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FINITENESS AND CONSTRUCTIBILITY IN LOCAL ANALYTIC GEOMETRY

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ABSTRACT. Using the Houzel finiteness theorem and the Whitney-Thom stratification theory we show, in local analytic geometry, that relatively constructible sheaves have coherent higher direct images.

INTRODUCTION

In 1953, Cartan-Serre and Schwartz proved that the cohomology spaces of a coherent analytic sheaf on a compact complex analytic manifold are finite-dimensional [7, 37]. This result was extended to the relative case by Grauert in 1960 who showed that the direct images sheaves $R^k f_* \mathcal{F}$, associated to a coherent analytic sheaf $\mathcal{F}$, are coherent provided that the holomorphic map $f: X \to S$ is proper [14]. It was only with the work of Kiehl-Verdier [24] that a proof similar to the absolute one was obtained (see also [9, 12, 28]). The proof was simplified and extended to a wider class of sheaves by Houzel [21]. The aim of this paper is to deduce from Houzel's theorem a practical criterion for the coherence of direct image sheaves, close in spirit to the work in [5, 6, 22, 36, 40]. Our formulation of finiteness theorems is based on the

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Whitney-Thom theory of stratified sets and mappings. One of the key results is the following theorem:

THEOREM 0.1. *Let* $f\colon X \to S$, $X_0 = f^{-1}(0)$, *be a standard representative of a holomorphic map-germ* $f\colon (\mathbf{C}^n, 0) \to (\mathbf{C}^k, 0)$ *satisfying the* a_f*-condition and* $\mathcal{K}^\bullet$ *a complex of coherent sheaves on* X. *If the complex* $\mathcal{K}^\bullet$ *is* f*-constructible then the direct image sheaves* $\mathbf{R}^k f_* \mathcal{K}^\bullet$ *are coherent and the canonical map* $\Gamma(X_0, \mathcal{K}^\bullet) \to K^\bullet := \mathcal{K}_0^\bullet$ *induces an isomorphism of graded* $\mathcal{O}_{S,0}$*-modules between* $(\mathbf{R}^\bullet f_* \mathcal{K}^\bullet)_0$ *and* $H^\bullet(K^\bullet)$.

In the statement of the theorem $\mathcal{K}_0^\bullet = K^\bullet$ is the stalk at the origin of the sheaf $\mathcal{K}^\bullet$. Here f-constructible means fibrewise constructible, a notion that we shall carefully explain in the sequel. The notion of standard representatives and Thom's a_f-condition will be recalled in Section 4.1.

A particular case is when f defines an isolated singularity and the complex is the relative de Rham complex. In particular for hypersurface singularities, i.e. for $k = 1$, we get the Brieskorn-Deligne coherence theorem [5]. The proof of the theorem is indeed similar to that of Brieskorn and Deligne.

The results of this paper might be well known to some specialists, but we think that a paper giving an elementary presentation of the subject together with simple criteria, based on stratification theory, of the abstract theorems might be of some use.

1. THE FINITENESS THEOREM IN THE ABSOLUTE CASE

1.1 STATEMENT OF THE THEOREM

Given a sheaf $\mathcal{F}$ on $\mathbf{C}^n$, we denote by $\mathcal{F}_0$ its stalk at the origin. We denote by $B_r \subset \mathbf{C}^n$ the closed ball of radius r centred at the origin and by $\mathring{B}_r$ its interior. In the absolute case Theorem 0.1 can be stated as follows.

THEOREM 1.1. *For any constructible complex* $\mathcal{K}^\bullet$ *of coherent analytic sheaves in* $B_r \subset \mathbf{C}^n$, *the cohomology spaces* $H^p(K^\bullet)$, $K^\bullet := \mathcal{K}_0^\bullet$, *are finite-dimensional vector spaces, for any* $p \geq 0$. *Moreover, for* $\varepsilon < r$ *small enough, the canonical mapping* $\mathcal{K}^\bullet(B_\varepsilon) \to \mathcal{K}_0^\bullet$ *induces an isomorphism*

$$H^p(K^\bullet) \approx \mathbf{H}^p(B_\varepsilon, \mathcal{K}^\bullet), \quad \forall p \geq 0 .$$

The constructibility of the complex $\mathcal{K}^\bullet$ means that the cohomology sheaves $\mathcal{H}^k(\mathcal{K}^\bullet)$ are locally constant on the stratum of some Whitney stratification, i.e., there exists a sheaf E_U obtained from some vector space E such that

$$\mathcal{H}(\mathcal{K}^\bullet)_{|U} \approx E_U \,,$$

in any sufficiently small open subset U of a stratum. We do not assume, a priori, the vector space E to be finite-dimensional, but it follows from the theorem that it is.

The proof of this theorem is a simple variant of the Cartan-Serre-Schwartz proof for the finiteness of coherent cohomology on a compact complex analytic manifold. Although it is quite elementary, it contains in essence all the ingredients involved in the proofs of more sophisticated results. We first give an example of an application.

1.2 Finiteness of de Rham cohomology of an isolated singularity

Consider the complex $\Omega^\bullet_X$ of Kähler differentials on a Stein complex variety $X \subset \mathbf{C}^n$. For instance, if X is a hypersurface then the terms of the complex are $\Omega^k_X = \Omega^k_{\mathbf{C}^n}/(df \wedge \Omega^{k-1}_{\mathbf{C}^n} + f\Omega^k_{\mathbf{C}^n})$, where f is a generator of the ideal of X, the differential of the complex being induced by the de Rham differential.

The Poincaré lemma states that at the smooth points of X, the complex is a resolution of the constant sheaf $\mathbf{C}_X$, therefore if X has isolated singular points the complex is constructible. Applying Theorem 1.1, we get the following result :

Proposition 1.2. *If $(X,0) \subset (\mathbf{C}^n,0)$ is the germ of a variety with an isolated singular point at the origin then the complex of Kähler differentials has finite-dimensional cohomology spaces.*

If X is a hypersurface then, as conjectured by Brieskorn, these cohomology spaces are all zero, except possibly for $j = 0,\, n-1,\, n$ (see [38]).

For non-isolated singularities, the complex $\Omega^\bullet_X$ might be non-constructible unless we make additional assumptions on the existence of a complex analytic stratification of the variety.

2. Constructible complexes of sheaves

2.1 Statement of the result

We use the notation of Theorem 1.1. The aim of this section is to prove the following result:

PROPOSITION 2.1. *For ε small enough, the restriction mappings* $r\colon \mathcal{K}^\bullet(B_\varepsilon) \to \mathcal{K}^\bullet(B_{\varepsilon'})$, $r'\colon \mathcal{K}^\bullet(B_\varepsilon) \to \mathcal{K}^\bullet(\mathring{B}_\varepsilon)$ *are quasi-isomorphisms for any* $\varepsilon' < \varepsilon$.

The proof of the proposition will use properties of Whitney stratifications, a notion that we will now recall.

2.2 Whitney stratified spaces

A (C^∞) *stratification* of a subset $X \subset \mathbf{R}^n$ is a decomposition of this set into disjoint C^∞ manifolds. A stratification is said to be *locally finite* if every point admits a neighbourhood which intersects finitely many strata. We now define *Whitney stratifications.*

A pair of C^∞ submanifolds $U, V \subset \mathbf{R}^n$, $\dim V < \dim U$, satisfies the *Whitney condition* if the following property holds: for any pair of sequences $(x_i), (y_i)$ in the submanifolds U and V both converging to the same point, such that:

1. the sequence of secants $(x_i y_i)$ converges to a line L, and
2. the sequence of spaces tangent to U at x_i converges to an affine subspace $A \subset \mathbf{R}^n$,

the line L is contained in the affine subspace A.

DEFINITION 2.2. A locally finite stratification $\bigcup_{i=1}^m X_i$ of a subset $X \subset \mathbf{R}^n$ is called a *Whitney stratification* if for any stratum X_j lying on the closure of a stratum X_i the pair (X_i, X_j) satisfies the Whitney condition.

These definitions are due to Whitney [43] (see also [13], Chapter 1). Whitney proved the existence of such a stratification for real semi-analytic sets; constructive proofs were given in [29, 41].

