

Functors for Long dimodules and Yetter–Drinfeld modules in a weak setting

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ABSTRACT – In this paper, for two weak Hopf monoids H and B with invertible antipode, we define a functor between the category of left-left $H \otimes B$ -Yetter–Drinfeld modules and the category of H - B -Long dimodules. We also show that, if moreover H is quasitriangular and B is coquasitriangular, this functor is a retraction of the well-known injective functor between left-left H - B -Long dimodules and left-left $H \otimes B$ -Yetter–Drinfeld modules.

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1. Introduction

Let R be a commutative fixed ring with unit and let H be a commutative and cocommutative Hopf algebra in the non-strict symmetric monoidal category of R -Mod. In order to study the Brauer group of H -dimodule algebras, Long introduced in [10] the notion of Long H -dimodule. Later, the notion was extended by considering two arbitrary Hopf algebras H and B with bijective antipode, introducing the category of left-left H - B -Long dimodules, denoted by ${}^B_H\text{Long}$.

On the other hand, to characterize bialgebras B such that $B \otimes H$ with the smash product structure is a bialgebra, Radford introduced in [14] conditions that subsequently

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give rise to the notion of Yetter–Drinfeld module on a bialgebra. The category of these modules was defined by Yetter in [16], denoted by ${}^H_H\text{YD}$ and called *crossed bimodule*.

It is a well-known fact that Yetter–Drinfeld modules over a Hopf algebra provide solutions of the quantum Yang–Baxter equation and then this category allows us to explain the relationship between different theories in mathematics and physics. Moreover, Yetter–Drinfeld modules play a central role in the theory of monoidal categories, allowing to categorize the concept of Drinfeld double (see [11]). Therefore, its generalization to broader contexts is very interesting. For example, in this sense, the category of Yetter–Drinfeld modules was studied in the context of weak Hopf algebras (see [8, 12]), Hopf quasigroups (see [6]) and Hom-Hopf algebras (see [15]).

Obviously, it is also interesting to know the connection between Yetter–Drinfeld modules and other categories. In the classic case of Hopf algebras, it is a well-known fact that, if the Hopf algebras H and B are quasitriangular and coquasitriangular, respectively, ${}^B_H\text{Long}$ is a braided monoidal subcategory of ${}^{H\otimes B}_{H\otimes B}\text{YD}$ and as a consequence of the above, the categories of Long dimodules provide non-trivial examples of solutions of the quantum Yang–Baxter equation. All this makes it interesting to obtain similar relations in the most popular generalizations of Hopf algebras, namely Hopf quasigroups [17] and weak Hopf algebras [5].

This paper is a continuation of [5]. By working again in the weak Hopf algebra setting, in the main result (Theorem 3.6) we define a functor between ${}^{H\otimes B}_{H\otimes B}\text{YD}$ and ${}^B_H\text{Long}$. If moreover H is quasitriangular and B coquasitriangular, we show in Theorem 4.4 that this functor is a retraction of the one defined in [5].

2. Preliminaries

Recall that a monoidal category is a category \mathcal{C} equipped with a tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, a unit object K of \mathcal{C} and a family of natural isomorphisms

$$\begin{aligned} a_{M,N,P} &: (M \otimes N) \otimes P \rightarrow M \otimes (N \otimes P), \\ r_M &: M \otimes K \rightarrow M, \quad l_M : K \otimes M \rightarrow M, \end{aligned}$$

in \mathcal{C} (called associativity, right unit and left unit constraints, respectively) satisfying the pentagon axiom and the triangle axiom. A monoidal category is called strict if the associativity, right unit and left unit constraints are identities. On the other hand, a strict monoidal category \mathcal{C} is braided if it has a natural family of isomorphisms $c_{M,N} : M \otimes N \rightarrow N \otimes M$ such that the equalities

$$\begin{aligned} c_{M,N\otimes P} &= (\text{id}_N \otimes c_{M,P}) \circ (c_{M,N} \otimes \text{id}_P), \\ c_{M\otimes N,P} &= (c_{M,P} \otimes \text{id}_N) \circ (\text{id}_M \otimes c_{N,P}) \end{aligned}$$

hold for all M, N in \mathbf{C} , where id_M, id_N and id_P denote the corresponding identity morphisms. If $c_{N,M} \circ c_{M,N} = \text{id}_{M \otimes N}$, for all M, N in \mathbf{C} , we will say that \mathbf{C} is symmetric.

From now on \mathbf{C} denotes a strict symmetric monoidal category with tensor product \otimes , unit object K and natural isomorphism of symmetry c . Taking into account that every non-strict monoidal category is monoidally equivalent to a strict one (see [9]), we can assume without loss of generality that the category is strict and, as a consequence, the results contained in this paper remain valid for every non-strict symmetric monoidal category, what would include for example the categories of vector spaces over a field \mathbb{F} , or the one of left modules over a commutative ring R . In what follows, for simplicity of notation, given objects M, N, P in \mathbf{C} and a morphism $f : M \rightarrow N$, we write $P \otimes f$ for $\text{id}_P \otimes f$ and $f \otimes P$ for $f \otimes \text{id}_P$. We also assume that in \mathbf{C} every idempotent morphism splits, i.e., for any morphism $q : Y \rightarrow Y$ such that $q \circ q = q$ there exists an object Z , called the image of q , and morphisms $i : Z \rightarrow Y, p : Y \rightarrow Z$ such that $q = i \circ p$ and $p \circ i = \text{id}_Z$. The pair of morphisms p and i will be called a factorization of q . Note that Z, p and i are unique up to isomorphism. The categories satisfying this property constitute a broad class that includes, among others, the categories with epi-monic decomposition for morphisms and categories with (co)equalizers.

In this section, we recall some basic definitions and well-known facts about monoids, comonoids, weak bimonoids and weak Hopf monoids in \mathbf{C} that we shall need later.

A monoid in \mathbf{C} is a triple $A = (A, \eta_A, \mu_A)$ where A is an object in \mathbf{C} and $\eta_A : K \rightarrow A$ (unit), $\mu_A : A \otimes A \rightarrow A$ (product) are morphisms in \mathbf{C} such that

$$\mu_A \circ (A \otimes \eta_A) = \text{id}_A = \mu_A \circ (\eta_A \otimes A), \quad \mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A).$$

Given two monoids $A = (A, \eta_A, \mu_A)$ and $B = (B, \eta_B, \mu_B)$, a morphism $f : A \rightarrow B$ in \mathbf{C} is a monoid morphism if

$$\mu_B \circ (f \otimes f) = f \circ \mu_A, \quad f \circ \eta_A = \eta_B.$$

Also, if A, B are monoids in \mathbf{C} , the object $A \otimes B$ is also a monoid in \mathbf{C} where $\eta_{A \otimes B} = \eta_A \otimes \eta_B$ and $\mu_{A \otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B)$. Note that, if A and B are commutative monoids, so is $A \otimes B$.

A comonoid in \mathbf{C} is a triple $D = (D, \varepsilon_D, \delta_D)$ where D is an object in \mathbf{C} and $\varepsilon_D : D \rightarrow K$ (counit), $\delta_D : D \rightarrow D \otimes D$ (coproduct) are morphisms in \mathbf{C} such that

$$(\varepsilon_D \otimes D) \circ \delta_D = \text{id}_D = (D \otimes \varepsilon_D) \circ \delta_D, \quad (\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D.$$

If $D = (D, \varepsilon_D, \delta_D)$ and $E = (E, \varepsilon_E, \delta_E)$ are comonoids, $f : D \rightarrow E$ is a comonoid morphism if

$$(f \otimes f) \circ \delta_D = \delta_E \circ f, \quad \varepsilon_E \circ f = \varepsilon_D.$$

If D, E are comonoids in \mathbf{C} , then $D \otimes E$ is a comonoid in \mathbf{C} where $\varepsilon_{D \otimes E} = \varepsilon_D \otimes \varepsilon_E$ and $\delta_{D \otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E)$. Note that, if D and E are cocommutative comonoids, so is $D \otimes E$.

Finally, if A is a monoid, C is a comonoid and $f : C \rightarrow A, g : C \rightarrow A$ are morphisms in \mathbf{C} , we define the convolution product $f * g : C \rightarrow A$ of f and g by $f * g = \mu_A \circ (f \otimes g) \circ \delta_C$.

