L'Enseignement Mathématique (2) 55 (2009), 127–137

THE BIG PICARD THEOREM AND OTHER RESULTS ON RIEMANN SURFACES

by Pablo ARÉS-GASTESI and T. N. VENKATARAMANA

ABSTRACT. In this paper we provide a new proof of the Big Picard Theorem, based on some simple observations about mappings between Riemann surfaces.

1. INTRODUCTION

Isolated singularities of holomorphic functions are of three types: removable singularities, when the function can be extended to a holomorphic function at the singular point; poles, if the function behaves locally like $z \mapsto 1/z^n$; and essential singularities, where the behaviour of the function is difficult to control. The Casorati-Weierstrass theorem, from around 1868 (see [9]), says that the image of any neighbourhood of an essential singularity is dense in the complex plane. The proof of this result is elementary, based simply on the characterisation of isolated singularities of holomorphic functions. The Big Picard Theorem is a deeper result which states that the image of a neighbourhood of an essential singularity covers the whole complex plane, except for perhaps one point. There are several proofs of this theorem, using different techniques: the original 1879 proof of Picard [10, pp. 19 and 27] uses elliptic modular functions, others use Bloch's theorem and normal families (see for example [3], [6] or [8]) or the Schwarz-Pick theorem and estimates on the Poincaré metric of certain plane domains [1]; see [12, p. 240] for further references. In this article we give a new proof of the Big Picard Theorem based on basic facts of complex analysis, the theory of covering spaces and the observation that there is no holomorphic mapping from the punctured disc to an annulus that is injective at the fundamental group level (Proposition 2.1).

We have tried to make the article accessible to a wide audience; to this end, we have recalled some basic facts of complex analysis and given references to classical texts. The organisation of the article is as follows: in Section 2 we prove the above-mentioned fact about mappings from the punctured disc to an annulus. In Section 3 we give a proof of Picard's Theorem; for this we recall some facts on the modular group and the modular function. In Section 4 we study properties of discrete groups of Möbius transformations; we do this using Lie group techniques. In particular we show that a covering from the punctured disc to a Riemann surface, if restricted to a small enough punctured disc, is a finite covering (Proposition 4.5). With this result we can provide another proof of Picard's Theorem.

We thank the referee for several very relevant comments which greatly improved the exposition. The statement of Proposition 2.1 given below was suggested by the referee and R. Narasimhan (in a private communication). We learned recently that the proof given in the present paper was also found by Madhav Nori a while ago.

2. A LEMMA ON RIEMANN SURFACES

We start by setting up some notation. The Riemann sphere (extended complex plane), the upper half plane and the unit disc will be denoted, respectively, as follows:

$$\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}\,, \quad \mathbf{H} = \{z \in \mathbf{C}\,;\, \mathrm{Im}(z) > 0\}\,, \quad \mathbf{D} = \{z \in \mathbf{C}\,;\, |z| < 1\}\,.$$

It is an easy consequence of Schwarz's lemma that the group $\mathrm{Aut}(\mathbf{D})$ of biholomorphic self-mappings of the unit disc consists of the Möbius transformations of the form $A(z) = (az+b)/(\overline{b}z+\overline{a})$, where a and b are complex numbers satisfying $|a|^2 - |b|^2 = 1$ [1, p. 1]. Using the Cayley transform $T(z) = \frac{z-i}{z+i} \colon \mathbf{H} \to \mathbf{D}$, it is easy to see that the group $\mathrm{Aut}(\mathbf{H})$ consists of the Möbius transformations with real coefficients; that is, transformations of the form $A(z) = \frac{az+b}{cz+d}$, with a, b, c, d real numbers satisfying $ad - bc = 1$. We can then identify $\mathrm{Aut}(\mathbf{H})$ with the quotient group $\mathrm{PSL}_2(\mathbf{R}) = \mathrm{SL}_2(\mathbf{R})/\{\pm Id\}$, where $\mathrm{SL}_2(\mathbf{R})$ is the group of square matrices of order 2 with real coefficients and determinant equal to 1, and Id denotes the identity matrix. (Similarly, $\mathrm{Aut}(\widehat{\mathbf{C}})$ can be identified with $\mathrm{PSL}_2(\mathbf{C})$.) The advantage of this identification is that it allows us to use Lie group techniques to obtain results on complex analysis (Lemma 4.1). We will denote the elements of $\mathrm{PSL}_2(\mathbf{R})$ by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$.

