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Functors for Long dimodules and Yetter–Drinfeld modules in a weak setting

José Nicanor Alonso Álvarez (*) – Ramón González Rodríguez (**)

Abstract – In this paper, for two weak Hopf monoids H and B with invertible antipode, we define a functor between the category of left-left $H \otimes B$-Yetter–Drinfeld modules and the category of H-B-Long dimodules. We also show that, if moreover H is quasitriangular and B is coquasitriangular, this functor is a retraction of the well-known injective functor between left-left H-B-Long dimodules and left-left $H \otimes B$-Yetter–Drinfeld modules.

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1. Introduction

Let R be a commutative fixed ring with unit and let H be a commutative and cocommutative Hopf algebra in the non-strict symmetric monoidal category of R-Mod. In order to study the Brauer group of H-dimodule algebras, Long introduced in [10] the notion of Long H-dimodule. Later, the notion was extended by considering two arbitrary Hopf algebras H and B with bijective antipode, introducing the category of left-left H-B-Long dimodules, denoted by ${}_H^B\mathsf{Long}$.

On the other hand, to characterize bialgebras B such that $B \otimes H$ with the smash product structure is a bialgebra, Radford introduced in [14] conditions that subsequently

(*) *Indirizzo dell'A.*: Departamento de Matemáticas, Facultade de Ciencias Económicas e Empresariais, Universidade de Vigo, Campus Universitario Lagoas-Marcosende, 36280 Vigo, Spain; jnalonso@uvigo.es

(**) *Indirizzo dell'A.*: Departamento de Matemática Aplicada II, Escola de Enxeñaría de Telecomunicación, Universidade de Vigo, Campus Universitario Lagoas-Marcosende, 36280 Vigo, Spain; rgon@dma.uvigo.es

give rise to the notion of Yetter–Drinfeld module on a bialgebra. The category of these modules was defined by Yetter in [16], denoted by ${}_H^H\mathsf{YD}$ and called *crossed bimodule*.

It is a well-known fact that Yetter–Drinfeld modules over a Hopf algebra provide solutions of the quantum Yang–Baxter equation and then this category allows us to explain the relationship between different theories in mathematics and physics. Moreover, Yetter–Drinfeld modules play a central role in the theory of monoidal categories, allowing to categorize the concept of Drinfeld double (see [11]). Therefore, its generalization to broader contexts is very interesting. For example, in this sense, the category of Yetter–Drinfeld modules was studied in the context of weak Hopf algebras (see [8, 12]), Hopf quasigroups (see [6]) and Hom-Hopf algebras (see [15]).

Obviously, it is also interesting to know the connection between Yetter–Drinfeld modules and other categories. In the classic case of Hopf algebras, it is a well-known fact that, if the Hopf algebras H and B are quasitriangular and coquasitriangular, respectively, ${}_H^B\mathsf{Long}$ is a braided monoidal subcategory of ${}_{H\otimes B}^{H\otimes B}\mathsf{YD}$ and as a consequence of the above, the categories of Long dimodules provide non-trivial examples of solutions of the quantum Yang–Baxter equation. All this makes it interesting to obtain similar relations in the most popular generalizations of Hopf algebras, namely Hopf quasigroups [17] and weak Hopf algebras [5].

This paper is a continuation of [5]. By working again in the weak Hopf algebra setting, in the main result (Theorem 3.6) we define a functor between ${}_{H\otimes B}^{H\otimes B}\mathsf{YD}$ and ${}_H^B\mathsf{Long}$. If moreover H is quasitriangular and B coquasitriangular, we show in Theorem 4.4 that this functor is a retraction of the one defined in [5].

2. Preliminaries

Recall that a monoidal category is a category C equipped with a tensor product functor $\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$, a unit object K of C and a family of natural isomorphisms

$$\mathsf{a}_{M,N,P} : (M \otimes N) \otimes P \to M \otimes (N \otimes P),$$
$$\mathsf{r}_M : M \otimes K \to M, \quad \mathsf{l}_M : K \otimes M \to M,$$

in C (called associativity, right unit and left unit constraints, respectively) satisfying the pentagon axiom and the triangle axiom. A monoidal category is called strict if the associativity, right unit and left unit constraints are identities. On the other hand, a strict monoidal category C is braided if it has a natural family of isomorphisms $c_{M,N} : M \otimes N \to N \otimes M$ such that the equalities

$$c_{M,N\otimes P} = (\mathrm{id}_N \otimes c_{M,P}) \circ (c_{M,N} \otimes \mathrm{id}_P),$$
$$c_{M\otimes N,P} = (c_{M,P} \otimes \mathrm{id}_N) \circ (\mathrm{id}_M \otimes c_{N,P})$$

hold for all M, N in C, where id_M, id_N and id_P denote the corresponding identity morphisms. If $c_{N,M} \circ c_{M,N} = \mathrm{id}_{M\otimes N}$, for all M, N in C, we will say that C is symmetric.

From now on C denotes a strict symmetric monoidal category with tensor product $\otimes$, unit object K and natural isomorphism of symmetry c. Taking into account that every non-strict monoidal category is monoidally equivalent to a strict one (see [9]), we can assume without loss of generality that the category is strict and, as a consequence, the results contained in this paper remain valid for every non-strict symmetric monoidal category, what would include for example the categories of vector spaces over a field $\mathbb{F}$, or the one of left modules over a commutative ring R. In what follows, for simplicity of notation, given objects M, N, P in C and a morphism $f : M \to N$, we write $P \otimes f$ for $\mathrm{id}_P \otimes f$ and $f \otimes P$ for $f \otimes \mathrm{id}_P$. We also assume that in C every idempotent morphism splits, i.e., for any morphism $q : Y \to Y$ such that $q \circ q = q$ there exists an object Z, called the image of q, and morphisms $i : Z \to Y$, $p : Y \to Z$ such that $q = i \circ p$ and $p \circ i = \mathrm{id}_Z$. The pair of morphisms p and i will be called a factorization of q. Note that Z, p and i are unique up to isomorphism. The categories satisfying this property constitute a broad class that includes, among others, the categories with epi-monic decomposition for morphisms and categories with (co)equalizers.

In this section, we recall some basic definitions and well-known facts about monoids, comonoids, weak bimonoids and weak Hopf monoids in C that we shall need later.

A monoid in C is a triple $A = (A, \eta_A, \mu_A)$ where A is an object in C and $\eta_A : K \to A$ (unit), $\mu_A : A \otimes A \to A$ (product) are morphisms in C such that

$$\mu_A \circ (A \otimes \eta_A) = \mathrm{id}_A = \mu_A \circ (\eta_A \otimes A), \quad \mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A).$$

Given two monoids $A = (A, \eta_A, \mu_A)$ and $B = (B, \eta_B, \mu_B)$, a morphism $f : A \to B$ in C is a monoid morphism if

$$\mu_B \circ (f \otimes f) = f \circ \mu_A, \quad f \circ \eta_A = \eta_B.$$

Also, if A, B are monoids in C, the object $A \otimes B$ is also a monoid in C where $\eta_{A\otimes B} = \eta_A \otimes \eta_B$ and $\mu_{A\otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B)$. Note that, if A and B are commutative monoids, so is $A \otimes B$.

A comonoid in C is a triple $D = (D, \varepsilon_D, \delta_D)$ where D is an object in C and $\varepsilon_D : D \to K$ (counit), $\delta_D : D \to D \otimes D$ (coproduct) are morphisms in C such that

$$(\varepsilon_D \otimes D) \circ \delta_D = \mathrm{id}_D = (D \otimes \varepsilon_D) \circ \delta_D, \quad (\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D.$$

If $D = (D, \varepsilon_D, \delta_D)$ and $E = (E, \varepsilon_E, \delta_E)$ are comonoids, $f : D \to E$ is a comonoid morphism if

$$(f \otimes f) \circ \delta_D = \delta_E \circ f, \quad \varepsilon_E \circ f = \varepsilon_D.$$

If D, E are comonoids in C, then $D \otimes E$ is a comonoid in C where $\varepsilon_{D\otimes E} = \varepsilon_D \otimes \varepsilon_E$ and $\delta_{D\otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E)$. Note that, if D and E are cocommutative comonoids, so is $D \otimes E$.

