}_K = \mathbb{C}[[t]]$, after replacing K by a finite extension $\mathbb{C}((t^{1/n}))$. So let $(X, g) \in \mathcal{M}(K)$ be a pair defined over K. By Theorem 1.1 there exists, after extending K, a smooth proper algebraic space $\mathcal{X}$ over $\mathcal{O}_K = \mathbb{C}[[t]]$ with K3 fibers, and g extends to a birational map $g \colon \mathcal{X} \dashrightarrow \mathcal{X}$. The period map gives an extension of the morphism $\mathrm{Spec}\, \mathbb{C}((t)) \to \mathcal{M}^\sigma$ to $\mathrm{Spec}\, \mathbb{C}[[t]] \to \overline{\mathcal{M}^\sigma}$. By [15, Proposition 2.2], $g \colon \mathcal{X} \dashrightarrow \mathcal{X}$ is defined over the complement of a closed subspace of $\mathcal{X}$ of codimension ≥ 2, and the birational map $g_0 := g|_{X_0}$ extends to an automorphism $\widetilde{g}_0$ of X_0. This $\widetilde{g}_0$ also satisfy $\rho(\widetilde{g}_0) = \zeta_p$, but the action on H^2 is possibly different from σ. ∎

EXAMPLE 6.2. Let $p = 11$. By [1, Theorem 7.3], there are three non-conjugate isometries $\sigma_1, \sigma_2, \sigma_3 \in \mathrm{O}(L_{K3})$ of order 11, respectively with $S(\sigma_1) \cong U$, $S(\sigma_2) \cong U(11)$, $S(\sigma_3) \cong U \oplus A_{10}$, and the geometric points of $\mathcal{M}^{\sigma_1}$ consist of elliptic K3 surfaces $y^2 = x^3 + ax + t^{11} - b$ with an automorphism $g(x, y, t) = (x, y, \zeta_{11} t)$, parametrized by $\{(a^3 : b^2) \in \mathbb{P}^1 : 4a^3 + 27b^2 \neq 0\}$. We use Kodaira's notation for types $\mathrm{I}_n, \mathrm{II}, \dots$ of singular fibers of elliptic surfaces (not be confused with type I, II, III of Kulikov models). This elliptic surface has one singular fiber of type II at $t = \infty$, and 22 of type I_1 at $4a^3 + 27(t^{11} - b)^2 = 0$, unless $a = 0$, in which case it has 12 singular fiber of type II. The boundary of the compactification $\overline{\mathcal{M}^{\sigma_1}}$ consists of one point ($4a^3 + 27b^2 = 0$), and this is $\mathcal{M}^{\sigma_3}$. At this point the elliptic surface has one singular fiber of type II, 11 of type I_1, and one of type I_{11}.

The following is also a direct consequence of Theorem 1.1.

PROPOSITION 6.3. *Assume the existence of a Kulikov model (after field extension) for any K3 surface defined over a number field. Take an isometry σ with order $p \geq 5$. Then any K3 surface corresponding to a point of $\mathcal{M}^\sigma(\bar{\mathbb{Q}})$ has everywhere potential good reduction.*

In [16, Theorem 6.3] the author proved (conditionally) everywhere potential good reduction of K3 surfaces with complex multiplications, which however gives only isolated examples. To the contrary, Proposition 6.3 can be applied to positive dimensional family: for $p = 5, 7, 11$ there exists $\sigma \in O(L_{K3})$ of order p with dim $\mathcal{M}^\sigma = 4, 2, 1$, respectively.