DEFINITION 2.1. A weak bimonoid H is an object in \mathbf{C} with a monoid structure (H, η_H, μ_H) and a comonoid structure $(H, \varepsilon_H, \delta_H)$ satisfying the following conditions:

- (a1) $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ \delta_{H \otimes H}$,
- (a2) $\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes \delta_H \otimes H)$
 $= (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes H)$,
- (a3) $(\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$
 $= (H \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$.

Let H be a weak bimonoid. If there exists a morphism $\lambda_H : H \rightarrow H$ in \mathbf{C} (called the antipode of H) such that the equalities

- (a4) $\text{id}_H * \lambda_H = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$,
- (a5) $\lambda_H * \text{id}_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$,
- (a6) $\lambda_H * \text{id}_H * \lambda_H = \lambda_H$

hold, we will say that H is a weak Hopf monoid.

For a weak bimonoid H , the morphisms Π_H^L (target), Π_H^R (source), $\overline{\Pi}_H^L$ and $\overline{\Pi}_H^R$ are defined by

$$\begin{aligned} \Pi_H^L &= ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H), \\ \Pi_H^R &= (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)), \\ \overline{\Pi}_H^L &= (H \otimes (\varepsilon_H \circ \mu_H)) \circ ((\delta_H \circ \eta_H) \otimes H), \\ \overline{\Pi}_H^R &= ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)). \end{aligned}$$

These morphisms are idempotent and satisfy the identities

- (1) $\Pi_H^L \circ \overline{\Pi}_H^L = \Pi_H^L, \quad \Pi_H^L \circ \overline{\Pi}_H^R = \overline{\Pi}_H^R,$
 $\Pi_H^R \circ \overline{\Pi}_H^L = \overline{\Pi}_H^L, \quad \Pi_H^R \circ \overline{\Pi}_H^R = \Pi_H^R,$
- (2) $\overline{\Pi}_H^L \circ \Pi_H^L = \overline{\Pi}_H^L, \quad \overline{\Pi}_H^L \circ \Pi_H^R = \Pi_H^R,$
 $\overline{\Pi}_H^R \circ \Pi_H^L = \Pi_H^L, \quad \overline{\Pi}_H^R \circ \Pi_H^R = \overline{\Pi}_H^R.$

If H_L denotes the image of the target morphism Π_H^L , and $p_H^L : H \rightarrow H_L$ and $i_H^L : H_L \rightarrow H$ are the morphisms such that

$$i_H^L \circ p_H^L = \Pi_H^L, \quad p_H^L \circ i_H^L = \text{id}_{H_L},$$

then the triples

$$\begin{aligned} (H_L, \eta_{H_L} = p_H^L \circ \eta_H, \mu_{H_L} = p_H^L \circ \mu_H \circ (i_H^L \otimes i_H^L)), \\ (H_L, \varepsilon_{H_L} = \varepsilon_H \circ i_H^L, \delta_{H_L} = (p_H^L \otimes p_H^L) \circ \delta_H \circ i_H^L) \end{aligned}$$

determine a monoid and a comonoid, respectively.

In the weak monoid setting, for the morphisms target and source, we have the following identities:

- (3) $\Pi_H^L \circ \mu_H \circ (H \otimes \Pi_H^L) = \Pi_H^L \circ \mu_H,$
 $\Pi_H^R \circ \mu_H \circ (\Pi_H^R \otimes H) = \Pi_H^R \circ \mu_H,$
- (4) $(H \otimes \Pi_H^L) \circ \delta_H \circ \Pi_H^L = \delta_H \circ \Pi_H^L,$
 $(\Pi_H^R \otimes H) \circ \delta_H \circ \Pi_H^R = \delta_H \circ \Pi_H^R,$
- (5) $\mu_H \circ (H \otimes \Pi_H^L) = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H),$
- (6) $(H \otimes \Pi_H^L) \circ \delta_H = (\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H),$
- (7) $\mu_H \circ (H \otimes \overline{\Pi}_H^L) = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (\delta_H \otimes H),$
- (8) $(H \otimes \overline{\Pi}_H^R) \circ \delta_H = (\mu_H \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)),$

which will be useful in what follows.

Moreover, if H is a weak Hopf monoid in \mathbb{C} , then the antipode λ_H is unique, antimultiplicative, anticomultiplicative and leaves the unit and the counit invariant, i.e.,

- (9) $\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H},$
 $\delta_H \circ \lambda_H = c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H,$
- (10) $\lambda_H \circ \eta_H = \eta_H, \quad \varepsilon_H \circ \lambda_H = \varepsilon_H.$

Also, it is easy to show that for the convolution product the morphisms target and source satisfy the equalities

- (11) $\Pi_H^L = \text{id}_H * \lambda_H, \quad \Pi_H^R = \lambda_H * \text{id}_H, \quad \Pi_H^L * \text{id}_H = \text{id}_H = \text{id}_H * \Pi_H^R,$
 $\Pi_H^R * \lambda_H = \lambda_H = \lambda_H * \Pi_H^L, \quad \Pi_H^L * \Pi_H^L = \Pi_H^L, \quad \Pi_H^R * \Pi_H^R = \Pi_H^R$

and

- (12) $\Pi_H^L = \lambda_H \circ \overline{\Pi}_H^L = \overline{\Pi}_H^R \circ \lambda_H, \quad \Pi_H^R = \overline{\Pi}_H^L \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^R.$

Finally, by [2, Proposition 2.15] and the condition of symmetric category for \mathbb{C} , we have the identities:

$$(13) \quad \begin{aligned} \mu_H \circ c_{H,H} \circ (\Pi_H^L \otimes \Pi_H^R) &= \mu_H \circ (\Pi_H^L \otimes \Pi_H^R), \\ \mu_H \circ c_{H,H} \circ (\Pi_H^R \otimes \Pi_H^L) &= \mu_H \circ (\Pi_H^R \otimes \Pi_H^L), \end{aligned}$$

$$(14) \quad \begin{aligned} (\Pi_H^L \otimes \Pi_H^R) \circ c_{H,H} \circ \delta_H &= (\Pi_H^L \otimes \Pi_H^R) \circ \delta_H, \\ (\Pi_H^R \otimes \Pi_H^L) \circ c_{H,H} \circ \delta_H &= (\Pi_H^R \otimes \Pi_H^L) \circ \delta_H. \end{aligned}$$

LEMMA 2.2. *Let H be a weak Hopf monoid in \mathbb{C} such that its antipode is an isomorphism. The following equalities hold:*

$$(15) \quad \begin{aligned} \mu_H \circ (H \otimes (\Pi_H^L \circ \mu_H \circ (H \otimes \lambda_H) \circ c_{H,H})) \circ (\delta_H \otimes H) \\ = \mu_H \circ (H \otimes (\Pi_H^L * (\lambda_H^{-2} \circ \Pi_H^R))), \end{aligned}$$

$$(16) \quad \begin{aligned} (\mu_H \otimes H) \circ (H \otimes (c_{H,H} \circ (H \otimes \lambda_H) \circ \delta_H \circ \Pi_H^L)) \circ \delta_H \\ = (H \otimes (\Pi_H^L * (\lambda_H^{-2} \circ \Pi_H^R))) \circ \delta_H, \end{aligned}$$

where $\lambda_H^{-2} = \lambda_H^{-1} \circ \lambda_H^{-1}$.