Finally, if A is a monoid, C is a comonoid and $f : C \to A$, $g : C \to A$ are morphisms in C, we define the convolution product $f * g : C \to A$ of f and g by $f * g = \mu_A \circ (f \otimes g) \circ \delta_C$.

Definition 2.1. A weak bimonoid H is an object in C with a monoid structure (H, η_H, μ_H) and a comonoid structure $(H, \varepsilon_H, \delta_H)$ satisfying the following conditions:

(a1) $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ \delta_{H\otimes H}$,

(a2) $\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes \delta_H \otimes H)$
$= (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes H)$,

(a3) $(\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$
$= (H \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$.

Let H be a weak bimonoid. If there exists a morphism $\lambda_H : H \to H$ in C (called the antipode of H) such that the equalities

(a4) $\mathrm{id}_H * \lambda_H = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$,

(a5) $\lambda_H * \mathrm{id}_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$,

(a6) $\lambda_H * \mathrm{id}_H * \lambda_H = \lambda_H$

hold, we will say that H is a weak Hopf monoid.

For a weak bimonoid H, the morphisms Π_H^L (target), Π_H^R (source), $\overline{\Pi}_H^L$ and $\overline{\Pi}_H^R$ are defined by

$$\Pi_H^L = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H),$$
$$\Pi_H^R = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)),$$
$$\overline{\Pi}_H^L = (H \otimes (\varepsilon_H \circ \mu_H)) \circ ((\delta_H \circ \eta_H) \otimes H),$$
$$\overline{\Pi}_H^R = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)).$$

These morphisms are idempotent and satisfy the identities

$$\Pi_H^L \circ \overline{\Pi}_H^L = \Pi_H^L, \quad \Pi_H^L \circ \overline{\Pi}_H^R = \overline{\Pi}_H^R, \tag{1}$$
$$\Pi_H^R \circ \overline{\Pi}_H^L = \overline{\Pi}_H^L, \quad \Pi_H^R \circ \overline{\Pi}_H^R = \Pi_H^R,$$

$$\overline{\Pi}_H^L \circ \Pi_H^L = \overline{\Pi}_H^L, \quad \overline{\Pi}_H^L \circ \Pi_H^R = \Pi_H^R, \tag{2}$$
$$\overline{\Pi}_H^R \circ \Pi_H^L = \Pi_H^L, \quad \overline{\Pi}_H^R \circ \Pi_H^R = \overline{\Pi}_H^R.$$

If H_L denotes the image of the target morphism Π_H^L, and $p_H^L : H \to H_L$ and $i_H^L : H_L \to H$ are the morphisms such that

$$i_H^L \circ p_H^L = \Pi_H^L, \quad p_H^L \circ i_H^L = \mathrm{id}_{H_L},$$

then the triples

$$(H_L, \eta_{H_L} = p_H^L \circ \eta_H, \mu_{H_L} = p_H^L \circ \mu_H \circ (i_H^L \otimes i_H^L)),$$
$$(H_L, \varepsilon_{H_L} = \varepsilon_H \circ i_H^L, \delta_{H_L} = (p_H^L \otimes p_H^L) \circ \delta_H \circ i_H^L)$$

determine a monoid and a comonoid, respectively.

In the weak monoid setting, for the morphisms target and source, we have the following identities:

$$\Pi_H^L \circ \mu_H \circ (H \otimes \Pi_H^L) = \Pi_H^L \circ \mu_H, \tag{3}$$
$$\Pi_H^R \circ \mu_H \circ (\Pi_H^R \otimes H) = \Pi_H^R \circ \mu_H,$$
$$(H \otimes \Pi_H^L) \circ \delta_H \circ \Pi_H^L = \delta_H \circ \Pi_H^L, \tag{4}$$
$$(\Pi_H^R \otimes H) \circ \delta_H \circ \Pi_H^R = \delta_H \circ \Pi_H^R,$$
$$\mu_H \circ (H \otimes \Pi_H^L) = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H), \tag{5}$$
$$(H \otimes \Pi_H^L) \circ \delta_H = (\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H), \tag{6}$$
$$\mu_H \circ (H \otimes \overline{\Pi}_H^L) = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (\delta_H \otimes H), \tag{7}$$
$$(H \otimes \overline{\Pi}_H^R) \circ \delta_H = (\mu_H \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)), \tag{8}$$

which will be useful in what follows.

Moreover, if H is a weak Hopf monoid in C, then the antipode λ_H is unique, antimultiplicative, anticomultiplicative and leaves the unit and the counit invariant, i.e.,

$$\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}, \tag{9}$$
$$\delta_H \circ \lambda_H = c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H,$$
$$\lambda_H \circ \eta_H = \eta_H, \quad \varepsilon_H \circ \lambda_H = \varepsilon_H. \tag{10}$$

Also, it is easy to show that for the convolution product the morphisms target and source satisfy the equalities

$$\Pi_H^L = \mathrm{id}_H * \lambda_H, \quad \Pi_H^R = \lambda_H * \mathrm{id}_H, \quad \Pi_H^L * \mathrm{id}_H = \mathrm{id}_H = \mathrm{id}_H * \Pi_H^R, \tag{11}$$
$$\Pi_H^R * \lambda_H = \lambda_H = \lambda_H * \Pi_H^L, \quad \Pi_H^L * \Pi_H^L = \Pi_H^L, \quad \Pi_H^R * \Pi_H^R = \Pi_H^R$$

and

$$\Pi_H^L = \lambda_H \circ \overline{\Pi}_H^L = \overline{\Pi}_H^R \circ \lambda_H, \quad \Pi_H^R = \overline{\Pi}_H^L \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^R. \tag{12}$$

Finally, by [2, Proposition 2.15] and the condition of symmetric category for C, we have the identities:

$$\mu_H \circ c_{H,H} \circ (\Pi_H^L \otimes \Pi_H^R) = \mu_H \circ (\Pi_H^L \otimes \Pi_H^R), \tag{13}$$
$$\mu_H \circ c_{H,H} \circ (\Pi_H^R \otimes \Pi_H^L) = \mu_H \circ (\Pi_H^R \otimes \Pi_H^L),$$
$$(\Pi_H^L \otimes \Pi_H^R) \circ c_{H,H} \circ \delta_H = (\Pi_H^L \otimes \Pi_H^R) \circ \delta_H, \tag{14}$$
$$(\Pi_H^R \otimes \Pi_H^L) \circ c_{H,H} \circ \delta_H = (\Pi_H^R \otimes \Pi_H^L) \circ \delta_H.$$

Lemma 2.2. *Let H be a weak Hopf monoid in C such that its antipode is an isomorphism. The following equalities hold:*

$$\mu_H \circ (H \otimes (\Pi_H^L \circ \mu_H \circ (H \otimes \lambda_H) \circ c_{H,H})) \circ (\delta_H \otimes H) \tag{15}$$
$$= \mu_H \circ (H \otimes (\Pi_H^L * (\lambda_H^{-2} \circ \Pi_H^R))),$$
$$(\mu_H \otimes H) \circ (H \otimes (c_{H,H} \circ (H \otimes \lambda_H) \circ \delta_H \circ \Pi_H^L)) \circ \delta_H \tag{16}$$
$$= (H \otimes (\Pi_H^L * (\lambda_H^{-2} \circ \Pi_H^R))) \circ \delta_H,$$

where $\lambda_H^{-2} = \lambda_H^{-1} \circ \lambda_H^{-1}$.