EXAMPLE 6.4. Consider again a K3 surface of the form $y^2 = x^3 + ax + t^{11} - b$, this time defined over a number field K. Then by the previous proposition it has potential good reduction at any prime of K. We can also show potential good reduction directly. If the residue characteristic is equal to 11 then this is done in [15, Example 6.8]. In other characteristics we proceed as follows. After extending K, we can find $a_i \in \mathcal{O}_K$ such that

$$\mathcal{E} = (F(x', y') = y'^2 + a_1 x' y' + a_3 y' + x'^3 + a_2 x'^2 + a_4 x' + a_6 = 0) \subset \mathbb{P}^2_{\mathcal{O}_K}$$

is a minimal Weierstrass model over $\mathcal{O}_K$ of the elliptic curve $(y^2 = x^3 + ax - b)$, and that the special fiber $E_0 = \mathcal{E} \otimes_{\mathcal{O}_K} k$ of $\mathcal{E}$ is either an elliptic curve or a nodal curve. Then the special fiber of

$$\mathcal{X} = (F(x', y') + t^{11} = 0)$$

is smooth or has one A_{10} singularity, according to E_0 being smooth or nodal respectively. Applying Artin's simultaneous resolution (after extending K) we achieve good reduction.

7. Examples

7.1 – *Theorems 1.1 and 1.3 are optimal*

The following two examples show that we cannot weaken the assumptions on m and E in Theorems 1.1 and 1.3.

EXAMPLE 7.1 (Type III). Assume char $k \neq 2$. Consider the family $\mathcal{X}'$ over $\mathcal{O}_K$ of quartic surfaces given by $t(w^4 + x^4 + y^4 + z^4) + wxyz = 0$, where t is a uniformizer

of $\mathcal{O}_K$. This is not a Kulikov model, since it has non-regular points, but we can perform small blow-ups to obtain a Kulikov model $\mathcal{X}$. (For example, we can resolve the singularity at $t = w = x = y^4 + z^4$ by blowing-up either the ideal (t, w) or (t, x) at a neighborhood.) Then the special fiber is of Type III (whose dual graph is a tetrahedron). The symmetric group $\mathfrak{S}_4$ acts on X (over K) naturally and its action on the 2-forms is given by sgn: $\mathfrak{S}_4 \twoheadrightarrow \{\pm 1\}$. Hence $m(X)$ is divisible by 2 (and by Theorem 1.1 we have $m(X) = 2$).

EXAMPLE 7.2 (Type II). Assume again char $k \neq 2$. Let E be an imaginary quadratic field. Let C_1 be an elliptic curve with complex multiplication by an order of E, and C_2 an elliptic curve with multiplicative reduction. Let $\mathcal{C}_i$ be the minimal regular models of C_i over $\mathcal{O}_K$. By extending K, we may assume that $(C_1)_0$ is smooth and that $(C_2)_0$ has an even number of components. Let ι be the multiplication-by-(-1) map on $\mathcal{A} = \mathcal{C}_1 \times_{\mathcal{O}_K} \mathcal{C}_2$ (then Fix(ι) is finite étale over $\mathcal{O}_K$), and $\mathcal{X}$ the blow-up of $\mathcal{A}/\iota$ at the image of Fix(ι). Then $\mathcal{X}$ is a Kulikov model of $X = \text{Km}(C_1 \times C_2)$ with Type II degeneration, $T(X_{\mathbb{C}})_{\mathbb{Q}} \cong H^1((C_1)_{\mathbb{C}}, \mathbb{Q}) \otimes H^1((C_2)_{\mathbb{C}}, \mathbb{Q})$ and $\text{End}_{\text{HS}} T(X)_{\mathbb{Q}} \supset E$ (moreover, by Theorem 1.3, we have $\text{End}_{\text{HS}} T(X)_{\mathbb{Q}} = E$).

In particular, if we take C_1 to be the elliptic curve with an automorphism of order 4 (resp. 6), then we have an example of Type II degeneration with $m(X)$ divisible by 4 (resp. 6).

7.2 – Some automorphisms of K3 surfaces of finite order in positive characteristic

The following collection of examples shows that in general e (and e_i) in Lemma 2.4 (statements (2) and (3)) may be nonzero.

EXAMPLE 7.3. Considering the obvious degree constraint, i.e. deg $\Phi_{p^e} \leq 22$, in Lemma 2.4 the condition p^e with $e \neq 0$ can occur only if p^e belongs to the set

$$\{2^e \ (e \leq 5), \ 3^e \ (e \leq 3), \ 5^e \ (e \leq 2), 7, 11, 13, 17, 19\}.$$

We give examples for $p^e = 2, 2^2, 3, 5, 7, 11$. All examples are automorphisms of finite order. For the other cases we do not know whether examples exist.

