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ON TOPOLOGICAL PROPERTIES OF THE FORMAL POWER SERIES SUBSTITUTION GROUP

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ABSTRACT. Certain topological properties of the group $\mathcal{J}(\mathbf{k})$ of formal one-variable power series with coefficients in a commutative topological unitary ring $\mathbf{k}$ are considered. We show, in particular, that in the case of $\mathbf{k} = \mathbf{Z}$ equipped with the discrete topology, in spite of the fact that the group $\mathcal{J}(\mathbf{Z})$ has continuous monomorphisms into compact groups, it cannot be embedded into a locally compact group. In the case where $\mathbf{k} = \mathbf{Q}$ the group $\mathcal{J}(\mathbf{Q})$ has no continuous monomorphisms into a locally compact group. In the last part of the paper the compressibility property for topological groups is considered. This property is valid for $\mathcal{J}(\mathbf{k})$ for a number of rings, in particular for the group $\mathcal{J}(\mathbf{Z})$.

Let $\mathbf{k}$ be a commutative ring with identity element. We further suppose that $\mathbf{k}$ is a topological ring, and if a topology is not indicated explicitly we suppose it to be discrete. Consider the set $\mathcal{J}(\mathbf{k})$ of all formal power series in the variable x with coefficients in $\mathbf{k}$ of the form:

$$(1)\quad f(x) = x + \alpha_1 x^2 + \alpha_2 x^3 + \cdots = x(1 + \alpha_1 x + \alpha_2 x^2 + \dots), \qquad \alpha_n \in \mathbf{k}.$$

This set becomes a group under the operation of substitution of series, then removing parentheses and collecting similar terms (cf. [15]): $f \circ g = f\big(g(x)\big)$. The general algebraic properties of the group $\mathcal{J}(\mathbf{k})$ are studied in [15].

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The group $\mathcal{J}(\mathbf{k})$ is a topological group in a natural way. Namely the set isomorphism $j\colon \mathbf{k}^{\mathbf{N}} \to \mathcal{J}(\mathbf{k})$, given by (1), defines the product topology on $\mathcal{J}(\mathbf{k})$. This topology coincides with the inverse limit topology of the groups $\mathcal{J}^m(\mathbf{k})$ as follows.

Even if $\mathbf{k}$ is not a priori a topological ring we can supply it with the discrete topology, as mentioned above. This defines the 0-dimensional inverse limit topology on $\mathcal{J}(\mathbf{k})$, cf. [2]. The latter is the strongest topology on $\mathcal{J}(\mathbf{k})$ among all natural topologies apart from the discrete one.

The group $\mathcal{J}(\mathbf{k})$, especially in case of a field of positive characteristic, has been of high interest in the last 15 years. Let $\mathbf{k} = \mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$ be a prime field. Known as the *Nottingham group*, $\mathcal{J}(\mathbf{Z}_p)$ is a pro-p group with remarkable properties. It is finitely presented [11], hereditarily just-infinite group of finite width [18, 14]. $\mathcal{J}(\mathbf{Z}_p)$ is known to be a universal pro-p group [6]. This means that every countably based pro-p group embeds into $\mathcal{J}(\mathbf{Z}_p)$ as a topological group. The Nottingham group $\mathcal{J}(\mathbf{Z}_p)$ can be viewed as a finite index subgroup of the automorphism group of the local field $\mathbf{Z}_p((x))$. This together with its universality links $\mathcal{J}(\mathbf{Z}_p)$ to Galois theory of local fields, see [12, 24] for more details. See also [8] for a panorama of properties of $\mathcal{J}(\mathbf{F}_q)$ where $\mathbf{F}_q$ is a finite field of characteristic p containing $q = p^k$ elements.

Some important general properties of the group $\mathcal{J}(\mathbf{k})$ in the case of a ring $\mathbf{k}$ of zero characteristic, say $\mathbf{k} = \mathbf{Z}, \mathbf{Q}$ or $\mathbf{R}$, were studied in the paper [15]. The structure of $\mathcal{J}(\mathbf{k})$, as a topological group, is completely different from the compact ring case. A number of questions which are trivial in the compact ring case become substantial for $\mathcal{J}(\mathbf{k})$ if $\mathbf{k}$ is not compact. For example, it is known [3] that any continuous action of the group $\mathcal{J}(\mathbf{k})$ on any compact space has an invariant probability Borel measure. This assertion, obvious for $\mathcal{J}(\mathbf{Z}_p)$, turns out to be a non-trivial fact even for $\mathcal{J}(\mathbf{Z})$ or $\mathcal{J}(\mathbf{R})$. In this paper we try to understand the topological group structure of $\mathcal{J}(\mathbf{k})$ for a non compact ring $\mathbf{k}$, in particular for $\mathbf{k} = \mathbf{Z}, \mathbf{Q}$ or $\mathbf{R}$. Some properties of these topological groups were obtained recently in [2, 3].