Proof. Identity (15) follows from

$$\mu_H \circ (H \otimes (\Pi_H^L \circ \mu_H \circ (H \otimes \lambda_H) \circ c_{H,H})) \circ (\delta_H \otimes H)$$
$$\stackrel{(3)}{=} \mu_H \circ (H \otimes (\Pi_H^L \circ \mu_H \circ (H \otimes (\Pi_H^L \circ \lambda_H)) \circ c_{H,H})) \circ (\delta_H \otimes H)$$
$$\stackrel{(1)}{=} \mu_H \circ (H \otimes (\Pi_H^L \circ \mu_H \circ (H \otimes (\Pi_H^L \circ \overline{\Pi}_H^L \circ \lambda_H)) \circ c_{H,H})) \circ (\delta_H \otimes H)$$
$$\stackrel{(3)}{=} \mu_H \circ (H \otimes (\Pi_H^L \circ \mu_H \circ (H \otimes (\overline{\Pi}_H^L \circ \lambda_H)) \circ c_{H,H})) \circ (\delta_H \otimes H)$$
$$\stackrel{(12)}{=} \mu_H \circ (H \otimes (\Pi_H^L \circ \mu_H \circ (H \otimes \Pi_H^R) \circ c_{H,H})) \circ (\delta_H \otimes H)$$
$$\stackrel{(*)}{=} \mu_H \circ (H \otimes (\Pi_H^L \circ \mu_H \circ c_{H,H})) \circ (((H \otimes \Pi_H^R) \circ \delta_H) \otimes H)$$
$$\stackrel{(12)}{=} \mu_H \circ (H \otimes (\Pi_H^L \circ \mu_H \circ c_{H,H})) \circ (((H \otimes (\lambda_H \circ \overline{\Pi}_H^R)) \circ \delta_H) \otimes H)$$
$$\stackrel{(8)}{=} \mu_H \circ (H \otimes (\Pi_H^L \circ \mu_H \circ c_{H,H})) \circ (\mu_H \otimes \lambda_H \otimes H)$$
$$\circ (H \otimes (\delta_H \circ \eta_H) \otimes H)$$
$$\stackrel{(*)}{=} \mu_H \circ (\mu_H \otimes (\Pi_H^L \circ \mu_H)) \circ (H \otimes c_{H,H} \otimes \lambda_H) \circ (H \otimes H \otimes (\delta_H \circ \eta_H))$$
$$\stackrel{(10),(9)}{=} \mu_H \circ (\mu_H \otimes (\Pi_H^L \circ \mu_H)) \circ (H \otimes c_{H,H} \otimes \lambda_H)$$
$$\circ (H \otimes H \otimes ((\lambda_H^{-1} \otimes \lambda_H^{-1}) \circ c_{H,H} \circ \delta_H \circ \eta_H))$$
$$\stackrel{(*)}{=} \mu_H \circ (\mu_H \otimes H)$$
$$\circ (H \otimes (c_{H,H} \circ ((\Pi_H^L \circ \mu_H) \otimes \lambda_H^{-1}) \circ (H \otimes (\delta_H \circ \eta_H))))$$

$$\begin{aligned}
&\overset{(8)}{=} \mu_H \circ (H \otimes (\mu_H \circ c_{H,H} \circ (\Pi_H^L \otimes (\lambda_H^{-1} \circ \overline{\Pi}_H^R)) \circ \delta_H))\\
&\overset{(12)}{=} \mu_H \circ (H \otimes (\mu_H \circ c_{H,H} \circ (\Pi_H^L \otimes (\overline{\Pi}_H^L \circ \lambda_H^{-1})) \circ \delta_H))\\
&\overset{(1)}{=} \mu_H \circ (H \otimes (\mu_H \circ c_{H,H} \circ (\Pi_H^L \otimes (\Pi_H^R \circ \overline{\Pi}_H^L \circ \lambda_H^{-1})) \circ \delta_H))\\
&\overset{(13)}{=} \mu_H \circ (H \otimes (\mu_H \circ (\Pi_H^L \otimes (\Pi_H^R \circ \overline{\Pi}_H^L \circ \lambda_H^{-1})) \circ \delta_H))\\
&\overset{(1)}{=} \mu_H \circ (H \otimes (\mu_H \circ (\Pi_H^L \otimes (\overline{\Pi}_H^L \circ \lambda_H^{-1})) \circ \delta_H))\\
&\overset{(12)}{=} \mu_H \circ (H \otimes (\Pi_H^L * (\lambda_H^{-2} \circ \Pi_H^R))),
\end{aligned}$$

where the three equations marked with $(*)$ follow by naturality of c.

Identity (16) follows from

$$\begin{aligned}
&(\mu_H \otimes H) \circ (H \otimes (c_{H,H} \circ (H \otimes \lambda_H) \circ \delta_H \circ \Pi_H^L)) \circ \delta_H\\
&\overset{(4)}{=} (\mu_H \otimes H) \circ (H \otimes (c_{H,H} \circ (H \otimes (\lambda_H \circ \Pi_H^L)) \circ \delta_H \circ \Pi_H^L)) \circ \delta_H\\
&\overset{(2)}{=} (\mu_H \otimes H) \circ (H \otimes (c_{H,H} \circ (H \otimes (\lambda_H \circ \overline{\Pi}_H^R \circ \Pi_H^L)) \circ \delta_H \circ \Pi_H^L)) \circ \delta_H\\
&\overset{(12)}{=} (\mu_H \otimes H) \circ (H \otimes (c_{H,H} \circ (H \otimes (\Pi_H^R \circ \Pi_H^L)) \circ \delta_H \circ \Pi_H^L)) \circ \delta_H\\
&\overset{(4)}{=} (\mu_H \otimes H) \circ (H \otimes (c_{H,H} \circ (H \otimes \Pi_H^R) \circ \delta_H \circ \Pi_H^L)) \circ \delta_H\\
&\overset{(12)}{=} (\mu_H \otimes H) \circ (H \otimes (c_{H,H} \circ (H \otimes (\overline{\Pi}_H^L \circ \lambda_H)) \circ \delta_H \circ \Pi_H^L)) \circ \delta_H\\
&\overset{(*)}{=} ((\mu_H \circ (H \otimes \overline{\Pi}_H^L)) \otimes H) \circ (H \otimes (c_{H,H} \circ (H \otimes \lambda_H) \circ \delta_H \circ \Pi_H^L)) \circ \delta_H\\
&\overset{(7)}{=} (H \otimes (\varepsilon_H \circ \mu_H) \otimes H) \circ (\delta_H \otimes (c_{H,H} \circ (H \otimes \lambda_H) \circ \delta_H \circ \Pi_H^L)) \circ \delta_H\\
&\overset{(10),(9)}{=} (H \otimes (\varepsilon_H \circ \mu_H \circ c_{H,H} \circ (\lambda_H^{-1} \otimes \lambda_H^{-1})) \otimes H)\\
&\qquad \circ (\delta_H \otimes (c_{H,H} \circ (H \otimes \lambda_H) \circ \delta_H \circ \Pi_H^L)) \circ \delta_H\\
&\overset{(*)}{=} (H \otimes ((H \otimes (\varepsilon_H \circ \mu_H)) \circ ((\delta_H \circ \Pi_H^L) \otimes \lambda_H^{-1}) \circ c_{H,H})) \circ (\delta_H \otimes H) \circ \delta_H\\
&\overset{(7)}{=} (H \otimes (\mu_H \circ (\Pi_H^L \otimes (\overline{\Pi}_H^L \circ \lambda_H^{-1})) \circ c_{H,H} \circ \delta_H)) \circ \delta_H\\
&\overset{(12)}{=} (H \otimes (\mu_H \circ (\Pi_H^L \otimes (\lambda_H^{-1} \circ \overline{\Pi}_H^R)) \circ c_{H,H} \circ \delta_H)) \circ \delta_H\\
&\overset{(2)}{=} (H \otimes (\mu_H \circ (\Pi_H^L \otimes (\lambda_H^{-1} \circ \overline{\Pi}_H^R \circ \Pi_H^R)) \circ c_{H,H} \circ \delta_H)) \circ \delta_H\\
&\overset{(14)}{=} (H \otimes (\mu_H \circ (\Pi_H^L \otimes (\lambda_H^{-1} \circ \overline{\Pi}_H^R \circ \Pi_H^R)) \circ \delta_H)) \circ \delta_H\\
&\overset{(2)}{=} (H \otimes (\mu_H \circ (\Pi_H^L \otimes (\lambda_H^{-1} \circ \overline{\Pi}_H^R)) \circ \delta_H)) \circ \delta_H\\
&\overset{(12)}{=} (H \otimes (\Pi_H^L * (\lambda_H^{-2} \circ \Pi_H^R))) \circ \delta_H,
\end{aligned}$$

where the two equations marked with $(*)$ follow again by naturality of c. ■

Let H be a weak bimonoid in C. Then,

$$H^{\rm op} = (H, \eta_H, \mu_H \circ c_{H,H}, \varepsilon_H, \delta_H), \quad H^{\rm cop} = (H, \eta_H, \mu_H, \varepsilon_H, c_{H,H} \circ \delta_H)$$

are weak bimonoids in C. Therefore so is

$$(H^{\rm op})^{\rm cop} = (H, \eta_H, \mu_H \circ c_{H,H}, \varepsilon_H, c_{H,H} \circ \delta_H).$$