PROOF. Identity (15) follows from

$$\begin{aligned} &\mu_H \circ (H \otimes (\Pi_H^L \circ \mu_H \circ (H \otimes \lambda_H) \circ c_{H,H})) \circ (\delta_H \otimes H) \\ &\stackrel{(3)}{=} \mu_H \circ (H \otimes (\Pi_H^L \circ \mu_H \circ (H \otimes (\Pi_H^L \circ \lambda_H)) \circ c_{H,H})) \circ (\delta_H \otimes H) \\ &\stackrel{(1)}{=} \mu_H \circ (H \otimes (\Pi_H^L \circ \mu_H \circ (H \otimes (\Pi_H^L \circ \overline{\Pi}_H^L \circ \lambda_H)) \circ c_{H,H})) \circ (\delta_H \otimes H) \\ &\stackrel{(3)}{=} \mu_H \circ (H \otimes (\Pi_H^L \circ \mu_H \circ (H \otimes (\overline{\Pi}_H^L \circ \lambda_H)) \circ c_{H,H})) \circ (\delta_H \otimes H) \\ &\stackrel{(12)}{=} \mu_H \circ (H \otimes (\Pi_H^L \circ \mu_H \circ (H \otimes \Pi_H^R) \circ c_{H,H})) \circ (\delta_H \otimes H) \\ &\stackrel{(*)}{=} \mu_H \circ (H \otimes (\Pi_H^L \circ \mu_H \circ c_{H,H})) \circ (((H \otimes \Pi_H^R) \circ \delta_H) \otimes H) \\ &\stackrel{(12)}{=} \mu_H \circ (H \otimes (\Pi_H^L \circ \mu_H \circ c_{H,H})) \circ (((H \otimes (\lambda_H \circ \overline{\Pi}_H^R)) \circ \delta_H) \otimes H) \\ &\stackrel{(8)}{=} \mu_H \circ (H \otimes (\Pi_H^L \circ \mu_H \circ c_{H,H})) \circ (\mu_H \otimes \lambda_H \otimes H) \\ &\quad \circ (H \otimes (\delta_H \circ \eta_H) \otimes H) \\ &\stackrel{(*)}{=} \mu_H \circ (\mu_H \otimes (\Pi_H^L \circ \mu_H)) \circ (H \otimes c_{H,H} \otimes \lambda_H) \circ (H \otimes H \otimes (\delta_H \circ \eta_H)) \\ &\stackrel{(10),(9)}{=} \mu_H \circ (\mu_H \otimes (\Pi_H^L \circ \mu_H)) \circ (H \otimes c_{H,H} \otimes \lambda_H) \\ &\quad \circ (H \otimes H \otimes ((\lambda_H^{-1} \otimes \lambda_H^{-1}) \circ c_{H,H} \circ \delta_H \circ \eta_H)) \\ &\stackrel{(*)}{=} \mu_H \circ (\mu_H \otimes H) \\ &\quad \circ (H \otimes (c_{H,H} \circ ((\Pi_H^L \circ \mu_H) \otimes \lambda_H^{-1}) \circ (H \otimes (\delta_H \circ \eta_H)))) \end{aligned}$$

$$\begin{aligned}
 &\stackrel{(8)}{=} \mu_H \circ (H \otimes (\mu_H \circ c_{H,H} \circ (\Pi_H^L \otimes (\lambda_H^{-1} \circ \overline{\Pi}_H^R)) \circ \delta_H)) \\
 &\stackrel{(12)}{=} \mu_H \circ (H \otimes (\mu_H \circ c_{H,H} \circ (\Pi_H^L \otimes (\overline{\Pi}_H^L \circ \lambda_H^{-1})) \circ \delta_H)) \\
 &\stackrel{(1)}{=} \mu_H \circ (H \otimes (\mu_H \circ c_{H,H} \circ (\Pi_H^L \otimes (\Pi_H^R \circ \overline{\Pi}_H^L \circ \lambda_H^{-1})) \circ \delta_H)) \\
 &\stackrel{(13)}{=} \mu_H \circ (H \otimes (\mu_H \circ (\Pi_H^L \otimes (\Pi_H^R \circ \overline{\Pi}_H^L \circ \lambda_H^{-1})) \circ \delta_H)) \\
 &\stackrel{(1)}{=} \mu_H \circ (H \otimes (\mu_H \circ (\Pi_H^L \otimes (\overline{\Pi}_H^L \circ \lambda_H^{-1})) \circ \delta_H)) \\
 &\stackrel{(12)}{=} \mu_H \circ (H \otimes (\Pi_H^L * (\lambda_H^{-2} \circ \Pi_H^R))),
 \end{aligned}$$

where the three equations marked with (*) follow by naturality of c .

Identity (16) follows from

$$\begin{aligned}
 &(\mu_H \otimes H) \circ (H \otimes (c_{H,H} \circ (H \otimes \lambda_H) \circ \delta_H \circ \Pi_H^L)) \circ \delta_H \\
 &\stackrel{(4)}{=} (\mu_H \otimes H) \circ (H \otimes (c_{H,H} \circ (H \otimes (\lambda_H \circ \Pi_H^L)) \circ \delta_H \circ \Pi_H^L)) \circ \delta_H \\
 &\stackrel{(2)}{=} (\mu_H \otimes H) \circ (H \otimes (c_{H,H} \circ (H \otimes (\lambda_H \circ \overline{\Pi}_H^R \circ \Pi_H^L)) \circ \delta_H \circ \Pi_H^L)) \circ \delta_H \\
 &\stackrel{(12)}{=} (\mu_H \otimes H) \circ (H \otimes (c_{H,H} \circ (H \otimes (\Pi_H^R \circ \Pi_H^L)) \circ \delta_H \circ \Pi_H^L)) \circ \delta_H \\
 &\stackrel{(4)}{=} (\mu_H \otimes H) \circ (H \otimes (c_{H,H} \circ (H \otimes \Pi_H^R) \circ \delta_H \circ \Pi_H^L)) \circ \delta_H \\
 &\stackrel{(12)}{=} (\mu_H \otimes H) \circ (H \otimes (c_{H,H} \circ (H \otimes (\overline{\Pi}_H^L \circ \lambda_H)) \circ \delta_H \circ \Pi_H^L)) \circ \delta_H \\
 &\stackrel{(*)}{=} ((\mu_H \circ (H \otimes \overline{\Pi}_H^L)) \otimes H) \circ (H \otimes (c_{H,H} \circ (H \otimes \lambda_H) \circ \delta_H \circ \Pi_H^L)) \circ \delta_H \\
 &\stackrel{(7)}{=} (H \otimes (\varepsilon_H \circ \mu_H) \otimes H) \circ (\delta_H \otimes (c_{H,H} \circ (H \otimes \lambda_H) \circ \delta_H \circ \Pi_H^L)) \circ \delta_H \\
 &\stackrel{(10), (9)}{=} (H \otimes (\varepsilon_H \circ \mu_H \circ c_{H,H} \circ (\lambda_H^{-1} \otimes \lambda_H^{-1})) \otimes H) \\
 &\quad \circ (\delta_H \otimes (c_{H,H} \circ (H \otimes \lambda_H) \circ \delta_H \circ \Pi_H^L)) \circ \delta_H \\
 &\stackrel{(*)}{=} (H \otimes ((H \otimes (\varepsilon_H \circ \mu_H)) \circ ((\delta_H \circ \Pi_H^L) \otimes \lambda_H^{-1}) \circ c_{H,H})) \circ (\delta_H \otimes H) \circ \delta_H \\
 &\stackrel{(7)}{=} (H \otimes (\mu_H \circ (\Pi_H^L \otimes (\overline{\Pi}_H^L \circ \lambda_H^{-1})) \circ c_{H,H} \circ \delta_H)) \circ \delta_H \\
 &\stackrel{(12)}{=} (H \otimes (\mu_H \circ (\Pi_H^L \otimes (\lambda_H^{-1} \circ \overline{\Pi}_H^R)) \circ c_{H,H} \circ \delta_H)) \circ \delta_H \\
 &\stackrel{(2)}{=} (H \otimes (\mu_H \circ (\Pi_H^L \otimes (\lambda_H^{-1} \circ \overline{\Pi}_H^R \circ \Pi_H^R)) \circ c_{H,H} \circ \delta_H)) \circ \delta_H \\
 &\stackrel{(14)}{=} (H \otimes (\mu_H \circ (\Pi_H^L \otimes (\lambda_H^{-1} \circ \overline{\Pi}_H^R \circ \Pi_H^R)) \circ \delta_H)) \circ \delta_H \\
 &\stackrel{(2)}{=} (H \otimes (\mu_H \circ (\Pi_H^L \otimes (\lambda_H^{-1} \circ \overline{\Pi}_H^R)) \circ \delta_H)) \circ \delta_H \\
 &\stackrel{(12)}{=} (H \otimes (\Pi_H^L * (\lambda_H^{-2} \circ \Pi_H^R))) \circ \delta_H,
 \end{aligned}$$

where the two equations marked with (*) follow again by naturality of c . ■

Let H be a weak bimonoid in \mathcal{C} . Then,

$$H^{\text{op}} = (H, \eta_H, \mu_H \circ c_{H,H}, \varepsilon_H, \delta_H), \quad H^{\text{cop}} = (H, \eta_H, \mu_H, \varepsilon_H, c_{H,H} \circ \delta_H)$$

are weak bimonoids in \mathcal{C} . Therefore so is

$$(H^{\text{op}})^{\text{cop}} = (H, \eta_H, \mu_H \circ c_{H,H}, \varepsilon_H, c_{H,H} \circ \delta_H).$$

Note that

$$\begin{aligned} \Pi_{H^{\text{op}}}^L &= \overline{\Pi}_H^R, & \Pi_{H^{\text{op}}}^R &= \overline{\Pi}_H^L, \\ \Pi_{H^{\text{cop}}}^L &= \overline{\Pi}_H^L, & \Pi_{H^{\text{cop}}}^R &= \overline{\Pi}_H^R. \end{aligned}$$

If H is a weak Hopf monoid and the antipode λ_H is an isomorphism, H^{op} and H^{cop} are weak Hopf monoids in \mathcal{C} with antipode $\lambda_{H^{\text{op}}} = \lambda_{H^{\text{cop}}} = \lambda_H^{-1}$. Then, under these conditions, $(H^{\text{op}})^{\text{cop}}$ is a weak Hopf monoid with antipode $\lambda_{(H^{\text{op}})^{\text{cop}}} = \lambda_H$.