We shall say that two Whitney stratified sets intersect *transversally* if their strata intersect pairwise transversally. We denote by $B_\varepsilon \subset \mathbf{R}^n$ the closed ball centred at the origin of radius ε. A direct consequence of the definition is the following

PROPOSITION 2.3. *Let $X \subset \mathbf{R}^n$ be a Whitney stratified subset, then there exists ε_0 such that the boundaries of the balls B_ε, $\varepsilon < \varepsilon_0$, intersect X transversally.*

Proof. If such an ε_0 did not exist, we could construct a sequence (x_i) lying on a stratum, such that the affine space T_i tangent to the stratum of x_i at the point x_i is also tangent to the boundary of the ball $B_{1/i}$ of radius $1/i$ at the point x_i. In particular the secant $(0x_i)$ is perpendicular to T_i. This contradicts the Whitney condition. □

2.3 INTEGRABLE VECTOR FIELDS

A C^∞ *vector field defined on a Whitney stratified topological space* is given by the collection of C^∞ vector fields on each stratum. It will be called *integrable* if it has a continuous flow. An example, in $\mathbf{R}^2$, of a non-continuous integrable vector field is given by

$$\begin{cases} \frac{x\partial_y - y\partial_x}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$$

for the stratification consisting of the origin and its complement. The flow of this vector field in $\mathbf{R}^2 \setminus \{0\}$ is given, in polar coordinates r, θ, by:

$$\varphi\colon (r,\theta,t) \mapsto \big(r\cos(\theta + \frac{t}{r}),\, r\sin(\theta + \frac{t}{r})\big)$$

and indeed we get that $\lim_{r\to 0} \varphi(r,\theta,t) = (0,0)$.

Another typical example, in $\mathbf{R}^3$, of an integrable vector field is given by

$$\begin{cases} \partial_z + \frac{x\partial_y - y\partial_x}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ \partial_z & \text{otherwise} \end{cases}$$

for the stratification consisting of the z-axis and its complement. This vector field has the following properties:

1. on each stratum its flow is an isometry (for the standard Euclidean metric);
2. the orthogonal projection on the z-axis commutes with the vector field (evaluating the vector field at a point and projecting it is the same as taking the vector field at the projected point on the z-axis).

These two properties – existence of a Riemannian metric for which the flow is an isometry and existence of a projection which commutes with the vector field – imply the existence of a continuous flow [42] (see also [32], Proposition 10.1).

2.4 Thom's first isotopy theorem

We give a variant of Thom's first isotopy theorem which is contained in the proof of the original statement rather than stated as a result on its own (see for instance [32], Propositions 7.1, 9.1 and 10.1).

THEOREM 2.4. *Let X be a Whitney stratified subset and let $f\colon X \to]0,1[$ be a surjective mapping. If the restriction of f to any stratum of X defines a submersion then any vector field on $]0,1[$ lifts to an integrable vector field on*[1] X.

EXAMPLE 2.5. Consider the real algebraic singular surface

$$S = \{(x,y,z) \in \mathbf{R}^3 : xy(x+y)(x-(1+z^2)y) = 0\}\,.$$

The slices of this surface by the planes $\{z = k\}$ consist of four lines. The cross-ratio of these four lines varies with the constant k. Around the singular set the embedded surface S is not locally diffeomorphic to a product embedded in $\mathbf{R}^3$: the differential of such a diffeomorphism at a point $(0,0,z)$ would be a linear mapping which sends four lines to other four lines having possibly a different cross-ratio (independently of the order of these lines). We assert that Thom's theorem implies the existence of a homeomorphism which sends the surface $S \cap \{|z| \leq R\}$ to the product $(S \cap \{z=0\}) \times [-R,R]$ for any $R > 0$. The stratification of $\mathbf{R}^3$ defined by $X_0 = \mathbf{R}^3 \setminus S$, $X_1 = S \setminus \{x=y=0\}$, $X_2 = \{x=y=0\}$ is a Whitney stratification. Consider the projection $\mathbf{R}^3 \to \mathbf{R}$, $(x,y,z) \mapsto z$. According to Theorem 2.4, the vector field ∂_z lifts to an integrable vector field θ on $\mathbf{R}^3$ with the above Whitney stratification. As S is invariant under the maps $(x,y,z) \mapsto (\lambda x, \lambda y, z)$, $\lambda > 0$, the pair $(\mathbf{R}^3, S)$ retracts on a pair $(T, S \cap T)$, where T is a tubular neighbourhood of the z-axis. Put $T_R = T \cap \{|z| \leq R\}$. As T_R is compact, the local flow of the vector field θ gives a global flow on T_R which induces a homeomorphism of the pairs $(T_R, S \cap T_R)$ and $(\mathbf{R}^3, (S \cap \{z=0\}) \times [-R,R])$ for any $R > 0$. This proves the assertion.

COROLLARY 2.6. *Consider a Whitney stratification of the ball $B_{\varepsilon_0} \subset \mathbf{R}^n$ such that the boundaries of the balls B_ε, $\varepsilon < \varepsilon_0$, intersect the strata transversally. For any ε, $\varepsilon' \in]0,\varepsilon_0[$, $\varepsilon' < \varepsilon$, there exists a homeomorphism $h_{\varepsilon,\varepsilon'}\colon B_\varepsilon \to B_{\varepsilon'}$ isotopic to the identity which sends each stratum to itself.*

[1]) It is of course sufficient to check this property for one non-vanishing vector field on the interval $]0,1[$.

Proof. The transversality assumption implies that the map

$$f\colon B_{\varepsilon_0}\setminus\{0\}\to\,]0,1[$$

obtained by restricting the Euclidean norm on $\mathbf{R}^n$ divided by ε_0 satisfies the conditions of the above theorem. $\square$

2.5 Proof of Proposition 2.1

We consider only the case of the mapping $r\colon \mathcal{K}^\bullet(B_\varepsilon)\to\mathcal{K}^\bullet(B_{\varepsilon'})$, the other case being quite similar.

We apply the considerations of the previous subsection to $\mathbf{C}^n\approx\mathbf{R}^{2n}$ with the stratification given by a constructible complex of coherent analytic sheaves $\mathcal{K}^\bullet$ defined in a neighbourhood $U\subset\mathbf{C}^n$ of the origin. According to Proposition 2.3, we can find a ball $B_{\varepsilon_0}\subset U$ such that all strata in U are transverse to the boundary of the ball B_ε for any $\varepsilon<\varepsilon_0$.

By Corollary 2.6, there exists a homeomorphism $\varphi\colon B_\varepsilon\to B_{\varepsilon'}$, $\varepsilon'<\varepsilon$, which is isotopic to the identity and preserves the stratification.

Choose an acyclic covering $U=(U_i)$ of B_ε; its image $U'=(U_i')$, $U_i'=\varphi(U_i)$, is an acyclic covering of $B_{\varepsilon'}$. As the cohomology sheaves of $\mathcal{K}^\bullet$ are locally constant on the strata and φ is isotopic to the identity, we have vector space isomorphisms $\mathcal{H}^q(\mathcal{K}^\bullet)(U_i)\approx\mathcal{H}^q(\mathcal{K}^\bullet)(U_i')$ for each i.

Consider the spectral sequences $E_0^{p,q}(B_\varepsilon)=\mathcal{C}^p(U,\mathcal{K}^q)$ and $E_0^{p,q}(B_{\varepsilon'})=\mathcal{C}^p(U',\mathcal{K}^q)$ for the hypercohomology of the complex $\mathcal{K}^\bullet$. Here, as usual, $\mathcal{C}^\bullet(\,\cdot\,)$ stands for the Čech resolution.

We have a vector space isomorphism $\mathcal{H}^q(\mathcal{K}^\bullet)(U_i)\approx\mathcal{H}^q(\mathcal{K}^\bullet)(U_i')$ on each small open subset U_i. Therefore the restriction mapping induces an isomorphism between the E_1-terms of the hypercohomology spectral sequences:

$$E_1^{p,q}(B_\varepsilon)=\mathcal{C}^p(U,\mathcal{H}^q(\mathcal{K}^\bullet))\approx\mathcal{C}^p(U',\mathcal{H}^q(\mathcal{K}^\bullet))=E_1^{p,q}(B_{\varepsilon'})\,.$$

This shows that the hypercohomology spaces $\mathbf{H}^\bullet(B_\varepsilon,\mathcal{K}^\bullet)$ and $\mathbf{H}^\bullet(B_{\varepsilon'},\mathcal{K}^\bullet)$ are isomorphic.

As B_ε is Stein, Cartan's Theorem B implies that the cohomology sheaves $\mathcal{H}^q(\mathcal{K}^\bullet)$ vanish for $q>0$. Therefore, the spectral sequence degenerates and we get the isomorphisms:

$$\mathbf{H}^p(B_\varepsilon,\mathcal{K}^\bullet)\approx H^p(\mathcal{K}^\bullet(B_\varepsilon))\,,\quad \mathbf{H}^p(B_{\varepsilon'},\mathcal{K}^\bullet)\approx H^p(\mathcal{K}^\bullet(B_{\varepsilon'}))\,.$$

This shows that the restriction mapping r is a quasi-isomorphism and concludes the proof of the proposition. $\square$

3. Riesz theory for nuclear Fréchet morphisms

We now come to the functional analytic argument of the proof: Proposition 2.1 and the fact that r is nuclear imply the finite-dimensionality of the cohomology. Thus we will now explain what nuclearity means and why it implies the finiteness of the cohomology. First we recall some basic notions of functional analysis.

3.1 The category of Fréchet spaces

We consider only vector spaces over the field of complex numbers.