Note that

$$\Pi^L_{H^{\rm op}} = \overline{\Pi}^R_H, \quad \Pi^R_{H^{\rm op}} = \overline{\Pi}^L_H,$$
$$\Pi^L_{H^{\rm cop}} = \overline{\Pi}^L_H, \quad \Pi^R_{H^{\rm cop}} = \overline{\Pi}^R_H.$$

If H is a weak Hopf monoid and the antipode λ_H is an isomorphism, $H^{\rm op}$ and $H^{\rm cop}$ are weak Hopf monoids in $\mathcal{C}$ with antipode $\lambda_{H^{\rm op}} = \lambda_{H^{\rm cop}} = \lambda_H^{-1}$. Then, under these conditions, $(H^{\rm op})^{\rm cop}$ is a weak Hopf monoid with antipode $\lambda_{(H^{\rm op})^{\rm cop}} = \lambda_H$.

If H and B are weak bimonoids in C, so is the tensor product $H \otimes B$. In this case, the monoid-comonoid structure is the one of $H \otimes B$ and

$$\Pi^L_{H\otimes B} = \Pi^L_H \otimes \Pi^L_B, \quad \Pi^R_{H\otimes B} = \Pi^R_H \otimes \Pi^R_B.$$

Then, if H and B are weak Hopf monoids in C, so is the tensor product $H \otimes B$, with $\lambda_{H\otimes B} = \lambda_H \otimes \lambda_B$. Note that $(H \otimes B)_L = H_L \otimes B_L$.

3. Yetter–Drinfeld modules and Long dimodules

Definition 3.1. Let H be a weak Hopf monoid in C. We say that a pair (M, φ_M) is a left H-module if M is an object in C and $\varphi_M : H \otimes M \to M$ is a morphism in C satisfying the following conditions:

$$\varphi_M \circ (\eta_H \otimes M) = {\rm id}_M, \quad \varphi_M \circ (H \otimes \varphi_M) = \varphi_M \circ (\mu_H \otimes M). \tag{17}$$

If (M, φ_M) and (N, φ_N) are left H-modules, a morphism $f : M \to N$ in C is a morphism of left H-modules if

$$\varphi_N \circ (H \otimes f) = f \circ \varphi_M \tag{18}$$

holds.

For two left H-modules (M, φ_M) and (N, φ_N), the morphism $\varphi_{M\otimes N} : H \otimes M \otimes N \to M \otimes N$ is defined by

$$\varphi_{M\otimes N} = (\varphi_M \otimes \varphi_N) \circ (H \otimes c_{H,M} \otimes N) \circ (\delta_H \otimes M \otimes N).$$

Then, $\varphi_{M\otimes N}$ satisfies the equality

$$\varphi_{M\otimes N} \circ (H \otimes \varphi_{M\otimes N}) = \varphi_{M\otimes N} \circ (\mu_H \otimes M \otimes N)$$

and $\nabla_{M\otimes N} = \varphi_{M\otimes N} \circ (\eta_H \otimes M \otimes N) : M \otimes N \to M \otimes N$ is an idempotent morphism. Let $M \boxdot N$ be the image of $\nabla_{M\otimes N}$ and let $p_{M\otimes N} : M \otimes N \to M \boxdot N$, $i_{M\otimes N} : M \boxdot N \to M \otimes N$ be the morphisms such that

$$i_{M\otimes N} \circ p_{M\otimes N} = \nabla_{M\otimes N}, \quad p_{M\otimes N} \circ i_{M\otimes N} = \mathrm{id}_{M\boxdot N}.$$

The object $M \boxdot N$ is a left H-module with action

$$\varphi_{M\boxdot N} = p_{M\otimes N} \circ \varphi_{M\otimes N} \circ (H \otimes i_{M\otimes N}) : H \otimes M \boxdot N \to M \boxdot N$$

and the equalities

$$\varphi_{M\otimes N} \circ (H \otimes \nabla_{M\otimes N}) = \varphi_{M\otimes N} = \nabla_{M\otimes N} \circ \varphi_{M\otimes N}$$

hold. Also, if (M, φ_M), (N, φ_N) and (P, φ_P) are left H-modules, we have that

$$(M \otimes \nabla_{N\otimes P}) \circ (\nabla_{M\otimes N} \otimes P) = (\nabla_{M\otimes N} \otimes P) \circ (M \otimes \nabla_{N\otimes P}).$$

For two morphisms $f : (M, \varphi_M) \to (M', \varphi_{M'})$ and $g : (N, \varphi_N) \to (N', \varphi_{N'})$ of left H-modules,

$$f \boxdot g = p_{M'\otimes N'} \circ (f \otimes g) \circ i_{M\otimes N} : M \boxdot N \to M' \boxdot N'$$

is a morphism of left H-modules between $(M \boxdot N, \varphi_{M\boxdot N})$ and $(M' \boxdot N', \varphi_{M'\boxdot N'})$. Also, the following identity holds:

$$(f \otimes g) \circ \nabla_{M\otimes N} = \nabla_{M'\otimes N'} \circ (f \otimes g).$$

DEFINITION 3.2. Let H be a weak Hopf monoid in the category C. We say that a pair (M, ρ_M) is a left H-comodule in the category C if M is an object in C and $\rho_M : M \to H \otimes M$ is a morphism in C satisfying the following conditions:

$$(\varepsilon_H \otimes M) \circ \rho_M = \mathrm{id}_M, \quad (H \otimes \rho_M) \circ \rho_M = (\delta_H \otimes M) \circ \rho_M.$$

If (M, ρ_M) and (N, ρ_N) are left H-comodules, a morphism $f : M \to N$ in C is a morphism of left H-comodules if

$$\rho_N \circ f = (H \otimes f) \circ \rho_M \tag{19}$$

holds.

For two left H-comodules (M, ρ_M) and (N, ρ_N) the morphism $\rho_{M\otimes N} : M \otimes N \to H \otimes M \otimes N$ defined by $\rho_{M\otimes N} = (\mu_H \otimes M \otimes N) \circ (H \otimes c_{M,H} \otimes N) \circ (\rho_M \otimes \rho_N)$ satisfies the equality

$$(H \otimes \rho_{M\otimes N}) \circ \rho_{M\otimes N} = (\delta_H \otimes M \otimes N) \circ \rho_{M\otimes N}.$$

Then, as a consequence, $\nabla'_{M\otimes N} = (\varepsilon_H \otimes M \otimes N) \circ \rho_{M\otimes N} : M \otimes N \to M \otimes N$ is an idempotent morphism. Let $M \odot N$ be the image of $\nabla'_{M\otimes N}$ and let $p'_{M\otimes N} : M \otimes N \to M \odot N$, $i'_{M\otimes N} : M \odot N \to M \otimes N$ be the morphisms such that

$$i'_{M\otimes N} \circ p'_{M\otimes N} = \nabla'_{M\otimes N}, \quad p'_{M\otimes N} \circ i'_{M\otimes N} = \mathrm{id}_{M\odot N}.$$

The object $M \odot N$ is a left H-comodule with coaction

$$\rho_{M\odot N} = (H \otimes p'_{M\otimes N}) \circ \rho_{M\otimes N} \circ i'_{M\otimes N} : M \odot N \to H \otimes (M \odot N)$$

and the equalities

$$(H \otimes \nabla'_{M\otimes N}) \circ \rho_{M\otimes N} = \rho_{M\otimes N} = \rho_{M\otimes N} \circ \nabla'_{M\otimes N}$$

hold. Also, if (M, ρ_M), (N, ρ_N) and (P, ρ_P) are left H-comodules, we have that

$$(M \otimes \nabla'_{N\otimes P}) \circ (\nabla'_{M\otimes N} \otimes P) = (\nabla'_{M\otimes N} \otimes P) \circ (M \otimes \nabla'_{N\otimes P}).$$

For two morphisms of left H-comodules $f : (M, \rho_M) \to (M', \rho_{M'})$ and $g : (N, \rho_N) \to (N', \rho_{N'})$,

$$f \odot g = p'_{M'\otimes N'} \circ (f \otimes g) \circ i'_{M\otimes N} : M \odot N \to M' \odot N'$$

is a morphism of left H-comodules between $(M \odot N, \rho_{M\odot N})$ and $(M' \odot N', \rho_{M'\odot N'})$. Also, the following identity holds:

$$(f \otimes g) \circ \nabla'_{M\otimes N} = \nabla'_{M'\otimes N'} \circ (f \otimes g).$$

Following [8, 12], we recall the notion of a left-left Yetter–Drinfeld module in the weak Hopf monoid setting.