Note that the groups $\mathcal{J}(\mathbf{Z})$ and $\mathcal{J}(\mathbf{R})$ arise naturally in algebraic topology. The well-known Landweber-Novikov algebra (the special subalgebra of stable cohomology operations in complex cobordisms) can be expressed as the algebra of integer left invariant differential operators on $\mathcal{J}(\mathbf{R})$, see [5] for more details. See also [1] for relations of $\mathcal{J}(\mathbf{R})$ with symplectic topology and $\mathcal{J}(\mathbf{C})$ with some problems of the theory of analytic functions.

We say that a topological group G *admits a continuous monomorphism into a topological group* H if there exists a continuous group-monomorphism $\phi\colon G \longrightarrow H$. Moreover, if such a monomorphism $\phi\colon G \longrightarrow H$ is at the same time a homeomorphism onto its image we say that the topological group G is *embedded* into the topological group H. From this point of view the group $\mathcal{J}(\mathbf{Z})$, although being neither compact nor locally compact, does admit a continuous monomorphism into a compact group.

If we supply $\mathbf{Z}$ with the p-adic topology and if we embed $\mathbf{Z}$ into the ring $\mathbf{Z}_{(p)}$ of p-adic integers for a prime number p, we can obtain the simplest and most natural continuous monomorphism of $\mathcal{J}(\mathbf{Z})$ into a compact group. The compactness of the ring $\mathbf{Z}_{(p)}$ in the p-adic topology implies the compactness of the group $\mathcal{J}(\mathbf{Z}_{(p)})$. Moreover, $\mathcal{J}(\mathbf{Z})$ is an everywhere dense subgroup which lies inside.

It is easy to see that $\mathcal{J}(\mathbf{Z})$ is a residually finite group. Moreover it is approximated by finite p-groups for any given prime p. Thus one can obtain other interesting continuous monomorphisms of $\mathcal{J}(\mathbf{Z})$ into compact groups, for example into the group $\mathcal{J}(\mathbf{Z}_p)$.

As mentioned above, every countably based pro-p-group can be embedded into $\mathcal{J}(\mathbf{Z}_p)$. The universality of $\mathcal{J}(\mathbf{Z}_p)$ and the approximability of $\mathcal{J}(\mathbf{Z})$ by finite p-groups easily imply that $\mathcal{J}(\mathbf{Z})$ admits many different continuous monomorphisms into $\mathcal{J}(\mathbf{Z}_p)$. Unlike the monomorphism into $\mathcal{J}(\mathbf{Z}_{(p)})$ described above, most of these monomorphisms are unfortunately only implicit.

One of the principal purposes of this paper is to show that $\mathcal{J}(\mathbf{Z})$, supplied with the natural topology, admits no embedding into a locally compact group.

THEOREM 1. *Let* $\mathbf{k}$ *be an infinite discrete commutative ring and let the group* $\mathcal{J}(\mathbf{k})$ *be endowed with the inverse limit topology. Then* $\mathcal{J}(\mathbf{k})$ *does not admit any embedding into a locally compact group.*

The following result shows that if we pass from $\mathbf{Z}$ to the fraction field $\mathbf{Q}$ the situation becomes much more rigid.

THEOREM 2. *Let* $\mathbf{k}$ *be a topological ring containing the field of rational numbers. Then the group* $\mathcal{J}(\mathbf{k})$ *does not admit a continuous monomorphism into a locally compact group.*

Here we do not make any assumptions about either the topology of the ring $\mathbf{k}$ or topological properties of the inclusion $\mathbf{Q} \subset \mathbf{k}$.

Finally we note that the problem of embedding into locally compact groups for $\mathcal{J}(\mathbf{Z})$ (or $\mathcal{J}(\mathbf{k})$ for rings of general type) is interesting from both the viewpoint of topology and that of dynamics, in particular when the amenability of such groups is considered. So if $\mathcal{J}(\mathbf{k})$ acts on a compact space and if we try to find an invariant measure for such an action, the absence of embeddings of $\mathcal{J}(\mathbf{k})$ into locally compact groups means that the classical techniques, see [13] or [21], cannot be applied. See also [3] for more details.

In the second part of the paper the property of compressibility of topological groups is considered. We say that a topological group G is *compressible* if G can be embedded into each neighborhood of its unit element. The compressibility is a fractal-like property which is not intrinsic to Lie groups. There are many natural examples of compressible groups: for example the group of homeomorphisms of the n-dimensional ball that preserve pointwise the boundary sphere possesses this property.

Any compact compressible group is of dimension either zero or infinity (Theorem 3.5). However there exist many different examples of compressible topological groups of arbitrary dimension (see Example 3.6 and Proposition 4.6).

The compressibility of $\mathcal{J}(\mathbf{Z}_p)$ is known [7]. We show (Proposition 4.3) that $\mathcal{J}(\mathbf{k})$ is compressible for a large number of base rings $\mathbf{k}$, for example, for the ring of integers $\mathbf{Z}$. The compressibility of a topological group G also proves to be an obstruction for embedding G into a locally compact group, see Proposition 3.4 below.

