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G_2 AND THE ROLLING DISTRIBUTION

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INTRODUCTION

Consider two balls of different radii, r and R, rolling along each other, without slipping or spinning. The configuration space for this system is a 5-dimensional manifold $Q = \mathrm{SO}_3 \times S^2$ on which the no-slip/no-spin condition defines a rank 2 distribution $D_\rho \subset TQ$ (depending on the radii ratio $\rho = R/r$), the *rolling distribution*.

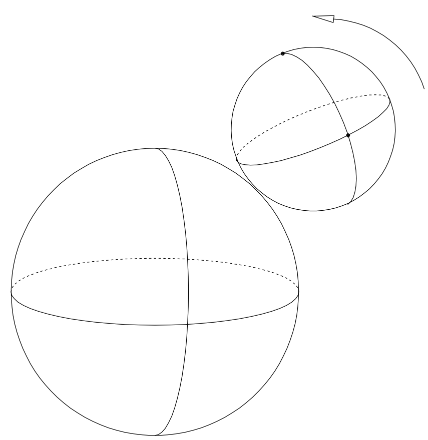

FIGURE 1
Rolling a ball on another ball

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Now D_ρ is a non-integrable distribution (unless the balls are of equal size, i.e. unless $\rho = 1$) admitting an obvious 6-dimensional transitive symmetry group $SO_3 \times SO_3$, arising from the isometry groups of each ball. But for balls whose radii are in the ratio $3:1$ or $1:3$, and only for these ratios, something strange happens: the *local* symmetry group of the distribution increases from $SO_3 \times SO_3$ to G_2, a 14-dimensional non-compact Lie group.

More precisely, let $\mathfrak{g}_2$ be the real *split form* of the 14-dimensional exceptional complex simple Lie algebra $\mathfrak{g}_2^{\mathbf{C}}$. An explicit matrix realization of $\mathfrak{g}_2$, appearing in Élie Cartan's 1894 thesis [5], is given by the set of 7×7 real matrices of the form

$$\begin{pmatrix} A & \Omega_{\mathbf{c}} & -2\mathbf{b} \\ \Omega_{\mathbf{b}} & -A^t & -2\mathbf{c} \\ \mathbf{c}^t & \mathbf{b}^t & 0 \end{pmatrix},$$

where $A \in \mathfrak{sl}_3(\mathbf{R})$ (real 3×3 traceless matrices), $\mathbf{b}, \mathbf{c} \in \mathbf{R}^3$ (column vectors), and where for each $\mathbf{u} = (u_1, u_2, u_3)^t \in \mathbf{R}^3$ we let $\Omega_{\mathbf{u}}$ denote the antisymmetric 3×3 matrix

$$\Omega_{\mathbf{u}} = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}.$$

Corresponding to $\mathfrak{g}_2$ is a closed connected subgroup $G_2 \subset SO_{3,4}$, a non-compact simple Lie group preserving a quadratic form on $\mathbf{R}^7$ of signature type $(3,4)$.

Furthermore, G_2 contains a 6-dimensional maximal compact subgroup $K \subset G_2$, a double-cover of $SO_3 \times SO_3$, with Lie algebra $\mathfrak{K} \subset \mathfrak{g}_2$ consisting of matrices of the form

$$\begin{pmatrix} \Omega_{\mathbf{a}} & \Omega_{\mathbf{b}} & -2\mathbf{b} \\ \Omega_{\mathbf{b}} & \Omega_{\mathbf{a}} & -2\mathbf{b} \\ \mathbf{b}^t & \mathbf{b}^t & 0 \end{pmatrix}, \quad \mathbf{a}, \mathbf{b} \in \mathbf{R}^3.$$

(The isomorphism $\mathfrak{K} \cong \mathfrak{so}_3 \oplus \mathfrak{so}_3$ is not so obvious; see Appendix C below.)

Next, for any open subset $U \subset Q$, let $\mathfrak{aut}(U, D_\rho|_U)$ be the Lie algebra of vector fields on U which preserve the restriction of D_ρ to U. Then we have

THEOREM 1. *If the radii ratio of the balls is $\rho = 3$ or $\rho = 1/3$, then $\mathfrak{aut}(U, D_\rho|_U) \cong \mathfrak{g}_2$ for any sufficiently small open $U \subset Q$. For any other ratio (other than $1:1$), $\mathfrak{aut}(U, D_\rho|_U) \cong \mathfrak{so}_3 \oplus \mathfrak{so}_3$ for all open sets $U \subset Q$.*

To get an actual D-preserving action of G_2, for radius ratio $3:1$ or $1:3$, one needs to lift D_ρ to the universal (double) cover $\widetilde{Q} = S^3 \times S^2$ (see Section 7 below).

This theorem was communicated to us by Robert Bryant for whom it is in essence contained in É. Cartan's notoriously difficult *Five Variables Paper* [4] of 1910. R. Bryant wrote to us:

> "Cartan himself gave a geometric description of the flat G_2-structure as the differential system that describes space curves of constant torsion 2 or $1/2$ in the standard unit 3-sphere. (See the concluding remarks of Section 53 in Paragraph XI in the *Five Variables Paper*.) One can easily transform the rolling balls problem (for arbitrary ratios of radii) into the problem of curves in the 3-sphere of constant torsion and, in this guise, one can recover the $3:1$ or $1:3$ ratio as Cartan's torsion 2 or 1/2 with a minimum of fuss. Thus, one could say that Cartan's calculation essentially covers the rolling ball case."

As far as we know, the only available proofs of this beautiful and mysterious theorem use the sophisticated *Cartan method of equivalence* or its variants such as those due to Tanaka [14] and his school. The group G_2 (or rather its Lie algebra) appears in the Cartan method of equivalence applied to the rolling distribution at the end of a rather lengthy and involved calculation (to put it mildly), and one is left somewhat puzzled at the appearance of G_2 in this context. Our primary goal in this article is to shed some light on this theorem in a direct manner, without appealing to Cartan's method of equivalence, by showing that the surprising appearance of G_2 as a symmetry group of a certain distribution is in fact rather natural, if one is familiar with some basic facts on Lie groups and algebras.

To this end, we provide two different constructions of the rolling distribution for radius ratio $3:1$, both with built-in G_2-invariance. The first construction is in terms of the root diagram of $\mathfrak{g}_2$, in the spirit of Section 4 of Bryant's lecture notes [3]. The second construction is in terms of *split* octonions, for which G_2 serves as the automorphism group, and can in fact be traced back to É. Cartan's 1894 thesis [5], although Cartan does not mention octonions there. The price we pay for avoiding the Cartan method of equivalence is that we can thus prove only part of Theorem 1, namely that $G_2 \subset \mathrm{Aut}(\widetilde{Q}, \widetilde{D}_\rho)$ for radii ratio $\rho = 3$ or $\rho = 1/3$, but we do not prove that G_2 is in fact the full automorphism group for these radius ratios. We hope the reader finds it worthwhile.

A secondary purpose is to correct an error appearing in the book [12] by one of us. We had mistakenly said there that the symmetry group for the rolling distribution for a ball on a plane (ratio $1:\infty$) was G_2. In fact, it is $\mathrm{SO}_3 \times E_2$, where E_2 is the (3-dimensional) group of Euclidean motions of $\mathbf{R}^2$, i.e. there are no "non-obvious" symmetries in this case.

A tertiary purpose is to give a feel for the simplest exceptional Lie algebra $\mathfrak{g}_2$ and its associated Lie groups, and to provide a refresher course on roots and weights.

FURTHER RESULTS. The G_2-action on $\widetilde{Q}$ does not descend to the rolling configuration space Q, but its restriction to the maximal compact $K \subset G_2$ does descend. This descended action of K forms the $2:1$ cover of the obvious symmetry group $\mathrm{SO}_3 \times \mathrm{SO}_3$ (the kernel of the cover acting trivially on Q). These facts are proved below in Section 7. For radii ratio $1:1$ the rolling distribution is integrable hence admits an infinite-dimensional symmetry group.

STRUCTURE OF THE PAPER. In the next section (Section 1) we describe the background and a wider context for the problem, with references to the literature. In Section 2 we describe the distributions associated with the rolling of balls, noting their $\mathrm{SO}_3 \times \mathrm{SO}_3$ *obvious* symmetries.

In Section 3 we describe a general set-up for G-homogeneous distributions, G a Lie group, in terms of *group-theoretic data* (G, H, W), where $H \subset G$ is a closed subgroup representing the isotropy subgroup, and where $W \subset \mathfrak{g}/\mathfrak{h}$ is an $\mathrm{Ad}(H)$-invariant subspace encoding the distribution. Using this data one can easily compare G-homogeneous distributions. We then identify the data for the rolling distributions (Q, D_ρ) with respect to the group $G = \mathrm{SO}_3 \times \mathrm{SO}_3$.

In Section 4 we use the root diagram of G_2 to give our first construction of a G_2-homogeneous distribution data (G_2, P, W). Here $P \subset G_2$ is a maximal parabolic subgroup. The identification of the resulting G_2-homogenous distribution on G_2/P with the rolling distribution on $\widetilde{Q}$ for radius ratios $\rho = 3$ or $\rho = 1/3$ is done by calculating the group-theoretic data with respect to the maximal compact subgroup $K \cong \mathrm{SO}_4$. This amounts to the embedding of $\mathfrak{so}_3 \oplus \mathfrak{so}_3$ in $\mathfrak{g}_2$ and is the subject of Section 5 (and of Appendix B), which forms the heart of this article.

In Section 6 we give a second construction of the rolling distribution with a natural G_2-action. Here we use the fact that G_2 is the automorphism group of the algebra of *split* octonions $\widetilde{\mathbf{O}}$ (analogous to the better-known fact that the compact form of G_2 is the automorphism group of the usual octonions, also called *Cayley numbers*). We consider the representation of G_2 on the 7-dimensional space of imaginary octonions. This action preserves a quadratic form of signature $(3,4)$ and we let C be the corresponding (ray) projectivized null cone. There is a rank 2 distribution on C defined solely in terms of octonion multiplication so it is automatically G_2-invariant. We then extract the G_2-homogeneous distribution data corresponding to this construction in order to identify it with the first construction.

In Section 7 we prove that the G_2-action on $\widetilde{Q}$ does not descend to Q.

Appendix C is historical. Following suggestions by R. Bryant we looked into Cartan's thesis and found that much of the content of Section 6, and hence of the rolling distribution, already appears there.

A CONFESSION. Despite all our efforts, the "3" of the radius ratio $3:1$ remains mysterious. In this article, it comes out of the calculations needed for the embedding of $\mathfrak{so}_3 \oplus \mathfrak{so}_3$ into $\mathfrak{g}_2$ (Section 5 and Appendix B). Somehow, we believe one should be able to "see" the $3:1$ ratio in the geometry of the root diagram of $\mathfrak{g}_2$, without calculations, just as we were able to see in it the distribution data $(\mathfrak{g}_2, \mathfrak{p}, W)$, but we cannot quite accomplish it, and so we resort to a tedious calculation with the structure constants of $\mathfrak{g}_2$.

AN OPEN PROBLEM. Find a geometric or dynamical interpretation for the "3" of the $3:1$ ratio.

For work in this direction see Agrachev [1] and also Kaplan and Levstein [11].

ACKNOWLEDGEMENTS. Robert Bryant has been crucial, at various key steps along the way, in steering us in the right direction. Martin Weissmann supplied us with key information regarding G_2, and the crucial Vogan reference [16].

CONTENTS

1. HISTORY AND BACKGROUND

1.1 ON DISTRIBUTIONS

Here a *distribution* means a linear subbundle of the tangent bundle of a manifold. Mathematicians usually first encounter the integrable and the contact distributions. Both have infinite-dimensional symmetry groups. Cartan investigated rank 2 and 3 distributions in 5 dimensions in detail, in his famous *Five Variables Paper* [4]. He showed there (among many other results) that the generic distribution of rank 2 or 3 in 5 dimensions has *no* continuous local symmetries.

The distributions Cartan investigated are those whose growth vector is everywhere $(2,3,5)$. To say that a distribution D *has growth* $(2,3,5)$ *at a point* p means the following. Let X, Y be locally defined vector fields spanning the distribution near p and set $Z = [X,Y]$. Then $X(p)$, $Y(p)$, $Z(p)$ span a 3-dimensional subspace of the tangent space of the manifold — this is the "3" of $(2,3,5)$ —, while $\{X(p), Y(p), Z(p), [X,Z](p), [Y,Z](p)\}$ span the entire 5-dimensional tangent space — the "5" of $(2,3,5)$. The $(2,3,5)$ growth condition is an open condition on germs of distributions: if it holds at a point, it holds in a neighbourhood of that point.

Cartan's work was purely local. He worked out the complete set of local invariants for $(2,3,5)$ distributions. The invariants Cartan constructed are certain symmetric covariant tensors defined on the distribution, and can be thought of as extensions of the Riemann curvature tensor. For the distribution's symmetry group to act transitively all of Cartan's invariants must be constant. To get the maximal dimensional symmetry group, the Cartan invariants must all vanish, in which case we call the distribution *flat*. Any flat distribution is locally diffeomorphic to that of the *Carnot group distribution* associated to the unique graded nilpotent Lie group $\mathfrak{n} = \mathfrak{n}_{2,3,5}$ of growth $(2,3,5)$, and the local symmetry algebra of such a distribution is $\mathfrak{g}_2$. Here, by the *local symmetry algebra* of a distribution, we mean the algebra of vector fields X satisfying $[X, \Gamma(D)] \subset \Gamma(D)$ where $\Gamma(D)$ is the sheaf of local sections of vector fields tangent to the distribution.

As mentioned in the above quote from Bryant, Cartan [4] presented several geometric realizations of the flat case. Bryant and Hsu [2] (see Section 3.4) pointed out the rolling incarnation of G_2. A $(2,3,5)$ distribution will arise whenever one rolls one Riemannian surface on another, provided their Gaussian curvatures are not equal. The Cartan invariants vanish if and only if the ratio of their curvatures are $1:9$, hence the magic $1:3$ radii for spheres. We could also achieve the maximal local symmetry algebra $\mathfrak{g}_2$ by rolling two hyperbolic planes along each other, provided their "radii" are in the ratio $1:3$. More history, and more instances of the flat G_2 system are explained in Byrant [3].