The elements of $\mathrm{PSL}_2(\mathbf{R})$ can be classified by their conjugacy classes as follows:

1. the identity;
2. hyperbolic elements, conjugate to $\left[\begin{smallmatrix} k & 0 \\ 0 & 1/k \end{smallmatrix}\right]$, with k real, $k > 1$;
3. elliptic elements, conjugate to $\left[\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right]$, with $0 < \theta < \pi$;
4. unipotent or parabolic transformations, conjugate to $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right]$ (these two transformations are conjugate in $\mathrm{Aut}(\widehat{\mathbf{C}})$ but not in $\mathrm{Aut}(\mathbf{H})$).

Different values of k or θ in the above list (within the given ranges) yield different conjugacy classes.

Let A be a parabolic transformation of $\mathrm{PSL}_2(\mathbf{R})$, say $A(z) = z + 1$; let G denote the subgroup (of $\mathrm{PSL}_2(\mathbf{R})$) generated by A. The quotient Riemann surface $\mathbf{H}/G$ can be identified with the punctured disc $\mathbf{D}^* = \mathbf{D}\backslash\{0\}$; the exponential function provides the covering map

$$\begin{aligned} \pi\colon \mathbf{H} &\to \mathbf{D}^* \\ z &\mapsto \exp(2\pi i z). \end{aligned}$$

For the case of a hyperbolic element, take $A = \left[\begin{smallmatrix} k & 0 \\ 0 & 1/k \end{smallmatrix}\right]$; then $\mathbf{H}/G$ is conformally equivalent to an annulus, $\mathbf{A}_r = \{z \in \mathbf{C}\,;\, r < |z| < 1\}$. The covering map is again given by an exponential mapping:

$$\begin{aligned} \pi\colon \mathbf{H} &\to \mathbf{A}_r \\ z &\mapsto \exp(\pi i \log z / \log k) \end{aligned}$$

(where the argument of $\log z$ is chosen between 0 and π). The numbers k and r are related by $r = \exp(-\pi^2/\log k)$. See, for example, [4, IV.6.4 and IV.6.8].

Since parabolic and hyperbolic transformations are not conjugate in $\mathrm{PSL}_2(\mathbf{R})$ (or in $\mathrm{PSL}_2(\mathbf{C})$), there does not exist a biholomorphic mapping between the punctured disc and an annulus. However the next proposition shows that more is true.

PROPOSITION 2.1. *A holomorphic map from the punctured disc* $\mathbf{D}^*$ *to an annulus* $\mathbf{A}_r$ *(*$r > 0$*) induces the trivial map between fundamental groups.*

Proof. Let $f\colon \mathbf{D}^* \to \mathbf{A}_r$ be a holomorphic mapping; denote by γ the generator of the fundamental group of the punctured disc. Since f is bounded, its singularity at the origin is removable and so f extends to a function from the unit disc $\mathbf{D}$ to the closure of the annulus $\mathbf{A}_r$. However non-constant holomorphic functions are open mappings, so $f(\mathbf{D})$ actually lies in $\mathbf{A}_r$. Since γ is homotopically trivial in $\mathbf{D}$ we have that $f(\gamma)$ is homotopically trivial in $\mathbf{A}_r$, and thus f induces the trivial mapping between fundamental groups.

3. Picard's Theorem

The well-known Picard Theorems are results about the range of entire functions or of functions with essential singularities. More precisely, the Little Theorem says that the range of a non-constant entire function can miss at most one point in $\mathbf{C}$ (for example, a polynomial is surjective while the exponential mapping misses the origin). The Big Picard Theorem on the other hand, states that the same result holds for a function with an (isolated) essential singularity, in any neighbourhood of the singularity.

