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ROOTS OF COMPLEX POLYNOMIALS AND FOCI OF REAL ALGEBRAIC CURVES

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ABSTRACT. We give a new proof of results of B. Z. Linfield presenting the roots of the derivative of a complex polynomial as the foci of a certain real algebraic curve in the complex plane $\mathbf{C}$.

1. INTRODUCTION

A nice and old theorem due to J. Siebeck ([12]), also ascribed to F. J. van den Berg ([13]), asserts that if $f \in \mathbf{C}[z]$ is a polynomial of the third degree whose roots z_1, z_2, z_3, viewed as points in the complex plane, are not aligned, then the roots of its derivative df/dz are the foci of the unique conic C (an ellipse in fact) which is tangent to the sides of the triangle $z_1z_2z_3$ at their midpoints. For a nice geometric proof based on the focal properties of conics, see [1]. Other proofs may be found in [10], 1.2.2.

Many generalizations and other proofs of Siebeck's theorem appeared in the first quarter of the 20th century, the reader may see M. Marden's paper [8] and references therein [1]). Among them, Linfield's paper [7] is worth noting, not only because it deals with polynomials of arbitrary degree (and even rational functions), but especially because it obtains the real algebraic curve playing the role of the conic C in Siebeck's result (*Siebeck curve*) as the curve enveloped by part of a certain polar curve in the dual plane. This gives a far clearer insight into the problem and allows one to cover the cases of particular positions of the roots.

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1) The original result of Siebeck has recently been named after Marden, which, in view of the careful historic quotations by Marden himself, makes no sense.

The present paper gives a precise statement and a new proof of Siebeck's theorem for polynomials of arbitrary degree following Linfield's approach in [7], written according to modern standards. We have in particular addressed a number of points, such as the zero-dimensional components and the uniqueness of the Siebeck curve, which received no mention in Linfield's paper. For the convenience of the reader, the easier case of non-aligned roots is presented first, in Section 6, while the somewhat more technical general case is dealt with in Section 7. Further properties of the Siebeck curve and its application to the location of the roots of the derivative, refining the Gauss-Lucas theorem, will appear in [2].

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2. PRELIMINARIES

We will deal with abstract, real or complex, projective planes with an already fixed system of homogeneous coordinates, and with algebraic curves C (in the sequel simply called *curves*) in them, defined by equations $F = 0$, F an homogeneous polynomial in the projective coordinates. The point with homogeneous coordinates (x_0, x_1, x_2) will be denoted $[x_0, x_1, x_2]$ and we will usually write $C\colon F = 0$ to indicate that the curve C has equation $F = 0$. A line t is said to be *tangent* to the curve C at a point $p \in C$ if and only if the intersection multiplicity of t and C at p is higher than the multiplicity of p on C. The point p is then called a *contact point* of t and C, many contact points being possible for the same tangent. The curve composed of (possibly repeated) curves $C_j\colon F_j = 0$, $j = 1, \dots, r$, which by definition has equation $\prod_{j=1}^r F_j = 0$, will be denoted by $C_1 + \cdots + C_r$.

Any real projective plane $\mathbf{P}_2$ will be viewed as embedded in its complex extension $\mathbf{CP}_2$, which is obtained by just allowing the homogeneous coordinates to take arbitrary complex values, not all zero. In this situation, the points of $\mathbf{P}_2$ are those which have real coordinates and will be called *real points*, while, as usual, the points in $\mathbf{CP}_2 - \mathbf{P}_2$ are called *imaginary points*. The (complex) *conjugate* of $[x_0, x_1, x_2] \in \mathbf{CP}_2$ is $[\bar{x}_0, \bar{x}_1, \bar{x}_2]$, the bar meaning complex conjugation. The fact that complex conjugation is an involutive automorphism of $\mathbf{C}$ over $\mathbf{R}$, and therefore preserves any type of algebraic relation, will be used without further mention in what follows.

Groups of points will be taken to be finite unordered lists of possibly repeated points $\mathbf{G} = \{p_j\}_{j=1,\dots,n}$, the number of times that a point is repeated being the *multiplicity* of the point in the group. Groups of points will be represented as formal sums $\mathbf{G} = \sum_{j=0}^{n} p_j$ or, showing the multiplicities, after a suitable renumbering, $\mathbf{G} = \sum_{j=0}^{r} \mu_j p_j$, $p_j \neq p_s$ if $j \neq s$, $\sum_{j=1}^{r} \mu_j = n$. The integer n is called the *degree*, and also the *number of points (counted according to multiplicities)* of $\mathbf{G}$. If C and C' are curves of a projective plane with no common component, then

$$C \cdot C' = \sum_{p \in C \cap C'} [C \cdot C']_p \, p \,,$$

$[\ \cdot\ ]_p$ meaning the intersection multiplicity at p, will be called the *intersection group* of C and C'. If the points of a group $\mathbf{G}$ belong to a projective line and have there homogeneous coordinates $p_j = [a_j, b_j]$, then $G = \prod_{j=1}^{n}(b_j x_0 - a_j x_1) = 0$ will be taken as an equation for $\mathbf{G}$. Conversely, any homogeneous polynomial $G \in \mathbf{R}[x_0, x_1]$, of degree n, is a product of n linear factors in $\mathbf{C}[x_0, x_1]$, and hence any equation $G = 0$ is the equation of a group of n, possibly imaginary, points.

We will consider objects composed of a curve C and a group of points $\mathbf{G}$ in the same plane, represented as $\mathbf{G} + C$: they will be called *augmented curves*. A useful convention is to consider both curves and groups of points as augmented curves (with empty zero-dimensional or one-dimensional part, respectively).

We will think of the field of complex numbers $\mathbf{C}$ as a (real) Euclidean plane, its metric structure being the one defined by the usual absolute value $|z| = \sqrt{z\bar{z}}$ of complex numbers. To avoid confusions, this Euclidean plane will be denoted by $\mathbf{E}$. Thus, as sets, $\mathbf{C} = \mathbf{E}$. If $f \in \mathbf{C}[z]$ is a polynomial, its roots, repeated according to their multiplicities, compose a group of points in $\mathbf{E}$ that will be denoted by $\mathbf{Z}(f)$.

Taking, as usual, (x, y) as the coordinates of the complex number $x + yi$ defines orthonormal coordinates on $\mathbf{E}$. We will denote by $\mathbf{P}(\mathbf{E})$ the projective closure of $\mathbf{E}$, namely the result of adding to $\mathbf{E}$ a line of improper points, each corresponding to a direction on $\mathbf{E}$. Thus $\mathbf{P}(\mathbf{E})$ is a real projective plane and we take on it the homogeneous coordinates associated to the above orthonormal coordinates x, y, so that the complex number $x + yi$ has homogeneous coordinates $(1, x, y)$ in $\mathbf{P}(\mathbf{E})$; equivalently, $x + yi = [1, x, y]$. We will also consider the complex extension of $\mathbf{P}(\mathbf{E})$, a further enlargement of the complex plane according to the sequence

$$\mathbf{C} = \mathbf{E} \subset \mathbf{P}(\mathbf{E}) \subset \mathbf{CP}(\mathbf{E}) \,.$$

In particular we will deal with the *cyclic* (or *circular*) points of the Euclidean plane $\mathbf{E}$: they are the (improper, imaginary and mutually conjugate) points $I = [0,1,i]$ and $J = [0,1,-i]$, which determine the metric structure of the Euclidean plane up to the choice of the unit of length.

We will not distinguish between an algebraic curve C in $\mathbf{E}$, defined by a non-homogeneous equation $g(x,y) = 0$, $g \in \mathbf{R}[x,y]$, and its projective closure in $\mathbf{P}(\mathbf{E})$, defined by the homogeneous equation $G = x_0^d\, g(x_1/x_0, x_2/x_0) = 0$, $d = \deg g$. Augmented curves of $\mathbf{E}$ will be those composed of a curve in $\mathbf{E}$ and a group of points all belonging to $\mathbf{E}$.

In the sequel we will write simply $\mathbf{P}$ for $\mathbf{P}(\mathbf{E})$; this will cause no confusion. As for any projective plane, the lines of $\mathbf{P}$ are the points of another projective plane $\mathbf{P}^\vee$, the *dual plane* of $\mathbf{P}$; coordinates in $\mathbf{P}^\vee$ may be taken so that the line of equation $wx_0 + ux_1 + vx_2 = 0$ in $\mathbf{P}$ has coordinates (w,u,v) in $\mathbf{P}^\vee$. Since the condition for the line of coordinates (w,u,v) to belong to the pencil p^*, of the lines through a fixed point $p = [c,a,b]$, is

$$wc + ua + vb = 0\,,$$

we see that in turn the lines of $\mathbf{P}^\vee$ are the pencils of lines of $\mathbf{P}$. Any inclusion $p \in \ell$, between a point and a line of $\mathbf{P}$, appears reversed, $\ell \in p^*$, in $\mathbf{P}^\vee$. Mapping $p^* \mapsto p$ is a projectivity through which the bidual space $(\mathbf{P}^\vee)^\vee$ is usually identified with $\mathbf{P}$.