Zelenko and Agrachev have been able to rederive Cartan's $(2,3,5)$ invariants using a perspective arising from geometric control theory. See [1] and references therein. Their construction is based on the singular curves. Every non-integrable rank 2 distribution in dimension n, $n > 3$, admits a special family of integral curves known as *singular* or *abnormal* [12]. These are the integral curves which admit no fixed endpoint local variations through integral curves. In the case of distributions of growth $(2,3,5)$ there is precisely one singular curve (up to reparameterization) through every point *in every* *direction* tangent to D. In the particular case of rolling one Riemannian surface along another, the singular curves correspond to rolling one geodesic along another. The foundation for Zelenko and Agrachev's reconstruction of Cartan's invariants is a kind of Jacobi field theory of singular curves.

Tanaka and his school have established a wonderful generalization of the passage from the flat nilpotent model $\mathfrak{n}_{2,3,5}$ to $\mathfrak{g}_2$. Associated to each point p of a manifold endowed with a non-integrable distribution there is a graded nilpotent Lie algebra $\mathfrak{m}(p)$ called by Tanaka and his school the *symbol algebra* of the distribution and by others the *nilpotentization* of the distribution. The dimension of $\mathfrak{m}(p)$ is that of the underlying manifold. Call the distribution *of type* $\mathfrak{m}$ if all the different algebras $\mathfrak{m}(p)$ are isomorphic to the same Lie algebra $\mathfrak{m}$, i.e. if the isomorphism type of the $\mathfrak{m}(p)$'s does not vary with p. Every $(2,3,5)$ distribution is of type $\mathfrak{n}_{2,3,5}$. Out of any given graded nilpotent $\mathfrak{m}$, Tanaka outlined a purely algebraic construction of another graded Lie algebra $\mathfrak{g} \supset \mathfrak{m}$ (possibly infinite-dimensional) called the *prolongation* of $\mathfrak{m}$. This $\mathfrak{g}$ represents, roughly speaking, the maximal possible symmetry of a distribution of type $\mathfrak{m}$: the symmetry algebra of any type-$\mathfrak{m}$ distribution, after being subjected to a grading process which changes its Lie algebraic structure, but not its dimension, becomes a subalgebra of $\mathfrak{g}$. The prolongation of the $(2,3,5)$ algebra is $\mathfrak{g}_2$, and this fact is an algebraic restatement of Cartan's work on the flat $(2,3,5)$ distributions. Tanaka's prolongation method yields a proof that $\mathrm{Aut}(\widetilde{Q},\widetilde{D}) \subset G_2$ (our Theorem 1) alternative to Cartan's proof. Yamaguchi [17] has classified all $\mathfrak{m}$'s whose $\mathfrak{g}$'s are simple. To each of these pairs $(\mathfrak{m},\mathfrak{g})$ is associated an intricate differential geometry. Most of these geometries have not been explored in any detail.

1.2 On G_2

The Lie algebra $\mathfrak{g}_2$ is the smallest of the exceptional simple Lie algebras. In 1894 Killing uncovered strong evidence of its existence by constructing the root lattice for $\mathfrak{g}_2$. But the theorem variously known as Serre's theorem, or Chevalley's theorem [13] which asserts that every root lattice is the root lattice of a Lie algebra had not yet been established, so the existence of $\mathfrak{g}_2$ was left hanging. Cartan established the existence of $\mathfrak{g}_2$ directly by constructing its 7-dimensional representation, a representation intimately connected with our second construction of $(\widetilde{Q},\widetilde{D})$. He did so in one page of his thesis [5], and we have devoted Appendix C to this page and to its connection with our second construction. In 1914 Cartan [6] showed that G_2 can be realized as the automorphism group of the octonions. For our split G_2 he used *split* *octonions*. The compact form of G_2 appears in the Berger list of potential holonomy groups of Riemannian metrics. In part because of its appearance in Berger's list, the compact G_2 has been popular among string theorists, but its popularity has faded by now in that rapidly changing field.

2. DISTRIBUTION FOR ROLLING BALLS

2.1 THE DISTRIBUTION

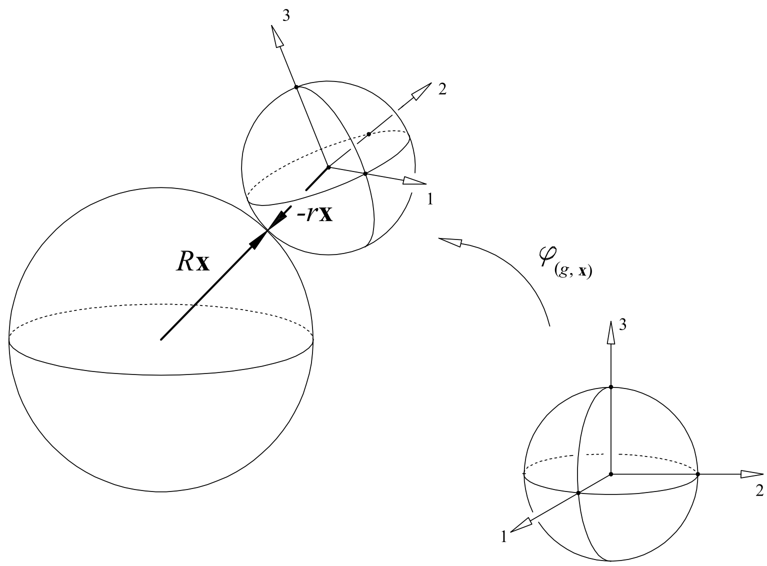

FIGURE 2
Rolling a ball on another ball

Take the first ball to be stationary, of radius R, with center at the origin of a Euclidean space called *inertial space*. Imagine the second ball, of radius r, in its own Euclidean space, with points on that ball called *material points*. Now roll the second ball on the first. We record the instantaneous position of the second ball relative to the first by an isometry (rigid motion) $\varphi_{(g,\mathbf{x})} \colon \mathbf{R}^3 \to \mathbf{R}^3$ mapping each material point $\mathbf{P}$ of the second ball to a point

$$\mathbf{p} = \varphi_{(g,\mathbf{x})}(\mathbf{P}) = g\,\mathbf{P} + (R+r)\,\mathbf{x}$$

of inertial space. Here $(g, \mathbf{x}) \in \mathrm{SO}_3 \times S^2$, $R\,\mathbf{x}$ is the point of contact of the two balls, $(R+r)\,\mathbf{x}$ is the center of the second ball, and $g \in \mathrm{SO}_3$ describes the rotation of the second ball relative to its initial position. See Figure 2. We have thus identified the configuration space Q for our rolling problem with the manifold $\mathrm{SO}_3 \times S^2$. For elementary, visceral accounts of rolling a ball on a plane, accessible to advanced undergraduates, we recommend [8] or [10].

Let $(g_t, \mathbf{x}_t) \in Q$ be a differentiable rolling motion. Let $\omega = \omega_t \in \mathbf{R}^3 \cong \mathfrak{so}_3$ be the angular velocity of the rolling ball relative to its center, measured with respect to inertial axes. In other words, if $\mathbf{P}$ is a material point fixed on the second ball, $\dot{\mathbf{P}} = 0$, and if we write $\mathbf{p}_t = g_t\,\mathbf{P}$, then $\dot{\mathbf{p}} = \dot{g}g^{-1}\mathbf{p} = \omega \times \mathbf{p}$. Then we have

PROPOSITION 1. *Let $Q = \mathrm{SO}_3 \times S^2$ be the configuration space of two rolling balls of radii R and r. Let $\rho = R/r$. Then a curve $(g_t, \mathbf{x}_t) \in Q$ describes a rolling motion without slipping or spinning if and only if*

(1) $(\rho+1)\,\dot{\mathbf{x}} = \omega \times \mathbf{x}$ *(no-slip condition),*

(2) $\langle \omega, \mathbf{x} \rangle = 0$ *(no-spin condition, i.e. ω needs to be tangent to the stationary ball at $R\,\mathbf{x}$).*

Proof. (1) The contact point between the two balls is $\mathbf{p} = R\,\mathbf{x}$ on the first ball, $\mathbf{P} = -g^{-1}r\,\mathbf{x}$ with respect to the second ball. For non-slip, their velocities must match: $\dot{\mathbf{p}} = g\,\dot{\mathbf{P}}$. Now $\dot{\mathbf{p}} = R\,\dot{\mathbf{x}}$ and

$$\dot{\mathbf{P}} = [-\frac{d}{dt}g^{-1}]r\,\mathbf{x} - g^{-1}r\,\dot{\mathbf{x}} = g^{-1}\dot{g}g^{-1}r\,\mathbf{x} - g^{-1}r\,\dot{\mathbf{x}} = g^{-1}r(\omega \times \mathbf{x} - \dot{\mathbf{x}})\,,$$

hence the no-slip condition $\dot{\mathbf{p}} = g\,\dot{\mathbf{P}}$ is equivalent to $R\,\dot{\mathbf{x}} = r(\omega \times \mathbf{x} - \dot{\mathbf{x}})$, from which (1) follows.

(2) Let $\mathbf{P}$ be a material point *fixed* on the second ball ($\dot{\mathbf{P}} = 0$). From the inertial point of view, which is to say, from the point of view of the first ball with its center at the origin of inertial space, the position of this material point is $\mathbf{p} = g\,\mathbf{P} + (R+r)\,\mathbf{x}$, and so its velocity

$$\dot{\mathbf{p}} = \dot{g}\,\mathbf{P} + (R+r)\,\dot{\mathbf{x}} = \dot{g}g^{-1}[\mathbf{p} - (R+r)\,\mathbf{x}] + (R+r)\,\dot{\mathbf{x}} = \omega \times [\mathbf{p} - (R+r)\,\mathbf{x}] + (R+r)\,\dot{\mathbf{x}}\,.$$

Using the no-slip equation, $(R+r)\,\dot{\mathbf{x}} = r\omega \times \mathbf{x}$, we get

$$\dot{\mathbf{p}} = \omega \times [\mathbf{p} - (R+r)\,\mathbf{x}] + r\omega \times \mathbf{x} = \omega \times (\mathbf{p} - R\,\mathbf{x})\,.$$

The equation $\dot{\mathbf{p}} = \omega \times (\mathbf{p} - R\,\mathbf{x})$ asserts that the instantaneous motion of the second ball is a rotation whose axis of rotation (a line) passes through $R\,\mathbf{x}$, the point of contact of the two balls, in the direction of ω and with angular velocity of magnitude $\|\omega\|$. The no-spin condition is that the second ball does not spin about the point of contact of the two balls, which is to say that ω should have no component orthogonal to the common tangent plane of the two balls, i.e. $\langle\omega, \mathbf{x}\rangle = 0$. $\square$

The two conditions in the last proposition define together a rank 2 distribution $D_\rho \subset TQ$, depending on the radius ratio $\rho = R/r$. This is the *rolling distribution*.

REMARK. The no-slip condition, $(\rho+1)\,\dot{\mathbf{x}} = \omega \times \mathbf{x}$, implies the following somewhat counter-intuitive result: as the moving ball rolls once around a great circle of the stationary ball, then upon returning it has rotated $\rho+1$ times around, not ρ times. It may help to play with two coins of the same monetary value ($\rho = 1$) in order to get convinced of this fact.

2.2 THE OBVIOUS SYMMETRY

The group $\mathrm{SO}_3 \times \mathrm{SO}_3$ acts on Q by $\varphi_{(g,\mathbf{x})} \mapsto g' \circ \varphi_{(g,\mathbf{x})} \circ g''^{-1}$, where $g', g'' \in \mathrm{SO}_3$. In terms of $(g, \mathbf{x})$ this action is

$$(g, \mathbf{x}) \mapsto (g' g g''^{-1}, g' \mathbf{x}), \quad g', g'' \in \mathrm{SO}_3 .$$

This action is transitive and preserves the rolling distribution D_ρ for any value of $\rho = R/r$. The proofs of these assertions are easy and left as exercises.

3. GROUP-THEORETIC DESCRIPTION OF THE ROLLING DISTRIBUTION

In the previous section we wrote down a family of distributions D_ρ on $Q = \mathrm{SO}_3 \times S^2$, depending on a positive real parameter ρ and admitting a transitive $\mathrm{SO}_3 \times \mathrm{SO}_3$-action. Our aim now is to show that for two values of ρ, $\rho = 3$ and $\rho = 1/3$, there is a G_2-action on the universal (double) cover $\widetilde{Q} = S^3 \times S^2$ which preserves the lifted distribution $\widetilde{D}_\rho \subset T\widetilde{Q}$ ('lifted' here means that the local diffeomorphism $\widetilde{Q} \to Q$ sends $\widetilde{D}_\rho$ to D_ρ). The G_2-action on $\widetilde{Q}$ does not descend to Q (we will show this in Section 7), but restricted to a maximal compact subgroup $K \subset G_2$ (a double-cover of $\mathrm{SO}_3 \times \mathrm{SO}_3$), the action does descend to Q, and in fact gives the $\mathrm{SO}_3 \times \mathrm{SO}_3$-action on Q.

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Now when working with homogeneous manifolds and distributions it is more convenient to work with the associated group-theoretic data, rather than the manifolds and distributions themselves. In what follows we give a general set-up for describing homogeneous distributions in terms of *group-theoretic data*. This general description is followed by the specific determination of the group-theoretic data for the rolling distributions (Q, D_ρ).

3.1 G-HOMOGENEOUS DISTRIBUTIONS

Let G be a Lie group. A *G-homogeneous distribution* is a pair (M, D) where M is a manifold on which G acts transitively and $D \subset TM$ is a G-invariant distribution. Fixing a base point $m_0 \in M$ with isotropy $H \subset G$ we obtain a G-equivariant identification $G/H \cong M$, where $gH \mapsto gm_0$. Differentiating the map $G \to M$, $g \mapsto gm_0$, at $g = e$ (the identity of G) we obtain a map $\mathfrak{g} \to T_{m_0}M$, called the *infinitesimal action* of $\mathfrak{g}$ at m_0, and an $\mathrm{Ad}(H)$-equivariant identification $\mathfrak{g}/\mathfrak{h} \cong T_{m_0}M$ where $\mathfrak{h}$, $\mathfrak{g}$ denote the Lie algebras of H, G (respectively). Under this identification, the distribution plane at m_0, $D_{m_0} \subset T_{m_0}M$, corresponds to an $\mathrm{Ad}(H)$-invariant subspace $W \subset \mathfrak{g}/\mathfrak{h}$.

In this way, every G-homogeneous distribution (M, D) corresponds to group-theoretic data (G, H, W), where $H \subset G$ is a closed subgroup with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ and $W \subset \mathfrak{g}/\mathfrak{h}$ is an H-invariant subspace. The adjoint action of G defines an equivalence relation on the set of pairs (H, W) so that different choices of base points on Q correspond to equivalent pairs $(H, W) \sim (H', W')$. Conversely, given the data (G, H, W), we can construct a G-homogeneous distribution (M, D) by letting G act by left translations on the right H-coset space $M := G/H$, and use this G-action to push the plane $D_{[e]} := W \subset \mathfrak{g}/\mathfrak{h} \cong T_{[e]}(G/H)$ around all of M so as to define the distribution $D \subset TM$.