The Little Theorem is easy to prove using the fact (see below about properties of the modular function) that the universal covering space of $\mathbf{C}\setminus\{0,1\}$ is the upper half plane, or equivalently the unit disc $\mathbf{D}$. Indeed, if $f\colon \mathbf{C} \to \mathbf{C}\setminus\{0,1\}$ is an entire function, then f will lift to a function $\widetilde{f}\colon \mathbf{C} \to \mathbf{D}$; by Liouville's theorem $\widetilde{f}$ is constant, and therefore f is also constant. In the case of the Big Picard Theorem the problem is that f is defined on the punctured disc, and it is not clear that it will lift to the unit disc; Proposition 2.1 gives us a way out of this difficulty.

Before we give the proof of Picard's Theorem we need to recall some results on the modular function (for more details see for example [2, Chapter 7] or [13, Chapter 16]). Let Γ denote the congruence subgroup mod 2 of $\mathrm{PSL}_2(\mathbf{Z})$; that is, the group of Möbius transformations $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ with integer coefficients, satisfying $ad - bc = 1$ and where a and d are odd and b and c are even (the transformation is congruent with the identity modulo 2). This group is generated by $A = \left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 1 & 0 \\ -2 & 1 \end{smallmatrix}\right]$. These two transformations, as well as their product $C = AB = \left[\begin{smallmatrix} -3 & 2 \\ -2 & 1 \end{smallmatrix}\right]$, are parabolic.

LEMMA 3.1. *Any parabolic element of Γ is conjugate in Γ to a power of A, B or C.*

Proof. A parabolic Möbius transformation $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ has a unique fixed point in $\widehat{\mathbf{C}}$, namely $\frac{a-d}{2c}$ when $c \neq 0$ and ∞ if $c = 0$. If γ is in Γ then its fixed point is in $\mathbf{Q} \cup \{\infty\}$. Let $\Gamma(z)$ denote the Γ-orbit of a point $z \in \widehat{\mathbf{C}}$; then $\mathbf{Q} \cup \{\infty\}$ can be written as the disjoint union of three orbits:

$$\mathbf{Q} \cup \{\infty\} = \Gamma(\infty) \sqcup \Gamma(0) \sqcup \Gamma(1)\,.$$

The fixed points of A, B and C are ∞, 0 and 1 respectively. If $\gamma \in \Gamma$ is a parabolic transformation with fixed point at ∞ then it is easy to see (by the conditions set on the coefficients of elements of Γ) that $\gamma = A^n$ for some integer n. Similarly if the fixed point of γ is 0 (respectively 1), then $\gamma = B^n$ (respectively $\gamma = C^n$). For the general case, assume that $\gamma(z_0) = z_0$ and $z_0 \in \Gamma(\infty)$; then there exists a $g \in \Gamma$ such that $g(\infty) = z_0$. Then we have $g^{-1}\gamma g = A^n$ because the left hand side of this equality is a parabolic transformation in Γ, with fixed point at ∞. Thus γ is conjugate in Γ to a power of A. The cases $z_0 \in \Gamma(0)$ and $z_0 \in \Gamma(1)$ are handled similarly.

The modular function λ is a holomorphic function defined on the upper half plane invariant under the group Γ; that is, $\lambda(\gamma(\tau)) = \lambda(\tau)$, for all $\gamma \in \Gamma$ and $\tau \in \mathbf{H}$. It turns out that λ is actually a covering map from the upper half plane to the plane minus two points, $\lambda\colon \mathbf{H} \to \mathbf{C}\backslash\{0,1\} = \widehat{\mathbf{C}}\backslash\{\infty,0,1\}$.

We now can prove Picard's Theorem.