If H and B are weak bimonoids in \mathcal{C} , so is the tensor product $H \otimes B$. In this case, the monoid-comonoid structure is the one of $H \otimes B$ and

$$\Pi_{H \otimes B}^L = \Pi_H^L \otimes \Pi_B^L, \quad \Pi_{H \otimes B}^R = \Pi_H^R \otimes \Pi_B^R.$$

Then, if H and B are weak Hopf monoids in \mathcal{C} , so is the tensor product $H \otimes B$, with $\lambda_{H \otimes B} = \lambda_H \otimes \lambda_B$. Note that $(H \otimes B)_L = H_L \otimes B_L$.

3. Yetter–Drinfeld modules and Long dimodules

DEFINITION 3.1. Let H be a weak Hopf monoid in \mathcal{C} . We say that a pair (M, φ_M) is a left H -module if M is an object in \mathcal{C} and $\varphi_M : H \otimes M \rightarrow M$ is a morphism in \mathcal{C} satisfying the following conditions:

$$(17) \quad \varphi_M \circ (\eta_H \otimes M) = \text{id}_M, \quad \varphi_M \circ (H \otimes \varphi_M) = \varphi_M \circ (\mu_H \otimes M).$$

If (M, φ_M) and (N, φ_N) are left H -modules, a morphism $f : M \rightarrow N$ in \mathcal{C} is a morphism of left H -modules if

$$(18) \quad \varphi_N \circ (H \otimes f) = f \circ \varphi_M$$

holds.

For two left H -modules (M, φ_M) and (N, φ_N) , the morphism $\varphi_{M \otimes N} : H \otimes M \otimes N \rightarrow M \otimes N$ is defined by

$$\varphi_{M \otimes N} = (\varphi_M \otimes \varphi_N) \circ (H \otimes c_{H,M} \otimes N) \circ (\delta_H \otimes M \otimes N).$$

Then, $\varphi_{M \otimes N}$ satisfies the equality

$$\varphi_{M \otimes N} \circ (H \otimes \varphi_{M \otimes N}) = \varphi_{M \otimes N} \circ (\mu_H \otimes M \otimes N)$$

and $\nabla_{M \otimes N} = \varphi_{M \otimes N} \circ (\eta_H \otimes M \otimes N) : M \otimes N \rightarrow M \otimes N$ is an idempotent morphism. Let $M \square N$ be the image of $\nabla_{M \otimes N}$ and let $p_{M \otimes N} : M \otimes N \rightarrow M \square N$, $i_{M \otimes N} : M \square N \rightarrow M \otimes N$ be the morphisms such that

$$i_{M \otimes N} \circ p_{M \otimes N} = \nabla_{M \otimes N}, \quad p_{M \otimes N} \circ i_{M \otimes N} = \text{id}_{M \square N}.$$

The object $M \square N$ is a left H -module with action

$$\varphi_{M \square N} = p_{M \otimes N} \circ \varphi_{M \otimes N} \circ (H \otimes i_{M \otimes N}) : H \otimes M \square N \rightarrow M \square N$$

and the equalities

$$\varphi_{M \otimes N} \circ (H \otimes \nabla_{M \otimes N}) = \varphi_{M \otimes N} = \nabla_{M \otimes N} \circ \varphi_{M \otimes N}$$

hold. Also, if (M, φ_M) , (N, φ_N) and (P, φ_P) are left H -modules, we have that

$$(M \otimes \nabla_{N \otimes P}) \circ (\nabla_{M \otimes N} \otimes P) = (\nabla_{M \otimes N} \otimes P) \circ (M \otimes \nabla_{N \otimes P}).$$

For two morphisms $f : (M, \varphi_M) \rightarrow (M', \varphi_{M'})$ and $g : (N, \varphi_N) \rightarrow (N', \varphi_{N'})$ of left H -modules,

$$f \square g = p_{M' \otimes N'} \circ (f \otimes g) \circ i_{M \otimes N} : M \square N \rightarrow M' \square N'$$

is a morphism of left H -modules between $(M \square N, \varphi_{M \square N})$ and $(M' \square N', \varphi_{M' \square N'})$. Also, the following identity holds:

$$(f \otimes g) \circ \nabla_{M \otimes N} = \nabla_{M' \otimes N'} \circ (f \otimes g).$$

DEFINITION 3.2. Let H be a weak Hopf monoid in the category \mathbf{C} . We say that a pair (M, ρ_M) is a left H -comodule in the category \mathbf{C} if M is an object in \mathbf{C} and $\rho_M : M \rightarrow H \otimes M$ is a morphism in \mathbf{C} satisfying the following conditions:

$$(\varepsilon_H \otimes M) \circ \rho_M = \text{id}_M, \quad (H \otimes \rho_M) \circ \rho_M = (\delta_H \otimes M) \circ \rho_M.$$

If (M, ρ_M) and (N, ρ_N) are left H -comodules, a morphism $f : M \rightarrow N$ in \mathbf{C} is a morphism of left H -comodules if

$$(19) \quad \rho_N \circ f = (H \otimes f) \circ \rho_M$$

holds.

For two left H -comodules (M, ρ_M) and (N, ρ_N) the morphism $\rho_{M \otimes N} : M \otimes N \rightarrow H \otimes M \otimes N$ defined by $\rho_{M \otimes N} = (\mu_H \otimes M \otimes N) \circ (H \otimes c_{M,H} \otimes N) \circ (\rho_M \otimes \rho_N)$ satisfies the equality

$$(H \otimes \rho_{M \otimes N}) \circ \rho_{M \otimes N} = (\delta_H \otimes M \otimes N) \circ \rho_{M \otimes N}.$$

Then, as a consequence, $\nabla'_{M \otimes N} = (\varepsilon_H \otimes M \otimes N) \circ \rho_{M \otimes N} : M \otimes N \rightarrow M \otimes N$ is an idempotent morphism. Let $M \odot N$ be the image of $\nabla'_{M \otimes N}$ and let $p'_{M \otimes N} : M \otimes N \rightarrow M \odot N$, $i'_{M \otimes N} : M \odot N \rightarrow M \otimes N$ be the morphisms such that

$$i'_{M \otimes N} \circ p'_{M \otimes N} = \nabla'_{M \otimes N}, \quad p'_{M \otimes N} \circ i'_{M \otimes N} = \text{id}_{M \odot N}.$$

The object $M \odot N$ is a left H -comodule with coaction

$$\rho_{M \odot N} = (H \otimes p'_{M \otimes N}) \circ \rho_{M \otimes N} \circ i'_{M \otimes N} : M \odot N \rightarrow H \otimes (M \odot N)$$

and the equalities

$$(H \otimes \nabla'_{M \otimes N}) \circ \rho_{M \otimes N} = \rho_{M \otimes N} = \rho_{M \otimes N} \circ \nabla'_{M \otimes N}$$

hold. Also, if (M, ρ_M) , (N, ρ_N) and (P, ρ_P) are left H -comodules, we have that

$$(M \otimes \nabla'_{N \otimes P}) \circ (\nabla'_{M \otimes N} \otimes P) = (\nabla'_{M \otimes N} \otimes P) \circ (M \otimes \nabla'_{N \otimes P}).$$

For two morphisms of left H -comodules $f : (M, \rho_M) \rightarrow (M', \rho_{M'})$ and $g : (N, \rho_N) \rightarrow (N', \rho_{N'})$,

$$f \odot g = p'_{M' \otimes N'} \circ (f \otimes g) \circ i'_{M \otimes N} : M \odot N \rightarrow M' \odot N'$$

is a morphism of left H -comodules between $(M \odot N, \rho_{M \odot N})$ and $(M' \odot N', \rho_{M' \odot N'})$. Also, the following identity holds:

$$(f \otimes g) \circ \nabla'_{M \otimes N} = \nabla'_{M' \otimes N'} \circ (f \otimes g).$$

Following [8, 12], we recall the notion of a left-left Yetter–Drinfeld module in the weak Hopf monoid setting.