On the level of Lie algebras, the data $(\mathfrak{g}, \mathfrak{h}, W)$ determines (M, D) up to a cover. If, as in our case of $\mathfrak{g} = \mathfrak{so}_3 \oplus \mathfrak{so}_3$, the simply connected Lie group G realizing $\mathfrak{g}$ is compact, then there are only finitely many homogeneous distributions (G, H, W) which realize the given Lie algebraic data $(\mathfrak{g}, \mathfrak{h}, W)$.

3.2 GROUP-THEORETIC DATA FOR THE ROLLING DISTRIBUTION

We now determine the data (G, H, W) corresponding to the rolling distributions (Q, D_ρ) of Section 2.1. Here $G = \mathrm{SO}_3 \times \mathrm{SO}_3$, $Q = \mathrm{SO}_3 \times S^2$, $\dim H = 1$, $\dim W = 2$ and the G-action on Q is given in Section 2.2. Identify the Lie algebra $\mathfrak{so}_3 \oplus \mathfrak{so}_3$ of $\mathrm{SO}_3 \times \mathrm{SO}_3$ with $\mathbf{R}^3 \times \mathbf{R}^3$, thought of as the set of pairs of angular velocities (ω', ω''), with Lie bracket given by the cross product:

$$[(\omega', \omega''), (\eta', \eta'')] = (\omega' \times \eta', \omega'' \times \eta'') .$$

The first factor ω' corresponds to the first (stationary) sphere, of radius R, while the second factor ω'' corresponds to the second (rolling) sphere of radius r.

Fix the base point to be $q_0 = (1, \mathbf{e}_3) \in \mathrm{SO}_3 \times S^2 = Q$. The isotropy at this base point is the circle subgroup H consisting of elements of the form (h, h), where h is a rotation around the $\mathbf{e}_3$ axis. Thus $\mathfrak{h} = \mathbf{R}(\mathbf{e}_3, \mathbf{e}_3) \subset \mathbf{R}^3 \times \mathbf{R}^3$. Using the standard metric on $\mathfrak{g} = \mathfrak{so}_3 \oplus \mathfrak{so}_3 = \mathbf{R}^3 \times \mathbf{R}^3$ we can identify $\mathfrak{g}/\mathfrak{h} \cong \mathfrak{h}^\perp$, so that the plane of the distribution at the base point is given by some 2-plane in $\mathfrak{h}^\perp$. Let us determine this 2-plane explicitly.

PROPOSITION 2. *The rolling distribution D_ρ on $\mathrm{SO}_3 \times S^2$ of Proposition* 1 *is given by the* 2*-plane $W_\rho \subset (\mathfrak{so}_3 \oplus \mathfrak{so}_3)/\mathfrak{h} \cong \mathfrak{h}^\perp$ defined by the equations*

$$\langle \omega', \mathbf{e}_3 \rangle = \langle \omega'', \mathbf{e}_3 \rangle = 0 , \qquad \rho\omega' + \omega'' = 0 ,$$

where $\rho = R/r$.

Proof. Since $\mathfrak{h} \subset \mathbf{R}^3 \times \mathbf{R}^3$ is generated by the vector $(\omega', \omega'') = (\mathbf{e}_3, \mathbf{e}_3)$, $\mathfrak{h}^\perp \subset \mathbf{R}^3 \times \mathbf{R}^3$ is given by the equation $\langle \omega', \mathbf{e}_3 \rangle + \langle \omega'', \mathbf{e}_3 \rangle = 0$, i.e.

$$\langle \omega' + \omega'', \mathbf{e}_3 \rangle = 0 .$$

From the formula for the $\mathrm{SO}_3 \times \mathrm{SO}_3$-action in §2.2 we compute the infinitesimal action at the base point $\mathfrak{so}_3 \oplus \mathfrak{so}_3 \to T_{q_0}Q = \mathfrak{so}_3 \times \mathbf{e}_3^\perp$ to be the map

$$(\omega', \omega'') \mapsto (\omega, \dot{\mathbf{x}}) ,$$

with

$$\omega = \omega' - \omega'', \quad \dot{\mathbf{x}} = \omega' \times \mathbf{e}_3 .$$

Substituting these into the rolling conditions at the base point (see §2.1),

$$\langle \omega, \mathbf{e}_3 \rangle = 0 , \quad (\rho + 1)\,\dot{\mathbf{x}} = \omega \times \mathbf{e}_3 ,$$

we obtain

$$\langle \omega' - \omega'', \mathbf{e}_3 \rangle = 0 , \quad [\rho\omega' + \omega''] \times \mathbf{e}_3 = 0 .$$

Adding the condition of orthogonality to $\mathfrak{h}$, $\langle \omega' + \omega'', \mathbf{e}_3 \rangle = 0$, we obtain the above equations. □

We have thus assembled the group-theoretic data $(\mathrm{SO}_3 \times \mathrm{SO}_3, H, W_\rho)$ corresponding to the rolling of two balls of radius ratio $\rho = R/r$.

3.3 Shrinking and inflating the group

The following observation will be key later on. Suppose that (M,D) is a G-homogeneous distribution and (G,H,W) the corresponding data, i.e. $H \subset G$ and $W \subset \mathfrak{g}/\mathfrak{h}$ is $\mathrm{Ad}(H)$-invariant. Let $G_1 \subset G$ be a subgroup for which the restriction of the G-action on M is still transitive. The corresponding *shrunk* data is (G_1,H_1,W_1), where $H_1 = H \cap G_1$ and $W_1 \subset \mathfrak{g}_1/\mathfrak{h}_1$ corresponds to W under the linear isomorphism $\mathfrak{g}_1/\mathfrak{h}_1 \to \mathfrak{g}/\mathfrak{h}$, induced by the diffeomorphism $G_1/H_1 \cong G/H$.

Now suppose we wish to reverse this process, i.e. we are given a G_1-homogeneous distribution (M,D) and we wish to extend the G_1-action to a larger group G. (In our case G_1 is a double-cover of the *obvious* $\mathrm{SO}_3 \times \mathrm{SO}_3$, and G is G_2.) Then in terms of group-theoretic data, this amounts to the following procedure: given the data (G_1,H_1,W_1), we need to embed it into the data (G,H,W) by finding an embedding of groups $G_1 \hookrightarrow G$ (injective homomorphism), which maps H_1 to the intersection of the image of G_1 with H, and such that the induced isomorphism $\mathfrak{g}_1/\mathfrak{h}_1 \cong \mathfrak{g}/\mathfrak{h}$ maps W_1 to W.

At the Lie algebra level, this discussion asserts that if we embed the Lie algebraic data $(\mathfrak{g}_1,\mathfrak{h}_1,W_1)$ into $(\mathfrak{g},\mathfrak{h},W)$, then, upon passing to a cover (if necessary), $(\widetilde{G}_1,\widetilde{H}_1,W_1)$ embeds into $(\widetilde{G},\widetilde{H},W)$, hence $\widetilde{G}$ acts on $(\widetilde{M},\widetilde{D})$, where $\widetilde{M} = \widetilde{G}_1/\widetilde{H}_1 = \widetilde{G}/\widetilde{H}$. We therefore obtain a *local* action of G on M, i.e. an embedding of $\mathfrak{g}$ in $\mathfrak{aut}(U,D|_U)$ for all sufficiently small $U \subset M$.

We thus see that the task of extending the $\mathrm{SO}_3 \times \mathrm{SO}_3$-action on the rolling distribution (Q,D) to a G_2-action on some cover $(\widetilde{Q},\widetilde{D})$ amounts to finding a (suitably chosen) embedding $\mathfrak{so}_3 \oplus \mathfrak{so}_3 \hookrightarrow \mathfrak{g}_2$.

4. A G_2-homogeneous distribution

We now describe the other main actor in this paper, a distribution with Lie algebraic data $(\mathfrak{g}_2,\mathfrak{p},W)$. Please see the root diagram of $\mathfrak{g}_2$ in Figure 3. This diagram will be explained immediately below. The decorations on the diagram will be explained a bit later.

4.1 A reminder of the meaning of the root diagram

The plane in which the diagram is drawn is the dual of a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}_2$. A *Cartan subalgebra* of a semi-simple Lie algebra $\mathfrak{g}$ is a maximal abelian subalgebra $\mathfrak{t} \subset \mathfrak{g}$ of semi-simple elements, i.e. each $\mathrm{ad}(T) \in \mathrm{End}(\mathfrak{g})$, $T \in \mathfrak{t}$, is diagonalizable. A given semi-simple Lie algebra $\mathfrak{g}$ has many

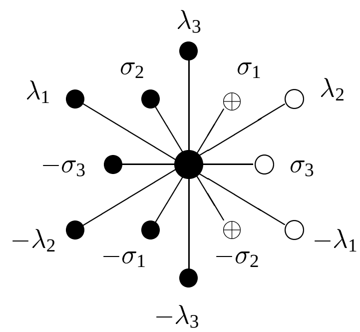

FIGURE 3
The root diagram of $\mathfrak{g}_2$

Cartan subalgebras, but they are all conjugate in $\mathfrak{g} \otimes \mathbf{C}$ and hence of the same dimension. The *rank* of $\mathfrak{g}$ is the dimension of any one of its Cartan subalgebras. The rank of $\mathfrak{g}_2$ is 2, accounting for the subscript 2 in G_2, and accounting for the fact that its root diagram is planar, so we can draw it in the manner of Figure 3. The root diagram of $\mathfrak{g}$ encodes the adjoint action of $\mathfrak{t}$ on $\mathfrak{g}$, from which one can recover the whole structure of $\mathfrak{g}$.

The commutativity of the Cartan subalgebra $\mathfrak{t}$ implies that the diagonalizable endomorphisms $\mathrm{ad}(T) \in \mathrm{End}(\mathfrak{g})$, $T \in \mathfrak{t}$, are *simultaneously* diagonalizable, resulting in a $\mathfrak{t}$-invariant decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha} \mathfrak{g}_{\alpha} \,,$$

where each $\mathfrak{g}_\alpha \subset \mathfrak{g}$ is a 1-dimensional subspace of $\mathfrak{t}$-common eigenvectors called a *root space*. The corresponding eigenvalue depends linearly on the acting element of $\mathfrak{t}$, so is given by a linear functional $\alpha \in \mathfrak{t}^*$, called a *root*. Thus

$$[T, X] = \alpha(T)X \,, \quad T \in \mathfrak{t} \,, \quad X \in \mathfrak{g}_\alpha \,.$$

When we draw the root diagram in $\mathfrak{t}^*$ we use the Killing metric in $\mathfrak{g}$ to determine the size of the roots and the angles between them. The *Killing metric* in $\mathfrak{g}$ is the bilinear form $\langle X, Y\rangle = \mathrm{tr}(\mathrm{ad}(X)\,\mathrm{ad}(Y))$. The form is non-degenerate (non-degeneracy is equivalent to semi-simplicity) and its restriction to $\mathfrak{t}$ is also non-degenerate as well. In fact, this restriction is positive-definite if all the roots are real, as can be arranged in our situation of a *split-real form*.

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For a general Cartan subalgebra of a (real) semi-simple algebra, $\mathrm{ad}(T)$ may have complex eigenvalues, hence roots may have complex values and the root space decomposition of $\mathfrak{g}$ requires complexifying $\mathfrak{g}$; however, in case $\mathfrak{g}$ is the so-called split-form of its complexification, as is the case for our G_2, one can choose a Cartan subalgebra with only real roots, and no complexification of $\mathfrak{g}$ is needed.

4.2 Example: the root diagram of $\mathfrak{g} = \mathfrak{sl}_3(\mathbf{R})$

We review the more familiar example of $\mathfrak{sl}_3(\mathbf{R})$ before proceeding to $\mathfrak{g}_2$. The Lie algebra $\mathfrak{sl}_3(\mathbf{R})$ is the vector space of 3 by 3 traceless real matrices with Lie bracket the usual matrix Lie bracket. It is the Lie algebra of the Lie group $\mathrm{SL}_3(\mathbf{R})$ of 3 by 3 real matrices with determinant 1. Like $\mathfrak{g}_2$, $\mathfrak{sl}_3(\mathbf{R})$ has rank 2, and is the non-compact split-real form of its complexification $\mathfrak{sl}_3(\mathbf{C})$.

As the Cartan subalgebra for $\mathfrak{sl}_3(\mathbf{R})$ we will take the subspace $\mathfrak{t} \subset \mathfrak{sl}_3(\mathbf{R})$ of traceless diagonal matrices,

$$\mathfrak{t} := \left\{ \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \;\middle|\; t_1 + t_2 + t_3 = 0,\ t_i \in \mathbf{R} \right\}.$$

Now $\mathfrak{sl}_3(\mathbf{R})$ has 6 roots (all real):

$$\alpha_{ij} := t_i - t_j \in \mathfrak{t}^*, \quad i \neq j, \quad i,j \in \{1,2,3\},$$

with corresponding root spaces

$$\mathfrak{g}_{\alpha_{ij}} = \mathbf{R}E_{ij},$$

where E_{ij} is the matrix whose ij entry is 1 and all of whose other entries are 0. The corresponding root space decomposition

$$\mathfrak{sl}_3 = \mathfrak{t} \oplus \sum_{i \neq j} \mathfrak{g}_{\alpha_{ij}}$$

is just the decomposition of a matrix as a diagonal matrix plus its off-diagonal terms. The metric induced on $\mathfrak{t}$ by the Killing metric is some multiple of the standard Euclidean metric, so that $\langle T, T' \rangle = c \sum_i t_i t_i'$ for some $c > 0$.

4.3 Reading the root diagram

Returning to the general semi-simple $\mathfrak{g}$, we observe that much of the structure of $\mathfrak{g}$ can be read off from its root diagram in a formula-free manner.

Here is the key observation. Let α, β be two roots with (non-zero) root vectors $E_\alpha \in \mathfrak{g}_\alpha$, $E_\beta \in \mathfrak{g}_\beta$, so that

$$[T, E_\alpha] = \alpha(T)E_\alpha\,, \quad [T, E_\beta] = \beta(T)E_\beta\,, \quad T \in \mathfrak{t}\,.$$

It then follows immediately from the Jacobi identity that

$$[T, [E_\alpha, E_\beta]] = (\alpha + \beta)(T)[E_\alpha, E_\beta]\,.$$

This means that

(1) if $\alpha + \beta \neq 0$ and is not a root then $[E_\alpha, E_\beta] = 0$;

(2) if $\alpha + \beta \neq 0$ and is a root then $[E_\alpha, E_\beta] \in \mathfrak{g}_{\alpha+\beta}$;

(3) if $\alpha + \beta = 0$, i.e. $\beta = -\alpha$, then $[E_\alpha, E_\beta] \in \mathfrak{t}$.