THEOREM 3.2. *If $f\colon \mathbf{D}^* \to X = \widehat{\mathbf{C}}\backslash\{\infty,0,1\}$ is holomorphic then f cannot have an essential singularity at the origin.*

Before we proceed with the proof we make two observations: firstly, given any three distinct points in $\widehat{\mathbf{C}}$ there exists a (unique) Möbius transformation that maps them to ∞, 0 and 1; thus these three points do not play any particular role in the above statement of Picard's theorem. Secondly, the radius of the disc around the singularity can be arbitrary since any two punctured discs are conformally equivalent.

Proof. Let c be a generator of the fundamental group of $\mathbf{D}^*$ (we again make an abuse of notation and use the same letter for a path and its homotopy class) and let γ be the element of Γ that corresponds to $f_*(c)$ under the covering defined by the modular function λ. By the lifting criterion [5, p. 61] the function f lifts to a map $\widetilde{f}\colon \mathbf{D}^* \to \mathbf{H}/\langle\gamma\rangle$. We consider the different possibilities for γ. If γ is the identity then $\widetilde{f}\colon \mathbf{D}^* \to \mathbf{H}$; but the upper half plane and the unit disc are conformally equivalent, so $\widetilde{f}$ will have a removable singularity at the origin and therefore f will also have a removable singularity. If γ were hyperbolic then $\mathbf{H}/\langle\gamma\rangle$ would be (conformally equivalent to) an annulus, and $\widetilde{f}$ would be a holomorphic mapping from the punctured disc to an annulus that induces an isomorphism between the fundamental groups. By Proposition 2.1 this is not possible. Since Γ has no elliptic elements we conclude that γ must be a parabolic transformation.

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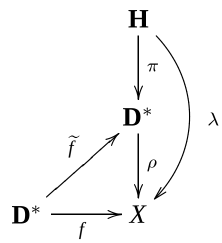
By Lemma 3.1, by conjugating within Γ, we may assume that γ is equal to a power of A, B or C, say $\gamma = A^n$, with $n \neq 0$. The function f lifts to the covering determined by A; we will denote the lift by $\widetilde{f}$ to simplify notation. Using the fact that $\mathbf{H}/\langle A \rangle$ is (conformally equivalent to) the punctured disc we get the following commutative diagram:



Here π is the covering given by the exponential function, $\tau \to \exp(\pi i \tau)$, and ρ is the holomorphic covering such that $\lambda = \rho \circ \pi$. Since $\widetilde{f}$ is bounded it has a removable singularity at the origin; moreover, since γ is not trivial we must have $\widetilde{f}(0) = 0$. From [2, Equation 27, p. 272], the function ρ has a power series expansion of the form $\rho(q) = 16q + \dots$ near the origin. Thus $z = 0$ is a removable singularity for both ρ and $\widetilde{f}$; hence f also has a removable singularity at the origin.

If we assume that $\gamma = B^n$ we have again that (the corresponding) ρ has a removable singularity at $z = 0$, while if $\gamma = C^n$ then ρ has a pole. In each case, the function f does not have an essential singularity at the origin.

4. Results about discrete groups and another proof of Picard's Theorem

The space of 2×2 matrices with real (or complex) coefficients has a norm given by $\|A\| = \max\{|a|, |b|, |c|, |d|\}$, where $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. This norm induces a (subspace) topology on $\mathrm{SL}_2(\mathbf{R})$ and a (quotient) topology on $\mathrm{PSL}_2(\mathbf{R})$. We will denote by $B(Id, \epsilon)$ the ball of centre the identity and radius ϵ; it consists of the transformations represented by a matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ such that $|a-1|$, $|b|$, $|c|$ and $|d-1|$ are bounded above by ϵ. A subgroup G of $\mathrm{PSL}_2(\mathbf{R})$ is called *discrete* if it is discrete in this topology; that is, there is no sequence of elements of G, say $A_n = \left[\begin{smallmatrix} a_n & b_n \\ c_n & d_n \end{smallmatrix}\right]$, $A_n \neq Id$, such that $a_n, d_n \to 1$ and $b_n, c_n \to 0$.