1. Embeddings of the group $\mathcal{J}(\mathbf{k})$

Let the base ring $\mathbf{k}$ be supplied with some topology; the case of the discrete topology is of special interest. The corresponding inverse limit topology on the group $\mathcal{J}(\mathbf{k})$ is considered.

The subgroups

$$\mathcal{J}_n(\mathbf{k}) = \{f(x) \in \mathcal{J}(\mathbf{k}) \big| \alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = 0\}, \quad n = 2, 3, \ldots$$

are normal and form a base of neighborhoods of the unity element. It is natural that $\mathcal{J}_1(\mathbf{k}) = \mathcal{J}(\mathbf{k})$. The following sequences of nilpotent groups will also be useful

$$\begin{aligned} &\mathcal{J}^m(\mathbf{k}) = \mathcal{J}(\mathbf{k}) \big/ \mathcal{J}_{m+1}(\mathbf{k}), \quad m = 1, 2, \ldots\ ; \\ &\mathcal{J}_n^m(\mathbf{k}) = \mathcal{J}_n(\mathbf{k}) \big/ \mathcal{J}_{m+1}(\mathbf{k}), \quad m \geq n\ . \end{aligned} \tag{2}$$

We divide the proof of the above results into several simpler steps. The verification of the following statements is not difficult and we shall omit some parts of the proofs.

PROPOSITION 1.1. *The mapping* $p_n: \mathcal{J}_n(\mathbf{k}) \to \mathbf{k}$, $p_n(f) = \alpha_n \in \mathbf{k}$ *induces an isomorphism of the topological groups* $\hat{p}_n: \mathcal{J}_n^n(\mathbf{k}) \to \mathbf{k}$, *where* $\mathbf{k}$ *is considered as an additive group.*

PROPOSITION 1.2. *If a monomorphism* $f: G \to H$ *of topological groups is a topological embedding, then for any closed normal subgroup* $H_1 \subset H$ *the subgroup* $G_1 = f^{-1}(H_1)$ *is a closed normal subgroup in* G, *and the induced homomorphism of factor groups*

$$\hat{f}: G/G_1 \to H/H_1$$

is a topological embedding.

PROPOSITION 1.3. *If* $X \subset Y$ *is an everywhere dense subset in a space* Y *having no isolated points then* X *has no isolated points either.*

Proof. Let $a \in X$ be an isolated point in X, that is, $\{a\}$ is an open subset of X. Choose such a neighborhood V_a of a in Y that $V_a \cap X = \{a\}$. Since a is not an isolated point in Y, there exists a point $b \in V_a \setminus \{a\} \subset Y \setminus X$. The set $(Y \setminus \{a\}) \cap V_a$ is a neighborhood of b and it does not contain any points in X. This contradicts the density of X in Y. $\square$

PROPOSITION 1.4. *If a compact group* G *has an isolated point, then it is finite.*

COROLLARY 1.5. *A topological group containing an infinite discrete subgroup cannot be embedded into a compact group.*

Theorem 1 immediately follows from the following statement.

THEOREM 1.6. *If the additive group of a topological ring* $\mathbf{k}$ *contains an infinite discrete subgroup, then the group* $\mathcal{J}(\mathbf{k})$ *cannot be embedded into a locally compact group.*

Proof. Let $f: \mathcal{J}(\mathbf{k}) \to \widetilde{H}$ be a topological embedding in a locally compact group $\widetilde{H}$. Consider a compact neighborhood $V \subset \widetilde{H}$ of the identity. Then $U = f^{-1}(V) \subset \mathcal{J}(\mathbf{k})$ is a neighborhood of the identity in $\mathcal{J}(\mathbf{k})$. So there exists an index m such that $\mathcal{J}_m(\mathbf{k}) \subset U$. The subset $f(\mathcal{J}_m(\mathbf{k})) \subset V$ is a subgroup, so its closure $H = \overline{f(\mathcal{J}_m(\mathbf{k}))}$ is a compact group. It is easy to see that the restriction mapping (which we denote by the same symbol) $f: \mathcal{J}_m(\mathbf{k}) \to H$ is a topological embedding. Since the subgroup $G_1 = \mathcal{J}_{m+1}(\mathbf{k}) \subset \mathcal{J}_m(\mathbf{k})$ is closed and normal in $\mathcal{J}_m(\mathbf{k})$, the closure of its image $H_1 = \overline{f(G_1)}$ is a closed and normal subgroup of the compact group H. Since the subgroup $f(\mathcal{J}_m(\mathbf{k}))$ is dense in H we have $f(G_1) = H_1 \cap f(\mathcal{J}_m(\mathbf{k}))$.

