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THE SURJECTIVITY OF THE COMBINATORIAL LAPLACIAN ON INFINITE GRAPHS

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ABSTRACT. Given a connected locally finite simplicial graph G with vertex set V, the combinatorial Laplacian $\Delta_G \colon \mathbf{R}^V \to \mathbf{R}^V$ is defined on the space of all real-valued functions on V. We prove that Δ_G is surjective if G is infinite.

1. INTRODUCTION

Let G be a connected locally finite graph with vertex set V. To simplify the exposition, we shall always assume that G is *simplicial*, that is, without loops and multiple edges. Two vertices $v, w \in V$ are called *adjacent*, and one then writes $v \sim w$, if $\{v, w\}$ is an edge of G.

The *combinatorial Laplacian* on G is the linear map $\Delta_G \colon \mathbf{R}^V \to \mathbf{R}^V$, where $\mathbf{R}^V$ is the vector space consisting of all real-valued functions on V, defined by

$$\Delta_G(f)(v) = f(v) - \frac{1}{\deg(v)} \sum_{v \sim w} f(w)$$

for all $f \in \mathbf{R}^V$ and $v \in V$. Here $\deg(v)$ denotes the *degree* of the vertex v, i.e. the number of vertices in G which are adjacent to v.

Note that Δ_G is never injective since all constant functions are in its kernel. As a consequence, when the graph G is finite, Δ_G is not surjective since, in this case, $\mathbf{R}^V$ is finite-dimensional. More precisely, when G is finite, Δ_G is selfadjoint for the inner product on $\mathbf{R}^V$ defined by

$$\langle f, g \rangle = \sum_{v \in V} \deg(v) f(v) g(v) \,,$$

its kernel coincides with the space of constant functions, and its image is the orthogonal complement of this kernel, that is, the hyperplane consisting of all $f \in \mathbf{R}^V$ such that $\sum_{v \in V} \deg(v) f(v) = 0$.

In the present paper, we shall establish the following result.

THEOREM 1.1. *Let G be an infinite, connected, locally finite simplicial graph with vertex set V. Then the combinatorial Laplacian $\Delta_G \colon \mathbf{R}^V \to \mathbf{R}^V$ is surjective.*

In the particular case when G is the Cayley graph of an infinite finitely generated group, this result was obtained in [3] by taking two steps. The first consisted in showing that the image of the Laplacian is closed in the prodiscrete topology (see below). The second distinguished two cases according to whether the group is amenable or not. The proof we present here for general graphs is simpler in the sense that we also first establish the closed image property of the Laplacian (Section 2) but do not need to introduce amenability considerations. Instead, we apply the maximum principle to finitely-supported functions on vertices in order to prove that the image of the Laplacian is also dense in the prodiscrete topology (Section 3). The image, being both closed and dense, must be equal to the whole space $\mathbf{R}^V$.

2. The closed image property

Let G be a connected locally finite simplicial graph with vertex set V.

The *prodiscrete topology* on $\mathbf{R}^V$ is the product topology obtained by taking the discrete topology on each factor $\mathbf{R}$ of $\mathbf{R}^V$. This topology is metrizable. Indeed, if $(\Omega_n)_{n \in \mathbf{N}}$ is a non-decreasing sequence of finite subsets of V whose union is V, then the metric δ on $\mathbf{R}^V$ defined by

$$\delta(f,g) = \sum_{n \in \mathbf{N}} \frac{1}{2^{n+1}} \delta_n(f,g) \quad \text{for all } f,g \in \mathbf{R}^V,$$

where $\delta_n(f,g) = 0$ if f and g coincide on Ω_n and $\delta_n(f,g) = 1$ otherwise, induces the prodiscrete topology on $\mathbf{R}^V$. Note that a base of neighborhoods of $f \in \mathbf{R}^V$ in the prodiscrete topology is provided by the sets

$$W_n(f) = \{ g \in \mathbf{R}^V : f|_{\Omega_n} = g|_{\Omega_n} \} .$$

The goal of this section is to prove that the image of Δ_G is closed in $\mathbf{R}^V$ in the prodiscrete topology (Lemma 2.3). The proof is analogous to the proof of the closed image property for linear cellular automata over groups whose alphabets are finite-dimensional vector spaces (see [6], [4], and [5]).

For completeness, let us first recall some elementary facts about projective sequences and the Mittag-Leffler condition (cf. [1], [2], and [5]).

A *projective sequence* of sets consists of a sequence $(X_n)_{n\in\mathbf{N}}$ of sets together with maps $u_{nm}\colon X_m \to X_n$ defined for all $n \le m$ satisfying the following conditions:

(PS-1) u_{nn} is the identity map on X_n for all $n \in \mathbf{N}$;

(PS-2) $u_{nk} = u_{nm} \circ u_{mk}$ for all $n, m, k \in \mathbf{N}$ such that $n \le m \le k$.

Such a projective sequence will be denoted by (X_n, u_{nm}) or simply (X_n).