The same is done with the lines of $\mathbf{CP}$, which are the points of the dual plane $\mathbf{CP}^\vee$ of $\mathbf{CP}$. Each line of $\mathbf{P}$ being identified to the line of $\mathbf{CP}$ with the same equation, we see that $\mathbf{P}^\vee \subset \mathbf{CP}^\vee$, and the latter appears as the complex extension of the former. In particular the improper line of $\mathbf{P}$, $L_\infty\colon x_0 = 0$, appears as the point of $\mathbf{P}^\vee$ with coordinates $(1,0,0)$, the pencils of lines through the cyclic points, I^*, J^*, are the lines of $\mathbf{CP}^\vee$ that have equations $u + iv = 0$ and $u - iv = 0$, and their intersection is L_∞, their only real point.

There is a one-to-one correspondence between curves of $\mathbf{CP}$ containing no lines and curves of $\mathbf{CP}^\vee$ containing no lines, so that the points of the curve C^* corresponding to a curve C of $\mathbf{CP}$ are the lines tangent to C and, conversely, the tangent lines to C^* are the pencils p^*, $p \in C$. The curve C^* is called the *envelope* of C, and C the curve *enveloped by* C^*. Also, C and C^* are said to be *dual* to each other. The degree of C^* is called the *class* of C: it may be viewed as the number of tangent lines to C going through any already fixed point p, counted with the multiplicities they have in the group $C^* \cdot p^*$. The latter is a group of lines in p^*, usually called the *group of tangents to C from p*. The reader may see [14], V.8.1 or [5], 5.1. In our case it is easy to see that if C has a real equation, then also C^* has a real equation, and conversely. Thus the bijection $C \leftrightarrow C^*$ restricts to a bijection between curves of $\mathbf{P}$ and curves of $\mathbf{P}^\vee$ containing no (real or imaginary) lines.

We define the envelope of a group of points $\mathbf{G} = p_1 + \cdots + p_m$, of $\mathbf{CP}$, to be the curve $\mathbf{G}^*$ of $\mathbf{CP}^\vee$ composed of the pencils of lines through the points of $\mathbf{G}$ taken with the same multiplicities, $\mathbf{G}^* = p_1^* + \cdots + p_m^*$. Then we extend the above bijection between curve and envelope to a bijection between the set of augmented curves of $\mathbf{CP}$ containing no line and the set of all curves of $\mathbf{CP}^\vee$, by taking as the *envelope* of an augmented curve $\mathcal{C} = \mathbf{G} + C$ the curve $\mathcal{C}^* = \mathbf{G}^* + C^*$, composed of the envelopes of $\mathbf{G}$ and C. The augmented curve $\mathcal{C}$ will be referred to as the augmented curve *enveloped* by $\mathcal{C}^\vee$. Obviously the degree of $\mathcal{C}^*$ equals the class of C plus the number of points of $\mathbf{G}$: we will call it the *class* of the augmented curve $\mathcal{C}$.

An augmented curve as above, $\mathcal{C} = \mathbf{G} + C$, is called *real* if and only if the curve C is real and for each point p belonging to $\mathbf{G}$, its complex conjugate also belongs to $\mathbf{G}$ and has the same multiplicity as p. The second condition is obviously satisfied if p is real. It is easy to check that real augmented curves have real envelopes and, conversely, each real curve of $\mathbf{P}^\vee$ envelops a real augmented curve of $\mathbf{P}$.

As defined above, a line ℓ is tangent to an augmented curve $\mathcal{C} = \mathbf{G} + C$ if and only if either ℓ is *tangent* to the curve C at a point q or ℓ contains a point q of $\mathbf{G}$. In both cases q will be called a *contact point* of ℓ and we will say that ℓ is tangent to $\mathcal{C}$ at q. Assume that $p \in \mathbf{P}$ does not belong to $\mathbf{G}$; then the *group of tangents* to $\mathcal{C}$ from p is defined to be $\mathcal{C}^* \cdot p^*$. It is well defined because in no case is $p^* \subset C^*$, and its elements are the tangents to $\mathcal{C}$ going through p.

3. Foci of algebraic curves

The aim of this section is to recall and reformulate the definition and some properties of the foci of algebraic curves, which belong to the today almost forgotten metric theory of algebraic curves; for more details, the reader may see Chapter X of [4], as well as the historical notes and references in [6]. We will continue to deal with the Euclidean plane $\mathbf{E}$, but of course, since any two Euclidean planes are isometric, the content of this section applies without changes to any Euclidean plane. Let C be a curve of $\mathbf{E}$ containing no real or imaginary line. For simplicity we will assume from now on that C is not tangent to the improper line, this being enough for our purposes.

The classical definition (due to Plücker) extends the usual one for central conics (see [11], V.9, for instance) by taking the *foci* of C to be the intersection points of the pairs of conjugate tangents to C from I and J. Equivalently, a point $q \in \mathbf{P}$ is a *focus* of C if and only if, in the dual plane, the line q^* joins two conjugate intersections of C^* with I^* and J^*. We will complete this definition by assigning multiplicities to the foci. Assume that the class of C is m, place ourselves in $\mathbf{CP}^\vee$ and write $C^* \cdot I^* = \ell_1 + \cdots + \ell_m$. The lines $\ell_1, \ldots, \ell_m$ are thus the tangent lines to C from I, repeated according to their multiplicities in $C^* \cdot I^*$. Since the equation of C^* may be taken real and those of the lines I^*, J^* mutually conjugate, the conjugates $\bar{\ell}_1, \ldots, \bar{\ell}_m$ of the above ℓ_j are the intersections of C^* and J^*, that is, the tangents to C from J (repeated according to their multiplicities in $C^* \cdot J^*$). Since C is assumed to be not tangent to L_∞, we have $L_\infty = I^* \cap J^* \notin C^*$ and therefore $\ell_j \neq \bar{\ell}_j$, $j = 1, \ldots, m$. Thus each pair $\ell_j, \bar{\ell}_j$ spans a real line of $\mathbf{CP}^\vee$ that does not contain L_∞, that is, a pencil of lines q_j^*, $q_j \in \mathbf{E}$. Each q_j is a focus according to the definition recalled above and we define $\Phi(C) = q_1 + \cdots + q_m$ to be the *focal group* of C.

The above definition applies without changes to any real augmented curve $\mathcal{C} = \mathbf{G} + C$ for which C is a curve of $\mathbf{P}$ containing no real or imaginary line and not tangent to L_∞, these augmented curves being called *non-parabolic* in the sequel. As the reader may easily check, the real points of $\mathbf{G}$ belong to the focal group of $\mathcal{C}$ with the same multiplicity they have in $\mathbf{G}$. In case all points of $\mathbf{G}$ are real, we have $\Phi(\mathbf{G} + C) = \mathbf{G} + \Phi(C)$. In particular the focal group of a group of real points is the group itself.

Most of the properties of foci follow from the next proposition, which is well known and widely used in the case of conics.

PROPOSITION 3.1. *Assume that $\mathcal{C} = \mathbf{G} + C$ is a non-parabolic augmented curve of class m and let $F = 0$ be a real homogeneous equation of the envelope of $\mathcal{C}$. Assume that $\mathbf{H}$ is a group of m points of $\mathbf{E}$ and that $H = 0$ is a real equation of the envelope of $\mathbf{H}$. Then $\mathbf{H}$ is the focal group of $\mathcal{C}$ if and only if there exist $\lambda \in \mathbf{R} - \{0\}$ and a homogeneous polynomial $P \in \mathbf{R}[w, u, v]$, of degree $m - 2$, for which*

$$F = \lambda H + (u^2 + v^2)P\,.$$

Proof. We place ourselves in $\mathbf{CP}^\vee$. Assume that $\mathbf{H} = \Phi(\mathcal{C})$. Then neither $\mathbf{H}^*$ nor $\mathbf{G}^*$ contains I^*, because both $\mathbf{H}$ and $\mathbf{G}$ are composed of proper points. By the definition of $\Phi(\mathcal{C})$, we have

$$\mathbf{H}^* \cdot I^* = \mathcal{C}^* \cdot I^*.$$

Take $\lambda = F(1,0,0)/H(1,0,0)$, which is finite and non-zero because $[1,0,0] = L_\infty$ does not belong to C^* or H^*, and is obviously real. Then the curve $D\colon F - \lambda H = 0$ of $\mathbf{CP}^\vee$ has degree m, intersects I^* at the points of $\mathcal{C}^* \cdot I^* = \mathbf{H}^* \cdot I^*$ with at least the multiplicities they have in the group and furthermore contains L_∞. By Bézout's theorem, D contains I^*. Therefore, since D is a real curve (or just repeating the above argument for J^*), D also contains J^* and hence contains the pair of lines $I^* + J^*$. Since $I^* + J^*$ has equation $u^2 + v^2 = 0$, we have

$$F - \lambda H = (u^2 + v^2)P\,,$$

$P \in \mathbf{C}[w,u,v]$, homogeneous and of degree $m-2$. By taking conjugates in the former equality, one sees that P must be real, which proves that the condition is necessary.