According to Proposition 1.2 the induced mapping of factor groups

$$\hat{f}\colon \mathcal{J}_m(\mathbf{k})/\mathcal{J}_{m+1}(\mathbf{k}) \to H/H_1$$

is a topological embedding. The isomorphism $\mathcal{J}_m(\mathbf{k})/\mathcal{J}_{m+1}(\mathbf{k}) \simeq \mathbf{k}$ of Proposition 1.1 together with Corollary 1.5 lead to a contradiction. This completes the proof. □

2. Homomorphisms of $\mathcal{J}(\mathbf{Q})$ into compact and locally compact groups

As we saw in the introduction there are many different continuous monomorphisms of $\mathcal{J}(\mathbf{Z})$ not only into locally compact groups but into compact groups as well. The situation radically changes if we pass from integer coefficients to rational ones.

Now we shall study the structure of continuous homomorphisms from $\mathcal{J}(\mathbf{Q})$ into locally compact groups. Theorem 2 stated in the introduction immediately follows from the following stronger result, which we shall prove in this section.

THEOREM 2.1. *Let G be a locally compact group and $\phi\colon \mathcal{J}(\mathbf{Q}) \longrightarrow G$ be a continuous homomorphism. Then there exists an integer m such that $\mathcal{J}_m(\mathbf{Q}) \subset \ker\phi$. If furthermore G is compact, then $\mathcal{J}_3(\mathbf{Q}) \subset \ker\phi$.*

We recall (compare [16]) that a group G is called (algebraically) *complete* if for any $g \in G$ and any natural integer k there exists an element $h \in G$ such that $h^k = g$. This property is also known [25] as the *divisibility* of a group.

LEMMA 2.2. *If $\mathbf{k}$ is a field of zero characteristic then all the groups $\mathcal{J}_n(\mathbf{k})$, $n = 1, 2, \ldots$ are complete. Moreover the operation of taking the root is uniquely defined.*

The proof is simple if we consider the exponential map and in fact it can be understood from section 4 of [15]. We only note that the group $\mathcal{J}_n(\mathbf{Q})$

is actually the *Malcev completion* of $\mathcal{J}_n(\mathbf{Z})$. In other words $\mathcal{J}_n(\mathbf{Q})$ is the minimal complete group containing $\mathcal{J}_n(\mathbf{Z})$.

LEMMA 2.3. *Let G be a compact Lie group and $\phi\colon \mathcal{J}_n(\mathbf{Q}) \longrightarrow G$ be a continuous homomorphism. Then* $\mathrm{Im}(\phi)$ *lies in a maximal torus of G.*

Proof. Without loss of generality we may assume G to be the unitary group $\mathbf{U}(k)$ for some k. Let $H = \overline{\phi(\mathcal{J}_n(\mathbf{Q}))}$ and let H_0 be the connected component of the identity of this group. Since H is a compact Lie group, the factor group H/H_0 is finite.

Consider the sequence of homomorphisms

$$\mathcal{J}_n(\mathbf{Q}) \xrightarrow{\phi} H \longrightarrow H/H_0 .$$

The above composition is trivial since $\mathcal{J}_n(\mathbf{Q})$ is a complete group, so H is a connected Lie subgroup of the unitary group $\mathbf{U}(k)$. Since there are no small subgroups in Lie groups [19, p. 107], there exists a number m such that $\mathcal{J}_m(\mathbf{Q}) \subset \ker\phi$. Thus the homomorphism ϕ factorizes through the homomorphism $\hat{\phi}\colon \mathcal{J}_n^m(\mathbf{Q}) \longrightarrow H$, so this implies the nilpotency of H. Applying the Lie theorem [20, p. 54] to H (sometimes this result is known as the *Kolchin-Malcev theorem* [16] and as the *Lie-Kolchin theorem* as well [26]), we obtain that all the matrices of H have a common eigenvector. This, together with H being a connected subgroup of $\mathbf{U}(k)$, implies that

$$H \subset \mathbf{U}(1) \times \mathbf{U}(k-1) \subset \mathbf{U}(k) .$$

Inductively applying the above Lie theorem k times we obtain that H lies in a maximal torus of $\mathbf{U}(k)$. □

By [15, theorem 3.4] for a field $\mathbf{k}$ of zero characteristic the equality

$$\mathcal{J}_{2n+1}(\mathbf{k}) = \overline{[\mathcal{J}_n(\mathbf{k}), \mathcal{J}_n(\mathbf{k})]}$$

holds. Hence we immediately obtain

COROLLARY 2.4. *Under the hypothesis of Lemma* 2.3*, the inclusion* $\mathcal{J}_{2n+1}(\mathbf{Q}) \subset \ker\phi$ *holds.*