This set of three conclusions permits us to see at a glance from the diagram a fair amount of the structure of $\mathfrak{g}$. In the last two cases one can further show that $[E_\alpha, E_\beta]$ is non-zero and determine, with some calculations, the actual bracket, as will be illustrated in Appendix B.

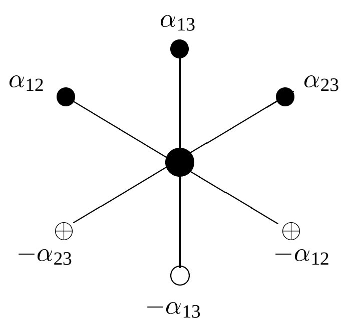

FIGURE 4
The root diagram of $\mathfrak{sl}_3$

4.4 EXAMPLE: READING THE ROOT DIAGRAM OF $\mathfrak{sl}_3$

Consider the subspace $\mathfrak{p} \subset \mathfrak{sl}_3$ spanned by $\mathfrak{t}$ and the root spaces corresponding to the roots marked with dark dots in Figure 4. The diagram, and properties (1) and (2), shows that $\mathfrak{p}$ is a 5-dimensional subalgebra. (The thick dot at the origin stands for the 2-dimensional Cartan subalgebra.) Indeed, $\mathfrak{p}$ is the subalgebra of upper triangular matrices (including diagonal ones), with corresponding subgroup $P \subset \mathrm{SL}_3$, the subgroup of upper triangular matrices with determinant 1. The quotient space $\mathrm{SL}_3(\mathbf{R})/P$ can be identified with the space F of full flags in $\mathbf{R}^3$. A *full flag* is a pair (l,π), where l is a line and π is a plane, and $l \subset \pi \subset \mathbf{R}^3$. The *standard flag* consisting of the x axis sitting inside the xy plane has isotropy group P. The tangent space to F at this base point is naturally identified with $\mathfrak{sl}_3/\mathfrak{p}$, represented in the root diagram by the remaining three light dots. Two of the light dots are marked $+$. The diagram, combined with properties (1), (2) and (3), shows that the root spaces corresponding to these roots span a $\mathfrak{p}$-invariant 2-dimensional subspace of $\mathfrak{sl}_3/\mathfrak{p}$ which Lie generates the root space associated with the third light dot. This means that we have on F an $\mathrm{SL}_3(\mathbf{R})$-invariant rank 2 *contact distribution*, i.e. a non-integrable distribution that Lie generates the tangent bundle.

This distribution can be geometrically interpreted as the *tautological* contact distribution on F ("l moves tangent to π"). This distribution is spanned by two vector fields, corresponding to the two $+$'s in Figure 4. One vector field generates the flow in which the line l spins within the plane π while that plane remains fixed. The other vector field generates the flow in which the plane π rotates about the line l while the line remains fixed.

4.5 Reading the $\mathfrak{g}_2$ diagram

Now let us draw conclusions in a similar fashion from the $\mathfrak{g}_2$ diagram. There are twelve roots in the diagram (Figure 3) and so 12 root spaces. The rank of $\mathfrak{g}_2$ is 2 and so the dimension of $\mathfrak{g}_2$ is $14 = 2 + 12$. Consider the 9-dimensional subspace $\mathfrak{p} \subset \mathfrak{g}_2$ spanned by $\mathfrak{t}$ and the root spaces associated with the roots marked by the dark dots in the diagram of Figure 3. Then the diagram shows that

- $\mathfrak{p}$ is closed under the Lie bracket, i.e. is a subalgebra (a so-called *parabolic subalgebra*, a subalgebra containing a Borel subalgebra).
- Let $P \subset G_2$ be the corresponding subgroup. It follows that G_2 has a 5-dimensional homogeneous space G_2/P, whose tangent space $\mathfrak{g}_2/\mathfrak{p}$ at a point is represented by the remaining 5 light dots.
- Two of the light dots are marked with $+$. The diagram shows that their root spaces generate a 2-dimensional $\mathfrak{p}$-invariant subspace $W \subset \mathfrak{g}_2/\mathfrak{p}$, hence a G_2-invariant rank 2 distribution on G_2/P.
- This distribution is of type $(2,3,5)$. Bracketing once gives the light dot marked with σ_3 and bracketing the root space for σ_3 with W again gives the remaining two light dots.

5. THE MAXIMAL COMPACT SUBGROUP OF G_2

In the previous sections we have assembled the ingredients for group-theoretic data $(\mathrm{SO}_3 \times \mathrm{SO}_3, H, W_\rho)$ and (G_2, P, W). Next, in order to define a G_2-action on some covering space of the rolling distribution (Q, D), following the outline of Section 3.3 ("Shrinking and inflating the group"), we need to embed the data $(\mathfrak{so}_3 \oplus \mathfrak{so}_3, \mathfrak{h}, W_\rho)$ in $(\mathfrak{g}_2, \mathfrak{p}, W)$, for $\rho = 3$ and $\rho = 1/3$. This amounts to the appropriate identification of $\mathfrak{so}_3 \oplus \mathfrak{so}_3$ as the maximal compact subalgebra of $\mathfrak{g}_2$.

5.1 FINDING MAXIMAL COMPACTS

How can we "see" a maximal compact subgroup of G_2 tangled within its root diagram? Let us look back again at the example of $\mathrm{SL}_3(\mathbf{R})$. Here the maximal compact subgroup is SO_3, with Lie algebra $\mathfrak{so}_3$, the set of 3×3 antisymmetric matrices. These are spanned by the vectors $E_{ij} - E_{ji}$, $i > j$. So we see that corresponding to each pair of *antipodal* roots $\pm\alpha_{ij}$ we have one generator of $\mathfrak{K}$, lying in the sum of the two corresponding root spaces.

More generally, for the *split* real form of any semi-simple Lie algebra (such as our $\mathfrak{g}_2$), the situation is similar: we get the Lie algebra $\mathfrak{K}$ of a maximal compact subgroup $K \subset G$ by taking the sum of 1-dimensional subspaces, one subspace for each pair of antipodal roots $\pm\alpha$. In fact, for a certain particularly nice choice of root vectors $E_\alpha \in \mathfrak{g}_\alpha$ (a *Weyl basis*) the sought-for line is $\mathbf{R}(E_\alpha - E_{-\alpha})$, as in the $\mathfrak{sl}_3$ case.

In the case of $\mathfrak{g}_2$ we thus have:

- $\mathfrak{K}$ is the sum of six 1-dimensional subspaces $\mathfrak{s}_i$, $\mathfrak{l}_i$, $i = 1, 2, 3$, where $\mathfrak{s}_i$ lies in the sum of the root spaces corresponding to the *short* roots $\pm\sigma_i$, and $\mathfrak{l}_i$ lies in the sum of the root spaces corresponding to the *long* roots $\pm\lambda_i$.
- The isotropy of the K-action, $K \cap P \subset K$, is given in the diagram by the vertical segment $\mathfrak{l}_3$.
- The distribution plane $W_1 \subset \mathfrak{K}/\mathfrak{l}_3$ corresponding to $W \subset \mathfrak{g}_2/\mathfrak{p}$ is generated by $\mathfrak{s}_1, \mathfrak{s}_2$ (mod $\mathfrak{l}_3$).

We now need to identify this *shrunk* data $(\mathfrak{K}, \mathfrak{l}_3, W_1)$ with $(\mathfrak{so}_3 \oplus \mathfrak{so}_3, \mathfrak{h}, W_\rho)$, for $\rho = 3$ or $\rho = 1/3$.

5.2 $\mathfrak{so}_3 \oplus \mathfrak{so}_3 \simeq \mathfrak{K}$

Our task is to define an embedding $\mathfrak{so}_3 \oplus \mathfrak{so}_3 \hookrightarrow \mathfrak{g}_2$ that maps the data $(\mathfrak{so}_3 \oplus \mathfrak{so}_3, \mathfrak{h}, W_\rho)$, for $\rho = 3$ or $\rho = 1/3$, to the data $(\mathfrak{K}, \mathfrak{l}_3, W_1)$. This entails the decomposition of $\mathfrak{K}$ into the direct sum of two ideals, each isomorphic to $\mathfrak{so}_3$. It would have been quite nice and simple if the sought-for decomposition of $\mathfrak{K}$ had been the decomposition into *long* ($\mathfrak{l}_i$) and *short* ($\mathfrak{s}_i$). But this is not the case. For the diagram shows that although the $\mathfrak{l}_i$ generate an $\mathfrak{so}_3$ subalgebra of $\mathfrak{K}$, this subalgebra is not an ideal, so is *not* one of the summands in the decomposition. And the $\mathfrak{s}_i$ do not even generate a subalgebra. We have to work harder.

PROPOSITION 3. *There is a basis $\{S_i, L_i \mid i = 1,2,3\}$ of $\mathfrak{K}$, with $S_i \in \mathfrak{s}_i$ and $L_i \in \mathfrak{l}_i$, such that*

$$[L_i, L_j] = \epsilon_{ijk} L_k\,, \quad [L_i, S_j] = \epsilon_{ijk} S_k\,, \quad [S_i, S_j] = \epsilon_{ijk}(\tfrac{3}{4} L_k - S_k)\,,$$

where ϵ_{ijk} is the totally antisymmetric tensor on 3 indices ($\epsilon_{ijk} = 1$ if ijk is a cyclic permutation of 123, -1 if it is anticyclic, and 0 otherwise).

The proof of this proposition is relegated to Appendix B. It consists of simple but tedious calculations which we could not "see" in the diagram. We tried. We were reduced to picking a reasonably nice basis for $\mathfrak{g}_2$ and calculating the corresponding structure constants with the help of Serre [13].

Continuing with the notation of the proposition, set

$$\mathbf{e}'_i := \frac{3L_i + 2S_i}{4}\,, \quad \mathbf{e}''_i := \frac{L_i - 2S_i}{4}\,, \quad i = 1,2,3\,.$$

These 6 vectors form a new basis for $\mathfrak{K}$ and satisfy the standard $\mathfrak{so}_3 \oplus \mathfrak{so}_3$ commutation relations

$$[\mathbf{e}'_i, \mathbf{e}'_j] = \epsilon_{ijk}\mathbf{e}'_k\,, \quad [\mathbf{e}''_i, \mathbf{e}''_j] = \epsilon_{ijk}\mathbf{e}''_k\,, \quad [\mathbf{e}'_i, \mathbf{e}''_j] = 0\,, \tag{1}$$

thus establishing the Lie algebra isomorphism $\mathfrak{so}_3 \oplus \mathfrak{so}_3 \simeq \mathfrak{K}$.

COROLLARY 1. *The map $\mathfrak{so}_3 \oplus \mathfrak{so}_3 \to \mathfrak{K}$ defined by $(\mathbf{e}_i, 0) \mapsto \mathbf{e}'_i$, $(0, \mathbf{e}_i) \mapsto \mathbf{e}''_i$, $i = 1,2,3$, is a Lie algebra isomorphism. It maps $\mathfrak{h} = \mathbf{R}(\mathbf{e}_3, \mathbf{e}_3)$ to $\mathfrak{l}_3 = \mathbf{R}L_3$. It maps the 2-plane in $\mathfrak{so}_3 \oplus \mathfrak{so}_3$ defined in the proposition of §4 for $\rho = 3$ to the 2-plane $\mathfrak{s}_1 \oplus \mathfrak{s}_2 \subset \mathfrak{K}$, thus mapping $W_\rho \subset \mathfrak{so}_3 \oplus \mathfrak{so}_3/\mathfrak{h}$ to $W_1 \subset \mathfrak{K}/\mathfrak{l}_3$. Interchanging the summands in $\mathfrak{so}_3 \oplus \mathfrak{so}_3$, i.e. mapping $(0, \mathbf{e}_i) \mapsto \mathbf{e}'_i$, $(\mathbf{e}_i, 0) \mapsto \mathbf{e}''_i$, corresponds to replacing $\rho = 3$ by $\rho = 1/3$.*

Proof of corollary. The first assertion is Equation (1) above. The rest is easily verified using the last proposition. $\square$

COROLLARY 2. *Let D_ρ be the rolling distribution on $Q = \mathrm{SO}_3 \times S^2$ for two balls of radius ratio $\rho = R/r$. Let $\widetilde{Q} = S^3 \times S^2$ equipped with the distribution $\widetilde{D}_\rho$ lifted to the double covering $\widetilde{Q} \to Q$. Then, for $\rho = 3$ or $\rho = 1/3$ there is an effective G_2-action on $(\widetilde{Q}, \widetilde{D}_\rho)$ whose restriction to the maximal compact group $K \subset G_2$ covers the obvious $\mathrm{SO}_3 \times \mathrm{SO}_3$-action on (Q, D_ρ).*

Proof. Let $\mathrm{SU}_2 \to \mathrm{SO}_3$ be the universal double-cover, such that $U_1 \subset \mathrm{SU}_2$ (the subgroup of diagonal elements) is mapped onto the subgroup of rotations around the $\mathbf{e}_3$-axis. Then $\mathrm{SU}_2 \times \mathrm{SU}_2 \to \mathrm{SO}_3 \times \mathrm{SO}_3$ is the universal (four-fold) cover. Let $G_1 = \mathrm{SU}_2 \times \mathrm{SU}_2 / \pm(1,1)$. Then $G_1 \to \mathrm{SO}_3 \times \mathrm{SO}_3$ is a double-cover of $\mathrm{SO}_3 \times \mathrm{SO}_3$. Let $H_1 \subset G_1$ be the image of U_1 under the diagonal embedding $\mathrm{SU}_2 \to \mathrm{SU}_2 \times \mathrm{SU}_2$ followed by the double-cover $\mathrm{SU}_2 \times \mathrm{SU}_2 \to \mathrm{SU}_2 \times \mathrm{SU}_2 / \pm(1,1)$. Then under the double covering $G_1 \to \mathrm{SO}_3 \times \mathrm{SO}_3$, $H_1 \subset G_1$ is mapped *isomorphically* onto $H \subset \mathrm{SO}_3 \times \mathrm{SO}_3$. Let $Q = \mathrm{SO}_3 \times \mathrm{SO}_3 / H = \mathrm{SO}_3 \times S^2$ and $\widetilde{Q} = G_1/H_1 = S^3 \times S^2$. Then we obtain a double-cover $\widetilde{Q} \to Q$ so that the G_1-action on $\widetilde{Q}$ covers the $\mathrm{SO}_3 \times \mathrm{SO}_3$ action on Q and preserves the distribution $\widetilde{D}_\rho$ on $\widetilde{Q}$ lifted from Q through the double covering $\widetilde{Q} \to Q$.