LEMMA 4.1. *There exists* $\epsilon > 0$ *such that if* A *and* B *are in* $B(Id, \epsilon)$ *and generate a discrete subgroup* G *of* $\mathrm{PSL}_2(\mathbf{R})$ *then* G *is cyclic.*

Proof. We first recall the concept of nilpotent group. Let H_1 and H_2 be two subgroups of a group H; denote by $[H_1, H_2]$ the subgroup of H generated by all elements of the form $h_1 h_2 h_1^{-1} h_2^{-1}$, where $h_1 \in H_1$ and $h_2 \in H_2$. Define $H^{(1)} = H$ and for $n > 1$ set $H^{(n)} = [H, H^{(n-1)}]$. A group is called nilpotent if there exists a positive integer n such that $H^n = \{Id\}$.

By a result of Zassenhaus and Kazhdan-Margulis [11, Theorem 8.16] there exists an $\epsilon > 0$ such that if A and B are in $B(Id, \epsilon)$ in $\mathrm{PSL}_2(\mathbf{R})$, then the group G generated by A and B is nilpotent (this holds for a general Lie group, not only for $\mathrm{PSL}_2(\mathbf{R})$). The lemma will be proved if we can show that any nilpotent discrete subgroup G of $\mathrm{PSL}_2(\mathbf{R})$ is in fact cyclic.

Let then G be nilpotent and non-trivial; let $n > 0$ be an integer such that

$$G = G^{(1)} \supsetneq \cdots \supsetneq G^{(n+1)} = \{Id\}.$$

Then $G^{(n)}$ is central in G but not trivial. There are three cases to consider:

1. Suppose $G^{(n)}$ contains a unipotent transformation, say $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ without loss of generality. Then G is a subgroup of translations, that is,

$$G \subset \left\{ \left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right] ; x \in \mathbf{R} \right\},$$

since these are the only elements that commute with $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. Thus G can be identified with a discrete subgroup of $\mathbf{R}$ and by standard results, G must be cyclic.

2. Assume that $G^{(n)}$ contains a hyperbolic element, say $\left[\begin{smallmatrix} k_0 & 0 \\ 0 & 1/k_0 \end{smallmatrix}\right]$. Then we have

$$G \subset \left\{ \left[\begin{smallmatrix} k & 0 \\ 0 & 1/k \end{smallmatrix}\right] ; k \in \mathbf{R},\ k > 0 \right\},$$

so G can be identified with a discrete subgroup of $\mathbf{R}^+$, the positive real numbers with multiplication as the operation. Again one easily obtains that G must be cyclic.

3. Finally, if G contains an elliptic element, then G is conjugate to a discrete subgroup of the circle group, S^1 (the group of rotations). By discreteness all elements of G have finite order and it is not difficult to see that G is cyclic.

The next result assumes the Uniformization Theorem; for the reader unfamiliar with it we state the necessary facts in the proof.

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THEOREM 4.2. *Let $f\colon \mathbf{D}^* \to S$ be a holomorphic mapping from the punctured disc into a Riemann surface $S = \widetilde{S}/\Gamma$, where $\widetilde{S}$ denotes the universal covering space of S. Denote by γ a generator of the subgroup of Γ corresponding to the image $f_*(\pi_1(\mathbf{D}^*))$ of the fundamental group of $\mathbf{D}^*$ under f. Then γ is the identity or unipotent.*

Proof. By the Uniformization Theorem the surface $\widetilde{S}$ is the Riemann sphere $\widehat{\mathbf{C}}$, the complex plane or the upper half plane, and the group Γ is a group of Möbius transformations acting on $\widetilde{S}$ (see [4, Theorem IV.5.6, p. 206]). If $\widetilde{S}$ is the Riemann sphere then Γ is the trivial group so there is nothing to prove. If $\widetilde{S}$ is the complex plane then Γ consists only of unipotent elements (actually all elements of Γ are translations of the form $z \mapsto z + \lambda$, $\lambda \in \mathbf{C}$), so again the theorem holds. Thus we may assume that $\widetilde{S} = \mathbf{H}$. By the lifting criterion the map f will lift to a map $\widetilde{f}\colon \mathbf{D}^* \to X = \mathbf{H}/\langle\gamma\rangle$. Since Γ does not contain any elliptic transformations (see for example [4, IV.6.5], or simply use the fact that elliptic elements have fixed points in $\mathbf{H}$ while covering transformations act fixed-point freely) we need to consider only the situation when γ is hyperbolic. But in that case X would be an annulus and $\widetilde{f}$ a holomorphic mapping from the punctured disc to the annulus that induces an isomorphism between the fundamental groups; Proposition 2.1 rules out this case, and the theorem is proved.