The *projective limit* $\varprojlim X_n$ of the projective sequence (X_n, u_{nm}) is the subset of $\prod_{n\in\mathbf{N}} X_n$ consisting of all the sequences $(x_n)_{n\in\mathbf{N}}$ which satisfy $x_n = u_{nm}(x_m)$ for all $n, m \in \mathbf{N}$ with $n \le m$.

Observe that if (X_n, u_{nm}) is a projective sequence of sets then it follows from (PS-2) that, for $n \in \mathbf{N}$ fixed, the sequence $(u_{nm}(X_m))_{m\ge n}$ is a non-increasing sequence of subsets of X_n. We say that the projective sequence (X_n, u_{nm}) satisfies the *Mittag-Leffler condition* if, for each $n \in \mathbf{N}$, the sequence $(u_{nm}(X_m))_{m\ge n}$ stabilizes, that is, there exists an integer $m_0 = m_0(n) \ge n$ such that $u_{nm}(X_m) = u_{nm_0}(X_{m_0})$ for all $m \ge m_0$.

LEMMA 2.1 (Mittag-Leffler). *If* (X_n, u_{nm}) *is a projective system of nonempty sets which satisfies the Mittag-Leffler condition, then its projective limit* $\varprojlim X_n$ *is not empty.*

Proof. Let (X_n, u_{nm}) be an arbitrary projective sequence of sets. The set $X'_n = \bigcap_{m\ge n} u_{nm}(X_m)$ is called the set of *universal elements* in X_n (cf. [7]). It is clear that the map u_{nm} induces by restriction a map $u'_{nm}\colon X'_m \to X'_n$ for all $n \le m$ and that (X'_n, u'_{nm}) is a projective sequence having the same projective limit as the projective sequence (X_n, u_{nm}).

Suppose now that all the sets X_n are nonempty and that the projective sequence (X_n, u_{nm}) satisfies the Mittag-Leffler condition. Then, for each $n \in \mathbf{N}$, there is an integer $m_0 = m_0(n) \ge n$ such that $u_{nm}(X_m) = u_{nm_0}(X_{m_0})$ for all $m \ge m_0$. It follows that $X'_n = u_{nm_0}(X_{m_0})$ so that, in particular, the set X'_n is not empty. We claim that the map $u'_{n,n+1}\colon X'_{n+1} \to X'_n$ is surjective for every $n \in \mathbf{N}$. Indeed, let us fix $n \in \mathbf{N}$ and suppose that $x'_n \in X'_n$. By the Mittag-Leffler condition, we can find an integer $p \ge n+1$ such that $u_{nk}(X_k) = u_{np}(X_p)$ and $u_{n+1,k}(X_k) = u_{n+1,p}(X_p)$ for all $k \ge p$. It follows that $X'_n = u_{np}(X_p)$ and $X'_{n+1} = u_{n+1,p}(X_p)$. Consequently, we can find $x_p \in X_p$ such that $x'_n = u_{np}(x_p)$. Setting $x'_{n+1} = u_{n+1,p}(x_p)$, we have $x'_{n+1} \in X'_{n+1}$ and

$$u'_{n,n+1}(x'_{n+1}) = u_{n,n+1}(x'_{n+1}) = u_{n,n+1} \circ u_{n+1,p}(x_p) = u_{np}(x_p) = x'_n .$$

This proves that $u'_{n,n+1}$ is onto. Now, as the sets X'_n are nonempty, we can construct by induction a sequence $(x'_n)_{n\in\mathbf{N}}$ such that $x'_n = u'_{n,n+1}(x'_{n+1})$ for all $n \in \mathbf{N}$. This sequence is in the projective limit $\varprojlim X'_n = \varprojlim X_n$. This shows that $\varprojlim X_n$ is not empty. $\square$

REMARK 2.2. In fact, the preceding proof shows that if (X_n, u_{nm}) is a projective system of nonempty sets satisfying the Mittag-Leffler condition and $X = \varprojlim X_n$ denotes its projective limit, then the natural projection map $\pi_n \colon X \to X'_n$ is surjective for every n (cf. [1, chapitre III, §7, no. 4, Proposition 5] and [2, chapitre II, §3, no. 5, Théorème 1 and Corollaire 1]).

LEMMA 2.3. *Let G be a connected locally finite simplicial graph with vertex set V. Then the image of the combinatorial Laplacian $\Delta_G \colon \mathbf{R}^V \to \mathbf{R}^V$ is closed in $\mathbf{R}^V$ in the prodiscrete topology.*

Proof. Let us fix a vertex $v_0 \in V$. For each $n \in \mathbf{N}$, let $B_n = \{v \in V : d_G(v_0, v) \leq n\}$ denote the closed ball of radius n centered at v_0 with respect to the graph metric d_G on V. Observe that Δ_G induces by restriction a linear map

$$\Delta_G^{(n)} \colon \mathbf{R}^{B_{n+1}} \to \mathbf{R}^{B_n} \tag{2.1}$$

for every $n \in \mathbf{N}$.