Next, consider each of the two Lie algebra isomorphisms $\mathfrak{so}_3 \oplus \mathfrak{so}_3 \cong \mathfrak{K}$ of the previous corollary (one for $\rho = 3$, another for $\rho = 1/3$). They each define a Lie group isomorphism $G_1 \cong K$ (see Vogan [16, p. 679] or Appendix A), which identifies $H_1 \cong K \cap P$, and G_1-equivariant identifications $(G_1/H_1, \widetilde{D}_\rho) \cong (G_2/P, D)$, and thus a G_2-action on $\widetilde{Q} = G_1/H_1$, extending the G_1-action and preserving $\widetilde{D}_\rho$, whose restriction to K covers the $\mathrm{SO}_3 \times \mathrm{SO}_3$-action on Q. $\square$

How we came up with the formulae for $\mathbf{e}'_i, \mathbf{e}''_i$: We first observed that L_3 generates the isotropy $H = P \cap K$ so that we should have $L_3 = \mathbf{e}'_3 + \mathbf{e}''_3$. Since everything is symmetric in $1, 2, 3$ we concluded that $L_i = \mathbf{e}'_i + \mathbf{e}''_i$, $i = 1, 2, 3$. Next, we noted that S_3 commutes with L_3 so that we should have $S_3 = a\,\mathbf{e}'_3 + b\,\mathbf{e}''_3$ for some constants a, b, and again by symmetry $S_i = a\,\mathbf{e}'_i + b\,\mathbf{e}''_i$, $i = 1, 2, 3$. Now by using the sought-after commutation relations for the $\mathbf{e}'_i, \mathbf{e}''_i$ and the known commutations for L_i, S_i we got that a, b are roots of the equation $x^2 + x - 3/4 = 0$, i.e. $a = 1/2$, $b = -3/2$. Hence,

$$L_i = \mathbf{e}'_i + \mathbf{e}''_i\,, \quad S_i = (\mathbf{e}'_i - 3\mathbf{e}''_i)/2\,, \quad i = 1, 2, 3\,.$$

Inverting these equations we obtained the above equations for $\mathbf{e}'_i, \mathbf{e}''_i$.

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6. Split octonions and the projective quadric realization of $\widetilde{Q}$

We present a second construction of the rolling distribution with its natural G_2-action. This construction is based on the *split octonions* $\widetilde{\mathbf{O}}$, an 8-dimensional real algebra whose automorphism group is our G_2. The rolling space $\widetilde{Q}$ will be the projectivized null-cone of imaginary octonions. The rolling distribution on this rolling space will be defined solely in terms of octonion multiplication, and is thus automatically G_2-invariant. This construction is very similar to the construction of G_2 which appeared in Cartan's 1894 thesis [5], although the octonions do not appear there, so the similarity is mysterious at first. (See our Appendix C where we dispell some of that mystery.) It was only in 1914 that Cartan described the relation of G_2 with octonions [6].

We begin with a description of $\widetilde{\mathbf{O}}$, following the treatment of [9], in the section titled "The Cayley-Dickson process" (p. 104). There further consequences and motivation can also be found. The split octonions $\widetilde{\mathbf{O}}$ are a real eight-dimensional algebra with unit and which is neither associative nor commutative. We identify $\widetilde{\mathbf{O}}$ with $\mathbf{H}^2$, the 2-dimensional quaternionic vector space. Its multiplication law is

$$(a,b)(c,d) = (ac + \bar{d}b, da + b\bar{c}), \qquad a,b,c,d \in \mathbf{H}, \tag{2}$$

where $\bar{q}$ denotes the usual quaternionic conjugate of a quaternion q. The unit $1 \in \widetilde{\mathbf{O}}$ is $(1,0) \in \mathbf{H}^2$.

The *automorphism group* of a real algebra A is defined to be the space of invertible real linear maps $g \colon A \to A$ satisfying $g(xy) = g(x)g(y)$ for all $x, y \in A$. G_2 is the automorphism group of $\widetilde{\mathbf{O}}$. See [6] or [9].

The unit 1 of any unital algebra is always invariant under its automorphism group, so the one-dimensional subspace $\mathbf{R} = \mathbf{R}1 \subset \widetilde{\mathbf{O}}$ is a G_2-invariant subspace. This subspace has an invariant complement:

$$\widetilde{\mathbf{O}} = \mathbf{R}1 \oplus V,$$

where $\mathbf{R}1 = Re(\widetilde{\mathbf{O}})$, $V = Im(\widetilde{\mathbf{O}})$. In quaternionic terms:

$$V = Im\ \widetilde{\mathbf{O}} = Im\ \mathbf{H} \oplus \mathbf{H} \subset \mathbf{H} \oplus \mathbf{H} = \widetilde{\mathbf{O}}. \tag{3}$$

To see the G_2-invariant nature of V, we use the split-octonion conjugation $x \mapsto \bar{x}$ defined by $x = (a,b) \in \widetilde{\mathbf{O}} \mapsto \bar{x} = (\bar{a}, -b)$ for $x \in \widetilde{\mathbf{O}}$. Then $x = Re(x) + Im(x)$ with $Re(x) = (x+\bar{x})/2 \in \mathbf{R}1$, and $Im(x) = (x-\bar{x})/2 \in V$. Also $x\bar{x} = -\langle x,x\rangle 1 \in \mathbf{R}1$, where $\langle x,y\rangle = Re(x\bar{y})$ defines an inner product of signature $(4,4)$ on $\widetilde{\mathbf{O}}$ which is invariant under the action of G_2. Then V is the orthogonal complement of $1 \in \widetilde{\mathbf{O}}$ relative to this G_2-invariant inner product, and is thus G_2-invariant. Alternatively, an element $x \in \widetilde{\mathbf{O}}$ lies in V if and only if $x^2 = \langle x, x\rangle 1$ (see [9], Lemma 6.67), providing another proof of the G_2-invariance of V. And V forms a 7-dimensional inner product space of signature $(3,4)$ relative to the restriction of $\langle\cdot,\cdot\rangle$. The G_2-action on V leaves this inner product invariant, so that G_2 is realized as a subgroup of $\mathrm{SO}_{3,4}$ through its representation on V.

The maximal compact subgroup of G_2 is $K \cong \mathrm{SO}_4 \cong (\mathrm{SU}_2 \times \mathrm{SU}_2)/\pm(1,1)$. See [16]. Upon restricting from G_2 to K, the representation V decomposes into irreducibles according to (3). In other words, thinking of SU(2) as unit quaternions, for $(q_1, q_2) \in \mathrm{SU}_2 \times \mathrm{SU}_2 = \widetilde{K}$ (the universal double-cover of K) and $(a, b) \in Im(\mathbf{H}) \oplus \mathbf{H} = V$ we have $(q_1, q_2)\cdot(a, b) = (q_1 a\bar{q}_1, q_1 b\bar{q}_2)$.

In quaternionic terms (3) the quadratic form associated to our $(3,4)$ inner product on V is

$$\langle(v, q), (v, q)\rangle = -|v|^2 + |q|^2 .$$

Note that K acts transitively on the product of spheres $S^2 \times S^3 \subset V$. Let $S(V)$ denote the space of rays through the origin in V, which is to say the orbit space for the $\mathbf{R}^+$-action on $V \setminus \{0\}$, where $\mathbf{R}^+$ acts by scalar multiplication. Let $C \subset S(V)$ be the set of *null* rays, i.e.

$$C := \text{null rays in } V = \{\mathbf{R}^+ x \subset V \mid \langle x, x\rangle = 0, x \neq 0\} \subset S(V) := \text{rays in } V .$$

Since G_2 preserves the inner product $\langle\cdot,\cdot\rangle$ on V, G_2 acts on C. Now C is diffeomorphic to $S^2 \times S^3 \cong S^3 \times S^2 = \widetilde{Q}$, as is seen by mapping $\mathbf{R}^+(q, v) \mapsto (q, v)/\|q\|$. This diffeomorphism commutes with the K-action, where the K-action on C arises by restriction of the action of G_2 on C, and the K-action on $\widetilde{Q} = S^3 \times S^2$ is the lifting from $Q = \mathrm{SO}_3 \times S^2$ of the $\mathrm{SO}_3 \times \mathrm{SO}_3$-action of Section 2.2.

We proceed to define a G_2-invariant distribution E on $\widetilde{Q} = C$. Given a point $\mathbf{R}^+ x = [x] \in C$, set

$$x^\perp = \{y \in V \mid \langle x, y\rangle = 0\}, \quad x^0 = \{y \in V \mid xy = 0\} .$$

Then

PROPOSITION 4. $\mathbf{R}x \subset x^0 \subset (x^0)^\perp \subset x^\perp \subset V$, *and the dimensions are* 1, 3, 4, 6, 7.

Proof. Use the definitions of the split octonion product and of the inner product above. □

Upon projectivizing, the nested sequence of subspaces of this proposition becomes

$$0 \subset x^0/\mathbf{R}x \subset (x^0)^{\perp}/\mathbf{R}x \subset x^{\perp}/\mathbf{R}x = T_{[x]}C \subset V/\mathbf{R}x\,,$$

of dimensions $0,2,3,5,6$. In particular $E_{[x]} := x^0/\mathbf{R}x$, has dimension 2 for all $[x] \in C$, and depends smoothly on $[x]$, thus defining a rank 2 distribution on C. This construction of (C,E) depends only on the algebraic structure of $\widetilde{\mathbf{O}}$, so that $G_2 = \mathrm{Aut}(\widetilde{\mathbf{O}})$ acts on C preserving the distribution E.

PROPOSITION 5. *The (ray) projective quadric $C \subset S(Im\,\widetilde{\mathbf{O}})$ is a 5-dimensional homogeneous space for G_2 which carries a G_2-invariant rank 2 distribution E with the same data (G_2,P,W) of Section 3.3, and so pushes down to the rolling distribution (Q,D_ρ) for radius ratios $\rho = 3$ or $\rho = 1/3$ under the two-to-one cover $C = S^3 \times S^2 \to Q = \mathrm{SO}_3 \times S^2$.*

Steps of the proof of Proposition 5. In the paragraph preceding the proposition we proved that C is a homogeneous space for G_2, that E is invariant under this G_2-action, and that C is diffeomorphic to $\widetilde{Q}$, K-equivariantly. It remains to prove that the $\mathfrak{g}_2$-data for (C,E) is the data $(\mathfrak{g}_2,\mathfrak{p},W)$ as described in the previous section. We will use the weights for the G_2-representation space $V = Im\,(\widetilde{\mathbf{O}})$.

WEIGHTS FOR THE 7-DIMENSIONAL REPRESENTATION. Here is the weight diagram for this representation:

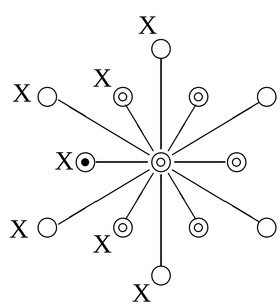

FIGURE 5

Weights and roots associated with the representation V

The weights of the representation V form a subset of the roots of $\mathfrak{g}_2$. In Figure 5 we redrew the root diagram of $\mathfrak{g}_2$, marking those roots which are weights for V with bull's-eyes. They are the six short roots and one zero root. The corresponding weight spaces V_w are all one-dimensional. The weight marked with a black dot corresponds to a *choice of base point* $c_0 \in C$. The meaning of the X's will be given below.

A REMINDER OF THE MEANING OF THE WEIGHT DIAGRAM. We begin generally. Let V be a representation of a semi-simple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{t}$. A *weight* for V is an element $w \in \mathfrak{t}^*$ such that there is a nonzero vector $v \in V$ with the property that $\zeta \cdot v = w(\zeta)\, v$ for all $\zeta \in \mathfrak{t}$ (a simultaneous eigenvector). The space of v's for a given weight w is called the *weight space* for w and is denoted V_w. If $w \in \mathfrak{t}^*$ is not a weight we set $V_w = 0$. For a finite-dimensional representation V of $\mathfrak{g}$ the set of weights is finite, and

$$V = \bigoplus_{w \in \mathfrak{t}^*} V_w \,.$$

The roots of $\mathfrak{g}$ are the weights of the adjoint representation, with the corresponding weight spaces called the *root spaces*, and denoted by $\mathfrak{g}_\alpha$.

From $\zeta\xi v = \xi\zeta v + [\zeta, \xi]v$ it follows that if $v \in V_w$ and $\xi \in \mathfrak{g}_\alpha$ then $\xi \cdot v \in V_{w+\alpha}$. In other words, $\mathfrak{g}_\alpha \cdot V_w \subset V_{w+\alpha}$, which implies the following

VANISHING WEIGHT CRITERION. *If w is a weight and α is a root such that $w + \alpha$ is not a weight, then $\mathfrak{g}_\alpha \cdot V_w = 0$.*

This is part of the proposition

$$\mathfrak{g}_\alpha \cdot V_w \neq 0 \iff w + \alpha \ \textit{is a weight.} \tag{4}$$

It follows that if, as in our case, all weight spaces are 1-dimensional, then $\mathfrak{g}_\alpha \cdot V_w = V_{w+\alpha}$ whenever $w + \alpha$ is a weight.

A BASIS AND MULTIPLICATION TABLE FOR $V = Im(\widetilde{\mathbf{O}})$. Let n be an imaginary quaternion. Then (n, n) and $(n, -n)$ are both null vectors in V. Take as a basis for V:

$$\begin{aligned} e_1 &= \tfrac{1}{2}(i, i)\,, \quad & e_2 &= \tfrac{1}{2}(j, j)\,, \quad & e_3 &= \tfrac{1}{2}(k, k)\,, \\ f_1 &= \tfrac{1}{2}(i, -i)\,, \quad & f_2 &= \tfrac{1}{2}(j, -j)\,, \quad & f_3 &= \tfrac{1}{2}(k, -k)\,, \end{aligned} \tag{5}$$

and

$$U = (0, 1)\,.$$

Then we have the multiplication table:

$$
\begin{aligned}
e_i^2 = f_i^2 &= 0 \\
e_i f_j = f_j e_i &= 0 \quad \text{if } i \neq j \\
e_i e_j = f_k\,; &\quad i,j,k \text{ a cyclic permutation of } 1,2,3 \\
f_i f_j = e_k\,; &\quad i,j,k \text{ a cyclic permutation of } 1,2,3 \\
e_i f_i &= -\tfrac{1}{2} + \tfrac{1}{2}U \\
f_i e_i &= -\tfrac{1}{2} - \tfrac{1}{2}U \\
e_i U &= e_i \\
f_i U &= -f_i\,.
\end{aligned}
$$

To complete the multiplication table, use that the conjugate of xy is $\overline{yx}$, and that if $x \in V$ then $\bar{x} = -x$. It follows that if $x, y \in V = Im(\widetilde{\mathbf{O}})$ and $yx = \bar{z}$ then $z = xy$. Thus, for example since $\bar{f_k} = -f_k$ we see that $e_j e_i = -f_k$, for i,j,k a cyclic permutation of $1,2,3$.