COROLLARY 4.3. *Let $S = \mathbf{H}/\Gamma$ be a Riemann surface with universal covering space the upper half plane. Suppose there is a punctured disc embedded in S and let $\gamma \in \Gamma$ be the element corresponding to a small loop around the puncture; then γ is unipotent.*

COROLLARY 4.4. *Let $S = \mathbf{H}/\Gamma$. Then there exists a covering from the punctured disc to S if and only if Γ contains unipotent transformations.*

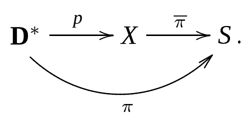
The following proposition is the key point in our second proof of the Big Picard Theorem.

PROPOSITION 4.5. *Let $\pi\colon \mathbf{D}^* \to S$ be a covering map, where S is a Riemann surface. Let $\mathbf{D}^*_\epsilon$ denote the punctured disc $\mathbf{D}^*_\epsilon = \{z \in \mathbf{C}\,;\, 0 < |z| < \epsilon\}$. Then there exists $\epsilon \in (0,1)$ such that the restriction of π to $\mathbf{D}^*_\epsilon$ is a finite covering onto its image.*

Proof. Since π is a covering, the universal covering space of S is the upper half plane, so we can write $S = \mathbf{H}/\Gamma$, for Γ a discrete subgroup of

$\mathrm{PSL}_2(\mathbf{R})$. By Theorem 4.2, the subgroup of Γ corresponding to $\pi_*(\pi_1(\mathbf{D}^*))$ is generated by a unipotent element g; without loss of generality we may assume that $g = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. Let $\mathcal{T}$ denote the group of translations in $\mathrm{PSL}_2(\mathbf{R})$; since Γ is discrete, $\Gamma \cap \mathcal{T}$ must be generated by a translation of the form $\left[\begin{smallmatrix} 1 & 1/m \\ 0 & 1 \end{smallmatrix}\right]$, for some positive integer m.

We can identify $\mathbf{D}^*$ (in the statement of the proposition) with the quotient $\mathbf{H}/\langle g \rangle$; let $X = \mathbf{H}/(\Gamma \cap \mathcal{T})$. Then the covering π factors through X:



Now X is also a punctured disc, and p is an m-to-1 covering (a power map, $p(z) = z^m$). The proposition will be proved if we can establish the following claim:

The mapping $\overline{\pi}\colon X \to S$ *is injective on* $\mathbf{D}^*_\epsilon$, *for some* $0 < \epsilon < 1$.

To prove the claim, assume that there are two sequences of points in X, say $\{z_n\}$ and $\{z'_n\}$, with $\overline{\pi}(z_n) = \overline{\pi}(z'_n)$ and such that $z_n \to 0$ and $z'_n \to 0$. Let $\tau_n = x_n + iy_n$ and $\tau'_n = x'_n + iy'_n$ be lifts of these points in $\mathbf{H}$; then $y_n \to +\infty$ and $y'_n \to +\infty$. Let $g_n = \frac{1}{\sqrt{y_n}} \left[\begin{smallmatrix} y_n & x_n \\ 0 & 1 \end{smallmatrix}\right]$; we have $g_n(i) = \tau_n$ and it is easy to see by direct computation that $g_n^{-1} g g_n \to Id$. Similarly, if g'_n corresponds to τ'_n, then $(g'_n)^{-1} g g'_n \to Id$.