Suppose that $g \in \mathbf{R}^V$ is in the closure of $\Delta_G(\mathbf{R}^V)$ in the prodiscrete topology. Then, for each $n \in \mathbf{N}$, there exists $f_n \in \mathbf{R}^V$ such that g and $\Delta_G(f_n)$ coincide on B_n. Consider, for each $n \in \mathbf{N}$, the affine subspace $X_n \subset \mathbf{R}^{B_{n+1}}$ defined by

$$X_n = (\Delta_G^{(n)})^{-1}(g|_{B_n}).$$

Observe that $X_n \neq \varnothing$ since $f_n|_{B_{n+1}} \in X_n$. Now, for all $n \leq m$, the restriction map $\mathbf{R}^{B_{m+1}} \to \mathbf{R}^{B_{n+1}}$ induces an affine map $u_{nm} \colon X_m \to X_n$. Conditions (PS-1) and (PS-2) are trivially satisfied so that (X_n, u_{nm}) is a projective sequence. We claim that this projective sequence satisfies the Mittag-Leffler condition. Indeed, for n fixed, as the sequence $u_{nm}(X_m)$, where $m = n, n+1, \dots$, is a non-increasing sequence of affine subspaces of the finite-dimensional vector space $\mathbf{R}^{B_{n+1}}$, it must stabilize. It follows from Lemma 2.1 that the projective limit $\varprojlim X_n$ is nonempty. Choose an element $(x_n)_{n\in\mathbf{N}} \in \varprojlim X_n$. We have $x_n \in \mathbf{R}^{B_{n+1}}$. Moreover, x_{n+1} coincides with x_n on B_{n+1} for all $n \in \mathbf{N}$. As $V = \cup_{n\in\mathbf{N}} B_{n+1}$, there exists a (unique) $f \in \mathbf{R}^V$ such that $f|_{B_{n+1}} = x_n$ for all n. We have $(\Delta_G(f))|_{B_n} = \Delta_G^{(n)}(x_n) = g|_{B_n}$ for all n since $x_n \in X_n$. Since $V = \cup_{n\in\mathbf{N}} B_n$, it follows that $\Delta_G(f) = g$. This shows that $\Delta(\mathbf{R}^V)$ is closed in $\mathbf{R}^V$ in the prodiscrete topology. $\square$

3. Surjectivity

We now prove Theorem 1.1.

Suppose that G is an infinite, locally finite, connected simplicial graph. We keep the notation introduced in the proof of Lemma 2.3. Let F_n denote the vector subspace of $\mathbf{R}^V$ consisting of all functions $f \in \mathbf{R}^V$ whose support is contained in B_n. Consider the linear map $u_n \colon F_n \to F_n$ defined by

$$u_n(f)(v) = \begin{cases} \Delta_G(f)(v) & \text{if } v \in B_n \\ 0 & \text{otherwise}, \end{cases}$$

for all $f \in F_n$ and $v \in V$.

By the maximum principle, u_n is injective. Indeed, if $f \in F_n$ satisfies $u_n(f) = 0$, then we have

$$|f(v)| = \left| \frac{1}{\deg(v)} \sum_{v \sim w} f(w) \right| \leq \frac{1}{\deg(v)} \sum_{v \sim w} |f(w)|,$$

for all $v \in B_n$. This implies that if $v \in B_n$ satisfies $|f(v)| = M$, where $M = \max |f|$, then $|f(w)| = M$ for all $w \in V$ with $v \sim w$. Therefore $|f|$ is constant on B_{n+1}. As G is infinite, there are points in B_{n+1} that are not in B_n. Consequently, f is identically zero.

Now the injectivity of u_n implies its surjectivity since F_n is finite-dimensional. It follows that for all $g \in \mathbf{R}^V$ and $n \in \mathbf{N}$, we can find $f \in F_n$ such that $\Delta_G(f)$ coincides with g on B_n. This shows that $\Delta_G(\mathbf{R}^V)$ is dense in $\mathbf{R}^V$ in the prodiscrete topology. As $\Delta_G(\mathbf{R}^V)$ is also closed by Lemma 2.3, we conclude that $\Delta_G(\mathbf{R}^V) = \mathbf{R}^V$. This completes the proof of Theorem 1.1.

Remark 3.1. More generally, the same proof yields, *mutatis mutandis*, the surjectivity of $L = \Delta_G + \lambda \operatorname{Id} \colon \mathbf{R}^V \to \mathbf{R}^V$ for every infinite, locally finite, connected simplicial graph G and any function $\lambda \colon V \to [0, +\infty)$ defined on the vertex set V of G (here Id is the identity map on $\mathbf{R}^V$). Indeed, L is linear and, for $f \in \mathbf{R}^V$ and $v \in V$, from $L(f)(v) = 0$ we deduce that

$$\begin{aligned} |f(v)| &= \left| \frac{1}{(1 + \lambda(v)) \deg(v)} \sum_{v \sim w} f(w) \right| \\ &\leq \frac{1}{(1 + \lambda(v)) \deg(v)} \sum_{v \sim w} |f(w)| \leq \frac{1}{\deg(v)} \sum_{v \sim w} |f(w)|, \end{aligned}$$

so that the maximum principle is also satisfied by L.

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