WEIGHTS FOR THE 7-DIMENSIONAL REPRESENTATION. To find the weights of the representation of G_2 on $V = Im(\widetilde{\mathbf{O}})$, we really find how the exponential T of the Cartan $\mathfrak{t}$ acts first, since it is easier. We use the general fact that if the roots and weights for the Cartan $\mathfrak{t}$ are real, then its *torus* $T = \exp(\mathfrak{t})$ (homeomorphic to a Euclidean space) acts on its weight spaces by scaling: if $\lambda = \exp(\xi) \in T$, with $\xi \in \mathfrak{t}$, then $\lambda e_w = \exp(w(\xi))\, e_w$ for $e_w \in V_w$. Now let $\lambda_1, \lambda_2, \lambda_3$ be nonzero reals with $\lambda_1\lambda_2\lambda_3 = 1$. Let $\alpha_i, \beta_i, \gamma_i$ and $\widetilde{\alpha}_i, \widetilde{\beta}_i, \widetilde{\gamma}_i$ be real exponents for $i = 1,2,3$ satisfying $\alpha_i + \beta_i + \gamma_i = 0$. Then the scaling transformation

$$
\begin{aligned}
e_i &\mapsto \lambda_1^{\alpha_i}\lambda_2^{\beta_i}\lambda_3^{\gamma_i} e_i\,, \\
f_i &\mapsto \lambda_1^{\widetilde{\alpha}_i}\lambda_2^{\widetilde{\beta}_i}\lambda_3^{\widetilde{\gamma}_i} f_i\,,
\end{aligned}
$$

together with $U \mapsto U$ preserves the multiplication table, and hence defines an element of G_2, provided

$$
\widetilde{\alpha}_i = -\alpha_i\,, \quad \widetilde{\beta}_i = -\beta_i\,, \quad \widetilde{\gamma}_i = -\gamma_i
$$

and provided that $(\alpha_i, \beta_i, \gamma_i)$ are multiples of the values from the following weight table:

	α_i	β_i	γ_i
$i = 1$	2	−1	−1
$i = 2$	−1	2	−1
$i = 3$	−1	−1	2

These scaling transformations generate a Cartan subgroup T of G_2, and the table gives the corresponding weights of the representation V. Thus for example e_1 is a weight vector with corresponding weight $(2,-1,-1)$ relative to $\mathfrak{t}$. Here we view $\mathfrak{t}$ as the collection of real vectors (a, b, c) with $a+b+c=0$. Looking at the inner products of these vectors we see that they are arranged on the weight diagram according to:

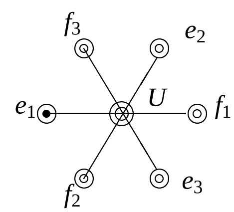

FIGURE 6
The weight space basis

We are now in a position to compute the $\mathfrak{g}_2$-data associated to (C, E).

WEIGHT VECTORS FOR NON-ZERO WEIGHTS ARE NULL VECTORS. Because the inner product is G_2-invariant, the $\mathfrak{g}_2$ action on V satisfies $\langle \xi x, x\rangle = 0$ for any $\xi \in \mathfrak{g}_2$, $x \in V$. Take x a weight vector with nonzero weight w, and take $\xi \in \mathfrak{t}$ with $w(\xi) \neq 0$. From $\langle \xi x, x\rangle = w(\xi)\langle x, x\rangle$ we have that x is a null vector.

COMPUTING THE ISOTROPY DATA. Set $c_0 = [e_1]$, the ray through e_1, as our base point in C. We now show that the isotropy group of the G_2-action on C at c_0 is P from the G_2 data (G_2, P, W), as constructed in Section 4.

We begin at the Lie algebra level, showing that $\mathfrak{g}_{c_0} = \mathfrak{p}$, where $\mathfrak{g}_{c_0} \subset \mathfrak{g}_2$ denotes the Lie algebra of the isotropy group at c_0. Now, $\mathfrak{g}_{c_0} = \{\xi \in \mathfrak{g}_2 \mid \xi \cdot e_1 = \lambda e_1 \text{ for some real number } \lambda\}$. Hence $\mathfrak{t}$ is contained in $\mathfrak{g}_{c_0}$.

Now, e_1 is a weight vector associated to the weight marked with a black dot in Figure 6, which is the root $-\sigma_3$ (see Figure 3). According to the *vanishing weight criterion* stated above, if α is a root for which $-\sigma_3+\alpha$ *is not a weight* then $\mathfrak{g}_\alpha \cdot e_1 = 0$. In other words, the sum of these $\mathfrak{g}_\alpha$'s is contained in $\mathfrak{g}_{c_0}$. In Figure 5 those roots α for which $-\sigma_3+\alpha$ is not a root are marked by X's. It follows now from the weight diagram that $\mathfrak{p} \subset \mathfrak{g}_{c_0}$ (see

Figure 5). Since there is no subalgebra of $\mathfrak{g}_2$ lying strictly between $\mathfrak{p}$ and all of $\mathfrak{g}_2$ we conclude that $\mathfrak{p} = \mathfrak{g}_{c_0}$.

It follows from this Lie algebra computation that the isotropy subgroup G_{c_0} *contains* P and has Lie algebra equalling the Lie algebra $\mathfrak{p}$ of P. P is the *connected* Lie subgroup of G_2 whose Lie algebra is $\mathfrak{p}$; thus, to show that $G_{c_0} = P$ is to show that G_{c_0} is connected. We use the homotopy exact sequence of the fibre bundle $G_{c_0} \to G_2 \to C = G_2/G_{[x]}$. This exact sequence is

$$\ldots \to \pi_1(C) \to \pi_0(G_{c_0}) \to \pi_0(G_2) \to \pi_0(C)\,.$$

Since C is simply connected and connected we get that $\pi_0(G_{c_0}) = \pi_0(G_2)$. Since $\pi_0(G_2) = 0$ we have the desired connectivity: $\pi_0(G_{c_0}) = 0$.

We have established that the isotropy part of the data for (C, D) is $P \subset G_2$.

COMPUTING THE DISTRIBUTION DATA. To complete the proof of Proposition 5 we now need to show that the infinitesimal $\mathfrak{g}_2$-action on C at c_0 maps $W \subset \mathfrak{g}_2/\mathfrak{p}$ to $E_{c_0} \subset T_{c_0}C$.

In Section 4 we saw that W is generated by x_1, y_2 (mod $\mathfrak{p}$), the root vectors associated to σ_1, $-\sigma_2$ (respectively), indicated by the pluses in Figure 3. (We follow the x, y notation from Figure 7 of Appendix B.) It follows from rule (4) that $x_1 \cdot e_1$, $y_2 \cdot e_1$ are weight vectors associated to the weights $\sigma_1 - \sigma_3$, $-\sigma_2 - \sigma_3$ (respectively), hence are multiples of the weight vectors f_2, f_3 (compare Figures 3 and 6). From the multiplication table following the description of our basis (5) we see that $(e_1)^0 = span\{e_1, f_2, f_3\}$, hence $W \cdot c_0 = E_{c_0}$, as required. $\square$

7. THE FULL ACTION DOES NOT DESCEND TO THE ROLLING SPACE

We now prove that the G_2-action on $\widetilde{Q}$ does not descend to Q. Observe that $Q = \mathbf{Z}_2 \backslash C$, where the $\mathbf{Z}_2 \subset K \subset G_2$ is generated by $\sigma = (\pm 1, 1)$. Use the following general fact about group actions. Suppose that a group G (here G_2) acts effectively on a set C and that $\Gamma \subset G$. ('Effectively' means that the only group element acting as the identity on C is the identity.) Then the action of an element $g \in G$ descends to the quotient space $\Gamma \backslash C$ if and only if $g \Gamma g^{-1} = \Gamma$. In particular, if Γ is not normal in G then the action of all of G does not descend to the quotient $\Gamma \backslash C$. Returning to our situation, we see that if the G_2-action were to descend then this $\mathbf{Z}_2$ generated by σ would have to be normal. But a discrete normal subgroup of a connected Lie group is central, and G_2 has no center. See Appendix A, or [16]. So our $\mathbf{Z}_2$ is not normal, and the G_2-action does not descend.

REMARK. Had we used lines instead of rays when constructing $C = \widetilde{Q}$, we would have arrived at a true projective quadric $Q_f \subset P(V)$. This Q_f is double-covered by $C = \widetilde{Q}$ and is diffeomorphic to $S^3 \times_{\pm I} S^2$, where the notation $\times_{\pm I}$ indicates that we divide out $C = S^3 \times S^2$ by the $\mathbf{Z}_2$-action generated by the involution $-I(v, h) = (-v, -h)$. The G_2-action on $\widetilde{Q}$ *does* descend to a G_2-action on Q_f since $-I$ commutes with the G_2-action on V. Thus $\widetilde{Q} = C$ double-covers two spaces, Q and Q_f, the distribution $\widetilde{D}$ on $\widetilde{Q}$ pushes down to both of these covered spaces, but the G_2-action on $\widetilde{Q}$ descends to only one of them, namely Q_f. In addition, Q_f is topologically distinct from Q. Indeed, since $\pi_1(\mathrm{SO}_3) = \mathbf{Z}_2$ there are precisely two topologically distinct SO_3-bundles over S^2, and both Q and Q_f are such bundles: Q is the trivial SO_3-bundle; Q_f is the other one. We find it curious that the action of G_2 on $\widetilde{Q}$ descends to this "false" rolling configuration space Q_f, but not to the real one, Q.

A. APPENDIX
COVERS: TWO G_2'S

How many connected Lie groups G are there (up to isomorphism) having a given finite-dimensional simple Lie algebra $\mathfrak{g}$ for their Lie algebra? There is at least one, the simply connected one, denoted $\widetilde{G}$. We can partially order all such groups G, writing $G < G'$ if there is a covering homomorphism from G' onto G. Then $\widetilde{G}$ is the largest such group. The smallest such group is the adjoint group, which is isomorphic to $\widetilde{G}/Z(\widetilde{G})$ where $Z(\widetilde{G})$ denotes the center of $\widetilde{G}$. (The *adjoint group* is, by definition, the image of $\widetilde{G}$ under the adjoint representation $\widetilde{G} \to \mathrm{Hom}(\mathfrak{g})$.) All other such groups are of the form $\widetilde{G}/\Gamma$ where Γ is a subgroup of $Z(\widetilde{G})$. So the lattice of such groups G is in one-to-one correspondence with the lattice of subgroups of $Z(\widetilde{G})$, except with the usual ordering on the lattice of subgroups reversed.

In the case of interest for the present paper, $\mathfrak{g} = \mathfrak{g}_2$, we will show that the center $Z(\widetilde{G}_2)$ of $\widetilde{G}_2$ is the two-element group $\mathbf{Z}_2$. Hence there are exactly two connected Lie groups with Lie algebra $\mathfrak{g}_2$: the simply connected group $\widetilde{G}_2$ and the adjoint group $\widetilde{G}_2/\mathbf{Z}_2$, which is the one we have been denoting as G_2.

We return to the general setting. The group $\widetilde{G}/\Gamma$ has fundamental group Γ, so that in particular the adjoint group $\mathrm{Ad}(\widetilde{G})$ has fundamental group $Z(\widetilde{G})$ equal to the center of $\widetilde{G}$. Now any finite-dimensional connected Lie group G

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deformation retracts onto its maximal compact subgroup K. It follows that if the maximal compact subgroup of $\mathrm{Ad}(\widetilde{G})$ has finite fundamental group, then the center $Z(\widetilde{G})$ of $\widetilde{G}$ is finite, being isomorphic to the fundamental group of this maximal compact.

We apply this logic to our setting. We saw above that the Lie algebra $\mathfrak{K}$ of the maximal compact K of any connected Lie group having Lie algebra $\mathfrak{g}_2$ is $\mathfrak{so}_3 \oplus \mathfrak{so}_3$. Now the connected Lie groups having Lie algebra $\mathfrak{so}_3 \oplus \mathfrak{so}_3$ all have finite fundamental groups, with either 1, 2 or 4 elements in them. It follows that the maximal compact of $\mathrm{Ad}(\widetilde{G}_2)$ has finite fundamental group, and so the center $Z(\widetilde{G}_2)$ of $\widetilde{G}_2$ is finite, with either 1, 2, or 4 elements in it. We will see that it has 2 elements. Being compact and central, $Z(\widetilde{G}_2)$ must lie in every maximal compact: $Z(\widetilde{G}_2) \subset \widetilde{K} \subset \widetilde{G}_2$, where $\widetilde{K}$ is the maximal compact of $\widetilde{G}_2$. Because $\widetilde{G}_2$ is simply connected and deformation retracts onto $\widetilde{K}$, we know that $\widetilde{K}$ is simply connected, and hence $\widetilde{K} \cong \mathrm{SU}_2 \times \mathrm{SU}_2$. Thus $Z(\widetilde{K})$ is the four element group $(\pm 1, \pm 1)$. Now the center $Z(\widetilde{K})$ of $\widetilde{K}$ need not be the center $Z(\widetilde{G}_2)$ of $\widetilde{G}_2$, but must contain it: $Z(\widetilde{G}_2) \subset Z(\widetilde{K})$. Indeed $Z(\widetilde{G}_2)$ is the subgroup of $Z(\widetilde{K})$ which acts (under the restriction of the adjoint action) trivially on $\mathfrak{g}_2$. A computation using roots and the restriction of the adjoint representation to $\widetilde{K}$ shows that this subgroup acting trivially is the two element group with elements $(1,1)$ and $-(1,1)$. See Vogan [16, p. 679]. Consequently $Z(\widetilde{G}_2) = \pm(1,1) = \mathbf{Z}_2$ as claimed.