Conversely, since $u^2 + v^2$ vanishes identically on I^*,

$$\mathbf{H}^* \cdot I^* = \mathcal{C}^* \cdot I^*, \tag{3.1}$$

and so, in particular,

$$\mathbf{H}^* \cap I^* = \mathcal{C}^* \cap I^*. \tag{3.2}$$

Take $q \in \mathbf{H}$ and $\ell = q^* \cap I^*$. By Equation (3.2), $\ell \in \mathcal{C}^*$ and so ℓ is a tangent to $\mathcal{C}$ from I. Since q is a proper point, $\ell \neq L_\infty$. The latter being the only real line in I^*, ℓ is imaginary and therefore different from its conjugate $\bar{\ell}$, which in turn obviously belongs to both q^*, $\mathcal{C}^*$ and J^*. Thus $\bar{\ell}$ is a tangent to $\mathcal{C}$ from J and $q = \ell \cap \bar{\ell}$ is a focus of $\mathcal{C}$.

Call ν the multiplicity of q in $\mathbf{H}$, which by definition is the multiplicity of q^* as an irreducible component of $\mathbf{H}^*$. Note first that no other $q_1^* \subset \mathbf{H}^*$ contains ℓ, as the same arguments used above would apply to q_1, giving $q_1 = \ell \cap \bar{\ell} = q$. Then, using Equation (3.1),

$$[\mathcal{C}^* \cdot I^*]_\ell = [\mathbf{H}^* \cdot I^*]_\ell = \nu[q^* \cdot I^*]_\ell = \nu\,,$$

and so the multiplicities of q in $\mathbf{H}$ and $\Phi(\mathbf{C})$ are the same. We have seen thus that all the points q of $\mathbf{H}$ belong to $\Phi(\mathbf{C})$ with the same multiplicities. Since both groups of points have the same degree, their equality follows. □

The reader may note that if λ and P, taken as in the above statement, are allowed to vary, then $\lambda H + (u^2 + v^2)P = 0$ describes the equations of the envelopes of all non-parabolic augmented curves with focal group H.

We close this section by showing a nice property of the foci of algebraic curves, due to Laguerre, that follows easily from Proposition 3.1. Its version for conics is better known (see [11], VII.5, for instance). We shall not use it in the sequel.

THEOREM 3.2 (Laguerre). *Assume that C is a non-parabolic algebraic curve of class m, p a point other than the foci of C and $\ell_1, \dots, \ell_m$ the lines joining p to the foci of C, repeated according to the multiplicities of the foci in the focal group. If $t_1 + \cdots + t_m$ is the group of tangents to C from p and we assume that all tangents t_j are real, then*

$$\sum_{j=1}^{m} \widehat{\ell_j t_j} = 0\,,$$

where $\widehat{\ell_j t_j}$ is the angle between the lines ℓ_j, t_j (in whatever manner the foci and the tangent lines are numbered).

Before proving Theorem 3.2 we introduce an auxiliary result concerning groups of points on a line:

LEMMA 3.3. *Suppose we have three different groups of m points of a complex projective line $\mathbf{P}_1$, say $\mathbf{A} = a_1 + \cdots + a_m$, $\mathbf{B} = b_1 + \cdots + b_m$ and $\mathbf{C} = c_1 + \cdots + c_m$ with linearly dependent equations. Assume also that $c_1 \neq c_2$ and that neither c_1 nor c_2 belongs to $\mathbf{A}$ or $\mathbf{B}$. Then*

$$\prod_{j=1}^{m} (a_j, b_j, c_1, c_2) = 1\,,$$

where (a_j, b_j, c_1, c_2) stands for the cross-ratio of a_j, b_j, c_1, c_2.

Proof. Take homogeneous coordinates on $\mathbf{P}_1$ such that $c_1 = [1, 0]$ and $c_2 = [0, 1]$. Then any equation of $\mathbf{C}$, $H = \sum_{s=0}^{m} h_s x_0^{m-s} x_1^s = 0$, has $h_0 = h_m = 0$.

As no point a_j is equal to c_1 or c_2, these points may be written $a_j = [\alpha_j, 1]$, $\alpha_j \neq 0$, and so we take

$$P = \prod_{j=1}^{m} (x_0 - \alpha_j x_1) = 0$$

as an equation for $\mathbf{A}$. Similarly, write $b_j = [\beta_j, 1]$, $\beta_j \neq 0$, and take

$$Q = \prod_{j=1}^{m} (x_0 - \beta_j x_1) = 0$$

as an equation for $\mathbf{B}$. Then we have

$$(a_j, b_j, c_1, c_2) = \frac{\beta_j}{\alpha_j}.$$

By hypothesis, there is a relation $H = \lambda P + \mu Q$, from which, by equating the coefficients of x_0^m and x_1^m on both sides, we get:

$$\lambda + \mu = 0 \quad \text{and} \quad \lambda \alpha_1 \dots \alpha_m + \mu \beta_1 \dots \beta_m = 0$$

and so

$$\frac{\beta_1}{\alpha_1} \cdots \frac{\beta_m}{\alpha_m} = 1$$

as claimed. □

Proof of Theorem 3.2. In the pencil p^*, of real and imaginary lines through p, we take $\mathbf{A} = C^* \cdot p^*$, the group of tangents to C from p and $\mathbf{B} = \Phi(C)^* \cdot p^*$, the group of lines projecting the foci from p. In case $\mathbf{A} = \mathbf{B}$ the claim is obviously satisfied, as the reader can see. Otherwise, notations being as in Proposition 3.1, take D to be the curve of $\mathbf{P}^\vee$ which has equation $(u^2 + v^2)P = 0$. The inclusion $p^* \subset D$ would imply, by Proposition 3.1, that $\mathbf{A} = \mathbf{B}$, which has been excluded. So we take $\mathbf{C} = D \cdot p^*$ and, by Proposition 3.1 again, the groups $\mathbf{A}, \mathbf{B}, \mathbf{C}$ satisfy the hypothesis of Lemma 3.3. In view of the definition of $\mathbf{C}$ we are allowed to take as c_1, c_2 the lines pI, pJ joining p to the cyclic points. Since these lines are imaginary, they do not belong to $\mathbf{A}$ or $\mathbf{B}$, and so we may apply Lemma 3.3. Laguerre's formula (see [11], IV.8, for instance) gives

$$\widehat{\ell_j t_j} = \frac{1}{2i} \log(\ell_j, t_j, pI, pJ),$$

and the assertion follows. □

4. Polar curves and polar groups

We recall the basic definitions and some easy facts relative to polar curves and polar groups of points. In order to deal with both cases together, the definition and first properties will be given in the n-dimensional case; the reader may assume that $n = 1, 2$.

Take a real n-dimensional projective space $\mathbf{P}_n$, with fixed homogeneous coordinates $x_0, \dots, x_n$. Assume given in $\mathbf{P}_n$ a point $p = [a_0, \dots, a_n]$ and a hypersurface V, with equation $F = 0$, $F \in \mathbf{R}[x_0, \dots, x_n]$, homogeneous and of degree $d > 1$. An easy computation shows that the equation

$$a_0 \frac{\partial F}{\partial x_0} + \cdots + a_n \frac{\partial F}{\partial x_n} = 0$$

is an identity if and only if p is a d-fold point of V (i.e., V is a cone with vertex p). Otherwise it defines a hypersurface of degree $d-1$ which is called the *polar* of V relative to p (and also the polar of p with respect to V); it will be denoted in the sequel by $\mathcal{P}_p(V)$.

It is straightforward to verify that the above definition does not depend on the coordinates and therefore the relationship between V, p and $\mathcal{P}_p(V)$ is invariant under projectivities. The following results on polars will be used in the forthcoming sections.

LEMMA 4.1. *$p \notin V$ if and only if $\mathcal{P}_p(V)$ is defined and $p \notin \mathcal{P}_p(V)$.*

Proof. Simply use Euler's formula

$$dF(a_0, \dots, a_n) = \sum_{j=0}^{n} a_j \frac{\partial F}{\partial x_j}(a_0, \dots, a_n). \qquad \square$$

REMARK 4.2. If p is the last vertex of the projective frame of reference, $p = [0, \dots, 0, 1]$, then the polar has equation $\partial F / \partial x_n = 0$. One may always assume this to be the case after a suitable choice of projective coordinates.