DEFINITION 3.3. Let H be a weak Hopf monoid in \mathcal{C} . We shall denote by ${}^H_H\text{YD}$ the category of left-left Yetter–Drinfeld modules over H . The objects of ${}^H_H\text{YD}$ are triples $M = (M, \psi_M, \gamma_M)$ where (M, ψ_M) is a left H -module and (M, γ_M) is a left H -comodule satisfying the following conditions:

$$(b1) \quad (\mu_H \otimes M) \circ (H \otimes c_{M,H}) \circ ((\gamma_M \circ \psi_M) \otimes H) \circ (H \otimes c_{H,M}) \circ (\delta_H \otimes M)$$

$$= (\mu_H \otimes \psi_M) \circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \otimes \gamma_M),$$

$$(b2) \quad (\mu_H \otimes \psi_M) \circ (H \otimes c_{H,H} \otimes M) \circ ((\delta_H \circ \eta_H) \otimes \gamma_M) = \gamma_M.$$

A morphism in ${}^H_H\text{YD}$ between (M, ψ_M, γ_M) and (N, ψ_N, γ_N) is a morphism $f : M \rightarrow N$ in \mathbb{C} such that (18) and (19) hold.

Let (M, ψ_M, γ_M) be a left-left Yetter–Drinfeld module. It is easy to show that the axiom (b2) is equivalent to

$$((\varepsilon_H \circ \mu_H) \otimes \psi_M) \circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \otimes \gamma_M) = \psi_M.$$

As a consequence of the previous identity we obtain that

$$(20) \quad \psi_M \circ (\Pi_H^L \otimes M) \circ \gamma_M = \text{id}_M$$

holds and by [3, (37) of Remark 1.11] we have the equality

$$(21) \quad (\Pi_H^R \otimes M) \circ \gamma_M = ((\Pi_H^R \circ \lambda_H) \otimes \psi_M) \circ ((c_{H,H} \circ \delta_H \circ \eta_H) \otimes M).$$

It is a well-known fact (see [8, Proposition 2.2]) that conditions (b1) and (b2) are equivalent to

$$(22) \quad \begin{aligned} \gamma_M \circ \psi_M &= (\mu_H \otimes M) \circ (H \otimes c_{M,H}) \\ &\quad \circ (((\mu_H \otimes \psi_M) \circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \otimes \gamma_M)) \otimes \lambda_H) \\ &\quad \circ (H \otimes c_{H,M}) \circ (\delta_H \otimes M). \end{aligned}$$

LEMMA 3.4. *Let H be a weak Hopf monoid in \mathbb{C} such that its antipode is an isomorphism. Let (M, ψ_M, γ_M) be a left-left Yetter–Drinfeld module. The following identity holds:*

$$(23) \quad \psi_M \circ ((\lambda_H^{-2} \circ \Pi_H^R) \otimes M) \circ \gamma_M = \text{id}_M.$$

PROOF. We have

$$\begin{aligned} &\psi_M \circ ((\lambda_H^{-2} \circ \Pi_H^R) \otimes M) \circ \gamma_M \\ &\stackrel{(21)}{=} \psi_M \circ ((\lambda_H^{-2} \circ \Pi_H^R \circ \lambda_H) \otimes \psi_M) \circ ((c_{H,H} \circ \delta_H \circ \eta_H) \otimes M) \\ &\stackrel{(*)}{=} \psi_M \circ ((\mu_H \circ (\overline{\Pi}_H^L \otimes H)) \circ c_{H,H} \circ \delta_H \circ \eta_H) \otimes M) \\ &\stackrel{(**)}{=} \psi_M \circ (\eta_H \otimes M) \\ &\stackrel{(17)}{=} \text{id}_M, \end{aligned}$$

where (*) follows by (17), (12) and the naturality of c , and (**) follows by (11) for H^{cop} . \blacksquare

If the antipode of the weak Hopf monoid H is an isomorphism, the category ${}^H_H\text{YD}$ is an example of a non-strict braided monoidal category. In the following paragraphs we will make a brief summary of its braided monoidal structure.

Let (M, ψ_M, γ_M) and (N, ψ_N, γ_N) be objects in ${}^H_H\text{YD}$. Then, for the morphisms $\nabla_{M \otimes N}$ and $\nabla'_{M \otimes N}$, defined in Definitions 3.1 and 3.2, by [2, Proposition 1.12 (iii)] we have that

$$\nabla_{M \otimes N} = \nabla'_{M \otimes N}.$$

Then, the tensor product of (M, ψ_M, γ_M) and (N, ψ_N, γ_N) is defined as the image of the idempotent morphism $\nabla_{M \otimes N}$, denoted by $M \boxtimes N$, with the following action and coaction:

$$\begin{aligned}\psi_{M \boxtimes N} &= p_{M \otimes N} \circ \psi_{M \otimes N} \circ (H \otimes i_{M \otimes N}), \\ \gamma_{M \boxtimes N} &= (H \otimes p_{M \otimes N}) \circ \gamma_{M \otimes N} \circ i_{M \otimes N}.\end{aligned}$$

The base object in ${}^H_H\text{YD}$ is H_L , which is a left-left Yetter–Drinfeld module over H with (co)module structure

$$\psi_{H_L} = p_H^L \circ \mu_H \circ (H \otimes i_H^L), \quad \gamma_{H_L} = (H \otimes p_H^L) \circ \delta_H \circ i_H^L.$$

The unit constrains are defined by

$$\begin{aligned}\iota_M &= \psi_M \circ (i_H^L \otimes M) \circ i_{H_L \otimes M} : H_L \boxtimes M \rightarrow M, \\ \tau_M &= \psi_M \circ c_{M, H} \circ (M \otimes (\overline{\Pi}_H^L \circ i_H^L)) \circ i_{M \otimes H_L} : M \boxtimes H_L \rightarrow M,\end{aligned}$$

and the associativity constrains $\alpha_{M, N, P} : (M \boxtimes N) \boxtimes P \rightarrow M \boxtimes (N \boxtimes P)$ are defined by

$$\alpha_{M, N, P} = p_{M \otimes (N \boxtimes P)} \circ (M \otimes p_{N \otimes P}) \circ (i_{M \otimes N} \otimes P) \circ i_{(M \boxtimes N) \otimes P},$$

where (P, ψ_P, γ_P) is a third object in the category of left-left Yetter–Drinfeld modules.

If $f : M \rightarrow M'$ and $g : N \rightarrow N'$ are morphisms in ${}^H_H\text{YD}$, then

$$f \boxtimes g = p_{M' \boxtimes N'} \circ (f \otimes g) \circ i_{M \otimes N} : M \boxtimes N \rightarrow M' \boxtimes N'$$

is a morphism in the same category and

$$(f' \boxtimes g') \circ (f \boxtimes g) = (f' \circ f) \boxtimes (g' \circ g)$$

holds, where $f' : M' \rightarrow M''$ and $g' : N' \rightarrow N''$ are morphisms in ${}^H_H\text{YD}$.