It is worth contrasting our situation to one in which the center of $\widetilde{G}$ is infinite. Take the case $\mathfrak{g} = \mathfrak{sl}_2(\mathbf{R})$. Then $\widetilde{G} = \widetilde{\mathrm{SL}}_2(\mathbf{R})$ and $\widetilde{G} = \mathrm{SL}_2(\mathbf{R})$, the usual matrix group consisting of two-by-two real matrices of unit determinant. The maximal compact subgroup of $\mathrm{SL}_2(\mathbf{R})$ is SO_2, and is isomorphic to the circle group S^1, which has infinite fundamental group $\mathbf{Z}$. It follows that the center of $\widetilde{\mathrm{SL}}_2(\mathbf{R})$ is the group of integers $\mathbf{Z}$. The Lie algebra of S^1 is the abelian algebra $\mathbf{R}$, and the simply connected Lie group with it for Lie algebra is the additive group $\mathbf{R}$ (sitting inside $\widetilde{\mathrm{SL}}_2(\mathbf{R})$ as the cover of SO_2). The maximal compact in $\widetilde{\mathrm{SL}}_2(\mathbf{R})$ is the identity group.

B. APPENDIX
THE ISOMORPHISM OF $\mathfrak{K}$ AND $\mathfrak{so}_3 \oplus \mathfrak{so}_3$ FROM PROPOSITION 3

We complete the proof of Proposition 3 from Section 5 with the explicit identification of $\mathfrak{so}_3 \oplus \mathfrak{so}_3$ as the Lie algebra $\mathfrak{K}$ of the maximal compact in $\mathfrak{g}_2$. We follow Serre [13, p. VI-11]: $\mathfrak{g}_2$ is Lie-generated by the elements x, y, h, X, Y, H, subject to the following relations, which one can read off the root diagram.

$$
\begin{array}{llll}
[x,y]=h\,, & [h,x]=2x\,, & [h,y]=-2y\,, & \\
[X,Y]=H\,, & [H,X]=2X\,, & [H,Y]=-2Y\,, & \\
[h,X]=-3X\,, & [h,Y]=3Y\,, & [H,x]=-x\,, & [H,y]=y\,,\\
[x,Y]=[X,y]=[h,H]=0\,, & & & \\
[\mathrm{ad}(x)]^4\,X=0\,, & [\mathrm{ad}(X)]^2\,x=0\,, & & \\
[\mathrm{ad}(y)]^4\,Y=0\,, & [\mathrm{ad}(Y)]^2\,x=0\,. & &
\end{array}
$$

Taking Lie brackets of the vectors x, y, h, X, Y, H we generate a complete set $\{x_i, X_i, y_i, Y_i \mid i=1,2,3\}$ of root vectors for $\mathfrak{g}_2$ which, together with the basis h, H for the Cartan subalgebra, form a basis for $\mathfrak{g}_2$ as follows:

$$
\begin{array}{l}
x_3=x\,,\ X_1=X\,,\ x_2=[x,X_1]\,,\quad x_1=[x,x_2]\,,\quad X_2=[x,x_1]\,,\quad X_3=[X_1,X_2];\\
y_3=y\,,\ Y_1=Y\,,\ y_2=-[y,Y_1]\,,\ y_1=-[y,y_2]\,,\ Y_2=-[y,y_1]\,,\ Y_3=-[Y_1,Y_2].
\end{array}
$$

We label each root in the diagram with the corresponding root vector.

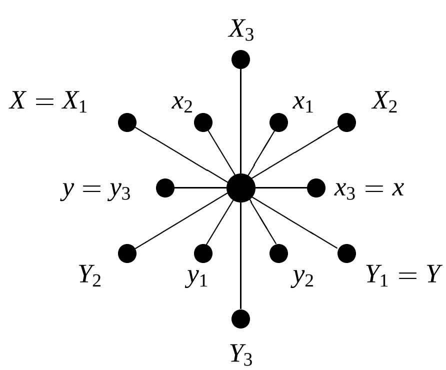

FIGURE 7
A basis for the Lie algebra of G_2

We end up with a "nice" basis with respect to which the structure constants are particularly pleasant; they are integers and have symmetry properties which greatly facilitate the work involved in their determination. Elementary $\mathfrak{sl}_2$ representation theory further facilitates the calculation. It helps to work with the root diagram nearby.

SYMMETRY PROPERTIES OF THE STRUCTURE CONSTANTS. Suppose α, β are two roots such that $\alpha+\beta$ is also a root. Let E_α, E_β be the corresponding root vectors, as chosen above. Then $[E_\alpha, E_\beta] = c_{\alpha,\beta} E_{\alpha+\beta}$, for some non-zero constant $c_{\alpha,\beta} \in \mathbf{Z}$. The nice feature of our base is that the structure constants satisfy

$$c_{-\alpha,-\beta} = -c_{\alpha,\beta} \,.$$

This cuts the amount of work involved in half, since you need only consider $\alpha > 0$ (the positive roots are the six dots in the last root diagram marked with x's or X's). Combining this with the obvious $c_{\alpha,\beta} = -c_{\beta,\alpha}$ (antisymmetry of Lie bracket) you obtain

$$c_{\alpha,-\beta} = c_{\beta,-\alpha} \,.$$

This cuts the amount of work in half again.

PROPOSITION 6. *The structure constants of $\mathfrak{g}_2$ with respect to the basis of root vectors $\{x_i, X_i, y_i, Y_i \mid i = 1,2,3\}$ and the Cartan algebra elements $\{h, H\}$ are given as follows. The basis elements are grouped in three sets: positive (three x's and three X's), negative (three y's and three Y's), and Cartan subalgebra elements (h and H).*

- *[Positive, positive] other than the ones given above, and those which are zero for obvious reasons from the root diagram (sum of roots which is not a root):*

$$[x_1, x_2] = X_3 \,.$$

- *[Positive, negative]*

$c_{\alpha,\beta}$	y_1	y_2	y_3	Y_1	Y_2	Y_3
x_1	1	4	-4	0	12	-12
x_2	4	1	-3	1	0	3
x_3	-4	-3	1	0	-3	0
X_1	0	1	0	1	0	-1
X_2	12	0	-3	0	1	36
X_3	-12	3	0	-1	36	1

The 1*'s on the diagonal stand for the relations* $[x_i, y_i] = h_i$, $[X_i, Y_i] = H_i$, *where, in terms of our basis* $\{h, H\}$ *for the Cartan subalgebra,*

$$h_1 = 8h + 12H\,, \qquad h_2 = h + 3H\,, \qquad h_3 = h\,,$$
$$H_1 = H\,, \qquad H_2 = 36(h + H)\,, \qquad H_3 = 36(h + 2H)\,.$$

- *[Cartan, anything] this is coded directly by the root diagram:*
  - ❄ $\mathrm{ad}(h)$ *has eigenvalues and eigenvectors*

eigenvalue	3	2	1	0	-1	-2	-3
eigenvectors	X_2, Y_1	x_3	x_1, y_2	X_3, Y_3, h, H	x_2, y_1	y_3	X_1, Y_2

  - ❄ $\mathrm{ad}(H)$ *has eigenvalues and eigenvectors*

eigenvalue	2	1	0	-1	-2
eigenvectors	X_1	X_3, x_2, y_3, Y_2	x_1, y_1, h, H	X_2, x_3, y_2, Y_3	Y_1

Proof. This is elementary, using only the Jacobi identity, but takes time. We will give as a typical example the calculation of $[x_1, x_2]$:

$$\begin{aligned}
[x_1, x_2] &= [x_1, [x, X]] && \text{(by definition of } x_2\text{)}\\
&= [x, [x_1, X]] + [X, [x, x_1]] && \text{(Jacobi identity)}\\
&= [X, [x, x_1]] && \text{(since } [x_1, X] = 0\text{)}\\
&= [X, X_2] = X_3 && \text{(by definitions of } X_2, X_3\text{)}.
\end{aligned}$$

The rest of the relations are derived in a similar fashion. □

Now we are ready to define the generators of the Lie algebra of a maximal compact subgroup $K \subset G_2$. Let

$$L_1 = X_1 - Y_1\,, \qquad L_2 = \frac{X_2 - Y_2}{6}\,, \qquad L_3 = \frac{X_3 - Y_3}{6}\,,$$
$$S_1 = \frac{x_1 - y_1}{4}\,, \qquad S_2 = \frac{x_2 - y_2}{2}\,, \qquad S_3 = \frac{x_3 - y_3}{2}\,.$$

Using the commutation relations of the last proposition one easily checks that

$$[L_i, L_j] = \epsilon_{ijk} L_k\,, \qquad [L_i, S_j] = \epsilon_{ijk} S_k\,, \qquad [S_i, S_j] = \epsilon_{ijk}(\tfrac{3}{4} L_k - S_k)\,.$$

NOTE. The strange-looking coefficients $2, 4, 6$ in the definition of the L_i, S_i are chosen precisely so that we get these pleasing commutation relations.

C. Appendix

The rolling distribution in Cartan's thesis

In É. Cartan's thesis [5, p. 146], we find the following constructions: consider $V = \mathbf{R}^7 = \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}$ with coordinates $(\mathbf{x}, \mathbf{y}, z)$, where $\mathbf{x}, \mathbf{y} \in \mathbf{R}^3$, $z \in \mathbf{R}$, and the following 15 linear vector fields (hence linear operators) on V:

- $X_{ii} = -x_i\partial_{x_i} + y_i\partial_{y_i} + \frac{1}{3}\sum_{j=1}^{3}(x_j\partial_{x_j} - y_j\partial_{y_j})$, $i = 1, 2, 3$.
- $X_{i0} = 2z\partial_{x_i} - y_i\partial_z - x_j\partial_{y_k} + x_k\partial_{y_j}$, $(i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$.
- $X_{0i} = -2z\partial_{y_i} + x_i\partial_z + y_j\partial_{x_k} - y_k\partial_{x_j}$, $(ijk) \in A_3$.
- $X_{ij} = -x_j\partial_{x_i} + y_i\partial_{y_j}$, $i \neq j$, $i, j = 1, 2, 3$.

Cartan makes the following claims without proofs:

(1) *The linear span of these* 15 *operators is a* 14*-dimensional Lie subalgebra* $\mathfrak{g} \subset \mathrm{End}(V)$ *isomorphic to* $\mathfrak{g}_2$.

(2) $\mathfrak{g}$ *preserves the quadratic form on* V *given by*

$$J = z^2 + \mathbf{x} \cdot \mathbf{y} .$$

(3) *The linear group* $G \subset \mathrm{GL}(V)$ *generated by* $\mathfrak{g}$ *acts transitively on the projectivized null cone of* J.

(4) G *preserves the system of* 6 *Pfaffian equations on* V, *given by the* 6 *components of*

$$\begin{cases} \alpha := z\,d\mathbf{x} - \mathbf{x}\,dz + \mathbf{y} \times d\mathbf{y} = 0 , \\ \beta := z\,d\mathbf{y} - \mathbf{y}\,dz + \mathbf{x} \times d\mathbf{x} = 0 , \end{cases}$$

which have as a consequence

$$\begin{cases} \gamma_1 := z\,dz + \mathbf{x} \cdot d\mathbf{y} = 0 , \\ \gamma_2 := z\,dz + \mathbf{y} \cdot d\mathbf{x} = 0 . \end{cases}$$

(5) G *preserves a* 5*-parameter family of* 3*-dimensional linear subspaces of* V, *contained in the null cone of* J,

$$\begin{cases} \mathbf{x} - z\,\mathbf{a} + \mathbf{b} \times \mathbf{y} = 0 , \\ \mathbf{y} - z\,\mathbf{b} + \mathbf{a} \times \mathbf{x} = 0 , \end{cases}$$

where

$$\mathbf{a} \cdot \mathbf{b} + 1 = 0 .$$

Our goal in this appendix is to sketch proofs of these claims, provide a minor correction in one place, relate Cartan's construction to the octonions, and show how they contain, in essence, the construction of the rolling distribution $\widetilde{Q}$ via projective geometry, as in Proposition 5 from Section 5.

C.1 ISOMORPHISM OF $\mathfrak{g}$ WITH $\mathfrak{g}_2$

PROPOSITION 7. *$\mathfrak{g}$ is a 14-dimensional Lie subalgebra of $\mathrm{End}(V)$, isomorphic to $\mathfrak{g}_2$, with a maximal compact subalgebra generated by*

$$\{X_{ij} - X_{ji} \mid i \neq j\} \quad \textit{and} \quad \{X_{i0} - X_{0i}\} \ .$$

Proof. It is convenient to put $\mathfrak{g}$ in block matrix form. For each $\mathbf{u} \in \mathbf{R}^3$ let $\Omega_{\mathbf{u}} \in \mathrm{End}(\mathbf{R}^3)$ be given by $\mathbf{v} \mapsto \mathbf{u} \times \mathbf{v}$; i.e.

$$\Omega_{\mathbf{u}} = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} .$$

Define the linear map $\rho\colon \mathfrak{sl}_3(\mathbf{R}) \times \mathbf{R}^3 \times \mathbf{R}^3 \to \mathrm{End}(V)$ by

$$\rho(A, \mathbf{b}, \mathbf{c}) = \begin{pmatrix} A & \Omega_{\mathbf{c}} & -2\mathbf{b} \\ \Omega_{\mathbf{b}} & -A^t & -2\mathbf{c} \\ \mathbf{c}^t & \mathbf{b}^t & 0 \end{pmatrix} .$$

Now ρ is clearly injective, hence its image is a 14-dimensional linear subspace of $\mathrm{End}(V)$. Denote the components of $A, \mathbf{b}, \mathbf{c}$ by a_{ij}, b_i, c_i (respectively), then it is easy to check that

$$\rho(A, \mathbf{b}, \mathbf{c}) = -\sum_{i,j} a_{ij} X_{ij} - \sum_i b_i X_{i0} + \sum_i c_i X_{0i} \ .$$

This shows that $\mathfrak{g}$ is the image of ρ and hence a 14-dimensional subspace of $\mathrm{End}(V)$.