LEMMA 4.3. *If $W \colon F_1 = 0$ is an irreducible component of multiplicity $\mu > 0$ of $V \colon F = 0$ (that is, F_1 is an irreducible factor of multiplicity μ of F) and $p \notin V$, then W is an irreducible component of multiplicity $\mu - 1$ of $\mathcal{P}_p(V)$.*

Proof. Assume that $F = F_1^{\mu} G$, with F_1 irreducible and not dividing G. By Remark 4.2 we may assume an equation of the polar to be

$$\frac{\partial F}{\partial x_n} = F_1^{\mu-1} \left(\mu \frac{\partial F_1}{\partial x_n} G + F_1 \frac{\partial G}{\partial x_n} \right) = 0 .$$

Since $p \notin V$, it does not belong to W either. Then, by Lemma 4.1, $\mathcal{P}_p(W)$ is defined and so $\partial F_1 / \partial x_n$ does not vanish identically. In the equality displayed above, F_1 does not divide $\partial F_1 / \partial x_n$ or G, because of its degree and the hypothesis. Thus F_1 is an irreducible factor of multiplicity $\mu - 1$ of the equation of $\mathcal{P}_p(V)$, as claimed. $\square$

Assume that p, ℓ and C are, respectively a point, a line and a curve of $\mathbf{P}_2$, with $p \in \ell$, $p \notin C$. Then $\ell \not\subset C$, $C \cdot \ell$ is a group of points of ℓ and we may consider its polar $\mathcal{P}_p(C \cdot \ell)$, which is defined by Lemma 4.1. Also $\mathcal{P}_p(C)$ is defined, for the same reason, and we have:

LEMMA 4.4. *Hypothesis and notations being as above, we have*

$$\mathcal{P}_p(C) \cdot \ell = \mathcal{P}_p(C \cdot \ell)\,.$$

Proof. Take coordinates with $p = [0,0,1]$ and $\ell\colon x_1 = 0$. Then x_0, x_2 may be taken as coordinates on ℓ and, relative to them, $C \cdot \ell$ has equation $F(x_0,0,x_2) = 0$. Then the assertion follows from Remark 4.2 and the obvious equality

$$\frac{\partial F}{\partial x_2}(x_0,0,x_2) = \frac{\partial F(x_0,0,x_2)}{\partial x_2}\,. \qquad \square$$

In the sequel we denote by $TC_q(C)$ the *tangent cone* to a plane curve C at one of its points q. The pencil q^* of the lines through q is a line of the dual plane $\mathbf{P}_2^\vee$ and in particular a one-dimensional projective space. Since the tangent cone $TC_q(C)$ is composed of lines through q counted with multiplicities, it is a group of points of the one-dimensional projective space q^*. It thus makes sense to consider, in q^*, the polar group of $TC_q(C)$ relative to any line through q. We have:

LEMMA 4.5. *Let C be a curve of $\mathbf{P}_2$ and q an e-fold point of C, $e > 1$. Assume that $p \neq q$ is a point of $\mathbf{P}_2$ such that the line qp does not belong to $TC_q(C)$. Then the polar $\mathcal{P}_p(C)$ is defined, q is a point of multiplicity $e-1$ of $\mathcal{P}_p(C)$ and*

$$\mathcal{P}_{qp}(TC_q(C)) = TC_q(\mathcal{P}_p(C))\,.$$

Proof. Take projective coordinates so that $q = [1,0,0]$ and $p = [0,0,1]$. Assume that C has degree d and equation

$$F = F_e x_0^{d-e} + \cdots + F_d = 0\,,$$

each F_j being a homogeneous polynomial in x_1, x_2. Then the tangent cone $TC_q(C)$ has equation $F_e = 0$. If F_e is written as a product of powers of distinct linear factors,

$$F_e(x_1,x_2) = \prod_{j=1}^{r} (u_j x_1 + v_j x_2)^{\mu_j}\,,$$

then $TC_q(C)$ is composed of the lines $\ell_j = [0, u_j, v_j]$, $j = 1, \dots, r$, with multiplicities μ_j. Then each ℓ_j has coordinates u_j, v_j in q^* and so

$$F_e(v, -u) = \prod_{j=1}^{r} (u_j v - v_j u)^{\mu_j} = 0$$

is an equation of $TC_q(C)$ as a group of points in q^*. The line qp has equation $x_1 = 0$, hence coordinates $(0, 1, 0)$ in $\mathbf{P}_2^\vee$ and thus coordinates $(1, 0)$ in q^*. The polar group $\mathcal{P}_{qp}(TC_q(C))$, which is well defined by Lemma 4.1, thus has equation

$$\frac{\partial F_e(v, -u)}{\partial u} = -\frac{\partial F_e}{\partial x_2}(v, -u)\,. \tag{4.1}$$

On the other hand, since we know from the above that $\partial F_e/\partial x_2$ is not identically zero, neither is

$$\frac{\partial F}{\partial x_2} = \frac{\partial F_e}{\partial x_2} x_0^{d-e} + \cdots + \frac{\partial F_d}{\partial x_2} = 0$$

identically. This expression can therefore be taken as an equation for $\mathcal{P}_p(C)$. Still using that $\partial F_e/\partial x_2$ is not identically zero, since it has degree $e - 1$, q has multiplicity $e - 1$ in $\mathcal{P}_p(C)$. Furthermore, an equation of $TC_q(\mathcal{P}_p(C))$ is

$$\frac{\partial F_e}{\partial x_2} = 0\,.$$

As argued for $TC_q(C)$, substituting $(v, -u)$ for (x_1, x_2) in the above equation yields an equation of $TC_q(\mathcal{P}_p(C))$ as a group of points of q^*. The result of the substitution is

$$\frac{\partial F_e}{\partial x_2}(v, -u) = 0\,;$$

comparing with Equation (4.1) concludes the proof. □

LEMMA 4.6. *Suppose we have a group* $\mathbf{G} = \mu_1 p_1 + \mu_2 p_2$, *of two different points of an affine line* $\mathbf{A}_1$, *and let* p_∞ *be the improper point of* $\mathbf{A}_1$. *Then* $\mathcal{P}_{p_\infty}(\mathbf{G}) = (\mu_1 - 1)p_1 + (\mu_2 - 1)p_2 + p$ *where* p *is the point dividing the segment* $p_1 p_2$ *in the ratio* μ_2/μ_1.

Proof. Take an affine coordinate x on $\mathbf{A}_1$ and the homogeneous coordinates x_0, x_1 associated to it ($x = x_1/x_0$). If p_1, p_2 have affine coordinates α_1, α_2, take

$$(x_1 - \alpha_1 x_0)^{\mu_1}(x_1 - \alpha_2 x_0)^{\mu_2} = 0$$

as an equation of $\mathbf{G}$. Since $p_\infty = [0, 1]$, $\mathcal{P}_{p_\infty}(\mathbf{G})$ has equation

$$(x_1 - \alpha_1 x_0)^{\mu_1 - 1}(x_1 - \alpha_2 x_0)^{\mu_2 - 1}\big(\mu_1(x_1 - \alpha_2 x_0) + \mu_2(x_1 - \alpha_1 x_0)\big) = 0\,,$$

which, using the affine coordinate, is

$$(x - \alpha_1)^{\mu_1 - 1}(x - \alpha_2)^{\mu_2 - 1}\big(\mu_1(x - \alpha_2) + \mu_2(x - \alpha_1)\big) = 0\,.$$

Then there is in $\mathcal{P}_{p_\infty}(\mathbf{G})$ a single point p other than p_1, p_2, and its affine coordinate α satisfies

$$\mu_1(\alpha - \alpha_2) + \mu_2(\alpha - \alpha_1) = 0\,,$$

as stated. □

5. Linfield's theorem

THEOREM 5.1 (Linfield). *Assume that $\mathcal{D}$ is a non-parabolic augmented curve whose focal group is the group of roots of a polynomial $f \in \mathbf{C}[z]$, $d = \deg f > 1$. Then the polar relative to the improper line of the envelope of $\mathcal{D}$ envelops a non-parabolic augmented curve $\mathcal{C}$, of class $d-1$, whose focal group is the group of roots of df/dz.*

In particular, if $\mathcal{D}$ is simply a group of points, we have:

COROLLARY 5.2. *Assume that $\mathbf{G}$ is the group of roots of a polynomial $f \in \mathbf{C}[z]$, $d = \deg f > 1$. Then the polar relative to the improper line of the envelope of $\mathbf{G}$ envelops a non-parabolic augmented curve $\mathcal{C}$, of class $d-1$, whose focal group is the group of roots of df/dz.*

Proof of Theorem 5.1. Denote, as before, by w, u, v the coordinates on $\mathbf{P}^\vee$, consider the ring homomorphism

$$\begin{aligned}\psi\colon \mathbf{R}[w,u,v] &\longrightarrow \mathbf{C}[z]\\ F(w,u,v) &\longmapsto F(z,-1,-i)\end{aligned}$$

and note the following easy facts:

(1) For any $F \in \mathbf{R}[w,u,v]$, it clearly follows from the definition that

$$\frac{d}{dz}\psi(F) = \psi\left(\frac{\partial F}{\partial w}\right).$$

(2) $\psi(u^2 + v^2) = 0$; therefore if $F = 0$ is an equation of the envelope of a non-parabolic augmented curve $\mathcal{C}$, and $H = 0$ an equation of the envelope of the focal group of $\mathcal{C}$, then by Proposition 3.1, $\psi(F) = \lambda\psi(H)$, $\lambda \in \mathbf{R} - \{0\}$.