Let p^e be one of $2, 2^2, 3, 5, 7, 11$. Define an integer n as in Table 1 below. Let $K = \mathbb{Q}_p(\zeta_{p^e n})$ if $e = 1$, and $K = \mathbb{Q}_p(\zeta_{3p^e n})$ if $p^e = 2^2$. If $p^e = 2, 2^2, 3, 7, 11$, let $\mathcal{X}$ (over $\mathcal{O}_K$) be the elliptic K3 surface defined by the corresponding equation in Table 1 (with the E_8 singularity at $t = \infty$ resolved in the standard way if $p^e = 2, 2^2, 7$). If $p^e = 5$, let $\mathcal{X}$ (over $\mathcal{O}_K$) be the blow-up of the double sextic surface defined by the equation below at the non-smooth locus ($w = x = G_5(y) = 0$). Define $g_1, g_2 \in \text{Aut}(\mathcal{X})$ as in Table 1.

p^e	n	equation	$g_1(x,y,t)$	$g_2(x,y,t)$	char. poly.
2	21	$G_2(y) = x^3 + t^7$	$(x, -y+1, t)$	$(\zeta_{21}^7 x, y, \zeta_{21}^{15} t)$	$\Phi_1^{10}\,\Phi_{42}$
2^2	7	$H(x,y) + t^7 = 0$	$(h(x,y), -t)$	$(x, y, \zeta_7 t)$	$\Phi_1^{10}\,\Phi_{28}$
3	22	$y^2 = G_3(x) + t^{11}$	$(\zeta_3 x + 1, y, t)$	$(x, -y, \zeta_{22}^{12} t)$	$\Phi_1^2\,\Phi_{66}$
5	8	$w^2 = x(x^4 + G_5(y))$	$(x, \zeta_5 y + 1, w)$	$(\zeta_8^2 x, y, \zeta_8 w)$	$\Phi_1^2\,\Phi_5\,\Phi_{40}$
7	6	$y^2 = x^3 + G_7(t)$	$(x, y, \zeta_7 t + 1)$	$(\zeta_6^4 x, -y, t)$	$\Phi_1^{10}\,\Phi_{42}$
11	1	$y^2 = x^3 + x^2 + G_{11}(t)$	$(x, y, \zeta_{11} t + 1)$	id	$\Phi_1^2\,\Phi_{11}^2$

TABLE 1. Equations of surfaces and automorphisms

In Table 1, G_p and H are defined as follows:

- $G_p(z) = \prod_{i=0}^{p-1}(z - a_i) \in \mathbb{Z}_p[\zeta_p][z]$, with $a_i = (\zeta_p^i - 1)/(\zeta_p - 1)$. It satisfies $G_p(\zeta_p z + 1) = G_p(z)$ and $G_p(z) \equiv z^p - z \pmod{(\zeta_p - 1)}$ (since $a_i \equiv i$).
- $H(x,y) = y^2 + a_1 xy + a_3 y + x^3 + a_2 x^2 + a_4 x + a_6 = 0$ and

$$h: (x,y) \mapsto (-x + b_2, \zeta_4^{-1} y + b_1 x + b_3),$$

with $a_i, b_i \in \mathcal{O}_K$, are equations of the Néron model of an elliptic curve with an automorphism acting on the 1-forms by ζ_4^{-1}. (For example, one can take

$$a_1 = 3 - \sqrt{3}, \qquad a_3 = 2 - \sqrt{3}, \qquad a_2 = a_4 = a_6 = 0$$
$$b_2 = -(2 - \sqrt{3}), \qquad b_1 = \zeta_{12}(1 - \zeta_{12})\sqrt{3}, \qquad b_3 = \zeta_{12}^{-1}(\zeta_{12} - 1)^3,$$

with $\sqrt{3} = \zeta_{12} + \zeta_{12}^{-1}$.)