A topological vector space E is called *locally convex* if its topology is generated by a set of continuous semi-norms $(p_n),\ n \in \Omega$, that is, the subsets $V_{n,\varepsilon} = \{x \in E : p_n(x) < \varepsilon\}$ form a fundamental system of 0-neighbourhoods. The morphisms of the category of locally convex vector spaces are the continuous linear mappings.

A locally convex topological vector space E is called a *Fréchet space* (or an F-*space*) if it is complete and if the topology of E can be generated by a countable set of semi-norms. Fréchet spaces form a subcategory of the category of locally convex spaces. These definitions are of course standard [2].

EXAMPLE 3.1. Consider the vector space $\mathcal{O}_{\mathbf{C}}(D)$ of holomorphic functions on the open disk $D \subset \mathbf{C}$. Each compact subset $K \subset D$ defines a semi-norm $p_K(f) = \sup_{x \in K} |f(x)|$, which is, in fact, a norm if K has a non empty interior. The topology is generated by a countable set of semi-norms constructed as follows. Consider the sequence (K_n) of closed disks of radius $1 - 1/n$ centred at the origin. The set of norms $\{p_{K_n},\ n \in \mathbf{N}\}$ induces the same topology as the set of semi-norms $\{p_K, K \text{ compact}\}$. The Cauchy formula implies that this topology is complete, thus these semi-norms induce a Fréchet space structure on the vector space $\mathcal{O}_{\mathbf{C}}(D)$. The supremum norm on compact subsets induces, in a similar way, a Fréchet space structure on the algebra of holomorphic functions on an open subset of $\mathbf{C}^n$.

CONVENTION. In the sequel, we will always endow the algebra of holomorphic functions on an open subset of $\mathbf{C}^n$ with the above mentioned Fréchet space structure.

3.2 The Mackey property

A subset of a locally convex topological vector space is called *bounded* if all semi-norms are bounded on it.

Definition 3.2 ([30, 11]). A sequence (x_n) in a locally convex space E *converges to zero in the sense of Mackey* if there exists a bounded subset $B \subset E$ such that for any $\varepsilon > 0$ there exists N with $x_n \in \varepsilon B$ provided that $n \geq N$.

Proposition 3.3. *Let (x_n) be a sequence in a Fréchet space E. The following conditions are equivalent:*

1. (x_n) *converges to zero in E;*
2. (x_n) *converges to zero in E in the sense of Mackey.*

Proof. Let us show that (1) $\implies$ (2). Assume that the sequence (x_n) converges to zero in E. As (x_n) is bounded, we can choose a sequence of increasing semi-norms $p_1, \dots, p_k, \dots$, defining the topology of E such that $p_k(x_n) \leq 1$ for all $k, n \in \mathbf{N}$. Denote by B_k the unit ball for the semi-norm p_k. The subset $B = \bigcap_k kB_k$ is bounded and for any $m > 0$, we can find N such that $x_n \in 1/mB, \forall n \geq N$. To see this, choose N such that

$$p_m(x_n) < \frac{1}{m}, \quad \forall n \geq N.$$

As the sequence (p_k) increases, we also have $p_k(x_n) < \frac{1}{m}$ for $k < m$ and $n \geq N$. As $x_n \in B_k$ and $B_k \subset 1/m(kB_k)$ for $m \leq k$, we conclude that $x_n \in 1/mB$, for all $n \geq N$.

The implication (2) $\implies$ (1) is in fact independent of the assumption that the topology is Fréchet. We have to show that, for any semi-norm p of E, we have $\lim_n p(x_n) = 0$. As B is bounded, the quantity $\alpha = \sup_{y \in B} p(y)$ is finite. That (x_n) converges to zero in the sense of Mackey means that there exists a sequence (ε_n), $\varepsilon_n \in \mathbf{R}_{\geq 0}$ which converges to zero and such that $x_n \in \varepsilon_n B$. We get that $p(x_n) \leq \varepsilon_n \alpha$ and the left-hand side tends to zero as n goes to infinity. □

Definition 3.4. A locally convex space E for which the notions of Mackey convergence and usual convergence agree is said to have the *Mackey property*.

Proposition 3.3 asserts that Fréchet spaces have the Mackey property.

3.3 TOPOLOGICAL TENSOR PRODUCTS

DEFINITION 3.5. The *topological tensor product* of two locally convex spaces E,F, denoted by $E \mathbin{\widehat{\otimes}} F$, consists in the set of expressions of the type $v = \sum_i \lambda_i x_i \otimes y_i$, where the sequences (x_i) and (y_i) are bounded and $\sum_i |\lambda_i| < \infty$.

PROPOSITION 3.6. *If E, F are locally convex vector spaces and if F has the Mackey property then any element of the topological tensor product $E \mathbin{\widehat{\otimes}} F$ can be written as $v = \sum \alpha_i x_i \otimes y_i$, where (x_i) is bounded, (y_i) converges to zero, and $\sum_i |\alpha_i| < \infty$*

Proof. Take $v = \sum \lambda_i x_i \otimes y_i \in E \mathbin{\widehat{\otimes}} F$ and write $\lambda_i = a_i b_i$, where a_i is summable and b_i tends to zero. We have

$$v = \sum_{i \geq 0} a_i x_i \otimes b_i y_i \,,$$

where $(b_i y_i)$ converges to zero in the sense of Mackey and therefore, by Proposition 3.3, converges to zero in F. $\square$

THEOREM 3.7 ([16, 17]). *The topological tensor product of two Fréchet spaces is complete and Hausdorff, thus it is also a Fréchet space.*

EXAMPLE 3.8. Let $U \subset \mathbf{C}^n$, $V \subset \mathbf{C}^k$ be two open subsets; then both topological vector spaces $\mathcal{O}_{\mathbf{C}^n}(U) \mathbin{\widehat{\otimes}} \mathcal{O}_{\mathbf{C}^k}(V)$ and $\mathcal{O}_{\mathbf{C}^{n+k}}(U \times V)$ are completions of the space of polynomials $\mathbf{C}[z_1, \dots, z_{n+k}]$ and induce the same topology on it; hence they are isomorphic.

The *strong dual* E' of a locally convex topological vector space E is the topological dual together with the topology induced by the semi-norms

$$p_B(u) = \sup_{x \in B} |u(x)| \,,$$

where B runs over the bounded subsets of E. For instance if E is Banach, this is the topological dual with the operator-norm topology. In view of the definition of nuclear morphisms, we recall the following result:

THEOREM 3.9 ([16, 17]). *The topological tensor product of the strong dual of a Fréchet space with a Fréchet space is complete and Hausdorff.*

3.4 Nuclear morphisms

DEFINITION 3.10. A morphism $u\colon E \to F$ of Fréchet spaces is called *nuclear* if it lies in the image of the morphism

$$E' \mathbin{\widehat{\otimes}} F \to L(E,F)\,, \quad \sum \lambda_i \xi_i \otimes y_i \mapsto [x \mapsto \sum \lambda_i \xi_i(x) y_i]\,.$$

If E, F are finite-dimensional then all linear mappings are nuclear. This is of course no longer the case in general Fréchet spaces: nuclear morphisms are limits of finite range mappings and are therefore compact.

EXAMPLE 3.11. Take $E = \mathcal{O}_{\mathbf{C}}(D)$, $F = \mathcal{O}_{\mathbf{C}}(D')$, where D, D' are open disks centred at the origin such that the radius r of the disk $D' \subset \mathbf{C}$ is strictly smaller than the radius R of D.

The restriction mapping $\rho\colon \mathcal{O}_{\mathbf{C}}(D) \to \mathcal{O}_{\mathbf{C}}(D')$ is nuclear. To see this, define the linear forms a_n

$$a_n\colon \mathcal{O}_{\mathbf{C}}(D) \to \mathbf{C}\,, \quad f \mapsto \frac{1}{2i\pi} \int_\gamma \frac{f(z)}{z^{n+1}} dz\,,$$

where γ is a path in $D \setminus D'$ which turns counterclockwise around the origin. For any holomorphic function $f \in \mathcal{O}_{\mathbf{C}}(D)$, the Hadamard criterion states that

$$\overline{\lim}\, |a_n(f)|^{-1/n} \geq R\,.$$

Therefore the sequence $(k^n a_n(f))$ is bounded for any $0 < k < R$. The Banach-Steinhaus theorem implies that the sequence of linear maps $(k^n a_n)$ is bounded for the strong topology. Choose $k \in]r, R[$; the equality

$$\rho = \sum_{n \geq 0} \lambda^n (k^n a_n) \otimes \frac{z^n}{r^n}\,, \quad \lambda = \frac{r}{k}$$

shows that ρ is a nuclear morphism. This result extends to arbitrary Stein neighbourhoods [24] (see also [23]).