(3) If $z_1, \dots, z_d$ are complex numbers and $F = 0$ is an equation of the envelope of the group of points of $\mathbf{E}$ which they compose, then $\psi(F)$ has roots $z_1, \dots, z_d$. For, in case $d = 1$, if $z_1 = a + bi$, then, up to a non-zero real factor, $F = w + av + bu$ and so $\psi(F) = z - (a + bi)$. The case $d > 1$ follows because ψ is a ring homomorphism.

To conclude the proof, assume that $F = 0$ is an equation for the envelope of $\mathcal{D}$. By (2) and (3) above, $\psi(F) = cf$ for a suitable $c \in \mathbf{C} - \{0\}$. Since $\mathcal{D}$ is assumed to be non-parabolic, $L_\infty \notin \mathcal{D}^*$; as seen in Lemma 4.1, $L_\infty \notin \mathcal{P}_{L_\infty}(\mathcal{D}^*)$, hence $\mathcal{C}$ is also non-parabolic. An equation of $\mathcal{P}_{L_\infty}(\mathcal{D}^*)$ being $\partial F/\partial w = 0$, on the one hand $\psi(\partial F/\partial w) = c\, df/dz$, by (2), while on the other hand, by (1), $\psi(\partial F/\partial w) = \psi(H)$ where H is a suitable equation of the envelope of the focal group of $\mathcal{C}$. Since, by (3), $\psi(H)$ has the focal group as group of roots, the claim follows. □

REMARK 5.3. It follows from Theorem 5.1 that for $1 \leq r \leq d - 1$, the r-th order iterated polar, relative to the improper line, of the envelope of $\mathcal{D}$ in Theorem 5.1 (or $\mathbf{G}$ in Corollary 5.2) envelops a non-parabolic augmented curve, of class $d - r$, whose focal group is the group of roots of $d^r f/dz^r$.

6. THE NICEST CASE

The envelope of the augmented curve $\mathcal{C}$ of Corollary 5.2 is the polar, relative to the improper line, of a curve composed of real lines. Due to this, $\mathcal{C}$ has a number of special properties that provide an alternative presentation of $\mathcal{C}$. Since for each multiple root z_j of f, say of multiplicity μ_j, the pencil z_j^* appears as a component of multiplicity $\mu_j - 1$ of $\mathcal{C}^*$ (by Lemma 4.3), in the sequel we will discard these obvious components of $\mathcal{C}^*$ and focus our attention on the remaining curve $\mathcal{S}^*$ and its enveloped augmented curve $\mathcal{S}$. In this section we will deal with the case of non-aligned roots. The next theorem is a direct generalization of Siebeck's result quoted in the introduction:

THEOREM 6.1. *Assume that $f(z) \in \mathbf{C}[z]$ has distinct roots $z_1, \dots, z_m$, $m > 1$, with respective multiplicities $\mu_1, \dots, \mu_m$, no three of the z_i being (as points of the complex plane) aligned. For each pair j, s, $1 \leq j < s \leq m$, let $p_{j,s}$ be the point which divides the segment with extremities z_j, z_s in the ratio μ_s/μ_j (i.e., $p_{j,s}z_j/p_{j,s}z_s = \mu_s/\mu_j$). Then:*

(1) *In the complex plane there is a unique augmented curve* $\mathcal{S}$, *of class* $m-1$, *tangent to each of the lines* z_jz_s, $1 \leq j < s \leq m$, *at the point* $p_{j,s}$.

(2) $\mathcal{S}$ *is non-parabolic and its foci agree, multiplicities included, with the roots of the derivative* df/dz *other than* $z_1, \ldots, z_m$. *In other words,*

$$\mathbf{Z}(df/dz) = \Phi(\mathcal{S}) + \sum_{j=1}^{m}(\mu_j - 1)z_j\,.$$

Proof. Take $\mathcal{C}^* = \mathcal{P}_{L_\infty}(\mu_1 z_1^* + \cdots + \mu_m z_m^*)$. By Corollary 5.2, the enveloped augmented curve $\mathcal{C}$ is non-parabolic and its focal group is the group of roots of df/dz. On the one hand, the roots z_j, $j = 1, \ldots, m$, of f appear with multiplicities $\mu_j - 1$ in the group of roots of df/dz. On the other hand, by Lemma 4.3, each pencil z_j^* appears as a component of multiplicity $\mu_j - 1$ of $\mathcal{C}^*$. Then we write

$$\mathcal{C}^* = (\mu_1 - 1)z_1^* + \cdots + (\mu_m - 1)z_m^* + \mathcal{S}^*$$

and take $\mathcal{S}$ to be the augmented curve enveloped by $\mathcal{S}^*$. Then $\mathcal{S}$ is non-parabolic too and the focal group $\Phi(\mathcal{C})$ is composed of the points z_j, with multiplicities $\mu_j - 1$, $j = 1, \ldots, m$, plus the focal group of $\mathcal{S}$: the latter is thus the group of roots of df/dz other than the z_j, $j = 1, \ldots, m$, and assertion (2) is established.

Regarding assertion (1), denote by $\ell_{j,s}$ the line of $\mathbf{E}$ joining z_j, z_s, $1 \leq j < s \leq m$, and call its improper point $q_{j,s}$. Since no three z_j are aligned, the $\ell_{j,s}$ are all different and so each $\ell_{j,s}$ is a singular point of $\mu_1 z_1^* + \cdots + \mu_m z_m^*$ at which the latter has tangent cone

$$TC_{\ell_{j,s}}(\mu_1 z_1^* + \cdots + \mu_m z_m^*) = \mu_j z_j^* + \mu_s z_s^*\,.$$

The line (of $\mathbf{P}^\vee$) joining $\ell_{j,s}$ and L_∞ is $q_{j,s}^* \neq z_j^*, z_s^*$. Then, by Lemmas 4.5 and 4.3, the tangent cone to the polar $\mathcal{C}^*$ at $\ell_{j,s}$ is

$$\mathcal{P}_{q_{j,s}^*}(\mu_j z_j^* + \mu_s z_s^*) = (\mu_j - 1)z_j^* + (\mu_s - 1)z_s^* + t_{j,s}\,, \tag{6.1}$$

where $t_{j,s}$ is a line of $\mathbf{P}^\vee$ through $\ell_{j,s}$, $t_{j,s} \neq z_j^*, z_s^*$. By omitting the components $(\mu_j - 1)z_j^*$ and $(\mu_s - 1)z_s^*$, it follows that

$$TC_{\ell_{j,s}}(\mathcal{S}^*) = t_{j,s}\,,$$

and so $\ell_{j,s}$ is a simple point of $\mathcal{S}^*$ at which the tangent line is $t_{j,s}$. Dualizing, $\ell_{j,s}$ is tangent to $\mathcal{S}$ as claimed and, its contact point being already named $p_{j,s}$, we have $t_{j,s} = p_{j,s}^*$.

Equality (6.1) may thus be rewritten

$$\mathcal{P}_{q_{j,s}^*}(\mu_j z_j^* + \mu_s z_s^*) = (\mu_j - 1)z_j^* + (\mu_s - 1)z_s^* + p_{j,s}^*,$$

or, equivalently, by biduality and the projective invariance of the polarity relationship,

$$\mathcal{P}_{q_{j,s}}(\mu_j z_j + \mu_s z_s) = (\mu_j - 1)z_j + (\mu_s - 1)z_s + p_{j,s}.$$

The last equality and Lemma 4.6 guarantee that $p_{j,s}$ belongs to the segment with endpoints z_j, z_s and divides it in the ratio μ_s/μ_j.