Then $\mathcal{X}$ is smooth proper over $\mathcal{O}_K$ with K3 fibers, $g_1, g_2 \in \mathrm{Aut}(\mathcal{X})$ commute and are of orders p^e and n respectively, moreover, $\rho(g_1|_{X_K}) = \zeta_{p^e}$ and $\rho(g_2|_{X_K}) = \zeta_n$. Hence $g := g_1 g_2$ is of order $p^e n$, and $m(g|_{X_K}) = p^e n$, $m(g|_{X_0}) = n$. Therefore, g acts on $T_l(X_K)$ by a power of $\Phi_{p^e n}$, hence also on $T_l(X_0)$.

Moreover, we can determine the characteristic polynomial of g on $H^2_{\text{ét}}$ completely (although we do not need this). For $p^e = 2, 4, 3, 7$, one observes that the sublattice of $\mathrm{NS}(X_K)$ generated by the zero section and the components of the singular fibers already has rank $22 - \phi(p^e n)$, and hence $H^2_{\text{ét}}$ of X_K is generated up to torsion by these curves and $T_l(X_K)$. Since the action of g on the classes of those curves are trivial, we obtain the characteristic polynomial $\Phi_1^{22 - \phi(p^e n)} \Phi_{p^e n}$. For $p^e = 5$, similarly $H^2_{\text{ét}}$ is generated by $T_l(X_K)$, the five exceptional curves, and the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$. Since g_1 and g_2 act respectively on the classes of the exceptional curves transitively and trivially, we obtain the characteristic polynomial $\Phi_1^2 \Phi_5 \Phi_{40}$. Finally, for $p^e = 11$, it is

proved by Dolgachev–Keum [4, Lemma 2.3(i)] that the characteristic polynomial of an order 11 automorphism of a K3 surface in characteristic 11 is always $\Phi_1^2 \Phi_{11}^2$.

It remains to show that X_0 is not supersingular. For the case $p^e = 11$ this is checked in [27, Section 3.2]. Assume $p^e \neq 11$. By [20, Theorem 2.1] (see Remark 2.2), if g is an automorphism of a supersingular K3 surface Y, then $m(g)$ should divide $p^{\sigma_0} + 1$, where σ_0 is the Artin invariant of Y (which is a positive integer ≤ 10). For the cases $p^e \neq 11$, we observe that no such integer σ_0 exists, and hence X_0 are not supersingular.

As the reader might have noticed, the generic fibers of $\mathcal{X}$ for $p^e = 2, 2^2, 3, 5, 7$ are the well-known examples of automorphisms g of characteristic zero K3 surfaces with $\mathrm{ord}(g) = \mathrm{ord}(\rho(g)) = 42, 28, 66, 40, 42$, given by Kondo in [11, Section 3] (order 28, 42, 66) and Machida–Oguiso in [14, Proposition 4(15)] (order 40). (To check this, use the equality $G_p(z) = (\zeta_p - 1)^{-p}(z'^p - 1)$, where $z' = (\zeta_p - 1)z + 1$.)

EXAMPLE 7.4. In [9], Keum classified possible finite orders of automorphisms of K3 surfaces over each characteristic $\neq 2, 3$. Since the problem is still open for characteristic 2 and 3, we note some examples in these characteristics, although these might be known to experts.

Case $p^e = 2^2$ in the previous example gives an automorphism of order 28 of a K3 surface in characteristic 2. By replacing 7 with 9 or 11, we also obtain automorphisms of order 36 or 44, respectively.

Case $p^e = 2$ gives order 42 in characteristic 2, and by replacing 7 with 11 we obtain order 66. However these two examples are almost written in [9, Example 3.6].

Case $p^e = 3$ gives an automorphism of order 66 of a K3 surface in characteristic 3. By replacing 11 with 7, 8 and 10 (and then resolving the singularity at $t = \infty$ in the standard way) we also obtain automorphisms of order 42, 48 and 60.

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