Consider a coherent analytic sheaf $\mathcal{F}$ on $\mathbf{C}^n$ and take a presentation of this sheaf

$$\mathcal{O}_{\mathbf{C}^n}^{n_1} \to \mathcal{O}_{\mathbf{C}^n}^{n_0} \to \mathcal{F} \to 0\,.$$

This exact sequence induces a Fréchet structure on the vector space $\mathcal{F}(U)$ for any Stein neighbourhood $U \subset \mathbf{C}^n$. This structure is independent of the choice of the presentation [7] (see also [10], Proposition 4).

The proof of the following proposition is a generalisation of our previous example.

PROPOSITION 3.12 ([7, 24]). *For any coherent analytic sheaf* $\mathcal{F}$ *on* $\mathbf{C}^n$ *and any Stein neighbourhoods* U, U' *such that the closure of* U' *is a compact subset of* U, *the restriction mapping* $\mathcal{F}(U) \to \mathcal{F}(U')$ *is nuclear.*

3.5 NUCLEAR SPACES

DEFINITION 3.13. A locally convex vector space E is called *nuclear* if any morphism from E to a Banach space is nuclear.

THEOREM 3.14 ([16, 17]). *The Fréchet space of holomorphic functions on a polydisk is a nuclear space.*

Consequently one may argue directly that the restriction mappings of Example 3.11 and of Proposition 3.12 are nuclear because they factorise by a morphism to a Banach space.

The following theorem is a consequence of Grothendieck's characterisation of nuclear spaces, namely the coincidence [2]) of the inductive topological tensor product with the projective one (see also [24]).

THEOREM 3.15. *For any nuclear Fréchet space* E, *the functor* $\widehat{\otimes}_{\mathbf{C}} E$ *is an exact functor from the category of Fréchet spaces to itself.*

3.6 THE SCHWARTZ PERTURBATION THEOREM

The following theorem was proved by Schwartz for the more general case of compact operators.

THEOREM 3.16 ([37]). *Let* $f\colon E \to F$ *be a surjective morphism of Fréchet spaces. For any nuclear morphism* $u\colon E \to F$, *the morphism* $f + u$ *has a finite-dimensional cokernel.*

In particular, if the identity mapping $I\colon E \to E$ is nuclear, then by taking $f = I$ and $u = -I$ we get that E is finite-dimensional. As nuclear morphisms are compact, this last assertion is a particular case of Riesz's theorem on the non-compactness of 0-neighbourhoods in infinite-dimensional Fréchet spaces.

[2]) In Grothendieck's original treatment, the coincidence of topologies is taken as the definition of nuclear spaces, while Definition 3.13 is stated as a theorem.

COROLLARY 3.17 ([37]). *Let $M^\bullet, N^\bullet$ be two complexes of Fréchet spaces. If there exists a nuclear quasi-isomorphism $u\colon M^\bullet \to N^\bullet$, then the complexes have finite-dimensional cohomology.*

Proof. We apply Theorem 3.16 to the maps
$$f^k\colon M^k \times N^{k-1} \to N^k\,, \quad (\alpha,\beta) \mapsto u(\alpha) + d\beta\,. \qquad \square$$

We prove Theorem 3.16 following Kiehl-Verdier and Houzel. The next result was proved by Houzel under much more general assumptions; note that for Banach spaces the proof of the lemma is obvious.

LEMMA 3.18 ([21]). *Any nuclear morphism $u\colon E \to E$ of a Fréchet space to itself can be written as $u = u' + u''$ where u' is of finite range, i.e.* $\dim \operatorname{Im} u' < +\infty$, *and $I + u''$ is invertible, where $I\colon E \to E$ denotes the identity mapping.*

Proof. Write $u = \sum_{i>0} \lambda_i \xi_i \otimes x_i$. As the sequences (ξ_i) and (x_i) are bounded there exists $M \in \mathbf{R}$ such that
$$|\xi_i(x_j)| < M\,, \quad \forall\, i, j\,.$$
As the sequence (λ_i) converges to zero, we can find N such that for $i > N$ we have
$$|\lambda_i| < \frac{1}{2M}\,.$$
We assert that the sequence $\big((u'')^k\big)$ consisting of iterates of the nuclear morphism $u'' = \sum_{i>N} \lambda_i \xi_i \otimes x_i$ is bounded. Indeed, a direct computation shows that the k-th iterate of the map u'' is given by the formula
$$(u'')^k = \sum_{i_1,\dots,i_k>0} \lambda_{i_1}\dots\lambda_{i_k} \xi_{i_2}(x_{i_1})\dots\xi_{i_k}(x_{i_{k-1}})\xi_{i_1} \otimes x_{i_k}\,.$$
Thus, for any semi-norm $p\colon E \to \mathbf{R}$, we have
$$|(u'')^k(x)| \le \sup_{i,j} |\xi_i(x)\, p(x_j)| \,\Big(\frac{1}{2}\Big)^{k-1} \sum_{i>0} |\lambda_i|\,.$$
This proves the assertion. The above inequality shows that the sequence
$$v_n = \sum_{k=0}^{n} (u'')^k$$
is pointwise convergent, the Banach-Steinhaus theorem implies that it is uniformly convergent and therefore its limit defines the inverse of $I + u''$. This concludes the proof of the lemma. $\square$

This lemma implies the Schwartz perturbation theorem in case f is the identity mapping. Indeed, take u, u', u'' as in the lemma; since the map $I+u''$ is invertible, the map $I+u = (I+u'')+u'$ has a cokernel of finite dimension.

Now consider a surjective morphism $f\colon E \to F$ of Fréchet spaces. We assert that any nuclear morphism u factors through f by a nuclear morphism v:

$$\begin{tikzcd} E \arrow[d, dotted, "v"'] \arrow[dr, "u"] & \\ E \arrow[r, "f"] & F \end{tikzcd}$$

As u is nuclear, we can write

$$u = \sum_{i>0} \lambda_i \xi_i \otimes y_i \,,$$

where (ξ_i) and (y_i) are bounded sequences and λ_i is summable. As F is Fréchet, by Proposition 3.6, we may assume that (y_i) converges to zero. The Banach open mapping theorem implies that (y_i) lifts to a sequence (x_i) which tends to zero. We define the nuclear morphism v by the formula

$$v = \sum_{i>0} \lambda_i \xi_i \otimes x_i \,.$$

The map $I + v$, and consequently the map $f \circ (I + v) = f + u$, have a finite-dimensional cokernel. This completes the proof of Theorem 3.16. $\square$

3.7 Proof of Theorem 1.1

As the restriction mapping $\mathcal{K}^\bullet(\mathring{B}_\varepsilon) \to \mathcal{K}^\bullet(\mathring{B}_{\varepsilon'})$ is a nuclear quasi-isomorphism (Propositions 2.1 and 3.12), Corollary 3.17 applies. This shows that the cohomology spaces of the complex $\mathcal{K}^\bullet(\mathring{B}_\varepsilon)$ are finite-dimensional vector spaces or equivalently that it is quasi-isomorphic to a complex $\mathcal{L}^\bullet$ of finite-dimensional constant sheaves $\mathcal{L}^k \approx \mathbf{C}^{n_k}$.

As $\mathcal{K}^\bullet(\mathring{B}_\varepsilon)$ is quasi-isomorphic to $\mathcal{K}^\bullet(\mathring{B}_{\varepsilon'})$, we can construct an exhaustive sequence of compact neighbourhoods of the origin (B_{ε_n}) such that $\mathcal{K}^\bullet(\mathring{B}_{\varepsilon_n})$ is quasi-isomorphic to $\mathcal{K}^\bullet(\mathring{B}_{\varepsilon_{n+1}})$. In the limit $n \to \infty$, we get that the complex $K^\bullet = \mathcal{K}^\bullet_0$ is quasi-isomorphic to the stalk of the complex $\mathcal{L}^\bullet$ at the origin. We have isomorphisms of vector spaces

$$\mathbf{H}^p(B_\varepsilon, \mathcal{K}^\bullet) \approx H^p(\mathcal{K}^\bullet(B_\varepsilon)) \approx H^p(K^\bullet)\,, \quad \forall p \geq 0\,,$$

where the first isomorphism follows from Cartan's Theorem B. This concludes the proof of Theorem 1.1. $\square$

4. Relatively constructible sheaves

We now explain the notion of f-constructibility introduced in Theorem 0.1.

4.1 Stratified mappings

Definition 4.1. A continuous map between Whitney stratified spaces $f\colon X \to S$, $X = \bigcup_j X_j$, $S = \bigcup_j S_j$ is said to be *stratified* if it maps stratum into stratum, and if the restriction of f to each stratum is a submersion.

Definition 4.2. A stratified map $f\colon X \to S$, $X \subset \mathbf{R}^k$, satisfies the *a_f-condition* if for any sequence of points (x_i) in a stratum X_j converging to a point x in an adjacent stratum $X_{j'}$, for which the affine subspaces $\ker df_{|X_j}(x_i)$ converge to a limit $A \subset \mathbf{R}^k$, we have the inclusion $\ker df_{|X_{j'}}(x) \subset A$.