To show that $\mathfrak{g}$ is a Lie algebra one calculates that

$$[\rho(A, \mathbf{b}, \mathbf{c}), \rho(A', \mathbf{b}', \mathbf{c}')] = \rho(A'', \mathbf{b}'', \mathbf{c}'') ,$$

where

$$\begin{aligned} A'' &= [A, A'] + \mathbf{c}\mathbf{b}'^t - \mathbf{c}'\mathbf{b}^t - 2\mathbf{b}\mathbf{c}'^t + 2\mathbf{b}'\mathbf{c}^t + [\mathbf{b} \cdot \mathbf{c}' - \mathbf{c} \cdot \mathbf{b}']I , \\ \mathbf{b}'' &= A\mathbf{b}' - A'\mathbf{b} + 2\mathbf{c} \times \mathbf{c}' , \\ \mathbf{c}'' &= -A^t\mathbf{c}' + A'^t\mathbf{c} + 2\mathbf{b} \times \mathbf{b}' . \end{aligned}$$

These formulae show that the subspace $\mathfrak{K} \subset \mathfrak{g}$ given by

$$\mathfrak{K} = \{\rho(\Omega_{\mathbf{a}}, \mathbf{b}, \mathbf{b}) \mid \mathbf{a}, \mathbf{b} \in \mathbf{R}^3\}$$

forms a 6-dimensional subalgebra. One can easily verify that $\mathfrak{K}$ is isomorphic to $\mathfrak{so}_3 \oplus \mathfrak{so}_3$ via $(\omega', \omega'') \mapsto \rho(\Omega_{\mathbf{a}}, \mathbf{b}, \mathbf{b})$, where

$$\mathbf{a} = \frac{\omega' + \omega''}{2} + \frac{\omega' - \omega''}{2\sqrt{2}} , \qquad \mathbf{b} = \frac{\omega'' - \omega'}{2\sqrt{2}} .$$

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The formulae also show that the subspace $\mathfrak{t} \subset \mathfrak{g}$ generated by $\rho(D, 0, 0)$, where D is a traceless diagonal matrix, is a 2-dimensional abelian subalgebra. We fix this as our Cartan subalgebra. Let $\alpha_i := a_{ii} \in \mathfrak{t}^*$, $i = 1, 2, 3$. Then the roots of $\mathfrak{g}$, relative to $\mathfrak{t}$, are $\pm\alpha_i$, $i = 1, 2, 3$, and $\pm(\alpha_i + \alpha_j)$, $i \neq j$. The corresponding root spaces are spanned by X_{i0} for α_i, X_{0i} for $-\alpha_i$, X_{ij}, $i > j$, for $\alpha_i + \alpha_j$, and X_{ij}, $i < j$, for $-(\alpha_i + \alpha_j)$. One now carefully draws these 14 roots in the plane $\mathfrak{t}^*$, using the Killing inner product $\langle D, D' \rangle = \mathrm{tr}(DD')$, and obtains the $\mathfrak{g}_2$ root diagram as in Section 4. $\square$

C.2 Invariance of J

Let $G_2 \subset \mathrm{GL}_7(\mathbf{R})$ be the subgroup generated by $\mathfrak{g}$.

Proposition 8. *J is G_2-invariant.*

Proof. This is equivalent to showing that every $X \in \mathfrak{g}$ is J-antisymmetric, i.e. that X anti-commutes with

$$\begin{pmatrix} 0 & I/2 & 0 \\ I/2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

One now easily checks that the set of J-antisymmetric matrices consists of the matrices of the form

$$\begin{pmatrix} A & \Omega_{\mathbf{c}} & -2\tilde{\mathbf{b}} \\ \Omega_{\mathbf{b}} & -A^t & -2\tilde{\mathbf{c}} \\ \tilde{\mathbf{c}}^t & \tilde{\mathbf{b}}^t & 0 \end{pmatrix},$$

where $A \in \mathrm{End}(\mathbf{R}^3)$ and $\mathbf{b}, \tilde{\mathbf{b}}, \mathbf{c}, \tilde{\mathbf{c}} \in \mathbf{R}^3$. Looking at the formula for $\rho(A, \mathbf{b}, \mathbf{c})$, we see that $\mathfrak{g}$ is the subset of the J-antisymmetric matrices satisfying $\mathrm{tr}\, A = 0$, $\mathbf{b} = \tilde{\mathbf{b}}$, $\mathbf{c} = \tilde{\mathbf{c}}$ (a codimension 7 condition). $\square$

C.3 G_2-invariance of the Pfaffian system

First some generalities. A *Pfaffian system* on a manifold M is given locally by the common kernels of a finite set of 1-forms,

$$\alpha_1 = \ldots = \alpha_m = 0.$$

Two sets of 1-forms

$$\{\alpha_1, \ldots, \alpha_m\}, \quad \{\beta_1, \ldots, \beta_n\},$$

give *equivalent systems* if one can express each element of one set as a linear combination (with coefficients in $C^\infty(M)$) of the elements of the other set. We write this as

$$\alpha_i \equiv 0 \mod \beta_1, \dots, \beta_n\,, \quad i = 1, \dots, m\,,$$

and similarly for the β's.

Consequently, if we want to prove that a system $\alpha_1 = \ldots = \alpha_m = 0$ is preserved by some diffeomorphism $f\colon M \to M$ we must show that

$$f^*\alpha_i \equiv 0 \mod \alpha_1, \dots, \alpha_m\,, \quad i = 1, \dots, m\,,$$

and if we want to show that the flow of some vector field X on M preserves the system we must show that

$$\mathcal{L}_X\alpha_i \equiv 0 \mod \alpha_1, \dots, \alpha_m\,, \quad i = 1, \dots, m\,.$$

Given such a system we can consider the common kernels $D_x \subset T_xM$ of the 1-forms at each point $x \in M$. This is well defined independently of the 1-forms chosen to represent the system. If $\dim D_x$ (the rank of the system) is constant we obtain a distribution $D \subset TM$ (a subbundle of the tangent bundle). But the rank may vary. For example, the system on $\mathbf{R}$ given by $x\,dx = 0$ has rank 1 at $x = 0$ and rank 0 for $x \neq 0$. However, if G acts on M preserving a Pfaffian system, then the rank must be constant along the G-orbits.

CARTAN'S PFAFFIAN SYSTEM. RANK JUMPS. A CORRECTION. Due to jumping of rank, as discussed in the last remark, the Pfaffian system which Cartan defined by the vanishing of the 6 components of α, β cannot be G_2-invariant, even when restricted to $\widetilde{C}$, the J null cone. For at $(\mathbf{e}_1, 0, 0) \in V$ the system reduces to $dx_2 = dx_3 = dz = 0$, and so has rank 4. On the other hand, at the point $(\mathbf{e}_1, \mathbf{e}_2, 0)$ the system is equivalent to $dy_1 = dx_2 = dz - dy_3 = dz + dx_3 = 0$, and so has rank 3. And both points lie in $\widetilde{C} \setminus \{0\}$, which is a single G_2-orbit, contradicting G_2-invariance. A related problem with Cartan's claim (4) is his claim that $\gamma_1 = \gamma_2 = 0$ is a consequence of $\alpha = \beta$. But this claim holds only on the $z \neq 0$ part of $\widetilde{C}$.

Both errors are fixed by imposing the extra equation $\gamma := \gamma_1 - \gamma_2 = 0$. Then, as in Section C.4, we do obtain a G_2-invariant system on V. Furthermore, as proved immediately below, the two equations $\gamma_1 = \gamma_2 = 0$ are indeed a consequence of $\alpha = \beta = 0, \gamma = 0$ on $\widetilde{C}$, and are a consequence of $\alpha = \beta = 0$ on the subset $z \neq 0$ of $\widetilde{C}$. So Cartan's claim is correct on the open dense set $z \neq 0$ of the null cone $\widetilde{C} \subset V$. (See also page 11 of Bryant's paper on geometric duality [3], where he adds the equation $\gamma = 0$ to $\alpha = \beta = 0$.)

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PROPOSITION 9. *The Pfaffian system on V given by $\alpha=\beta=0, \gamma=0$ is G_2-invariant. On $\widetilde{C}$ the system is equivalent to $\alpha=\beta=0, \gamma_1=\gamma_2=0$. On the subset $z\neq 0$ of $\widetilde{C}$ it is equivalent to $\alpha=\beta=0$.*

Proof. We prove the claims of the last two sentences first. Note that $\gamma_1+\gamma_2=dJ$. It follows that on $\widetilde{C}$, where $J=0$, we have that $\gamma_1=\gamma_2=0$ is a consequence of $\gamma := \gamma_1-\gamma_2=0$. Thus, restricted to $\widetilde{C}$, the system $\alpha=\beta=0, \gamma=0$ is equivalent to $\alpha=\beta=0, \gamma_1=\gamma_2=0$. Next, note that $\mathbf{x}\cdot\beta-\mathbf{y}\cdot\alpha=z\gamma$. It follows that on $z\neq 0$ the equation $\gamma=0$ is a consequence of $\alpha=\beta=0$.

It remains to establish G_2-invariance. We need to show that

$$\mathcal{L}_X\alpha_i \equiv \mathcal{L}_X\beta_j \equiv \mathcal{L}_X\gamma \equiv 0 \mod \alpha_i, \beta_j, \gamma\,,$$

for all $X=\rho(A,\mathbf{b},\mathbf{c})\in\mathfrak{g}$. Divide into 3 cases, corresponding to $(A,0,0)$, $(0,\mathbf{a},0)$ and $(0,0,\mathbf{b})$ in our coordinatization of $\mathfrak{g}$.

CASE 1: $X=\rho(A,0,0)$, $A\in\mathfrak{sl}_3(\mathbf{R})$.

LEMMA 1. *If $A\in\mathrm{End}(\mathbf{R}^3)$ and $\mathbf{u},\mathbf{v}\in\mathbf{R}^3$, then*

$$A(\mathbf{u}\times\mathbf{v})+A^t\mathbf{u}\times\mathbf{v}+\mathbf{u}\times A^t\mathbf{v}=\mathrm{tr}\,A(\mathbf{u}\times\mathbf{v})\,.$$

Proof. Divide in 2 cases. If $A^t=-A$ then $A=\Omega_{\mathbf{u}}$ for some $\mathbf{u}\in\mathbf{R}^3$, $\mathrm{tr}\,A=0$ and the identity reduces to the Jacobi identity for the cross product. If $A^t=A$ then can assume w.l.o.g. that A is diagonal and do an explicit easy calculation. □

Now, since $X(\mathbf{x},\mathbf{y},z)=(A\,\mathbf{x},-A^t\mathbf{y},0)$ and $\alpha=zd\mathbf{x}-\mathbf{x}dz+\mathbf{y}\times d\mathbf{y}$, we get, using the lemma and $\mathrm{tr}\,A=0$, that

$$\begin{aligned}\mathcal{L}_X\alpha &= zAd\mathbf{x}-A\,\mathbf{x}dz-A^t\mathbf{y}\times d\mathbf{y}-\mathbf{y}\times A^t d\mathbf{y}\\ &= A(zd\mathbf{x}-\mathbf{x}dz+\mathbf{y}\times d\mathbf{y})=A\alpha\equiv 0 \quad \mod \alpha\,.\end{aligned}$$

Similarly, $\mathcal{L}_X\beta=-A^t\beta\equiv 0 \pmod{\beta}$. Finally, $\mathcal{L}_X\gamma=(A\,\mathbf{x})\cdot d\mathbf{y}-\mathbf{x}\cdot(A^t d\mathbf{y})=0$.

CASE 2: $X=\rho(0,\mathbf{b},0)$, $\mathbf{b}\in\mathbf{R}^3$.

Here $X(\mathbf{x},\mathbf{y},z)=(-2\mathbf{b}z,\mathbf{b}\times\mathbf{x},\mathbf{b}\cdot\mathbf{y})$, and one calculates that

$$\mathcal{L}_X\alpha=-\mathbf{b}\gamma\,,\quad \mathcal{L}_X\beta=-\mathbf{b}\times\beta\,,\quad \mathcal{L}_X\gamma=-\mathbf{b}\cdot\beta\,.$$

CASE 3: $X = \rho(0, 0, \mathbf{c})$, $\mathbf{c} \in \mathbf{R}^3$. The proof for this case is very similar to the previous case. Just interchange $\mathbf{x}$ and $\mathbf{y}$, and $\mathbf{b}$ and $\mathbf{c}$.

This completes the proof of invariance, and hence the proof of the proposition.

C.4 RELATION WITH OCTONIONS

Recall the basis e_i, f_i, U of Section 6 for V (imaginary split octonions) with its consequent multiplication table. Make the change of basis $e_i \mapsto -e_i$, keeping f_i, U as they were, thus changing the signs of some entries of the multiplication table. Use this new basis $E_i = -e_i, f_i, U$ to identify V with $\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}$ by setting $(\mathbf{x}, \mathbf{y}, z) = \sum x_i E_i + \sum y_i f_i + zU \in V$. Referring to the multiplication table we compute

$$\begin{aligned}(\mathbf{x}, \mathbf{y}, z)(\mathbf{x}', \mathbf{y}', z') = \big(&-\mathbf{y} \times \mathbf{y}' - z\mathbf{x}' + z'\mathbf{x}, \\ &\mathbf{x} \times \mathbf{x}' + z\mathbf{y}' - z'\mathbf{y},\ \tfrac{1}{2}(\mathbf{x} \cdot \mathbf{y}' - \mathbf{x}' \cdot \mathbf{y})\big) + 1\{zz' + \tfrac{1}{2}(\mathbf{x} \cdot \mathbf{y}' - \mathbf{x}' \cdot \mathbf{y})\}\,.\end{aligned}$$

The last term is in the real part of the split octonions, and not in V. It follows from this formula that $(\mathbf{x}, \mathbf{y}, z)^2 = J$, of Cartan's claim (2) stated above. Multiplying out $(\mathbf{x}, \mathbf{y}, z)(d\mathbf{x}, d\mathbf{y}, dz)$ we find that

$$(\mathbf{x}, \mathbf{y}, z)(d\mathbf{x}, d\mathbf{y}, dz) = (\alpha, \beta, \tfrac{1}{2}(\gamma_1 - \gamma_2)) + 1\{\tfrac{1}{2}(\gamma_1 + \gamma_2)\}\,,$$

where $\alpha, \beta, \gamma_1, \gamma_2$ are as in Cartan's claim (4). It follows that any element of $G_2 = \mathrm{Aut}(\widetilde{\mathbf{O}})$ preserves J and preserves the Pfaffian system of Cartan's claim (4). The distribution D defined by this system is, upon restriction to the null cone $\{J = 0\} \setminus \{0\}$, precisely the distribution D which we defined in the final section of our paper: $D(\mathbf{x}, \mathbf{y}, z) := \{(\mathbf{a}, \mathbf{b}, c) : (\mathbf{x}, \mathbf{y}, z)(\mathbf{a}, \mathbf{b}, c) = 0\}$. It follows that Cartan's construction, pushed down to the space of rays using the $\mathbf{R}^+$-action, yields precisely our $\widetilde{Q}$.

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