Definition 3.3. Let H be a weak Hopf monoid in C. We shall denote by ${}^H_H\mathsf{YD}$ the category of left-left Yetter–Drinfeld modules over H. The objects of ${}^H_H\mathsf{YD}$ are triples $M = (M, \psi_M, \gamma_M)$ where (M, ψ_M) is a left H-module and (M, γ_M) is a left H-comodule satisfying the following conditions:

(b1) $(\mu_H \otimes M) \circ (H \otimes c_{M,H}) \circ ((\gamma_M \circ \psi_M) \otimes H) \circ (H \otimes c_{H,M}) \circ (\delta_H \otimes M)$
$= (\mu_H \otimes \psi_M) \circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \otimes \gamma_M)$,

(b2) $(\mu_H \otimes \psi_M) \circ (H \otimes c_{H,H} \otimes M) \circ ((\delta_H \circ \eta_H) \otimes \gamma_M) = \gamma_M$.

A morphism in ${}_{H}^{H}\mathsf{YD}$ between (M, ψ_M, γ_M) and (N, ψ_N, γ_N) is a morphism $f : M \to N$ in C such that (18) and (19) hold.

Let (M, ψ_M, γ_M) be a left-left Yetter–Drinfeld module. It is easy to show that the axiom (b2) is equivalent to

$$((\varepsilon_H \circ \mu_H) \otimes \psi_M) \circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \otimes \gamma_M) = \psi_M.$$

As a consequence of the previous identity we obtain that

$$\psi_M \circ (\Pi_H^L \otimes M) \circ \gamma_M = \mathrm{id}_M \tag{20}$$

holds and by [3, (37) of Remark 1.11] we have the equality

$$(\Pi_H^R \otimes M) \circ \gamma_M = ((\Pi_H^R \circ \lambda_H) \otimes \psi_M) \circ ((c_{H,H} \circ \delta_H \circ \eta_H) \otimes M). \tag{21}$$

It is a well-known fact (see [8, Proposition 2.2]) that conditions (b1) and (b2) are equivalent to

$$\begin{aligned} \gamma_M \circ \psi_M = {} & (\mu_H \otimes M) \circ (H \otimes c_{M,H}) \\ & \circ (((\mu_H \otimes \psi_M) \circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \otimes \gamma_M)) \otimes \lambda_H) \\ & \circ (H \otimes c_{H,M}) \circ (\delta_H \otimes M). \end{aligned} \tag{22}$$

Lemma 3.4. *Let H be a weak Hopf monoid in C such that its antipode is an isomorphism. Let (M, ψ_M, γ_M) be a left-left Yetter–Drinfeld module. The following identity holds:*

$$\psi_M \circ ((\lambda_H^{-2} \circ \Pi_H^R) \otimes M) \circ \gamma_M = \mathrm{id}_M. \tag{23}$$

Proof. We have

$$\begin{aligned} & \psi_M \circ ((\lambda_H^{-2} \circ \Pi_H^R) \otimes M) \circ \gamma_M \\ & \overset{(21)}{=} \psi_M \circ ((\lambda_H^{-2} \circ \Pi_H^R \circ \lambda_H) \otimes \psi_M) \circ ((c_{H,H} \circ \delta_H \circ \eta_H) \otimes M) \\ & \overset{(*)}{=} \psi_M \circ ((\mu_H \circ (\overline{\Pi}_H^L \otimes H) \circ c_{H,H} \circ \delta_H \circ \eta_H) \otimes M) \\ & \overset{(**)}{=} \psi_M \circ (\eta_H \otimes M) \\ & \overset{(17)}{=} \mathrm{id}_M, \end{aligned}$$

where $(*)$ follows by (17), (12) and the naturality of c, and $(**)$ follows by (11) for H^{cop}. ■

If the antipode of the weak Hopf monoid H is an isomorphism, the category ${}_{H}^{H}\mathsf{YD}$ is an example of a non-strict braided monoidal category. In the following paragraphs we will make a brief summary of its braided monoidal structure.

Let (M, ψ_M, γ_M) and (N, ψ_N, γ_N) be objects in ${}^H_H\mathsf{YD}$. Then, for the morphisms $\nabla_{M\otimes N}$ and $\nabla'_{M\otimes N}$, defined in Definitions 3.1 and 3.2, by [2, Proposition 1.12 (iii)] we have that

$$\nabla_{M\otimes N} = \nabla'_{M\otimes N}.$$

Then, the tensor product of (M, ψ_M, γ_M) and (N, ψ_N, γ_N) is defined as the image of the idempotent morphism $\nabla_{M\otimes N}$, denoted by $M \boxdot N$, with the following action and coaction:

$$\psi_{M\boxdot N} = p_{M\otimes N} \circ \psi_{M\otimes N} \circ (H \otimes i_{M\otimes N}),$$
$$\gamma_{M\boxdot N} = (H \otimes p_{M\otimes N}) \circ \gamma_{M\otimes N} \circ i_{M\otimes N}.$$

The base object in ${}^H_H\mathsf{YD}$ is H_L, which is a left-left Yetter–Drinfeld module over H with (co)module structure

$$\psi_{H_L} = p_H^L \circ \mu_H \circ (H \otimes i_H^L), \quad \gamma_{H_L} = (H \otimes p_H^L) \circ \delta_H \circ i_H^L.$$

The unit constrains are defined by

$$\mathfrak{l}_M = \psi_M \circ (i_H^L \otimes M) \circ i_{H_L\otimes M} : H_L \boxdot M \to M,$$
$$\mathfrak{r}_M = \psi_M \circ c_{M,H} \circ (M \otimes (\overline{\Pi}_H^L \circ i_H^L)) \circ i_{M\otimes H_L} : M \boxdot H_L \to M,$$

and the associativity constrains $\mathfrak{a}_{M,N,P} : (M \boxdot N) \boxdot P \to M \boxdot (N \boxdot P)$ are defined by

$$\mathfrak{a}_{M,N,P} = p_{M\otimes(N\boxdot P)} \circ (M \otimes p_{N\otimes P}) \circ (i_{M\otimes N} \otimes P) \circ i_{(M\boxdot N)\otimes P},$$

where (P, ψ_P, γ_P) is a third object in the category of left-left Yetter–Drinfeld modules.

If $f : M \to M'$ and $g : N \to N'$ are morphisms in ${}^H_H\mathsf{YD}$, then

$$f \boxdot g = p_{M'\boxdot N'} \circ (f \otimes g) \circ i_{M\otimes N} : M \boxdot N \to M' \boxdot N'$$

is a morphism in the same category and

$$(f' \boxdot g') \circ (f \boxdot g) = (f' \circ f) \boxdot (g' \circ g)$$

holds, where $f' : M' \to M''$ and $g' : N' \to N''$ are morphisms in ${}^H_H\mathsf{YD}$.

Finally, the braiding is defined by

$$t_{M,N} = p_{N\otimes M} \circ \tau_{M,N} \circ i_{M\otimes N} : M \boxdot N \to N \boxdot M,$$

where

$$\tau_{M,N} = (\psi_N \otimes M) \circ (H \otimes c_{M,N}) \circ (\gamma_M \otimes N) : M \otimes N \to N \otimes M.$$

Definition 3.5. Let H and B be weak Hopf monoids in C. A left-left H-B-Long dimodule (M, φ_M, ρ_M) is both a left H-module with action $\varphi_M : H \otimes M \to M$ and a left B-comodule with coaction $\rho_M : M \to B \otimes M$ satisfying the axiom

$$\rho_M \circ \varphi_M = (B \otimes \varphi_M) \circ (c_{H,B} \otimes M) \circ (H \otimes \rho_M). \tag{24}$$

A morphism between two left-left H-B-Long dimodules (M, φ_M, ρ_M) and (N, φ_N, ρ_N) is a morphism $f : M \to N$ of left H-modules and left B-comodules. Left-left H-B-Long dimodules and morphisms of left-left H-B-Long dimodules form a category, denoted as ${}^{B}_{H}\mathsf{Long}$.