Now we turn to the proof of Theorem 2.1. Let $U \subset G$ be a compact neighborhood of the identity. Since ϕ is continuous there exists n such that

$$\phi\big(\mathcal{J}_n(\mathbf{Q})\big) \subset U .$$

Thus we have the mapping $\phi\colon \mathcal{J}_n(\mathbf{Q}) \longrightarrow \widetilde{G}$ where $\widetilde{G} = \overline{\phi\big(\mathcal{J}_n(\mathbf{Q})\big)}$ is a compact group. According to Pontryagin's theorem [23, theorem 68], $\widetilde{G}$ is developed into an inverse sequence of compact Lie groups:

$$G_1 \xleftarrow{\pi_1^2} G_2 \xleftarrow{\pi_2^3} \dots \xleftarrow{\pi_{k-1}^k} G_k \xleftarrow{\pi_k^{k+1}} \dots \longleftarrow \widetilde{G}. \tag{3}$$

Let $\pi_k\colon \widetilde{G} \longrightarrow G_k$ be the projection of the limit group on the k-th term of (3). Let $\phi_k = \pi_k \circ \phi$, $k = 1, 2, \dots$. By applying Corollary 2.4 to each of the homomorphisms $\phi_k\colon \mathcal{J}_n(\mathbf{Q}) \longrightarrow G_k$ we get $\mathcal{J}_{2n+1}(\mathbf{Q}) \subset \ker \phi_k$ for all k. This implies

$$\mathcal{J}_{2n+1}(\mathbf{Q}) \subset \ker \phi\,, \tag{4}$$

so the first part of the theorem has been proved.

In the case of a compact group G we can set $n = 1$ in (4), which finishes the proof.

REMARK 2.5. If a topological ring $\mathbf{k}$ admits a continuous monomorphism into a compact ring $\mathbf{k}'$ then the group $\mathcal{J}(\mathbf{k})$ clearly admits a continuous monomorphism into the compact group $\mathcal{J}(\mathbf{k}')$. If we pass from the compact case to the locally compact one we have quite a different situation. The ring of rational numbers $\mathbf{Q}$ equipped, for example, with the discrete topology admits continuous monomorphisms into locally compact rings, for example, into the ring of p-adic numbers $\mathbf{Q}_{(p)}$. Theorem 2.1 shows that if we pass from $\mathbf{Q}$ to the group $\mathcal{J}(\mathbf{Q})$ the above property completely disappears. The latter group cannot admit any continuous monomorphism into any locally compact group. If we have a continuous monomorphism $\mathbf{Q} \longrightarrow \mathbf{Q}_{(p)}$ the corresponding continuous monomorphism $\mathcal{J}(\mathbf{Q}) \longrightarrow \mathcal{J}(\mathbf{Q}_{(p)})$ is well defined but $\mathcal{J}(\mathbf{Q}_{(p)})$ is not locally compact.

3. Compressible topological groups

DEFINITION 3.1. A topological group G is called *compressible* if for any neighborhood of the unit $e \in U \subset G$ there exists a homomorphic embedding $h_U\colon G \longrightarrow G$ such that $\operatorname{Im} h_U \subset U$. The corresponding embedding h_U is called a *compression*.

EXAMPLE 3.2. Let H be a topological group and let $\mathbf{G} = H^{\mathbf{N}}$ be a countable direct product. It is easy to see that the group $\mathbf{G}$ is compressible.

All right shifts

$$h_n(g_1, g_2, \dots) = (e, e, \dots, e, g_1, g_2, \dots), \quad n = 1, 2, \dots ,$$

where e is the identity of G and g_1 on the right hand side is in the $(n+1)$-th position, can be taken as corresponding compressions.

The situation can be radically different in the case of limit groups of inverse sequences of topological groups. For example, let us assume in (3) G_k to be S^1 for $k = 1, 2, \dots$ and let the homomorphism π_k^{k+1} be the p-fold covering map for all k. In this case the corresponding limit group $\widetilde{G}$ is the so-called *p-adic solenoid*, and it is not a compressible group.

EXAMPLE 3.3. Consider the group $G = Homeo_+[0,1]$ of homeomorphisms of the interval $[0,1]$ preserving the endpoints. We suppose that G is equipped with the natural topology of a metric space homeomorphism group.

Let $f \in G$. We assume

$$h_n\big(f\big)(x) = \frac{1}{n} f(nx), \quad 0 \le x \le \frac{1}{n}; \qquad h_n\big(f\big)(x) = x, \quad \frac{1}{n} \le x \le 1.$$

It is clear that h_n is an embedding of G into itself for any n. Moreover, for an arbitrary neighborhood $U \subset G$ of the identity $e = e(x) = x$, there exists n such that $\mathrm{Im} h_n \subset U$. Thus the mappings $h_n, \quad n = 1, 2, \dots,$ form a system of compressions on G. Note that the *Thompson group F* [9] is naturally embedded into $Homeo_+[0,1]$, and this embedding is compatible with the compressions $h_{2^k}, \quad k = 1, 2, \dots$. Thus $h_{2^k}, \quad k = 1, 2, \dots$, are the compressions of the group F if it is supplied not with the discrete topology but with the topology induced from the embedding $F \subset Homeo_+[0,1]$.