A map-germ satisfies the *a_f-condition* if it admits a stratified representative satisfying the a_f-condition.

These definitions are due to Thom [42] (see also [32]).

Definition 4.3 ([27]). A *standard representative* $f\colon X \to S$ of a holomorphic map-germ $f\colon (\mathbf{C}^n, 0) \to (\mathbf{C}^k, 0)$ satisfying the a_f-condition is a representative obtained as follows. Let $g\colon B_r \to T$, $T = g(B_r)$, be a holomorphic representative of the germ such that

1. the fibre $g^{-1}(0)$ is transverse to the boundaries of the balls B_ε for any $\varepsilon < r$;
2. the fibres of g intersect transversally the boundary of some ball B_ε above the closure $\bar{S}$ of a polydisk $S \subset T$ centred at the origin;
3. the map g satisfies the a_f-condition.

The standard representative $f\colon X \to S$ is obtained by restricting g to $X = g^{-1}(S) \cap \mathring{B}_\varepsilon$.

Remark that by a misuse of notation the same letter denotes the germ and a standard representative of it. We sometimes write $f\colon X \to S$, $X = Y \cap \mathring{B}_\varepsilon$ when we want to emphasize the radius of the ball used to define X.

4.2 Thom's second isotopy theorem

Thom's second isotopy theorem is a relative version of Thom's first isotopy theorem.

THEOREM 4.4 ([42, 32]). *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr] &]0,1[\times S \arrow[d, "\pi"] \\ &]0,1[\ , \end{tikzcd}$$

where f is a stratified mapping and π denotes projection to the first factor. If f satisfies Thom's a_f-condition then any C^∞ vector field on $]0,1[$ lifts to an integrable vector field on X tangent to the fibres of f.

For a proof of this theorem see [32], Propositions 11.3, 11.5 and 11.6. As for the first isotopy theorem, one usually states the theorem as a statement of local triviality induced by the flow of the vector field. We get the following relative variant of Corollary 2.6.

COROLLARY 4.5. *Let $f\colon X \to S$, $X = Y \cap \mathring{B}_\varepsilon$ be a standard representative of a germ. For any $\varepsilon' \in]0,\varepsilon[$ there exists a homeomorphism $h_{\varepsilon,\varepsilon'}$ isotopic to the identity which preserves the stratification and such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[rr, "h_{\varepsilon,\varepsilon'}"] \arrow[dr, "f"'] & & X \cap B_{\varepsilon'} \arrow[dl, "f_{|B_{\varepsilon'}}"] \\ & S & \end{tikzcd}$$

4.3 RELATIVE CONSTRUCTIBILITY

DEFINITION 4.6. Consider a stratified map $f\colon X \to S$ satisfying the a_f-condition. A sheaf $\mathcal{F}$ on X is called *f-constructible* if the following condition holds: each point $x \in X$ admits a neighbourhood U inside the stratum of x such that

$$\mathcal{F}_{|U} \approx f^{-1}(f_{|U})_*\mathcal{F}\,.$$

If f is the map to a point, an f-constructible sheaf is a constructible sheaf in the usual sense.

DEFINITION 4.7. Consider a stratified holomorphic mapping $f\colon X \to S$, $X \subset \mathbf{C}^n$, $S \subset \mathbf{C}^k$, satisfying the a_f-condition. A complex $(\mathcal{K}^\bullet,\delta)$ of $\mathcal{O}_X$-coherent sheaves is called *f-constructible* if its cohomology sheaves $\mathcal{H}^k(\mathcal{K}^\bullet)$ are f-constructible and if its differential is $f^{-1}\mathcal{O}_S$-linear.

The notion of f-constructibility extends to germs: given a holomorphic map-germ satisfying the a_f-condition $f\colon (X,0) \to (S,0)$, a complex $(K^\bullet, \delta)$ of $\mathcal{O}_{\mathbf{C}^n,0}$-coherent modules is called *f-constructible* if there exists a standard representative $f\colon X \to S$ and a complex $(\mathcal{K}^\bullet, \delta)$ of f-constructible $\mathcal{O}_X$-coherent sheaves such that $K^\bullet$ is the stalk at the origin of the complex $\mathcal{K}^\bullet$.

The notions of Theorem 0.1 have now been explained.

4.4 The relative de Rham complex

We give a simple example of relative constructibility: the relative de Rham complex for an isolated singularity.

We consider the relative de Rham complex $\Omega_f^\bullet$ associated to a holomorphic map $f\colon X \to S$, $X \subset \mathbf{C}^n$, $S \subset \mathbf{C}^k$ and assume that S is a smooth complex manifold.

For instance if $k = 1$ then the complex has terms $\Omega_f^j = \Omega_X^j/\Omega_X^{j-1} \wedge df$ and the differential is induced by the de Rham differential of $\Omega_X^\bullet$. The differential of the complex is obviously $f^{-1}\mathcal{O}_S$-linear:

$$\pi(d(f\alpha)) = \pi(df \wedge \alpha + fd\alpha) = \pi(fd\alpha)\,,$$

where $\pi\colon \Omega_X^\bullet \to \Omega_f^\bullet$ denotes the canonical projection. A flat holomorphic map-germ $f\colon (X,0) \to (\mathbf{C}^k,0)$ defines an *isolated singularity* if its special fibre has an isolated singular point at the origin and if it satisfies the a_f-condition.

Proposition 4.8. *The relative de Rham complex $\Omega_f^\bullet$ associated to a holomorphic map-germ $f\colon (X,0) \to (\mathbf{C}^k,0)$ defining an isolated singularity is f-constructible.*

Proof. Take a flat standard representative $f\colon X \to S$ of the germ. Consider the stratification consisting of the smooth points of the map $f\colon X \to S$ (smooth points of the fibres) and its complement. Stratify the map $f\colon X \to S$ by refining this stratification. At a regular point of f, the implicit function theorem shows that the relative de Rham complex is a resolution of the sheaf $f^{-1}\mathcal{O}_S$; it is therefore f-constant on the open strata (this statement is known as the *relative Poincaré lemma*).

As f is flat, the singular set of a fibre is either empty or finite. Hence the cohomology sheaves restricted to any non-open stratum are also f-constant (any sheaf restricted to a point is constant). Therefore the relative de Rham complex is f-constructible. $\square$

Achevé de composer le 5 juin 2009 à 10 : 22

For non isolated singularities the situation is of course much more delicate as the de Rham complex is not always constructible. Additional conditions implying constructibility are given in [1, 34, 40].

From Theorem 0.1 and Proposition 4.8, one deduces the following result.

THEOREM 4.9. *Let $f\colon X \to S$, $X_0 = f^{-1}(0)$, be a flat standard representative of a holomorphic map-germ $f\colon (\mathbf{C}^n, 0) \to (\mathbf{C}^k, 0)$. If f defines an isolated singularity then the direct image sheaves $\mathbf{R}^k f_* \Omega_f^\bullet$ are coherent and the canonical map $\Gamma(X_0, \Omega_f^\bullet) \to \Omega_{f,0}^\bullet$ induces an isomorphism of graded $\mathcal{O}_{S,0}$-modules between $(\mathbf{R}^\bullet f_* \Omega_f^\bullet)_0$ and $H^\bullet(\Omega_{f,0}^\bullet)$.*

If X is smooth and if the components of f define a complete intersection, these modules are all zero except possibly for $p = 0, \dim X - 1$ [5, 15, 31].

5. PROOF OF THEOREM 0.1

5.1 CONSTRUCTION OF THE CONTRACTION

PROPOSITION 5.1. *For any standard representative $f\colon X \to S$ and any $\varepsilon' < \varepsilon$, the restriction mapping $r\colon \mathcal{K}^\bullet(X) \to \mathcal{K}^\bullet(X')$, $X' = f^{-1}(S) \cap \mathring{B}_{\varepsilon'}$, is a quasi-isomorphism.*

Proof. The proof is similar to that of Proposition 1.7.

Choose an acyclic covering $U = (U_i)$ of X, find $h_{\varepsilon,\varepsilon'}$ as in Corollary 4.5 and put $U_i' = h_{\varepsilon,\varepsilon'}(U_i)$.

As the complex of sheaves is f-constructible, we get vector space isomorphisms

$$\mathcal{H}^q(\mathcal{K}^\bullet)(U_i) \approx (f_{|U_i})_* \mathcal{H}^q(\mathcal{K}^\bullet) f(U_i) \approx \mathcal{H}^q(\mathcal{K}^\bullet)(U_i')$$

on each small open subset U_i. Consequently, the corresponding hypercohomology spectral sequences show that the restriction mapping $r\colon \mathcal{K}^\bullet(X) \to \mathcal{K}^\bullet(X')$ is a quasi-isomorphism for any $\varepsilon' < \varepsilon$. □

5.2 FRÉCHET MODULES AND NUCLEAR MORPHISMS

An associative algebra A is called a *Fréchet algebra* if there exist seminorms $p_n\colon A \to \mathbf{R}$, $n \in \mathbf{N}$ defining a Fréchet space topology on A and such that

$$p_n(ab) \le p_n(a)\, p_n(b)\,, \quad \forall\, n, a, b\,.$$

For instance, the algebra of holomorphic functions on an open subset of $\mathbf{C}^n$ is a Fréchet algebra.