Lastly, to prove the uniqueness of an augmented curve subjected to the conditions of assertion (1), it is enough to prove the uniqueness of its envelope, which in turn follows directly from Lemma 6.2 below. □

LEMMA 6.2. *Assume there are given, in a real projective plane* $\mathbf{P}_2$*, lines* $L_1, \ldots, L_m$*,* $m \geq 2$*, no three concurrent. For each pair* s, j $(j < s)$*, write* $P_{j,s} = L_j \cap L_s$ *and assume we have fixed a line* $T_{j,s}$ *through* $P_{j,s}$*,* $T_{j,s} \neq L_j, L_s$*. Then there is at most one curve* C *of* $\mathbf{P}_2$*, of degree* $m-1$*, going through all the* $P_{j,s}$ *and having tangent* $T_{j,s}$ *at each* $P_{j,s}$*.*

Proof. Let C be a curve satisfying the above conditions. We begin by showing that C cannot contain any of the lines L_j. Indeed, up to renumbering the lines, assume that $C = rL_1 + C_1$, $\deg C_1 < m-1$ and $C_1 \not\supset L_1$. Then C_1 has to be tangent to each $T_{1,s}$ at $P_{1,s}$, $s = 2, \ldots m$, because L_1 is not. In particular $L_1 \cap C_1$ contains $P_{1,2}, \ldots, P_{1,m}$, in contradiction to the Bézout theorem.

Once we know that C contains no L_j, note that, for $j = 1, \ldots, m$ and again by Bézout's theorem, there are no intersections of C and L_j other than the $m-1$ points $P_{k,s}$ lying on L_j, the latter are simple intersections of C and L_j and so, in particular, non-singular points of C.

Assume now that two different curves $C\colon F = 0$ and $C'\colon F' = 0$ satisfy the above conditions. Call Θ the pencil of curves spanned by C and C', namely the family of the curves with equations

$$\lambda F + \lambda' F' = 0, \qquad (\lambda, \lambda') \in \mathbf{R}^2 - \{0, 0\}.$$

It is clear from these equations that any curve in Θ goes through any point shared by C and C', and so in particular through all the points $P_{j,s}$, and has at $P_{j,s}$ intersection multiplicity higher than one with $T_{i,j}$, because both C and C' have. It is also clear that for any point $P \in \mathbf{P}_2$ there is at least one curve in Θ going through P. If P is taken on L_j and different from all the $P_{k,s}$, we get a curve $C_j \in \Theta$ sharing more than $m-1$ points with L_j and hence (once again by Bézout's theorem) containing it. If $P_{k,s} \in L_j$, then both lines L_j and $T_{k,s}$ have intersection multiplicity with C_j higher than one, hence such a $P_{k,s}$ is a singular point of C_j. Assume now that two of the curves C_j, say, up to renumbering, C_1 and C_2, are different. Then C_1 and C_2 span Θ and both have $P_{1,2}$ as a singular point. It follows that all curves in Θ, in particular C, have $P_{1,2}$ as a singular point, in contradiction to what we have already proved for C. Thus $C_1 = \cdots = C_m$, which would be a curve of degree $m-1$ containing m different lines, a contradiction which proves the uniqueness of C. □

REMARK 6.3. If all roots of f are simple, then each $p_{j,s}$ is the midpoint of z_j, z_s.

REMARK 6.4. The number of conditions imposed on $\mathcal{S}$ in Theorem 6.1 (1) is $m(m-1)$, always larger than the number $m(m+1)/2-1$ of parameters (the ratios between the coefficients of the equation of its envelope) on which a general curve of class $m-1$ depends. Thus, the existence of $\mathcal{S}$ is a priori not clear.

REMARK 6.5. The augmented curve $\mathcal{S}$ of Theorem 6.1 will be called the *Siebeck curve of f*. It is important to retain that, besides its characterization in Theorem 6.1, the Siebeck curve of f is the augmented curve enveloped by

$$\mathcal{P}_{L_\infty}(\mu_1 z_1^* + \cdots + \mu_m z_m^*) - (\mu_1 - 1)z_1^* - \cdots - (\mu_m - 1)z_m^* .$$

EXAMPLE 6.6. Take $f = z^4 - 1$, whose roots $1, -1, i, -i$, all simple, are the vertices of a square. The envelope of the group of roots has equation

$$(w+u)(w-u)(w+v)(w-v) = w^4 - w^2u^2 - w^2v^2 + u^2v^2 = 0 .$$

Its polar relative to $L_\infty = [1, 0, 0]$ thus has equation

$$4w^3 - 2wu^2 - 2wv^2 = 0$$

and so splits into $w = 0$, the pencil of lines through the origin $O = 0$ of the complex plane, and the envelope $2w^2 - u^2 - v^2 = 0$ of the circle $C\colon x^2 + y^2 - 1/2 = 0$. The Siebeck curve of f is thus $\mathcal{S} = O + C$: C is tangent to the four sides of the square at their midpoints, while O is the midpoint of the two diagonals, according to the conditions of Theorem 6.1 (1). Since the focal group of C is $2O$, the focal group of $\mathcal{S}$ is $3O$ in accordance with $df/dz = 4z^3$.

The next example was given a direct proof in [9]:

EXAMPLE 6.7. Let f be a polynomial with only simple roots $z_1, \dots, z_m$. Assume that $z_1, \dots, z_m$ are the images by an affine map of the vertices of a regular m-gon. Take $i = 1, \dots, m$ and read the indices $\bmod m$. Then, for each s fixed, $1 \leq s < m/2$, all the segments $z_i z_{i+s}$ are tangent to an ellipse C_s at their midpoints and, furthermore, if m is even, all segments $z_i z_{i+m/2}$ have the same midpoint, called O in the sequel. Indeed, both properties are affine-invariant and obvious in the case of a regular polygon. It follows from Theorem 6.1 that the Siebeck curve of f is either $\mathcal{S} = C_1 + \cdots + C_{(m-1)/2}$, if m is odd, or $\mathcal{S} = O + C_1 + \cdots + C_{(m-2)/2}$ if m is even. The group of roots of df/dz is then

$$\mathbf{Z}(\frac{df}{dz}) = \Phi(\mathcal{S}) = \begin{cases} \Phi(C_1) + \cdots + \Phi(C_{(m-1)/2}) & \text{if } m \text{ is odd}, \\ O + \Phi(C_1) + \cdots + \Phi(C_{(m-2)/2}) & \text{if } m \text{ is even}. \end{cases}$$

EXAMPLE 6.8. Assume that f has simple roots $0, 1, 2i, 5+3i$. Since the roots are simple, the polar of the envelope of the group of roots is the envelope of the Siebeck curve $\mathcal{S}$ of f. A direct computation gives

$$\mathcal{S}^* : 4w^3 + 18w^2u + 15w^2v + 10wu^2 + 12wv^2 + 30wuv + 10u^2v + 6uv^2 = 0\,,$$

which is a non-singular cubic of $\mathbf{P}^\vee$. Then $\mathcal{S}$ is a sextic of $\mathbf{P}$ with 9 cusps, three of which are real, see Figure 1.

7. THE GENERAL CASE

From now on, we will no longer assume that no three roots of f are aligned. If three or more distinct roots of f lie on a line ℓ, there is still an augmented curve $\mathcal{S}$, determined by the roots of f, whose foci are the roots of df/dz other than the roots of f. The main difference with the case of Theorem 6.1 is that the lines containing three or more roots appear as multiple tangents to $\mathcal{S}$; their contact points are still determined by the roots of f lying on the line, but the determination is less explicit than in Theorem 6.1. The next definition will help to locate these contact points.

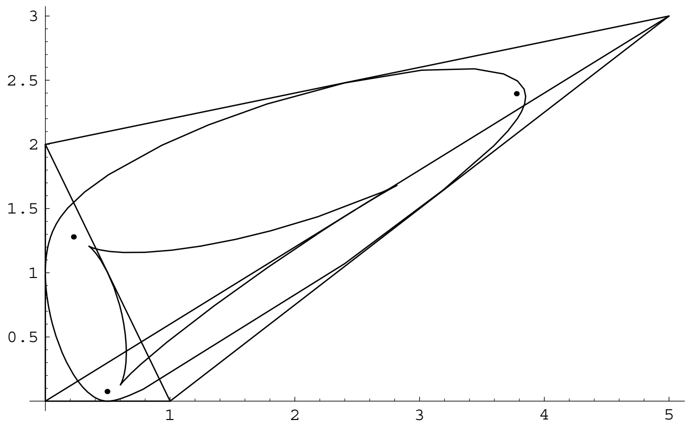

FIGURE 1
The lines joining the roots of f in Example 6.8,
the Siebeck curve $\mathcal{S}$ and its three foci

Assume that $\mathbf{G} = \sum_{j=1}^{r} \mu_j q_j$, with $q_j \neq q_s$ if $j \neq s$, is a group of points of a real affine line $\mathbf{A}^1$. If q_∞ is the improper point of $\mathbf{A}_1$, we call

$$\mathbf{H}(\mathbf{G}) = \mathcal{P}_{q_\infty}(\mathbf{G}) - \sum_{j=1}^{r}(\mu_j - 1)q_j$$

the harmonic group of $\mathbf{G}$; this makes sense by Lemma 4.3.