In [5] we can find many examples of Long dimodules in the weak setting. One of the main results proved in [5] asserts that ${}^{B}_{H}\mathsf{Long}$ is an example of a monoidal category (see [5, Theorem 1]). As in the Yetter–Drinfeld case, in the following paragraphs we will make a brief summary of its monoidal structure. The complete details can be found in [5, Lemmas 2–6, Propositions 1–3 and Theorem 1].

Let (M, φ_M, ρ_M) and (N, φ_N, ρ_N) be in ${}^{B}_{H}\mathsf{Long}$. The idempotent morphisms $\nabla_{M\otimes N}$ and $\nabla'_{M\otimes N}$, defined in Definitions 3.1 and 3.2, satisfy

$$\nabla'_{M\otimes N} \circ \nabla_{M\otimes N} = \nabla_{M\otimes N} \circ \nabla'_{M\otimes N}.$$

As a consequence, the morphism

$$\Omega_{M\otimes N} = \nabla'_{M\otimes N} \circ \nabla_{M\otimes N}$$

is idempotent and we have two morphisms $j_{M\otimes N} : M \times N \to M \otimes N$ and $q_{M\otimes N} : M \otimes N \to M \times N$ such that

$$q_{M\otimes N} \circ j_{M\otimes N} = \mathrm{id}_{M\times N}, \quad j_{M\otimes N} \circ q_{M\otimes N} = \Omega_{M\otimes N},$$

where $M \times N$ is the image of $\Omega_{M\otimes N}$. Then, the tensor product of (M, φ_M, ρ_M) and (N, φ_N, ρ_N) is defined as the image of the idempotent morphism $\Omega_{M\otimes N}$. It belongs to ${}^{B}_{H}\mathsf{Long}$ with H-module and B-comodule structures

$$\varphi_{M\times N} = q_{M\otimes N} \circ \varphi_{M\otimes N} \circ (H \otimes j_{M\otimes N}),$$
$$\rho_{M\times N} = (B \otimes q_{M\otimes N}) \circ \rho_{M\otimes N} \circ j_{M\otimes N},$$

respectively.

Moreover, if

$$f : (M, \varphi_M, \rho_M) \to (M', \varphi_{M'}, \rho_{M'}), \quad g : (N, \varphi_N, \rho_N) \to (N', \varphi_{N'}, \rho_{N'})$$

are morphisms in ${}^{B}_{H}\mathsf{Long}$, then

$$f \times g = q_{M' \times N'} \circ (f \otimes g) \circ j_{M \times N} : M \times N \to M' \times N'$$

is a morphism in ${}^{B}_{H}\mathsf{Long}$ between $(M \times N, \varphi_{M \times N}, \rho_{M \times N})$ and $(M' \times N', \varphi_{M' \times N'}, \rho_{M' \times N'})$.

If (M, φ_M, ρ_M), (N, φ_N, ρ_N) and (P, φ_P, ρ_P) are in ${}^{B}_{H}\mathsf{Long}$, the associativity constraint

$$a_{M,N,P} : (M \times N) \times P \to M \times (N \times P)$$

is defined by

$$a_{M,N,P} = q_{M \otimes (N \times P)} \circ (M \otimes q_{N \otimes P}) \circ (j_{M \otimes N} \otimes P) \circ j_{(M \times N) \otimes P}$$

and the base object is $H_L \otimes B_L$, where the action and the coaction are defined by

$$\varphi_{H_L \otimes B_L} = (p_H^L \circ \mu_H \circ (H \otimes i_H^L)) \otimes B_L,$$
$$\rho_{H_L \otimes B_L} = (c_{H_L,B} \otimes p_B^L) \circ (H_L \otimes (\delta_B \circ i_B^L)).$$

Finally, the unit constraints are $l_M : (H_L \otimes B_L) \times M \to M$ and $r_M : M \times (H_L \otimes B_L) \to M$, where

$$\begin{aligned} l_M &= ((\varepsilon_B \circ \mu_B) \otimes M) \circ (B \otimes (\rho_M \circ \varphi_M)) \\ &\quad \circ ((c_{H,B} \circ (i_H^L \otimes i_B^L)) \otimes M) \circ j_{(H_L \otimes B_L) \otimes M}, \\ r_M &= ((\varphi_M \circ c_{M,H}) \otimes (\varepsilon_B \circ \mu_B)) \circ (M \otimes c_{B,H} \otimes B) \\ &\quad \circ ((c_{B,M} \circ \rho_M) \otimes (\overline{\Pi}_H^L \circ i_H^L) \otimes i_B^L) \circ j_{M \otimes (H_L \otimes B_L)}. \end{aligned}$$

Theorem 3.6. *Let H and B be weak Hopf algebras such that their antipodes are isomorphisms. There exists a functor*

$$\mathsf{F} : {}^{H \otimes B}_{H \otimes B}\mathsf{YD} \to {}^{B}_{H}\mathsf{Long}$$

defined on objects by

$$\mathsf{F}((M, \psi_M, \gamma_M)) = (M, \chi_M = \psi_M \circ (H \otimes \eta_B \otimes M), \omega_M = (\varepsilon_H \otimes B \otimes M) \circ \gamma_M)$$

and by the identity on morphisms.

Proof. Let (M, ψ_M, γ_M) be in ${}^{H \otimes B}_{H \otimes B}\mathsf{YD}$. Then, using that (M, ψ_M) is a left $H \otimes B$-module, the unit properties and the naturality of c, we obtain that (M, χ_M) is a left H-module. Similarly, using that (M, γ_M) is a left $H \otimes B$-comodule, the counit properties and the naturality of c, we obtain that (M, ω_M) is a left B-comodule. The proof for equality (24) is the following:

$$
\begin{aligned}
&\omega_M \circ \chi_M \\
&\stackrel{\text{(a)}}{=} (((\varepsilon_H \otimes B) \circ \mu_{H\otimes B}) \otimes M) \circ (H \otimes B \otimes ((H \otimes c_{M,B}) \circ (c_{M,H} \otimes B))) \\
&\quad \circ (((\mu_{H\otimes B} \otimes \psi_M) \circ (H \otimes B \otimes c_{H\otimes B, H\otimes B} \otimes M) \circ (\delta_{H\otimes B} \otimes \gamma_M)) \otimes \lambda_H \otimes \lambda_B) \\
&\quad \circ (H \otimes B \otimes ((c_{H,M} \otimes B) \circ (H \otimes c_{B,M}))) \circ ((\delta_{H\otimes B} \circ (H \otimes \eta_B)) \otimes M) \\
&\stackrel{\text{(b)}}{=} (B \otimes \psi_M) \circ (c_{H,B} \otimes B \otimes M) \circ ((((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \\
&\quad \circ (\delta_H \otimes H)) \otimes ((\mu_B \otimes B) \circ (B \otimes c_{B,B}) \circ (\delta_B \otimes (\mu_B \circ (B \otimes \lambda_B) \circ c_{B,B})) \\
&\quad \circ (\delta_B \otimes B)) \otimes M) \circ (H \otimes (((\mu_H \circ (H \otimes \lambda_H) \circ c_{H,H}) \otimes B) \\
&\quad \circ (H \otimes c_{B,H})) \otimes B \otimes M) \circ (\delta_H \otimes \eta_B \otimes \gamma_M) \\
&\stackrel{\text{(c)}}{=} (B \otimes \psi_M) \circ (c_{H,B} \otimes B \otimes M) \circ ((\mu_H \circ (H \otimes (\Pi_H^L \circ \mu_H \circ (H \otimes \lambda_H) \circ c_{H,H}))) \\
&\quad \otimes ((\mu_B \otimes B) \circ (B \otimes (c_{B,B} \circ (B \otimes \lambda_B) \circ \delta_B)) \circ (\mu_B \otimes B) \circ (B \otimes c_{B,B}) \\
&\quad \circ ((\delta_B \circ \eta_B) \otimes B)) \otimes M) \circ (\delta_H \otimes \gamma_M) \\
&\stackrel{\text{(d)}}{=} (B \otimes \psi_M) \circ (c_{H,B} \otimes B \otimes M) \circ ((\mu_H \circ (H \otimes (\Pi_H^L \circ \mu_H \circ (H \otimes \lambda_H) \circ c_{H,H}))) \\
&\quad \otimes ((\mu_B \otimes B) \circ (B \otimes (c_{B,B} \circ (B \otimes \lambda_B) \circ \delta_B)) \circ (B \otimes \Pi_B^L) \circ \delta_B) \otimes M) \\
&\quad \circ (\delta_H \otimes \gamma_M) \\
&\stackrel{\text{(e)}}{=} (B \otimes \psi_M) \circ (c_{H,B} \otimes B \otimes M) \circ ((\mu_H \circ (H \otimes (\Pi_H^L * (\lambda_H^{-2} \circ \Pi_H^R)))) \\
&\quad \otimes ((B \otimes (\Pi_B^L * (\lambda_B^{-2} \circ \Pi_B^R))) \circ \delta_B) \otimes M) \circ (H \otimes \gamma_M) \\
&\stackrel{\text{(f)}}{=} (B \otimes (\psi_M \circ (\mu_{H\otimes B} \otimes M))) \circ (((c_{H,B} \otimes \Pi_B^L) \circ ((\mu_H \circ (H \otimes \Pi_H^L)) \otimes \delta_B)) \\
&\quad \otimes ((\lambda_H^{-2} \circ \Pi_H^R) \otimes (\lambda_B^{-2} \circ \Pi_B^R) \otimes M)) \circ (H \otimes ((\delta_{H\otimes B} \otimes M) \circ \gamma_M)) \\
&\stackrel{\text{(g)}}{=} (B \otimes \psi_M) \circ (((c_{H,B} \otimes \Pi_B^L) \circ ((\mu_H \circ (H \otimes \Pi_H^L)) \otimes \delta_B)) \\
&\quad \otimes (\psi_M \circ ((\lambda_H^{-2} \circ \Pi_H^R) \otimes (\lambda_B^{-2} \circ \Pi_B^R) \otimes M) \circ \gamma_M)) \circ (H \otimes \gamma_M) \\
&\stackrel{\text{(h)}}{=} (B \otimes \psi_M) \circ (((c_{H,B} \otimes \Pi_B^L) \circ ((\mu_H \circ (H \otimes \Pi_H^L)) \otimes \delta_B)) \otimes M) \circ (H \otimes \gamma_M) \\
&\stackrel{\text{(i)}}{=} (B \otimes (\psi_M \circ ((\mu_{H\otimes B} \circ (H \otimes \eta_B \otimes H \otimes B)) \otimes M))) \circ (c_{H,B} \otimes \Pi_H^L \otimes \Pi_B^L \otimes M) \\
&\quad \circ (H \otimes ((((\varepsilon_H \otimes B \otimes H \otimes B) \circ \delta_{H\otimes B}) \otimes M) \circ \gamma_M)) \\
&\stackrel{\text{(g)}}{=} (B \otimes \chi_M) \circ (c_{H,B} \otimes (\psi_M \circ (\Pi_H^L \otimes \Pi_B^L \otimes M) \circ \gamma_M)) \circ (H \otimes \omega_M) \\
&\stackrel{\text{(j)}}{=} (B \otimes \chi_M) \circ (c_{H,B} \otimes M) \circ (H \otimes \omega_M),
\end{aligned}
$$

where (a) follows by (22) for $H \otimes B$; (b) by naturality of c and the associativity of μ_H and μ_B; (c) by the coassociativity of δ_B, the naturality of c and (5); (d) by naturality of c and (6); (e) by (15) for H and (16) for B; (f) by naturality of c and the associativity of μ_H; (g) by the condition of left $H \otimes B$ (co)module for M; (h) by (23) for $H \otimes B$; (i) by naturality of c and the properties of η_B and ε_H; (j) by (20) for $H \otimes B$.

Therefore (M, χ_M, ω_M) is an object in ${}^{B}_{H}\mathsf{Long}$.

Finally, if f is a morphism in ${}^{H\otimes B}_{H\otimes B}\mathsf{YD}$ between the objects (M, ψ_M, γ_M) and (N, ψ_N, γ_N), it is immediate to obtain that f is a morphism in ${}^{B}_{H}\mathsf{Long}$ between the objects (M, χ_M, ω_M) and (N, χ_N, ω_N). Therefore, F is a functor between the categories ${}^{H\otimes B}_{H\otimes B}\mathsf{YD}$ and ${}^{B}_{H}\mathsf{Long}$. ∎

4. The retraction in the (co)quasitriangular case

In [5, Theorem 2] we proved that, if H is a quasitriangular weak Hopf monoid and B is a coquasitriangular weak Hopf monoid, there exists a functor

$$\mathsf{L} : {}^{B}_{H}\mathsf{Long} \to {}^{H\otimes B}_{H\otimes B}\mathsf{YD}$$

injective on objects and, consequently, ${}^{B}_{H}\mathsf{Long}$ can be identified with a subcategory of ${}^{H\otimes B}_{H\otimes B}\mathsf{YD}$. In this section, we will show that in this context the functor F, introduced at the end of the previous section, is a retraction of L. We will begin with a brief review of the fundamental properties of weak Hopf (co)quasitriangular monoids (see [4, 5] for the complete details).

The following definition is the monoidal version of the definition of quasitriangular weak Hopf monoid introduced by Nikshych, Turaev and Vainerman in [13].

Definition 4.1. Let H be a weak Hopf monoid. Let Ω_H and Ω'_H be the idempotent morphisms defined by $\Omega_H = \Omega^2_H \circ \Omega^1_H$ and $\Omega'_H = \Omega^4_H \circ \Omega^3_H$ where Ω^i_H are the idempotent morphisms defined by

$$\Omega^1_H = \mu_{H\otimes H} \circ ((c_{H,H} \circ \delta_H \circ \eta_H) \otimes H \otimes H) : H \otimes H \to H \otimes H,$$
$$\Omega^2_H = \mu_{H\otimes H} \circ (H \otimes H \otimes (\delta_H \circ \eta_H)) : H \otimes H \to H \otimes H,$$
$$\Omega^3_H = \mu_{H\otimes H} \circ (H \otimes H \otimes (c_{H,H} \circ \delta_H \circ \eta_H)) : H \otimes H \to H \otimes H,$$
$$\Omega^4_H = \mu_{H\otimes H} \circ ((\delta_H \circ \eta_H) \otimes H \otimes H) : H \otimes H \to H \otimes H.$$

We will say that H is a quasitriangular weak Hopf monoid if there exists a morphism $\sigma : K \to H \otimes H$ in C satisfying the following conditions:

(c1) $$\Omega_H \circ \sigma = \sigma,$$

(c2) $$(\delta_H \otimes H) \circ \sigma = (H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\sigma \otimes \sigma),$$

(c3) $$(H \otimes \delta_H) \circ \sigma = (\mu_H \otimes c_{H,H}) \circ (H \otimes c_{H,H} \otimes H) \circ (\sigma \otimes \sigma),$$

(c4) $$\mu_{H\otimes H} \circ (\sigma \otimes \delta_H) = \mu_{H\otimes H} \circ ((c_{H,H} \circ \delta_H) \otimes \sigma).$$

(c5) There exists a morphism $\overline{\sigma} : K \to H \otimes H$ such that

(c5.1) $\Omega'_H \circ \overline{\sigma} = \overline{\sigma}$,

(c5.2) $\sigma * \overline{\sigma} = c_{H,H} \circ \delta_H \circ \eta_H$,

(c5.3) $\overline{\sigma} * \sigma = \delta_H \circ \eta_H$.

We will say that a quasitriangular weak Hopf monoid H is triangular if moreover $\overline{\sigma} = c_{H,H} \circ \sigma$.