Here and in the sequel, we will use the word *module* over a ring A instead of *bimodule*, i.e., all the modules that we will consider have left and right multiplications by elements of A, and of course if A is commutative, left and right multiplications coincide.

A module E over a Fréchet algebra A will be called a *Fréchet module* if E has a Fréchet space topology for which left and right multiplication mappings are continuous.

The morphisms of the category of Fréchet A-modules are the continuous $\mathbf{C}$-linear mappings which are left A-linear. The space of morphisms from a Fréchet A-module E to a Fréchet A-module F is denoted by $L_A(E,F)$. We use the notation $E^\vee := L_A(E,\mathbf{C})$.

The *topological tensor product* $E \mathbin{\widehat{\otimes}_A} F$ is defined as the cokernel of the map

$$(E \mathbin{\widehat{\otimes}_{\mathbf{C}}} A \mathbin{\widehat{\otimes}_{\mathbf{C}}} F) \to (E \mathbin{\widehat{\otimes}_{\mathbf{C}}} F)\,, \quad m \otimes a \otimes n \mapsto ma \otimes n - m \otimes an\,.$$

A morphism of Fréchet A-modules $u\colon E \to F$ is said to be A-*nuclear* if it lies in the image of the morphism

$$E^\vee \mathbin{\widehat{\otimes}_A} F \to L_A(E,F)\,, \quad \sum \lambda_i \xi_i \otimes y_i \mapsto [x \mapsto \sum \lambda_i \xi_i(x) y_i]\,, \quad \lambda_i \in \mathbf{C}\,,$$

where λ_i is summable. These definitions are due to Kiehl-Verdier [24].

EXAMPLE 5.2. Take $A = \mathcal{O}_{\mathbf{C}}(S)$, $E = \mathcal{O}_{\mathbf{C}^2}(D \times S)$, $F = \mathcal{O}_{\mathbf{C}^2}(D' \times S)$ where D, D', S are disks centred at the origin in $\mathbf{C}$ such that the respective radii R, r of the disks $D, D' \subset \mathbf{C}$ satisfy $R > r$.

The restriction mapping $\rho\colon \mathcal{O}_{\mathbf{C}^2}(D \times S) \to \mathcal{O}_{\mathbf{C}^2}(D' \times S)$ is an $\mathcal{O}(S)$-nuclear morphism. Indeed, let us define the $\mathcal{O}(S)$-linear forms a_n

$$a_n\colon \mathcal{O}_{\mathbf{C}^2}(D \times S) \to \mathcal{O}_{\mathbf{C}}(S)\,, \quad f \mapsto \frac{1}{2i\pi} \int_\gamma \frac{f(z)}{z^{n+1}} dz\,,$$

where γ is a path in $D \setminus D'$ turning counter-clockwise around the origin. Choose $k \in \,]r, R[$; the equality

$$\rho = \sum_{n \geq 0} \lambda^n (k^n a_n) \otimes \frac{z^n}{r^n}\,, \quad \lambda = \frac{r}{k}$$

shows that the mapping ρ is $\mathcal{O}(S)$-nuclear.

The $\mathcal{O}(S)$-nuclearity of the morphism ρ can also be seen directly by using the isomorphism

$$\mathcal{O}_{\mathbf{C}^2}(D \times S) \approx \mathcal{O}_{\mathbf{C}}(D) \,\widehat{\otimes}\, \mathcal{O}_{\mathbf{C}}(S)\,.$$

We saw that the restriction mapping $\mathcal{O}_{\mathbf{C}}(D) \to \mathcal{O}_{\mathbf{C}}(D')$ is $\mathbf{C}$-nuclear, therefore by tensoring both sides by $\mathcal{O}_{\mathbf{C}}(S)$, we get an $\mathcal{O}_{\mathbf{C}}(S)$-nuclear morphism.

5.3 The Schwartz perturbation theorem for Fréchet modules

The formulation given by Houzel of the generalised Schwartz perturbation theorem involves vector spaces with bornologies rather than topological vector spaces; here and in the sequel we apply the theorem for the bornology consisting of bounded subsets of locally convex spaces.

THEOREM 5.3 ([21]). *Let $f\colon E \to F$ be a surjective morphism between A-Fréchet modules. For any A-nuclear morphism $u\colon E \to F$, the cokernel of the map $f+u$ is an A-module of finite type.*

The proof of this theorem is similar to the one we gave in the absolute case and will therefore be omitted; for details we refer to [21]. In case the identity mapping is nuclear, then by taking $f=I$ and $u=-I$ we get the following corollary.

COROLLARY 5.4. *If the identity mapping $I\colon E \to E$ is A-nuclear then E is a finite type A-module*

EXAMPLE 5.5. Let $S \subset \mathbf{C}^k$ be an open subset and take an $\mathcal{O}_{\mathbf{C}^k}(S)$-linear mapping $\varphi\colon \mathcal{O}^p_{\mathbf{C}^k}(S) \to \mathcal{O}^q_{\mathbf{C}^k}(S)$. Such a mapping is given by a $p \times q$ matrix with entries in $\mathcal{O}_{\mathbf{C}^k}(S)$. The identity mapping is an $\mathcal{O}_{\mathbf{C}^k}(S)$-nuclear morphism from $\ker\varphi$ to itself. Therefore $\ker\varphi$ is an $\mathcal{O}_{\mathbf{C}^k}(S)$-module of finite type.

As in the absolute case, the perturbation theorem implies the following result.

THEOREM 5.6 ([24, 21]). *Let A be a Fréchet algebra and $M^\bullet, N^\bullet$ two complexes of A-Fréchet modules. If there exists a nuclear quasi-isomorphism $u\colon N^\bullet \to M^\bullet$ then the complexes $M^\bullet$ and $N^\bullet$ are quasi-isomorphic to a complex of finite type free A-modules.*

5.4 ELEMENTARY FUNCTIONAL ANALYTIC PROPERTIES OF COHERENT SHEAVES

Following Kiehl-Verdier, we say that a morphism $u\colon E \to F$ between two A-Fréchet modules is *A-quasinuclear* if there exists a commutative diagram of Fréchet A-modules:

$$\begin{tikzcd} A \widehat{\otimes}_{\mathbf{C}} V \arrow[d, "\pi"] \arrow[dr, "v"] & \\ E \arrow[r, "u"] & F \end{tikzcd},$$

where V is a nuclear Fréchet space, π is surjective and v is A-nuclear. The reason for considering this notion is explained by the following proposition, which is a direct consequence of Proposition 3.12.

PROPOSITION 5.7 ([24]). *For any coherent analytic sheaf $\mathcal{F}$ on $\mathbf{C}^n \times \mathbf{C}^k$ and any Stein neighbourhoods U, U' such that the closure of U' is a compact subset of $U \subset \mathbf{C}^n$ and for any Stein neighbourhood $S \subset \mathbf{C}^k$, the restriction mapping $\mathcal{F}(U \times S) \to \mathcal{F}(U' \times S)$ is $\mathcal{O}(S)$-quasinuclear.*

The generalised Schwartz perturbation theorem extends to the case where the perturbation is given by a quasinuclear morphism rather than a nuclear morphism.

The following proposition is a direct consequence of Theorem 3.15.

PROPOSITION 5.8 ([24]). *If E is a free module over a nuclear Fréchet algebra A, i.e. E is isomorphic to a product $A \widehat{\otimes}_{\mathbf{C}} V$ where V is a nuclear Fréchet space, then the functor $\widehat{\otimes}_A E$ is exact.*

COROLLARY 5.9 ([21, 24]). *For any $\mathcal{O}_{\mathbf{C}^{n+k}}$-coherent sheaf $\mathcal{F}$, any polycylinders $S, S' \subset \mathbf{C}^k$, $S' \subset S$, and any Stein open subset $U \subset \mathbf{C}^n$, we have an isomorphism of Fréchet modules over the ring $\mathcal{O}_{\mathbf{C}^k}(S)$*

$$\mathcal{F}(U \times S) \,\widehat{\otimes}\, \mathcal{O}_{\mathbf{C}^k}(S') \approx \mathcal{F}(U \times S')\,.$$

We conclude by pointing out that Corollary 5.9 together with Example 5.5 imply Oka's theorem: *Any system of generators for the kernel of an $\mathcal{O}_{\mathbf{C}^k}$-linear mapping of sheaves $\varphi\colon \mathcal{O}^p_{\mathbf{C}^k} \to \mathcal{O}^q_{\mathbf{C}^k}$ at the origin induces a system of generators in a sufficiently small neighbourhood of the origin.*

Indeed, we saw that $(\ker\varphi)(S)$ is a module of finite type, and a system of generators for the $\mathcal{O}(S)$-module $(\ker\varphi)(S)$ induces a system of generators for the $\mathcal{O}(S')$-module $(\ker\varphi)(S') \approx (\ker\varphi)(S) \mathbin{\widehat{\otimes}}_{\mathbf{C}} \mathcal{O}_{\mathbf{C}^k}(S')$ for any polycylinders $S' \subset S$.