Compressibility turns out to be an obstruction for a group to be embeddable into a locally compact group.

PROPOSITION 3.4. *A compressible group containing an infinite discrete subgroup cannot be embedded into any locally compact group.*

Proof. Suppose the contrary and let $G \subset K$ where G is a compressible group and K is locally compact. Consider an arbitrary compact neighborhood of the identity $V \subset K$, and let $U = V \cap G$ be the corresponding neighborhood of the identity in G. Consider a compression $h_U \colon G \longrightarrow U \subset V$. It induces an embedding $h_U \colon G \longrightarrow \overline{h_U(G)}$ where the group $\overline{h_U(G)}$ is compact. This contradicts Corollary 1.5 since G contains an infinite discrete subgroup. $\square$

The homeomorphism group G in Example 3.3 contains some infinite discrete subgroups. The subgroup generated by the mapping $f(x) = x^2$ is precisely of this type, so the group G cannot be embedded into a locally compact group.

THEOREM 3.5. *Any compressible compact group is zero-dimensional or infinite-dimensional.*

Proof. Let G be a compressible compact group of finite positive dimension $n = \dim G > 0$. By a theorem due to Pontryagin [23, Theorem 69], there exists a zero-dimensional normal subgroup N in G such that the factor group G/N is a Lie group. Let $p_N: G \to G/N$ be the projection. Since Lie groups have no small subgroups there exists a neighborhood V of the identity $p_N(e)$ in G/N which does not contain any nontrivial subgroups. Consider the neighborhood of the identity $U = p_N^{-1}(V)$ in G and let $h_U: G \longrightarrow G$ be a corresponding compression. As the subgroup $H = p_N\big(h_U(G)\big)$ is in V, it is trivial. This implies that $h_U(G) \subset N$. The above inclusion contradicts the dimension monotonicity principle. So the group G cannot be of finite positive dimension. □

The compactness hypothesis in Theorem 3.5 is significant.

EXAMPLE 3.6. For any natural number n there exists an algebraically complete, connected, locally connected, separable, metrizable, compressible group G_n, such that $\dim G_n = n$. In fact, for any $n \in \mathbf{N}$, Keesling [17] constructed a connected, locally connected, algebraically complete group $K_n \subset \mathbf{R}^{n+1}$, such that $\dim K_n = \dim(K_n)^{\mathbf{N}} = n$. According to Example 3.2, $G_n = (K_n)^{\mathbf{N}}$ is just the desired group.

Similar examples can be constructed in the class of more special groups like the substitution groups $\mathcal{J}(\mathbf{k})$, see Proposition 4.6 below.

4. DILATION AND COMPRESSIONS ON THE GROUP $\mathcal{J}(\mathbf{k})$

For an arbitrary ring $\mathbf{k}$ the group $\mathcal{J}(\mathbf{k})$ admits two important topological endomorphisms which we describe in this section. For any element $f(x) \in \mathcal{J}(\mathbf{k})$ defined by Formula (1) and for any $t \in \mathbf{k}$ we set

$$\delta_t(f) = \frac{1}{t} f(tx) = x(1 + t\alpha_1 x + t^2\alpha_2 x^2 + \dots)\,. \tag{5}$$

The second part of equality (5) correctly defines δ_t for all $t \in \mathbf{k}$, and the first part of (5), which should be understood formally, implies that δ_t is an endomorphism of the group $\mathcal{J}(\mathbf{k})$. If t is not a zero divisor in $\mathbf{k}$ then δ_t is an algebraic monomorphism. Furthermore if $\mathbf{k}$ is discrete (the case we are interested in) then δ_t is an embedding of $\mathcal{J}(\mathbf{k})$ into itself. Let the topological endomorphism δ_t introduced in (5) be called a *dilation* of $\mathcal{J}(\mathbf{k})$ corresponding to a parameter $t \in \mathbf{k}$. For any $t, r \in \mathbf{k}$, (5) directly implies $\delta_t \circ \delta_r = \delta_{tr}$. Thus, if t is invertible in $\mathbf{k}$, then δ_t is an automorphism of $\mathcal{J}(\mathbf{k})$, and δ_t realizes an embedding of the (multiplicative) group of the invertible elements of the ring $\mathbf{k}$ into the automorphism group $Aut(\mathcal{J}(\mathbf{k}))$.

To define the other special mapping of $\mathcal{J}(\mathbf{k})$ into itself, we rewrite (1) as follows

$$f(x) = x(1 + xh(x)),$$

where $h(x) = \alpha_1 + \alpha_2 x + \dots$ is a formal power series with coefficients in $\mathbf{k}$. Furthermore, for any natural integer s we set

$$\Theta_s(f) = x\Big(1 + s^2 x^s h(s^2 x^s)\Big)^{\frac{1}{s}}, \tag{6}$$

where the right hand side means a binomial series expansion together with the raising of $h(s^2 x^s)$ to powers. To prove the correctness of (6) we use the following fact from elementary analysis.