For any quasitriangular weak Hopf monoid the morphism $\overline{\sigma}$ is unique and by [4, Lemma 3.5] and [5, Lemma 7] the following equalities hold:

$$\sigma * \overline{\sigma} * \sigma = \sigma, \quad \overline{\sigma} * \sigma * \overline{\sigma} = \overline{\sigma}$$

$$(\varepsilon_H \otimes H) \circ \sigma = (H \otimes \varepsilon_H) \circ \sigma = \eta_H. \tag{25}$$

Definition 4.2. Let B be a weak Hopf monoid. Let Γ_B and Γ'_B be the idempotent morphisms defined by $\Gamma_B = \Gamma^2_B \circ \Gamma^1_B$ and $\Gamma'_B = \Gamma^4_B \circ \Gamma^3_B$, where Γ^i_B are the idempotent morphisms

$$\Gamma^1_B = ((\varepsilon_B \circ \mu_B \circ c_{B,B}) \otimes B \otimes B) \circ \delta_{B\otimes B} : B \otimes B \to B \otimes B,$$
$$\Gamma^2_B = (B \otimes B \otimes (\varepsilon_B \circ \mu_B)) \circ \delta_{B\otimes B} : B \otimes B \to B \otimes B,$$
$$\Gamma^3_B = (B \otimes B \otimes (\varepsilon_B \circ \mu_B \circ c_{B,B})) \circ \delta_{B\otimes B} : B \otimes B \to B \otimes B,$$
$$\Gamma^4_B = ((\varepsilon_B \circ \mu_B) \otimes B \otimes B) \circ \delta_{B\otimes B} : B \otimes B \to B \otimes B.$$

We will say that B is a coquasitriangular weak Hopf monoid if there exists a morphism $\omega : B \otimes B \to K$ in C satisfying the following conditions:

$$\omega \circ \Gamma_B = \omega, \tag{d1}$$

$$\omega \circ (\mu_B \otimes B) = (\omega \otimes \omega) \circ (B \otimes c_{B,B} \otimes B) \circ (B \otimes B \otimes \delta_B), \tag{d2}$$

$$\omega \circ (B \otimes \mu_B) = (\omega \otimes \omega) \circ (B \otimes c_{B,B} \otimes B) \circ (\delta_B \otimes c_{B,B}), \tag{d3}$$

$$(\omega \otimes \mu_B) \circ \delta_{B\otimes B} = ((\mu_B \circ c_{B,B}) \otimes \omega) \circ \delta_{B\otimes B}. \tag{d4}$$

(d5) There exists a morphism $\overline{\omega} : B \otimes B \to$ such that

(d5.1) $\overline{\omega} \circ \Gamma'_B = \overline{\omega}$,

(d5.2) $\omega * \overline{\omega} = \varepsilon_B \circ \mu_B \circ c_{B,B}$,

(d5.3) $\overline{\omega} * \omega = \varepsilon_B \circ \mu_B$.

We will say that a coquasitriangular weak Hopf monoid B is cotriangular if moreover $\overline{\omega} = \omega \circ c_{B,B}$.

For any coquasitriangular weak Hopf monoid B, we obtain that $\overline{\omega}$ is unique and the following equalities hold (see [5, Lemma 8]):

$$\omega * \overline{\omega} * \omega = \omega, \quad \overline{\omega} * \omega * \overline{\omega} = \overline{\omega},$$
$$\omega \circ (\eta_B \otimes B) = \omega \circ (B \otimes \eta_B) = \varepsilon_B. \tag{26}$$

Example 4.3. There are many interesting examples in the literature of quasitriangular and coquasitriangular weak Hopf monoids. Let $\mathsf{G} = (\mathsf{G}_0, \mathsf{G}_1)$ be a finite groupoid such that its set of arrows G_1 is finite and let R be a commutative ring. Then, the groupoid algebra of G, denoted by $R[\mathsf{G}]$, is an example of triangular weak Hopf monoid in the category R-Mod. Since G_1 is finite, $R[\mathsf{G}]$ is free of a finite rank as an R-module. Hence $R[\mathsf{G}]$ is finite as object in the category R-Mod and $R[\mathsf{G}]^*$ is an example of a cotriangular weak Hopf monoid in R-Mod.

On the other hand, in [7] Andruskiewitsch and Natale proved that it is possible to construct a weak Hopf monoid $\mathbb{K}(\mathsf{G}, \mathsf{H})$ in the symmetric monoidal category of vector spaces over a field $\mathbb{K}$ by working with a matched pair of finite groupoids (G, H). Moreover, in [1] Aguiar and Andruskiewitsch proved the following result: A matched pair of rotations gives rise to a quasitriangular structure for the associated weak Hopf monoid $\mathbb{K}(\mathsf{G}, \mathsf{H})$. Also, by [1, Theorem 5.10] we know that there is an isomorphism of quasitriangular weak Hopf monoids between the Drinfeld double of $\mathbb{K}(\mathsf{G}, \mathsf{H})$ and the weak Hopf monoid of a suitable matched pair of groupoids.

Finally, in [13], for a weak Hopf monoid H in the symmetric monoidal category of vector spaces over an algebraically closed field, Nikshych, Turaev and Vainerman defined the Drinfeld double $D(H)$ of H and proved that $D(H)$ is a quasitriangular weak Hopf monoid (see [13, Proposition 6.2]).

The main result in [5] asserts the following: Let H be a quasitriangular weak Hopf monoid with morphism $\sigma : K \to H \otimes H$ and let B be a coquasitriangular weak Hopf monoid with morphism $\omega : B \otimes B \to K$. There exists a functor

$$\mathsf{L} : {}^{B}_{H}\mathsf{Long} \to {}^{H\otimes B}_{H\otimes B}\mathsf{YD}$$

defined on objects by

$$\mathsf{L}((M, \varphi_M, \rho_M)) = (M, \phi_M, \varrho_M),$$

where

$$\phi_M = \varphi_M \circ (H \otimes (\omega \circ c_{B,B}) \otimes M) \circ (H \otimes B \otimes \rho_M),$$
$$\varrho_M = (H \otimes (\rho_M \circ \varphi_M)) \circ ((c_{H,H} \circ \sigma) \otimes M),$$

and by the identity on morphisms. Moreover, the functor L is injective on objects and, consequently, ${}^{B}_{H}\mathsf{Long}$ can be identified with a subcategory of ${}^{H\otimes B}_{H\otimes B}\mathsf{YD}$. Moreover, by

[5, Lemmas 10–12] we can conclude that the category ${}^{B}_{H}\mathsf{Long}$ is a braided monoidal subcategory of ${}^{H\otimes B}_{H\otimes B}\mathsf{YD}$ if the antipodes of H and B are isomorphisms.

Theorem 4.4. *Let H be a quasitriangular weak Hopf monoid with morphism $\sigma : K \to H \otimes H$ and let B be a coquasitriangular weak Hopf monoid with morphism $\omega : B \otimes B \to K$. Then the functor F introduced in Theorem 3.6 is a retraction of the functor L.*

Proof. Obviously for morphisms there is nothing to prove. For objects we have the following: Let (M, φ_M, ρ_M) be an object in ${}^{B}_{H}\mathsf{Long}$. Then

$$(\mathsf{F} \circ \mathsf{L})((M, \varphi_M, \rho_M)) = (M, \varphi_M, \rho_M)$$

because

$$\mathsf{F}(\mathsf{L}((M, \varphi_M, \rho_M))) = \mathsf{F}((M, \phi_M, \varrho_M)) = (M, \chi_M, \omega_M),$$

where, by (26) and (25), we have that

$$\begin{aligned}\chi_M &= \phi_M \circ (H \otimes \eta_B \otimes M) = \varphi_M \circ (H \otimes (\omega \circ c_{B,B}) \otimes M) \circ (H \otimes \eta_B \otimes \rho_M)\\ &= \varphi_M \circ (H \otimes ((\varepsilon_B \otimes M) \circ \rho_M)) = \varphi_M\end{aligned}$$

and

$$\begin{aligned}\omega_M &= (\varepsilon_H \otimes B \otimes M) \circ \varrho_M = (\varepsilon_H \otimes (\rho_M \circ \varphi_M)) \circ ((c_{H,H} \circ \sigma) \otimes M)\\ &= \rho_M \circ \varphi_M \circ (\eta_H \otimes M) = \rho_M.\end{aligned}$$

■

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