EXAMPLE 7.1. In Lemma 4.6, the point p is a single-point harmonic group.

LEMMA 7.2. *The harmonic group of a group of points* $\mathbf{G} = \sum_{j=1}^{r} \mu_j q_j$, $q_1, \ldots, q_r$ *distinct points of a real affine line, consists of* $r-1$ *distinct points, all real and of multiplicity one. Furthermore, any two consecutive points of* $\mathbf{G}$ *have just one of the points of the harmonic group between them.*

Proof. If x is an affine coordinate on $\mathbf{A}_1$ and x_0, x_1 its corresponding homogeneous coordinates ($x = x_1/x_0$), then q_∞ has homogeneous coordinates $(0,1)$. If $G(x_0,x_1) = 0$ is an homogeneous equation of $\mathbf{G}$, then $g = G(1,x)$ is a polynomial of degree $d = \mu_1 + \cdots + \mu_r$ whose roots are the affine coordinates α_j of the q_j, each root α_j having multiplicity μ_j. Since the polar group has equation $\partial G/\partial x_1 = 0$, similarly the roots of $(\partial G/\partial x_1)(1,x) = dg/dx$ are the α_j with multiplicities $\mu_j - 1$, $j = 1, \ldots, r$, together with the affine coordinates of the points of the harmonic group, the multiplicity of each root equal to the multiplicity of the corresponding point in the harmonic group. Then Rolle's theorem ensures that there is at least one point of the harmonic group between any two consecutive points of $\mathbf{G}$. Since, by its definition, the harmonic group contains at most $d-1-\sum_{j=1}^{r}(\mu_j-1)=r-1$ points, there is just one point of the harmonic group between any two consecutive points of $\mathbf{G}$, there are no further (real or imaginary) points in the harmonic group, and all multiplicities are one, as required. □

THEOREM 7.3. *Assume that $f(z)\in\mathbf{C}[z]$ has $m>1$ distinct roots $z_1,\dots,z_m$, with respective multiplicities $\mu_1,\dots,\mu_m$. For each line ℓ joining two different roots of f, let $\mathbf{G}_\ell$ be the group of the roots of f lying on ℓ, counted with their multiplicities as roots. Then:*

(1) *In the complex plane there is a unique augmented curve $\mathcal{S}$, of class $m-1$ and tangent to each line ℓ joining two roots of f at each of the points of the harmonic group of $\mathbf{G}_\ell$.*

(2) *$\mathcal{S}$ is non-parabolic and $\mathbf{Z}(df/dz)=\Phi(\mathcal{S})+\sum_{j=1}(\mu_j-1)z_j$.*

Proof. As in the proof of Theorem 6.1, take

$$\mathcal{C}^*=\mathcal{P}_{L_\infty}(\mu_1 z_1^*+\dots+\mu_m z_m^*)=(\mu_1-1)z_1^*+\dots+(\mu_m-1)z_m^*+\mathcal{S}^*.$$

The arguments used there prove that the augmented curve $\mathcal{S}$, enveloped by $\mathcal{S}^*$, satisfies assertion (2).

Assume that ℓ is a line joining two roots of f and that, after a suitable renumbering, the roots of f on ℓ are $z_1,\dots,z_r$. Denote by q_∞ the improper point of ℓ. We will work in $\mathbf{P}^\vee$ for a while. Clearly, the tangent cone to $\mu_1 z_1^*+\dots+\mu_m z_m^*$ at ℓ is $\mathbf{G}_\ell^*=\mu_1 z_1^*+\dots+\mu_r z_r^*$ and L_∞ does not belong to it. Then, by Lemma 4.5, the tangent cone to $\mathcal{C}^*$ at ℓ is the polar (in the pencil of lines of $\mathcal{P}^\vee$ through ℓ) of $\mathbf{G}_\ell^*$ relative to the line q_∞^* joining ℓ to L_∞. By Lemma 4.3,

$$\mathcal{P}_{q_\infty^*}(\mathbf{G}_\ell^*)=(\mu_1-1)z_1^*+\dots+(\mu_r-1)z_r^*+t_1+\dots+t_{r-1}\,,$$

where $t_1,\dots,t_{r-1}$ are lines of $\mathbf{P}^\vee$ through ℓ, $t_j\neq z_s^*$, for $j=1,\dots,r-1$ and $s=1,\dots,r$. Hence $t_1+\dots+t_{r-1}$ is the tangent cone to $\mathcal{S}^*$ at ℓ and therefore $t_j=p_j^*$ where $p_1,\dots,p_{r-1}$ are the contact points of ℓ and $\mathcal{S}$.

The above equality thus reads

$$\mathcal{P}_{q_\infty^*}(\mathbf{G}_\ell^*)=(\mu_1-1)z_1^*+\dots+(\mu_r-1)z_r^*+p_1^*+\dots+p_{r-1}^*\,,$$

which, returning to $\mathbf{P}$ by identifying lines of $\mathbf{P}^\vee$ to points of $\mathbf{P}$ by biduality, gives

$$\mathcal{P}_{q_\infty}(\mathbf{G}_\ell) = (\mu_1 - 1)z_1 + \cdots + (\mu_r - 1)z_r + p_1 + \cdots + p_r\,.$$

This shows that the contact points $p_1, \dots, p_{r-1}$ are the points of the harmonic group of $\mathbf{G}_\ell$.

As in the proof of Theorem 6.1, the uniqueness of $\mathcal{S}$ follows from the next lemma, which is just a more general version of Lemma 6.2. □

LEMMA 7.4. *Let Λ be a set of m distinct lines of a real projective plane $\mathbf{P}_2$. Denote by Π the set of points belonging to at least two lines in Λ and, for each $P \in \Pi$, by Λ_P the set of lines in Λ going through P. For each $P \in \Pi$, denote by ν_P the number of lines in Λ_P and assume given $\nu_P - 1$ different lines through P, $T_{P,1}, \dots, T_{P,\nu_p-1}$, none in Λ. Then there is at most one curve C, of degree $m-1$, which, for all $P \in \Pi$, goes through P and has tangents $T_{P,1}, \dots, T_{P,\nu_p-1}$ at P.*

Proof. For each $L \in \Lambda$, the lines other than L in Λ being $m-1$ in number,

$$m - 1 = \sum_{P \in L} (\nu_P - 1)\,, \tag{7.1}$$

which is the number of prescribed tangents at the points on L.

Assume that C satisfies the conditions stated in the conclusion, and fix $L \in \Lambda$. As in the proof of Lemma 6.2, $L \not\subset C$, since if not, $C = rL + C'$, $\deg C' < m-1$, $L \not\subset C'$, by Equation (7.1), C' would have at least $m-1$ different tangents at points on L, contradicting Bézout's theorem.

The prescribed tangents at $P \in \Pi$ are $\nu_P - 1$ in number, hence the multiplicity $e_P(C)$, of C at P, is $e_P(C) \geq \nu_P - 1$. Then for any line $L \in \Lambda$, by Bézout and Equation (7.1),

$$m - 1 \geq \sum_{P \in L \cap \Pi} e_P(C) \geq \sum_{P \in L \cap \Pi} (\nu_P - 1) = m - 1\,.$$

This ensures that $e_P(C) = \nu_P - 1$ for all $P \in L \cap \Pi$ and, since L is arbitrary, also for all $P \in \Pi$.

Assume that there are two curves C, C' satisfying the stated conditions. Arguing as in the proof of Lemma 6.2, C, C' span a pencil Θ of curves of degree $m-1$ in which, for each $L \in \Lambda$, there is a curve C_L containing L. For any $P \in L$, L and $T_{P,1}, \dots, T_{P,\nu_P-1}$ have with C_L intersection multiplicity at P higher than $\nu_P - 1$. This forces $e_P(C_L) > \nu_P - 1$. If $C_L \neq C_{L'}$ then they span Θ and both have multiplicity higher than $\nu_P - 1$ at $P = L \cap L'$. Therefore all curves in Θ, in particular C, have multiplicity higher than $\nu_P - 1$ at P, contradicting what we have seen above.

Lastly, if all the C_L agree, a curve of degree $m-1$ would contain all of the m distinct lines in Λ, which is absurd. $\square$

REMARK 7.5. Theorem 7.3 is simply a more general version of Theorem 6.1, just note Example 7.1 and Lemma 4.6.