Since z_n and z'_n are Γ-equivalent there exist $\gamma_n \in \Gamma$ such that $\gamma_n(\tau_n) = \tau'_n$. Let $k_n = g_n^{-1}\gamma_n^{-1} g'_n$; then $k_n(i) = i$ and it is easy to see that $(g'_n)^{-1} g g'_n \to Id$ implies $g_n^{-1}\gamma_n^{-1} g \gamma_n g_n \to Id$ (conjugate from the upper half plane to the unit disc to do the computations; on $\mathbf{D}$ the element conjugate to k_n has the form $\left[\begin{smallmatrix} \lambda_n & 0 \\ 0 & \overline{\lambda_n} \end{smallmatrix}\right]$, with $|\lambda_n| = 1$).

For a fixed n let G_n denote the subgroup of $\mathrm{PSL}_2(\mathbf{R})$ generated by $g_n^{-1} g g_n$ and $g_n^{-1}\gamma_n^{-1} g \gamma_n g_n$. We have that G_n is discrete (since G_n is conjugate to a subgroup of Γ, which is discrete) and generated by elements close to the identity (when n is big enough). By Lemma 4.1 G_n is cyclic. But this implies that $g_n^{-1} g g_n$ and $g_n^{-1}\gamma_n^{-1} g \gamma_n g_n$ have the same fixed points, and therefore $\gamma(\infty) = \infty$; thus γ_n is of the form $\gamma_n = \frac{1}{\sqrt{\alpha_n}} \left[\begin{smallmatrix} \alpha_n & \beta_n \\ 0 & 1 \end{smallmatrix}\right]$, where α_n is a positive real number and β_n is a real number.

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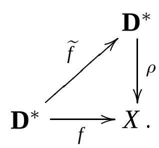
One could now use some results on Kleinian groups to conclude that $\alpha_n = 1$ (see, for example [7, I.D.4 and II.C.6]) but we prefer to give the following elementary argument. A simple computation shows that

$$\gamma_n^{-1}\begin{bmatrix}1 & 1/m\\ 0 & 1\end{bmatrix}\gamma_n = \begin{bmatrix}1 & \frac{1}{m\alpha_n}\\ 0 & 1\end{bmatrix}, \qquad \gamma_n\begin{bmatrix}1 & 1/m\\ 0 & 1\end{bmatrix}\gamma_n^{-1} = \begin{bmatrix}1 & \frac{\alpha_n}{m}\\ 0 & 1\end{bmatrix}.$$

These two transformations are in $\Gamma \cap \mathcal{T}$, and so α_n and $1/\alpha_n$ must be (positive) integers; that is, $\alpha_n = 1$. But then γ_n will be in $\Gamma \cap \mathcal{T}$ and therefore $z_n = z_n'$, since $X = \mathbf{H}/(\Gamma \cap \mathcal{T})$.

This proposition gives us a second way to prove the Big Picard Theorem without using the modular function, as follows. As in the proof given above, the function $f \colon \mathbf{D}^* \to X = \widehat{\mathbf{C}} \setminus \{\infty, 0, 1\}$ lifts to a function $\widetilde{f} \colon \mathbf{D}^* \to \mathbf{D}^*$ that gives a commutative diagram:



The function $\widetilde{f}$ has a removable singularity at the origin with $\widetilde{f}(0) = 0$; the problem is to determine the behaviour of the map ρ. By Proposition 4.5 there is an ϵ such that ρ is a finite covering from $\mathbf{D}_\epsilon^*$ onto its image. But then by the Casorati-Weierstrass theorem ρ cannot have an essential singularity at the origin, and neither can f.

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(Reçu le 28 juin 2007 ; version révisée reçue le 8 janvier 2008)

Pablo Arés-Gastesi
T. N. Venkataramana

School of Mathematics
Tata Institute of Fundamental Research
Mumbai 400005
India
e-mail : pablo@math.tifr.res.in, venky@math.tifr.res.in

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