LEMMA 4.1. *For any natural integer s, all positive index Taylor coefficients of the function*

$$u(z) = \Big(1 + s^2 z\Big)^{\frac{1}{s}}$$

are integers and divisible by s.

Proof. Let

$$u(z) = 1 + \beta_1 z + \beta_2 z^2 + \dots$$

be the expansion of $u(z)$ at zero. The integrality of β_k, $k = 1, 2, \dots$, and the divisibility of these coefficients by s will be proved by induction. By expanding $u(z)$ into a binomial series we obtain $\beta_1 = s$. Suppose that the statement is proved for all β_k, $k < n$, and we prove it for β_n. Setting $u_k(z) = \beta_1 z + \beta_2 z^2 + \dots + \beta_k z^k$ we obtain

$$\begin{aligned} u(z)^s &= \big(1 + u_{n-1}(z) + \beta_n z^n + \mathcal{O}(z^{n+1})\big)^s \\ &= \big(1 + u_{n-1}(z) + \beta_n z^n\big)^s + \mathcal{O}(z^{n+1}) \\ &= \big(1 + u_{n-1}(z)\big)^s + s\beta_n z^n + \mathcal{O}(z^{n+1}) \qquad (7) \\ &= 1 + s u_{n-1}(z) + \sum_{k=2}^{s} C_s^k (u_{n-1}(z))^k + s\beta_n z^n + \mathcal{O}(z^{n+1}). \end{aligned}$$

Since all the coefficients of the polynomial $u_{n-1}(z)$ are divisible by s, the coefficients of $\sum_{k=2}^{s} C_s^k (u_{n-1}(z))^k$ are divisible by s^2. By setting the sum of coefficients attached to z^n in the right hand side of (7) equal to zero, and using the functional equation $u(z)^s = 1 + s^2 z$ we obtain, for $s\beta_n$, an integer expression divisible by s^2. $\square$

REMARK 4.2. The above lemma is a particular case of a general result obtained by Eisenstein, see [22, sec. 8, chap. 3]. If a series $v(z) = \beta_1 z + \beta_2 z^2 + \dots$ having rational coefficients is an algebraic function in z then there exists a natural number T such that the series $v(Tz) = \beta_1 Tz + \beta_2 T^2 z^2 + \dots$ has integer coefficients. The smallest T which satisfies the above condition is the so-called *Eisenstein index*. It can be expressed by prime divisors of the coefficients of the equation for $v(z)$. Sometimes this expression can be rather complex, which we can see in the simplest case $\big(v(z) + 1\big)^s = 1 + z$ we are interested in. Lemma 4.1 shows that in the situation under consideration we can choose $T = s^2$. The quantity $T = s^2$ exceeds, as a rule, the corresponding Eisenstein index but it is sufficient for our purposes (cf. [22, sec. 8, chap. 3]).

It is clear that for an arbitrary natural integer s all power series having the structure

$$f(x) = x(1 + \alpha_1 x^s + \alpha_2 x^{2s} + \cdots + \alpha_n x^{ns} + \dots), \qquad \alpha_n \in \mathbf{k}$$

form a closed subgroup in $\mathcal{J}(\mathbf{k})$, which we denote by $\mathcal{J}_{(s)}(\mathbf{k})$. For any s we have the obvious inclusions $\mathcal{J}_{(s)}(\mathbf{k}) \subset \mathcal{J}_s(\mathbf{k})$. The subgroups $\mathcal{J}_{(s)}(\mathbf{k})$ have been closely studied in the case of a field $\mathbf{k}$ of positive characteristic, see, for example, [4, 7]. Some results obtained in this article are not related to phenomena of positive characteristic but they are of general nature and are subject to the compressibility scheme mentioned above.

PROPOSITION 4.3. *For an arbitrary natural integer s the mapping Θ_s is well defined, and it is a continuous homomorphism from the group $\mathcal{J}(\mathbf{k})$ into $\mathcal{J}_{(s)}(\mathbf{k})$. If $s = s1$ is not a zero divisor in $\mathbf{k}$, then Θ_s is a monomorphism. Moreover if $\mathbf{k}$ is discrete, then Θ_s is an embedding. If s is invertible in $\mathbf{k}$, then Θ_s is an isomorphism of $\mathcal{J}(\mathbf{k})$ onto the group $\mathcal{J}_{(s)}(\mathbf{k})$.*