REMARK 7.6. Still in the more general case of Theorem 7.3 we have

$$\mathcal{S}^* = \mathcal{P}_{L_\infty}(\mu_1 z_1^* + \cdots + \mu_m z_m^*) - (\mu_1 - 1)z_1^* - \cdots - (\mu_m - 1)z_m^*,$$

and the augmented curve $\mathcal{S}$ will be called the *Siebeck curve* of f.

COROLLARY 7.7 (of the proof of Theorem 7.3). *A line ℓ containing exactly $r > 1$ distinct roots of f is an ordinary singularity of multiplicity $r-1$ of $\mathcal{S}^*$ (a non-singular point if $r = 2$) with real tangents.*

Proof. The smooth and the ordinary singular points of a curve are those at which the number of tangents to the curve is equal to the multiplicity of the point. In the proof of Theorem 7.3 we have seen that, using the notation introduced there, the tangent cone to $\mathcal{S}^*$ at ℓ is $p_1^* + \cdots + p_{r-1}^*$ and also that $p_1 + \cdots + p_{r-1}$ is a harmonic group, so the claim follows from Lemma 7.2. $\square$

The singularities of a curve C^* in the dual plane are called *tangential singularities* of the enveloped (possibly augmented) curve C. The ordinary singularities of C^* are called *ordinary multiple tangents* of C: the number of contact points of an ordinary multiple tangent is equal to its multiplicity (as a point of C^*). In our case, a line ℓ containing exactly $r > 1$ distinct roots of f is either a non-singular tangent to $\mathcal{S}$, if $r = 2$, or an ordinary $(r-1)$-fold tangent with all its contact points real, if $r > 2$. Next is an example with a three-fold tangent.

EXAMPLE 7.8. Take $f = z(z+1)(z-2)(z+3)(z-1-i)$. Then the envelope of the roots is

$$w(u-w)(2u+w)(3u-w)(u+v+w) = 0$$

and so

$$\mathcal{S}^*: 6u^4 + 6u^3v + 2u^3w - 10u^2vw - 21u^2w^2 - 6uvw^2 - 4uw^3 + 4vw^3 + 5w^4 = 0,$$

which is a quartic of $\mathbf{P}^{\vee}$ with an ordinary triple point at $[0,0,1]$, hence rational. The Siebeck curve $\mathcal{S}$ is then a rational sextic of class four. See Figure 2.

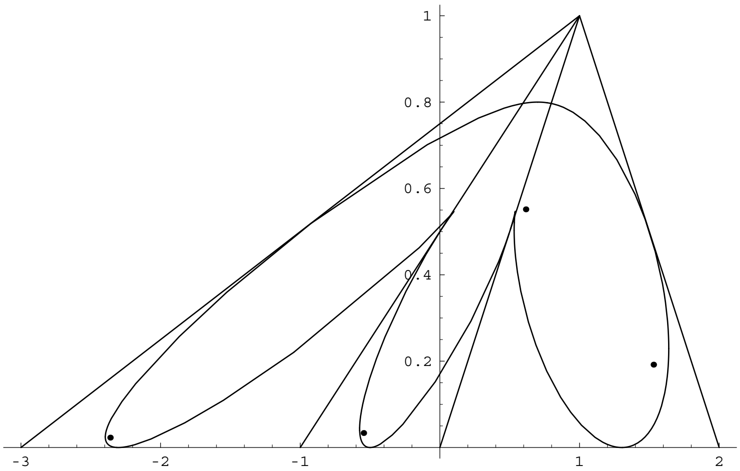

FIGURE 2
The lines joining the roots of f in Example 7.8, the Siebeck curve of f and its four foci

8. FURTHER PROPERTIES OF THE SIEBECK CURVE

A pencil of parallel lines p^*, p a (real) improper point of $\mathbf{E}$, is a one-dimensional projective space in which L_∞ is a distinguished element: taking L_∞ as the improper element defines on p^* a structure of an affine line. In what follows we will take all pencils of parallel lines endowed with this affine structure and use, in particular, the *betweenness* relation on parallel lines. The reader may note that if ℓ is any line of $\mathbf{E}$ transverse to p^* (that is, with $p \notin \ell$), then mapping each line of p^* to its intersection with ℓ is a projectivity which maps the improper line to the improper point, hence an affine map. In particular, regarding betweenness, L lies between L_1 and L_2 if and only if $L \cap \ell$ lies between $L_1 \cap \ell$ and $L_2 \cap \ell$, for any $L, L_1, L_2 \in p^* - \{L_\infty\}$.

Fix a pencil of parallel lines p^*, $p \in L_\infty$, of $\mathbf{E}$, and consider in it the group of lines $\mathbf{L}_p = \sum_{j=1}^m \mu_j z_j p$, composed of the lines through the roots of f in the direction of p, each counted with multiplicity equal to the sum of multiplicities of the roots of f lying on it. For each line L in $\mathbf{L}_p$, take r_L to be the number of different roots of f on L. Note that $r_L = 1$ for all L except in the case in which p is the improper point of a line joining two different roots of f. The main result in this section is:

PROPOSITION 8.1. *With the above notation, the group of tangents to $\mathcal{S}$ from an improper (real) point p is*

$$\mathbf{H}(\mathbf{L}_p) + \sum_{L \in \mathbf{L}_p} (r_L - 1)L\,,$$

where $\mathbf{H}(\mathbf{L}_p)$ denotes the harmonic group of $\mathbf{L}_p$.

Proof. For each line L joining p to one of the roots of f, we write $z_{L,1}, \ldots, z_{L,r_L}$ for the roots of f lying on L. By Lemma 4.4,

$$\mathcal{P}_{L_\infty}(\mu_1 z_1^* + \cdots + \mu_m z_m^*) \cdot p^* = \mathcal{P}_{L_\infty}((\mu_1 z_1^* + \cdots + \mu_m z_m^*) \cdot p^*) = \mathcal{P}_{L_\infty}(\mathbf{L}_p)\,.$$

On the one hand, by Remark 7.6,

$$\begin{aligned}
\mathcal{P}_{L_\infty}&(\mu_1 z_1^* + \ldots + \mu_m z_m^*) \cdot p^* \\
&= \mathcal{S}^* \cdot p^* + (\mu_1 - 1) z_1^* \cdot p^* + \cdots + (\mu_m - 1) z_m^* \cdot p^* \\
&= \mathcal{S}^* \cdot p^* + (\mu_1 - 1) z_1 p + \cdots + (\mu_m - 1) z_m p \\
&= \mathcal{S}^* \cdot p^* + \sum_{L \in \mathbf{L}_p} (\mu_{L,1} + \cdots + \mu_{L,r_L} - r_L) L\,.
\end{aligned}$$

On the other hand,

$$\mathcal{P}_{L_\infty}(\mathbf{L}_p) = H(\mathbf{L}_p) + \sum_{L \in \mathbf{L}_p} (\mu_{L,1} + \cdots + \mu_{L,r_L} - 1)L$$

and the conclusion follows. □

COROLLARY 8.2. *The Siebeck curve of f has no real tangential singularity other than the lines joining three or more distinct roots of f and so, in particular, no real tangential singularity at all if no three distinct roots of f are aligned.*

Proof. If a real line ℓ is a singular point of S^*, then it appears with multiplicity higher than one in any group $S^* \cdot p^*$ for any $p \in \ell$. The point p can be taken improper (and real), in which case Proposition 8.1 applies and shows that ℓ must be a line joining at least three roots of f. □

REMARK 8.3. Besides the multiple tangents of Corollary 8.2, $\mathcal{S}$ may have imaginary tangential singularities, for instance the tangents from O to C in Example 6.6.

From Corollaries 8.2 and 7.7 there follows:

COROLLARY 8.4. *All real tangential singularities of the Siebeck curve $\mathcal{S}$ of f are ordinary multiple tangents with the property that all their contact points are real.*

The tangents at the inflection points of a curve are non-ordinary tangential singularities ([14], V.8.1 or [5], 5.5), hence by Corollary 8.4:

COROLLARY 8.5. *A Siebeck curve has no real inflection points.*

The reader used to dealing with singularities and duality for plane curves will encounter no difficulty in deducing from Corollary 8.2 that all real branches of a Siebeck curve have class one (see for instance [3], Exercise 5.2), no two different branches having the same tangent, except for those tangent to one of the multiple tangents described in Corollary 7.7.

To conclude, the next corollary ensures that a Siebeck curve is bounded:

COROLLARY 8.6. *A Siebeck curve has no real improper points.*

Proof. If p is a real improper point of $\mathcal{S}$, then p^* is a line of $\mathbf{P}^\vee$ tangent to $\mathcal{S}^*$, and so either $p^* \subset S^*$ or $\mathcal{S}^* \cdot p^*$ contains at least one point (line of $\mathbf{P}$) with multiplicity higher than its multiplicity in $\mathcal{S}^*$. Proposition 8.1 shows that neither of these possibilities can occur. □

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