Finally, the braiding is defined by

$$t_{M, N} = p_{N \otimes M} \circ \tau_{M, N} \circ i_{M \otimes N} : M \boxtimes N \rightarrow N \boxtimes M,$$

where

$$\tau_{M, N} = (\psi_N \otimes M) \circ (H \otimes c_{M, N}) \circ (\gamma_M \otimes N) : M \otimes N \rightarrow N \otimes M.$$

DEFINITION 3.5. Let H and B be weak Hopf monoids in \mathcal{C} . A left-left H - B -Long dimodule (M, φ_M, ρ_M) is both a left H -module with action $\varphi_M : H \otimes M \rightarrow M$ and a left B -comodule with coaction $\rho_M : M \rightarrow B \otimes M$ satisfying the axiom

$$(24) \quad \rho_M \circ \varphi_M = (B \otimes \varphi_M) \circ (c_{H,B} \otimes M) \circ (H \otimes \rho_M).$$

A morphism between two left-left H - B -Long dimodules (M, φ_M, ρ_M) and (N, φ_N, ρ_N) is a morphism $f : M \rightarrow N$ of left H -modules and left B -comodules. Left-left H - B -Long dimodules and morphisms of left-left H - B -Long dimodules form a category, denoted as ${}^B_H\text{Long}$.

In [5] we can find many examples of Long dimodules in the weak setting. One of the main results proved in [5] asserts that ${}^B_H\text{Long}$ is an example of a monoidal category (see [5, Theorem 1]). As in the Yetter–Drinfeld case, in the following paragraphs we will make a brief summary of its monoidal structure. The complete details can be found in [5, Lemmas 2–6, Propositions 1–3 and Theorem 1].

Let (M, φ_M, ρ_M) and (N, φ_N, ρ_N) be in ${}^B_H\text{Long}$. The idempotent morphisms $\nabla_{M \otimes N}$ and $\nabla'_{M \otimes N}$, defined in Definitions 3.1 and 3.2, satisfy

$$\nabla'_{M \otimes N} \circ \nabla_{M \otimes N} = \nabla_{M \otimes N} \circ \nabla'_{M \otimes N}.$$

As a consequence, the morphism

$$\Omega_{M \otimes N} = \nabla'_{M \otimes N} \circ \nabla_{M \otimes N}$$

is idempotent and we have two morphisms $j_{M \otimes N} : M \times N \rightarrow M \otimes N$ and $q_{M \otimes N} : M \otimes N \rightarrow M \times N$ such that

$$q_{M \otimes N} \circ j_{M \otimes N} = \text{id}_{M \times N}, \quad j_{M \otimes N} \circ q_{M \otimes N} = \Omega_{M \otimes N},$$

where $M \times N$ is the image of $\Omega_{M \otimes N}$. Then, the tensor product of (M, φ_M, ρ_M) and (N, φ_N, ρ_N) is defined as the image of the idempotent morphism $\Omega_{M \otimes N}$. It belongs to ${}^B_H\text{Long}$ with H -module and B -comodule structures

$$\begin{aligned} \varphi_{M \times N} &= q_{M \otimes N} \circ \varphi_{M \otimes N} \circ (H \otimes j_{M \otimes N}), \\ \rho_{M \times N} &= (B \otimes q_{M \otimes N}) \circ \rho_{M \otimes N} \circ j_{M \otimes N}, \end{aligned}$$

respectively.

Moreover, if

$$f : (M, \varphi_M, \rho_M) \rightarrow (M', \varphi_{M'}, \rho_{M'}), \quad g : (N, \varphi_N, \rho_N) \rightarrow (N', \varphi_{N'}, \rho_{N'})$$

are morphisms in ${}^B_H\text{Long}$, then

$$f \times g = q_{M' \times N'} \circ (f \otimes g) \circ j_{M \times N} : M \times N \rightarrow M' \times N'$$

is a morphism in ${}^B_H\text{Long}$ between $(M \times N, \varphi_{M \times N}, \rho_{M \times N})$ and $(M' \times N', \varphi_{M' \times N'}, \rho_{M' \times N'})$.

If (M, φ_M, ρ_M) , (N, φ_N, ρ_N) and (P, φ_P, ρ_P) are in ${}^B_H\text{Long}$, the associativity constraint

$$a_{M,N,P} : (M \times N) \times P \rightarrow M \times (N \times P)$$

is defined by

$$a_{M,N,P} = q_{M \otimes (N \times P)} \circ (M \otimes q_{N \otimes P}) \circ (j_{M \otimes N} \otimes P) \circ j_{(M \times N) \otimes P}$$

and the base object is $H_L \otimes B_L$, where the action and the coaction are defined by

$$\begin{aligned} \varphi_{H_L \otimes B_L} &= (p_H^L \circ \mu_H \circ (H \otimes i_H^L)) \otimes B_L, \\ \rho_{H_L \otimes B_L} &= (c_{H_L, B} \otimes p_B^L) \circ (H_L \otimes (\delta_B \circ i_B^L)). \end{aligned}$$

Finally, the unit constraints are $l_M : (H_L \otimes B_L) \times M \rightarrow M$ and $r_M : M \times (H_L \otimes B_L) \rightarrow M$, where

$$\begin{aligned} l_M &= ((\varepsilon_B \circ \mu_B) \otimes M) \circ (B \otimes (\rho_M \circ \varphi_M)) \\ &\quad \circ ((c_{H, B} \circ (i_H^L \otimes i_B^L)) \otimes M) \circ j_{(H_L \otimes B_L) \otimes M}, \\ r_M &= ((\varphi_M \circ c_{M, H}) \otimes (\varepsilon_B \circ \mu_B)) \circ (M \otimes c_{B, H} \otimes B) \\ &\quad \circ ((c_{B, M} \circ \rho_M) \otimes (\bar{\Pi}_H^L \circ i_H^L) \otimes i_B^L) \circ j_{M \otimes (H_L \otimes B_L)}. \end{aligned}$$

THEOREM 3.6. *Let H and B be weak Hopf algebras such that their antipodes are isomorphisms. There exists a functor*

$$F : {}_{H \otimes B}^{H \otimes B} \text{YD} \rightarrow {}^B_H \text{Long}$$

defined on objects by

$$F((M, \psi_M, \gamma_M)) = (M, \chi_M = \psi_M \circ (H \otimes \eta_B \otimes M), \omega_M = (\varepsilon_H \otimes B \otimes M) \circ \gamma_M)$$

and by the identity on morphisms.

PROOF. Let (M, ψ_M, γ_M) be in ${}_{H \otimes B}^{H \otimes B} \text{YD}$. Then, using that (M, ψ_M) is a left $H \otimes B$ -module, the unit properties and the naturality of c , we obtain that (M, χ_M) is a left H -module. Similarly, using that (M, γ_M) is a left $H \otimes B$ -comodule, the counit properties and the naturality of c , we obtain that (M, ω_M) is a left B -comodule. The proof for equality (24) is the following:

$$\omega_M \circ \chi_M$$

$$\begin{aligned}
 &\stackrel{(a)}{=} (((\varepsilon_H \otimes B) \circ \mu_{H \otimes B}) \otimes M) \circ (H \otimes B \otimes ((H \otimes c_{M,B}) \circ (c_{M,H} \otimes B))) \\
 &\quad \circ (((\mu_{H \otimes B} \otimes \psi_M) \circ (H \otimes B \otimes c_{H \otimes B, H \otimes B} \otimes M)) \circ (\delta_{H \otimes B} \otimes \gamma_M)) \otimes \lambda_H \otimes \lambda_B \\
 &\quad \circ (H \otimes B \otimes ((c_{H,M} \otimes B) \circ (H \otimes c_{B,M}))) \circ ((\delta_{H \otimes B} \circ (H \otimes \eta_B)) \otimes M) \\
 &\stackrel{(b)}{=} (B \otimes \psi_M) \circ (c_{H,B} \otimes B \otimes M) \circ (((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \\
 &\quad \circ (\delta_H \otimes H)) \otimes ((\mu_B \otimes B) \circ (B \otimes c_{B,B}) \circ (\delta_B \otimes (\mu_B \circ (B \otimes \lambda_B) \circ c_{B,B}))) \\
 &\quad \circ (\delta_B \otimes B) \otimes M) \circ (H \otimes (((\mu_H \circ (H \otimes \lambda_H) \circ c_{H,H}) \otimes B) \\
 &\quad \circ (H \otimes c_{B,H})) \otimes B \otimes M) \circ (\delta_H \otimes \eta_B \otimes \gamma_M) \\
 &\stackrel{(c)}{=} (B \otimes \psi_M) \circ (c_{H,B} \otimes B \otimes M) \circ ((\mu_H \circ (H \otimes (\Pi_H^L \circ \mu_H \circ (H \otimes \lambda_H) \circ c_{H,H}))) \\
 &\quad \otimes ((\mu_B \otimes B) \circ (B \otimes (c_{B,B} \circ (B \otimes \lambda_B) \circ \delta_B))) \circ (\mu_B \otimes B) \circ (B \otimes c_{B,B}) \\
 &\quad \circ ((\delta_B \circ \eta_B) \otimes B) \otimes M) \circ (\delta_H \otimes \gamma_M) \\
 &\stackrel{(d)}{=} (B \otimes \psi_M) \circ (c_{H,B} \otimes B \otimes M) \circ ((\mu_H \circ (H \otimes (\Pi_H^L \circ \mu_H \circ (H \otimes \lambda_H) \circ c_{H,H}))) \\
 &\quad \otimes ((\mu_B \otimes B) \circ (B \otimes (c_{B,B} \circ (B \otimes \lambda_B) \circ \delta_B))) \circ (B \otimes \Pi_B^L) \circ \delta_B) \otimes M) \\
 &\quad \circ (\delta_H \otimes \gamma_M) \\
 &\stackrel{(e)}{=} (B \otimes \psi_M) \circ (c_{H,B} \otimes B \otimes M) \circ ((\mu_H \circ (H \otimes (\Pi_H^L * (\lambda_H^{-2} \circ \Pi_H^R)))) \\
 &\quad \otimes ((B \otimes (\Pi_B^L * (\lambda_B^{-2} \circ \Pi_B^R))) \circ \delta_B) \otimes M) \circ (H \otimes \gamma_M) \\
 &\stackrel{(f)}{=} (B \otimes (\psi_M \circ (\mu_{H \otimes B} \otimes M))) \circ (((c_{H,B} \otimes \Pi_B^L) \circ ((\mu_H \circ (H \otimes \Pi_H^L)) \otimes \delta_B)) \\
 &\quad \otimes ((\lambda_H^{-2} \circ \Pi_H^R) \otimes (\lambda_B^{-2} \circ \Pi_B^R) \otimes M)) \circ (H \otimes ((\delta_{H \otimes B} \otimes M) \circ \gamma_M)) \\
 &\stackrel{(g)}{=} (B \otimes \psi_M) \circ (((c_{H,B} \otimes \Pi_B^L) \circ ((\mu_H \circ (H \otimes \Pi_H^L)) \otimes \delta_B)) \\
 &\quad \otimes (\psi_M \circ ((\lambda_H^{-2} \circ \Pi_H^R) \otimes (\lambda_B^{-2} \circ \Pi_B^R) \otimes M) \circ \gamma_M)) \circ (H \otimes \gamma_M) \\
 &\stackrel{(h)}{=} (B \otimes \psi_M) \circ (((c_{H,B} \otimes \Pi_B^L) \circ ((\mu_H \circ (H \otimes \Pi_H^L)) \otimes \delta_B)) \otimes M) \circ (H \otimes \gamma_M) \\
 &\stackrel{(i)}{=} (B \otimes (\psi_M \circ ((\mu_{H \otimes B} \circ (H \otimes \eta_B \otimes H \otimes B)) \otimes M))) \circ (c_{H,B} \otimes \Pi_H^L \otimes \Pi_B^L \otimes M) \\
 &\quad \circ (H \otimes (((\varepsilon_H \otimes B \otimes H \otimes B) \circ \delta_{H \otimes B}) \otimes M) \circ \gamma_M) \\
 &\stackrel{(g)}{=} (B \otimes \chi_M) \circ (c_{H,B} \otimes (\psi_M \circ (\Pi_H^L \otimes \Pi_B^L \otimes M) \circ \gamma_M)) \circ (H \otimes \omega_M) \\
 &\stackrel{(j)}{=} (B \otimes \chi_M) \circ (c_{H,B} \otimes M) \circ (H \otimes \omega_M),
 \end{aligned}$$

where (a) follows by (22) for $H \otimes B$; (b) by naturality of c and the associativity of μ_H and μ_B ; (c) by the coassociativity of δ_B , the naturality of c and (5); (d) by naturality of c and (6); (e) by (15) for H and (16) for B ; (f) by naturality of c and the associativity of μ_H ; (g) by the condition of left $H \otimes B$ (co)module for M ; (h) by (23) for $H \otimes B$; (i) by naturality of c and the properties of η_B and ε_H ; (j) by (20) for $H \otimes B$.

Therefore (M, χ_M, ω_M) is an object in ${}^B_H\text{Long}$.

Finally, if f is a morphism in ${}^{H \otimes B}_{H \otimes B}\text{YD}$ between the objects (M, ψ_M, γ_M) and (N, ψ_N, γ_N) , it is immediate to obtain that f is a morphism in ${}^B_H\text{Long}$ between the objects (M, χ_M, ω_M) and (N, χ_N, ω_N) . Therefore, F is a functor between the categories ${}^{H \otimes B}_{H \otimes B}\text{YD}$ and ${}^B_H\text{Long}$. ■

4. The retraction in the (co)quasitriangular case

In [5, Theorem 2] we proved that, if H is a quasitriangular weak Hopf monoid and B is a coquasitriangular weak Hopf monoid, there exists a functor

$$L : {}^B_H\text{Long} \rightarrow {}^{H \otimes B}_{H \otimes B}\text{YD}$$

injective on objects and, consequently, ${}^B_H\text{Long}$ can be identified with a subcategory of ${}^{H \otimes B}_{H \otimes B}\text{YD}$. In this section, we will show that in this context the functor F , introduced at the end of the previous section, is a retraction of L . We will begin with a brief review of the fundamental properties of weak Hopf (co)quasitriangular monoids (see [4, 5] for the complete details).

The following definition is the monoidal version of the definition of quasitriangular weak Hopf monoid introduced by Nikshych, Turaev and Vainerman in [13].

DEFINITION 4.1. Let H be a weak Hopf monoid. Let Ω_H and Ω'_H be the idempotent morphisms defined by $\Omega_H = \Omega_H^2 \circ \Omega_H^1$ and $\Omega'_H = \Omega_H^4 \circ \Omega_H^3$ where Ω_H^i are the idempotent morphisms defined by

$$\Omega_H^1 = \mu_{H \otimes H} \circ ((c_{H,H} \circ \delta_H \circ \eta_H) \otimes H \otimes H) : H \otimes H \rightarrow H \otimes H,$$

$$\Omega_H^2 = \mu_{H \otimes H} \circ (H \otimes H \otimes (\delta_H \circ \eta_H)) : H \otimes H \rightarrow H \otimes H,$$

$$\Omega_H^3 = \mu_{H \otimes H} \circ (H \otimes H \otimes (c_{H,H} \circ \delta_H \circ \eta_H)) : H \otimes H \rightarrow H \otimes H,$$

$$\Omega_H^4 = \mu_{H \otimes H} \circ ((\delta_H \circ \eta_H) \otimes H \otimes H) : H \otimes H \rightarrow H \otimes H.$$

We will say that H is a quasitriangular weak Hopf monoid if there exists a morphism $\sigma : K \rightarrow H \otimes H$ in \mathcal{C} satisfying the following conditions:

$$(c1) \quad \Omega_H \circ \sigma = \sigma,$$

$$(c2) \quad (\delta_H \otimes H) \circ \sigma = (H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\sigma \otimes \sigma),$$

$$(c3) \quad (H \otimes \delta_H) \circ \sigma = (\mu_H \otimes c_{H,H}) \circ (H \otimes c_{H,H} \otimes H) \circ (\sigma \otimes \sigma),$$

$$(c4) \quad \mu_{H \otimes H} \circ (\sigma \otimes \delta_H) = \mu_{H \otimes H} \circ ((c_{H,H} \circ \delta_H) \otimes \sigma).$$

(c5) There exists a morphism $\bar{\sigma} : K \rightarrow H \otimes H$ such that

$$(c5.1) \quad \Omega'_H \circ \bar{\sigma} = \bar{\sigma},$$

$$(c5.2) \quad \sigma * \bar{\sigma} = c_{H,H} \circ \delta_H \circ \eta_H,$$

$$(c5.3) \quad \bar{\sigma} * \sigma = \delta_H \circ \eta_H.$$

We will say that a quasitriangular weak Hopf monoid H is triangular if moreover $\bar{\sigma} = c_{H,H} \circ \sigma$.