Proof. Lemma 4.1 immediately implies that Θ_s is well defined. If $\mathbf{k}$ is a discrete ring the continuity of the above mapping is obvious. In the general case we proceed as follows. If $f(x) = x(1+xh(x)+\mathcal{O}(x^{k+2}))$, where $h(x) \in \mathbf{k}[x]$ is a polynomial of degree k, then (6) implies

$$\Theta_s(f(x)) = x(1 + H(x^s) + \mathcal{O}(x^{s(k+2)})),$$

where $H(x) \in \mathbf{k}[x]$ is a polynomial of degree $k+1$. Moreover

$$H(x) + 1 = \left(1 + s^2xh(s^2x)\right)^{\frac{1}{s}} \mod x^{k+2}.$$

So the coefficients of $H(x)$ are expressed by the coefficients of $h(x)$ by means of universal integer polynomials. This implies the continuity of Θ_s. The formal presentation $\Theta_s = \theta_s \circ \delta_{s^2}$ implies that Θ_s is a homomorphism; here δ_{s^2} is the dilation with the parameter $s^2 = s^2 1$ and

$$\theta_s(f(x)) = f(x^s)^{\frac{1}{s}}. \tag{8}$$

If $f(x) = x(1 + \alpha_n x^n + \dots)$, where $\alpha_n \neq 0$, then we have from (6)

$$\Theta_s(f) = x\Big(1 + s^{2n-1}\alpha_n x^{ns} + \mathcal{O}(x^{ns+1})\Big).$$

This implies that Θ_s is a monomorphism in the case where s is not a zero divisor in $\mathbf{k}$. Finally, we note that for an arbitrary natural integer n, formula (6) implies the existence of the induced homomorphisms $\Theta_s^n \colon \mathcal{J}^{n-1}(\mathbf{k}) \to \mathcal{J}^{sn-1}(\mathbf{k})$. By virtue of all the above arguments, Θ_s^n is a monomorphism if s is not a zero divisor in $\mathbf{k}$. It remains to note that the discreteness of $\mathbf{k}$ implies the discreteness of the groups $\mathcal{J}^n(\mathbf{k})$, $n = 1, 2, \dots$, i.e. the monomorphisms Θ_s^n are topological embeddings. It is easy to see that

$$\Theta_s = \varprojlim_n \Theta_s^n,$$

which implies that Θ_s is an embedding. Finally, let s be invertible in $\mathbf{k}$, and let

$$\widehat{\theta}_s(f(x)) = \left(f(x^{\frac{1}{s}})\right)^s; \tag{9}$$

then the mapping

$$\widehat{\Theta}_s = \delta_{s^{-2}} \circ \widehat{\theta}_s \colon \mathcal{J}_{(s)}(\mathbf{k}) \longrightarrow \mathcal{J}(\mathbf{k})$$

is well defined. The continuity and the fact that $\widehat{\Theta}_s$ is a monomorphism can be verified in the same way as for Θ_s. Formulae (8) and (9) immediately imply the equality $\widehat{\Theta}_s \circ \Theta_s = \mathrm{id}$. This completes the proof. □

DEFINITION 4.4. We call the endomorphism Θ_s, introduced in (6), a *compression in the group* $\mathcal{J}(\mathbf{k})$ *with coefficient* s.

REMARK 4.5. If a ring $\mathbf{k}$ contains the field of rational numbers $\mathbf{Q}$, we can eliminate the dilation by simply setting $\Theta_s = \theta_s$, where θ_s is defined by (8).

PROPOSITION 4.6. *There exists a topological ring with unity* $\mathbf{k}$ *such that the substitution group* $\mathcal{J}(\mathbf{k})$ *is a one-dimensional separable metrizable compressible group.*

Proof. Let ℓ_2 be the Hilbert space of square-summable real sequences. Consider the *Erdős space* E which is the subset of ℓ_2 that consists of sequences of rational numbers. Endow E with a ring structure as follows. Let an element of ℓ_2 have coordinates $\mathbf{x} = (x_0, x_1, x_2, \dots)$, and write this element in the form $\mathbf{x} = (x_0, \bar{x})$, where $\bar{x}$ contains all the coordinates except x_0. To define a multiplication on ℓ_2 we set

$$\mathbf{xy} = (x_0 y_0\,,\ x_0\bar{y} + y_0\bar{x} + \bar{x}\bar{y})\,,$$

for two vectors $\mathbf{x} = (x_0, \bar{x})$ and $\mathbf{y} = (y_0, \bar{y})$, where $\bar{x}\bar{y}$ is the coordinate-wise multiplication. It is clear that ℓ_2 supplied with such a multiplication becomes an algebra with unit $\mathbf{1} = (1, \bar{0})$, and so E becomes a sub-ring with unit (and even a $\mathbf{Q}$-algebra). We take the topological ring E just defined as the base ring $\mathbf{k}$. The topological space E is one-dimensional and all its finite powers are homeomorphic to itself, so they are also one-dimensional. Its countable power $E^{\mathbf{N}}$ is also one-dimensional [10, examples 1.2.15 and 1.5.17]. Thus the substitution group $\mathcal{J}(E)$ yields the desired example. Since E is a $\mathbf{Q}$-algebra, the homomorphisms (8) form a system of compressions on $\mathcal{J}(E)$. □

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