5.5 Proof of Theorem 0.1, final step

LEMMA 5.10. *The restriction mapping $r\colon \mathcal{K}^\bullet(X) \to \mathcal{K}^\bullet(X')$ is an $\mathcal{O}_S(S)$-quasinuclear quasi-isomorphism.*

Proof. Recall that X is the intersection of a Stein open neighbourhood with some open ball B_ε. Define the complex of sheaves $\widetilde{\mathcal{K}}^\bullet$ in $B_\varepsilon \times S$ by the presheaf

$$\widetilde{\mathcal{K}}^\bullet(U \times V) = \mathcal{K}^\bullet(U \cap f^{-1}(V)).$$

Both complexes are isomorphic as complexes of Fréchet sheaves. Moreover, the restriction mapping $\widetilde{\mathcal{K}}^\bullet(B_\varepsilon \times S) \to \widetilde{\mathcal{K}}(B_{\varepsilon'} \times S)$ is $\mathcal{O}_S(S)$-quasinuclear (Proposition 5.7). This proves the lemma. $\square$

This lemma shows that Theorem 5.6 applies, therefore there exists a complex $\mathcal{L}^\bullet$ of free coherent $\mathcal{O}_S$-sheaves such that $\mathcal{L}^\bullet(S)$ is quasi-isomorphic to $\mathcal{K}^\bullet(X)$.

LEMMA 5.11. *The sheaf complexes $\mathcal{L}^\bullet$, $f_*\mathcal{K}^\bullet_{|X}$ are quasi-isomorphic.*

Proof. A mapping $u\colon M^\bullet \to L^\bullet$ of complexes induces a quasi-isomorphism between two complexes if and only if its mapping cone $C^\bullet(u)$ is exact. We apply this fact to the mapping cone of the quasi-isomorphism

$$u\colon \mathcal{L}^\bullet(S) \to \mathcal{K}^\bullet(X).$$

As the functor $\widehat{\otimes}\, \mathcal{O}_S(P)$ is exact for any polydisk $P \subset S$ (Proposition 5.8), the complex $C^\bullet(u) \mathbin{\widehat{\otimes}} \mathcal{O}_S(P)$ is also exact.

Using Corollary 5.9, we get that $C^\bullet(u) \mathbin{\widehat{\otimes}} \mathcal{O}_S(P)$ is the mapping cone of $u'\colon \mathcal{L}^\bullet(P) \to \mathcal{K}^\bullet(X \cap f^{-1}(P))$. Therefore, the complexes of sheaves $\mathcal{L}^\bullet$ and $f_*\mathcal{K}^\bullet_{|X}$ are quasi-isomorphic. This proves the lemma. $\square$

We assert that the complex $K^\bullet = \mathcal{K}^\bullet_0$ is quasi-isomorphic to the stalk of the complex $\mathcal{L}^\bullet$ at the origin.

Let (B_{ε_n}) be a fundamental sequence of neighbourhoods of the origin in $\mathbf{C}^n$, such that their intersection with the special fibre of f is transverse. As the map f satisfies the a_f-condition, we can find a fundamental sequence (S_n) of neighbourhoods of the origin in $\mathbf{C}^k$ such that the fibres of f intersect B_{ε_n} transversally above S_n.

Put $X_n = f^{-1}(S_n)$; we have the isomorphisms

$$\mathcal{L}^\bullet_{|S_n} \approx f_* \mathcal{K}^\bullet_{|X_n} \approx f_* \mathcal{K}^\bullet_{|X_n \cap B_{\varepsilon_n}} .$$

The first isomorphism is a consequence of the previous lemma and the second follows from the fact that the contraction is a quasi-isomorphism (Proposition 5.1).

In the limit $n \to \infty$, we get that the complex $K^\bullet = \mathcal{K}^\bullet_0$ is quasi-isomorphic to the complex $\mathcal{L}^\bullet_0$. This concludes the proof of the theorem. □

6. Finiteness theorem for coherent ind-analytic complexes

6.1 The generalised Schwartz perturbation theorem

We now state a variant of the theorem which includes most cases encountered in local analytic geometry. For this we need first to axiomatise the concepts introduced in the proof of the Schwartz perturbation theorem. A *topological algebra* A is a commutative algebra over $\mathbf{C}$ such that the underlying vector space is given a locally convex topology for which the algebra operations are continuous. An A-module E is a *topological A-module* if it carries a locally convex topology for which the module operations are continuous.

The following definition is adapted from [21].

Definition 6.1. A topological algebra A is called *multiplicatively convex* if any bounded subset of A is absorbed by a bounded subset invariant under multiplication.

For instance, Fréchet algebras are multiplicatively convex.

Let us now list the properties which are needed for the proof of the Schwartz perturbation theorem:

1. A, E, F should be complete;
2. E should be barrelled (if E is Fréchet this is the Banach-Steinhaus theorem);
3. A should be multiplicatively convex;
4. any bijective morphism from E to F should be an isomorphism (in case E, F are Fréchet this is the Banach open mapping theorem);
5. F should have the Mackey property.

The first three conditions are needed for the proof of the Houzel lemma, the last ones for reducing the proof to the case $f = I$. We thus get the following particular case of a theorem due to Houzel [21], the proof of which is similar to that of the Schwartz perturbation theorem (Theorem 3.16).

THEOREM 6.2. *Let $f\colon E \to F$ be a surjective morphism between topological A-modules. Assume that A, E, F satisfy the above listed properties; then for any A-quasinuclear morphism $u\colon E \to F$, the cokernel of the map $f+u$ is an A-module of finite type.*

6.2 INDUCTIVE LIMITS OF FRÉCHET SPACES

The category of Fréchet spaces is too restrictive to provide sufficient applications in local analytic geometry, for instance the vector space of holomorphic function germs does not admit such a topology. It is therefore necessary to introduce a more general class of objects, *LF-spaces*.

Consider a set of linear maps from Fréchet spaces to a fixed vector space $u_i\colon E_i \to E$, $i \in \Omega$ such that $\bigcup_{i\in\Omega} u_i(E_i) = E$. The *inductive limit topology* T of the vector space E is defined by

$$U \in T \iff \forall i \in \Omega\,,\ u_i^{-1}(U) \text{ is open in } E_i\,.$$

The category of LF-spaces is then a sub-category of the category of locally convex spaces.

Spaces of type LF are locally convex, *bornological* (bounded linear mappings coincide with continuous ones) and *barrelled* (pointwise bounded subset are uniformly bounded), but not always complete [25].

In case the E_i's form an increasing sequence of closed vector subspaces in E and the u_i's are the inclusions, the resulting LF-spaces are complete and satisfy the Banach open mapping theorem [8]. In fact, one has the following result.

THEOREM 6.3 ([26]). *Any complete LF-spaces E, F defined by limits of a countable set of Fréchet spaces satisfy the Banach open mapping theorem, that is, any bijective morphism from E to F is an isomorphism.*

PROPOSITION 6.4 ([19], Chapter 3). *For any compact subset $K \subset \mathbf{C}^n$, the LF-space $\mathcal{O}_{\mathbf{C}^n}(K) = \varinjlim \mathcal{O}_{\mathbf{C}^n}(U)$, $K \subset U$, of holomorphic functions restricted to $K \subset \mathbf{C}^n$ is complete and has the Mackey property. Moreover, the multiplication of functions defines a structure of multiplicatively convex algebra on $\mathcal{O}_{\mathbf{C}^n}(K)$.*

Proof. The proposition is a consequence of the following characterisation of bounded subsets due to Grothendieck ([19], Chapter 3, Proposition 5): a subset $B \subset \mathcal{O}_{\mathbf{C}^n}(K)$ is bounded provided that there exist a neighbourhood U containing K and a bounded subset $B' \subset C^0(\overline{U}) \cap \mathcal{O}_{\mathbf{C}^n}(U)$ which projects onto B via the restriction mapping $\mathcal{O}_{\mathbf{C}^n}(U) \to \mathcal{O}_{\mathbf{C}^n}(K)$. □

6.3 The sheaves $\mathcal{O}_{Y|X}$

Let $i\colon X \to Y$ be the inclusion of a complex analytic manifold X into another complex analytic manifold Y. The sheaf $i^{-1}\mathcal{O}_Y$ is denoted by $\mathcal{O}_{Y|X}$. If i is the inclusion of a submanifold $X \subset Y$ then $\mathcal{O}_{Y|X}$ is the sheaf of holomorphic functions on Y restricted to X. If Y is of the form $X \times T$, we denote simply by $\mathcal{O}_{X\times T|X}$ the sheaf obtained from the inclusion of $X \times \{0\}$ in $X \times T$. The stalk of the sheaf $\mathcal{O}_{X\times T|X}$ at a point x_0 is the space of germs of holomorphic functions in $X \times T$ at the point $x = x_0,\ t = 0$. These sheaves are frequently considered in microlocal analysis [35].