For any quasitriangular weak Hopf monoid the morphism $\bar{\sigma}$ is unique and by [4, Lemma 3.5] and [5, Lemma 7] the following equalities hold:

$$(25) \quad \begin{aligned} \sigma * \bar{\sigma} * \sigma &= \sigma, & \bar{\sigma} * \sigma * \bar{\sigma} &= \bar{\sigma} \\ (\varepsilon_H \otimes H) \circ \sigma &= (H \otimes \varepsilon_H) \circ \sigma = \eta_H. \end{aligned}$$

DEFINITION 4.2. Let B be a weak Hopf monoid. Let Γ_B and Γ'_B be the idempotent morphisms defined by $\Gamma_B = \Gamma_B^2 \circ \Gamma_B^1$ and $\Gamma'_B = \Gamma_B^4 \circ \Gamma_B^3$, where Γ_B^i are the idempotent morphisms

$$\begin{aligned} \Gamma_B^1 &= ((\varepsilon_B \circ \mu_B \circ c_{B,B}) \otimes B \otimes B) \circ \delta_{B \otimes B} : B \otimes B \rightarrow B \otimes B, \\ \Gamma_B^2 &= (B \otimes B \otimes (\varepsilon_B \circ \mu_B)) \circ \delta_{B \otimes B} : B \otimes B \rightarrow B \otimes B, \\ \Gamma_B^3 &= (B \otimes B \otimes (\varepsilon_B \circ \mu_B \circ c_{B,B})) \circ \delta_{B \otimes B} : B \otimes B \rightarrow B \otimes B, \\ \Gamma_B^4 &= ((\varepsilon_B \circ \mu_B) \otimes B \otimes B) \circ \delta_{B \otimes B} : B \otimes B \rightarrow B \otimes B. \end{aligned}$$

We will say that B is a coquasitriangular weak Hopf monoid if there exists a morphism $\omega : B \otimes B \rightarrow K$ in \mathbf{C} satisfying the following conditions:

$$(d1) \quad \omega \circ \Gamma_B = \omega,$$

$$(d2) \quad \omega \circ (\mu_B \otimes B) = (\omega \otimes \omega) \circ (B \otimes c_{B,B} \otimes B) \circ (B \otimes B \otimes \delta_B),$$

$$(d3) \quad \omega \circ (B \otimes \mu_B) = (\omega \otimes \omega) \circ (B \otimes c_{B,B} \otimes B) \circ (\delta_B \otimes c_{B,B}),$$

$$(d4) \quad (\omega \otimes \mu_B) \circ \delta_{B \otimes B} = ((\mu_B \circ c_{B,B}) \otimes \omega) \circ \delta_{B \otimes B}.$$

(d5) There exists a morphism $\bar{\omega} : B \otimes B \rightarrow K$ such that

$$(d5.1) \quad \bar{\omega} \circ \Gamma'_B = \bar{\omega},$$

$$(d5.2) \quad \omega * \bar{\omega} = \varepsilon_B \circ \mu_B \circ c_{B,B},$$

$$(d5.3) \quad \bar{\omega} * \omega = \varepsilon_B \circ \mu_B.$$

We will say that a coquasitriangular weak Hopf monoid B is cotriangular if moreover $\bar{\omega} = \omega \circ c_{B,B}$.

For any coquasitriangular weak Hopf monoid B , we obtain that $\bar{\omega}$ is unique and the following equalities hold (see [5, Lemma 8]):

$$(26) \quad \begin{aligned} \omega * \bar{\omega} * \omega &= \omega, & \bar{\omega} * \omega * \bar{\omega} &= \bar{\omega}, \\ \omega \circ (\eta_B \otimes B) &= \omega \circ (B \otimes \eta_B) = \varepsilon_B. \end{aligned}$$

EXAMPLE 4.3. There are many interesting examples in the literature of quasitriangular and coquasitriangular weak Hopf monoids. Let $G = (G_0, G_1)$ be a finite groupoid such that its set of arrows G_1 is finite and let R be a commutative ring. Then, the groupoid algebra of G , denoted by $R[G]$, is an example of triangular weak Hopf monoid in the category $R\text{-Mod}$. Since G_1 is finite, $R[G]$ is free of a finite rank as an R -module. Hence $R[G]$ is finite as object in the category $R\text{-Mod}$ and $R[G]^*$ is an example of a cotriangular weak Hopf monoid in $R\text{-Mod}$.

On the other hand, in [7] Andruskiewitsch and Natale proved that it is possible to construct a weak Hopf monoid $\mathbb{K}(G, H)$ in the symmetric monoidal category of vector spaces over a field \mathbb{K} by working with a matched pair of finite groupoids (G, H) . Moreover, in [1] Aguiar and Andruskiewitsch proved the following result: A matched pair of rotations gives rise to a quasitriangular structure for the associated weak Hopf monoid $\mathbb{K}(G, H)$. Also, by [1, Theorem 5.10] we know that there is an isomorphism of quasitriangular weak Hopf monoids between the Drinfeld double of $\mathbb{K}(G, H)$ and the weak Hopf monoid of a suitable matched pair of groupoids.

Finally, in [13], for a weak Hopf monoid H in the symmetric monoidal category of vector spaces over an algebraically closed field, Nikshych, Turaev and Vainerman defined the Drinfeld double $D(H)$ of H and proved that $D(H)$ is a quasitriangular weak Hopf monoid (see [13, Proposition 6.2]).

The main result in [5] asserts the following: Let H be a quasitriangular weak Hopf monoid with morphism $\sigma : K \rightarrow H \otimes H$ and let B be a coquasitriangular weak Hopf monoid with morphism $\omega : B \otimes B \rightarrow K$. There exists a functor

$$L : {}^B_H\text{Long} \rightarrow {}^{H \otimes B}_{H \otimes B}\text{YD}$$

defined on objects by

$$L((M, \varphi_M, \rho_M)) = (M, \phi_M, \varrho_M),$$

where

$$\begin{aligned} \phi_M &= \varphi_M \circ (H \otimes (\omega \circ c_{B,B}) \otimes M) \circ (H \otimes B \otimes \rho_M), \\ \varrho_M &= (H \otimes (\rho_M \circ \varphi_M)) \circ ((c_{H,H} \circ \sigma) \otimes M), \end{aligned}$$

and by the identity on morphisms. Moreover, the functor L is injective on objects and, consequently, ${}^B_H\text{Long}$ can be identified with a subcategory of ${}^{H \otimes B}_{H \otimes B}\text{YD}$. Moreover, by

[5, Lemmas 10–12] we can conclude that the category ${}^B_H\text{Long}$ is a braided monoidal subcategory of ${}^{H\otimes B}_H\text{YD}$ if the antipodes of H and B are isomorphisms.

THEOREM 4.4. *Let H be a quasitriangular weak Hopf monoid with morphism $\sigma : K \rightarrow H \otimes H$ and let B be a coquasitriangular weak Hopf monoid with morphism $\omega : B \otimes B \rightarrow K$. Then the functor F introduced in Theorem 3.6 is a retraction of the functor L .*

PROOF. Obviously for morphisms there is nothing to prove. For objects we have the following: Let (M, φ_M, ρ_M) be an object in ${}^B_H\text{Long}$. Then

$$(F \circ L)((M, \varphi_M, \rho_M)) = (M, \varphi_M, \rho_M)$$

because

$$F(L((M, \varphi_M, \rho_M))) = F((M, \phi_M, \varrho_M)) = (M, \chi_M, \omega_M),$$

where, by (26) and (25), we have that

$$\begin{aligned} \chi_M &= \phi_M \circ (H \otimes \eta_B \otimes M) = \varphi_M \circ (H \otimes (\omega \circ c_{B,B}) \otimes M) \circ (H \otimes \eta_B \otimes \rho_M) \\ &= \varphi_M \circ (H \otimes ((\varepsilon_B \otimes M) \circ \rho_M)) = \varphi_M \end{aligned}$$

and

$$\begin{aligned} \omega_M &= (\varepsilon_H \otimes B \otimes M) \circ \varrho_M = (\varepsilon_H \otimes (\rho_M \circ \varphi_M)) \circ ((c_{H,H} \circ \sigma) \otimes M) \\ &= \rho_M \circ \varphi_M \circ (\eta_H \otimes M) = \rho_M. \end{aligned} \quad \blacksquare$$

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