In the previous subsection, we saw that the space of global sections of the sheaf $\mathcal{O}_{Y|X}$, $Y \subset \mathbf{C}^n$, over a compact subset has an *LF-space* structure.

As inductive limits commute with topological tensor products, the topological vector space $\mathcal{O}_{\mathbf{C}^n}(K)$ is nuclear. Therefore most of the properties established for the sheaves $\mathcal{O}_X$ extend to the sheaves $\mathcal{O}_{Y|X}$.

As in the case of the sheaf of holomorphic functions, it follows from Theorem 6.2 that the sheaf $\mathcal{O}_{X\times T|X}$ is coherent, that is, the kernel of any morphism of sheaves of modules

$$\mathcal{O}^k_{X\times T|X} \to \mathcal{O}_{X\times T|X}$$

is finitely generated. This can also be deduced from Cartan's Theorem A and from the coherence of the sheaf $\mathcal{O}_{X\times T}$.

Following the general terminology [39], we say that a sheaf $\mathcal{F}$ on a space X is $\mathcal{O}_{X\times T|X}$*-coherent*, or that it is a *coherent ind-analytic sheaf*, if it is the cokernel of a morphism of $\mathcal{O}_{X\times T|X}$-modules:

$$\mathcal{O}^p_{X\times T|X} \to \mathcal{O}^n_{X\times T|X} \to \mathcal{F} \to 0\,.$$

The notion of f-constructibility extends trivially to complexes of $\mathcal{O}_{X\times T|X}$-coherent sheaves and to their stalks.

6.4 Local finiteness theorem

By Proposition 6.4, we may apply the generalised Schwartz perturbation theorem in the ind-analytic context and get the following result (we no longer consider direct image sheaves, as the conclusions are similar).

Achevé de composer le 5 juin 2009 à 10 : 22

THEOREM 6.5. *Let $f\colon (\mathbf{C}^n,0) \to (\mathbf{C}^k,0)$ be a holomorphic map-germ satisfying the a_f-condition. The cohomology spaces $H^p(K^\bullet)$ associated to a complex of f-constructible $\mathcal{O}_{\mathbf{C}^{n+N}|\mathbf{C}^n,0}$-coherent modules are $f^{-1}\mathcal{O}_{\mathbf{C}^{k+N}|\mathbf{C}^k,0}$-coherent modules, for any $p \geq 0$.*

REMARK 6.6. The algebra structure on $\mathcal{O}_{\mathbf{C}^{n+N}|\mathbf{C}^n}$ plays no role in the proof. Therefore, one may replace the condition « f-constructible $\mathcal{O}_{\mathbf{C}^{n+N}|\mathbf{C}^n,0}$-coherent modules » by « $f^{-1}\mathcal{O}_{\mathbf{C}^{k+N}|\mathbf{C}^k,0}$-topological modules isomorphic as *topological vector spaces* to $\mathcal{O}_{\mathbf{C}^{n+N}|\mathbf{C}^n,0}$ ».

Let us denote by $\mathfrak{M}$ the maximal ideal of the local ring $\mathcal{O}_{\mathbf{C}^k,0}$. We identify $\mathbf{C}^k$ to $\mathbf{C}^k \times \{0\} \subset \mathbf{C}^k \times \mathbf{C}^n$. To conclude, we give a proof of the following algebraic form of the division theorem due to Houzel and Serre.

PROPOSITION 6.7 ([20]). *Any $\mathcal{O}_{\mathbf{C}^{k+n},0}$-module M of finite type such that $M/\mathfrak{M}M$ is a finite-dimensional $\mathbf{C}$-vector space is itself an $\mathcal{O}_{\mathbf{C}^k,0}$-module of finite type.*

Proof. Consider the complex obtained from a projective resolution of M :

$$K^\bullet : \ \cdots \xrightarrow{\delta_2} \mathcal{O}^{n_1}_{\mathbf{C}^{k+n},0} \xrightarrow{\delta_1} \mathcal{O}^{n_0}_{\mathbf{C}^{k+n},0} \longrightarrow 0 \ , \ H^\bullet(K^\bullet) = \mathcal{O}^{n_0}_{\mathbf{C}^{k+n},0}/\operatorname{Im}\delta_1 \approx M \,.$$

This complex gives a complex $\mathcal{K}^\bullet$ of $\mathcal{O}_{X\times T}$-sheaves whose support is an analytic variety $V \subset X \times T$. Here $X \subset \mathbf{C}^n$ and $T \subset \mathbf{C}^k$ are small neighbourhoods of the origin in $\mathbf{C}^n$ and $\mathbf{C}^k$.

The dimension of $M/\mathfrak{M}M$ is equal to the intersection multiplicity of V with $X \times \{0\} = \{(x,t) : t = 0\}$. Assume that X and T are so small that the variety V intersects $X \times \{0\}$ only at the origin. Then the restriction of the complex $\mathcal{K}^\bullet$ to $X \times \{0\}$ defines an $\mathcal{O}_{X\times T|X,0}$-coherent sheaf complex on $X \subset \mathbf{C}^n$. The cohomology of the complex is supported at the origin, it is therefore constructible. Theorem 0.1 implies that $H^\bullet(K^\bullet) = M$ is a finite type module over the ring $\mathcal{O}_{\mathbf{C}^k,0}$ (here f is the mapping to a point). This proves the proposition. $\square$

6.5 THE BOUTET DE MONVEL DIVISION THEOREM

We now prove the Sato-Kashiwara-Kawai division theorem for pseudo-differential operators [35]. As in the commutative case, this theorem admits an algebraic version (or rather a generalisation) similar to the Houzel-Serre formulation of the division theorem [3] (see also [33], Chapter 3).

Denote by $\mathcal{E}(0)$ the sheaf of analytic pseudo-differential operators in $T^*\mathbf{C}^n \approx \mathbf{C}^{2n} = \{(q,p)\}$ of order 0. Let $\mathcal{E}'(0)$ be a subsheaf of operators which depend only on some of the variables, say $q_1, \dots, q_j, p_1, \dots, p_k$. We denote by $\mathcal{E}_0(0)$, $\mathcal{E}'_0(0)$ the stalks of the sheaves $\mathcal{E}(0)$, $\mathcal{E}'(0)$ at the point $x_0 \in \mathbf{C}^{2n}$ with coordinates $q_1 = \cdots = q_n = 0$, $p_1 = 1, p_2 = 0, \dots, p_n = 0$.

THEOREM 6.8 ([3, 33]). *For any coherent left $\mathcal{E}_0(0)$-module M the following assertions are equivalent:*

1. *the $\mathcal{O}_{\mathbf{C}^{j+k-1},0}$-module $M/\partial_{q_1}^{-1}M$ is of finite type;*
2. *the $\mathcal{E}'_0(0)$-left module M is of finite type.*

Proof. The module M is the stalk at the point $x_0 = (0, \dots, 0, 1, 0, \dots, 0)$ of a sheaf $\mathcal{M}$ of $\mathcal{E}_0(0)$-modules in $T^*\mathbf{C}^n \approx \mathbf{C}^{2n}$.

Consider the complex given by a resolution of $\mathcal{M}$

$$\mathcal{K}^\bullet : \ \cdots \xrightarrow{\delta_2} \mathcal{E}(0)^{n_1} \xrightarrow{\delta_1} \mathcal{E}(0)^{n_0} \longrightarrow 0\,, \quad \mathcal{H}^\bullet(\mathcal{K}^\bullet) = \mathcal{E}(0)^{n_0}/\operatorname{Im}\delta_1 \approx \mathcal{M}\,.$$

The support of $\mathcal{M}$ coincides with that of $\mathcal{M}/\partial_{q_1}^{-1}\mathcal{M}$, it is therefore an analytic subvariety $V \subset \mathbf{C}^{2n}$ ([35]; see also [33], Proposition 4.2.0).

The restriction of the sheaf $\mathcal{E}(0)$ to the complement of the zero section in $T^*\mathbf{C}^n \approx \mathbf{C}^{2n}$ is a sheaf of non-commutative Fréchet algebras [4]. Therefore the argument given in the proof of Proposition 6.7 applies *mutatis mutandis* to this situation. $\square$

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