J. Eur. Math. Soc. 14, 1275–1330

DOI 10.4171/JEMS/333

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Invariant weighted Wiener measures and almost sure global well-posedness for the periodic derivative NLS

Received January 6, 2011

Abstract. We construct an invariant weighted Wiener measure associated to the periodic derivative nonlinear Schrödinger equation in one dimension and establish global well-posedness for data living in its support. In particular almost surely for data in a Fourier–Lebesgue space $\mathcal{F}L^{s,r}(\mathbb{T})$ with $s \geq 1/2$, $2 < r < 4$, $(s-1)r < -1$ and scaling like $H^{1/2-\epsilon}(\mathbb{T})$, for small $\epsilon > 0$. We also show the invariance of this measure.

1. Introduction

In the past few years, methods such as those by J. Bourgain (high-low method, e.g. [5, 6]) on the one hand and by J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao (I-method or method of *almost conservation laws*, e.g. [15, 16, 17]) on the other, have been applied to study the global in time existence for dispersive equations at regularities which are right below or in between those corresponding to conserved quantities. It turns out, however, that for many dispersive equations and systems there still remains a gap between the local in time results and those that could be globally achieved. In those cases, it seems natural to return to one of Bourgain's early approaches for periodic dispersive equations (NLS, KdV, mKdV, Zakharov system) [3, 4, 5, 7, 8, 9] where global in time existence was studied in the almost sure sense via the existence and invariance of the associated Gibbs measure (cf. Lebowitz, Rose and Speer's and Zhidkov's works [30], [48]). More recently this approach has been used for example by N. Tzvetkov [44, 45] for subquintic radial nonlinear wave equation on the disc, N. Burq and N. Tzvetkov [12, 13] for subcubic and subquartic radial nonlinear wave equations on 3d ball, N. Burq,

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L. Thomann, and N. Tzvetkov [11] for the nonlinear Schrödinger equation with harmonic potential, and by T. Oh [33, 34, 35, 36] for the periodic KdV-type coupled systems, Schrödinger–Benjamin–Ono system and white noise for the KdV equation.

Failure to show global existence by Bourgain's high-low method or the I-method might come from certain 'exceptional' initial data set, and the virtue of the Gibbs measure is that it does not see that exceptional set. At the same time, the invariance of the Gibbs measure, just like the usual conserved quantities, can be used to control the growth in time of those solutions in its support and extend the local in time solutions to global ones almost surely. The difficulty in this approach lies in the actual construction of the associated Gibbs measure and in showing both its invariance under the flow and the almost sure global well-posedness, since, on the one hand, we need invariance to show global well-posedness, and on the other hand we need globally defined flow to discuss invariance.

Our goal in this paper is to construct an invariant weighted Wiener measure associated to the periodic derivative nonlinear Schrödinger equation DNLS in (2.1) in one dimension and establish global well-posedness for data living in its support. In particular almost surely for data in a Fourier–Lebesgue space $\mathcal{F}L^{s,r}$ defined in (2.2) below (cf. [27, 21, 14, 22]) and scaling like $H^{1/2-\epsilon}(\mathbb{T})$, for small $\epsilon > 0$. The motivation for this paper stems from the fact that by scaling DNLS should be well-posed for data in H^σ, $\sigma \geq 0$, but the results obtained so far are much weaker.

Local well-posedness is known for $\sigma \geq 1/2$ for the nonperiodic [40] and periodic [26] cases while global well-posedness is known for $\sigma \geq 1/2$ for the nonperiodic case ($\sigma > 1/2$ in [16] and $\sigma \geq 1/2$ in [31]) and for $\sigma > 1/2$ in the periodic case [47]. Furthermore, in the nonperiodic case the Cauchy initial value problem for DNLS is ill-posed for data in $H^\sigma(\mathbb{R})$, $\sigma < 1/2$ [40], [2], a strong indication that ill-posedness should also be expected in the periodic case in that range. Grünrock and Herr [22] have recently established local well-posedness for the periodic DNLS in Fourier–Lebesgue spaces $\mathcal{F}L^{s,r}$, which for appropriate choices of (s, r) scale like $H^\sigma(\mathbb{T})$ for any $\sigma > 1/4$. Their result is the starting point of this work (cf. Section 2 for a more detailed discussion).

The measure we construct is based on the energy functional rather than the Hamiltonian. Hence we simply refer to it as weighted Wiener measure rather than Gibbs measure since the name 'Gibbs measure' has traditionally been reserved for those weighted Wiener measures constructed using the Hamiltonian. By invariance of a measure μ we mean that if $\Phi(t)$ denotes the flow map associated to our nonlinear equation then $\Phi(t)$ is defined for all $t \in \mathbb{R}$, μ-almost surely and for all $f \in L^1(\mu)$ and all $t \in \mathbb{R}$,

$$\int f(\Phi(t)(\phi))\, \mu(d\phi) = \int f(\phi)\, \mu(d\phi).$$

In general terms our aim is to construct a well defined measure μ so that local well-posedness of the periodic DNLS holds in some space $\mathcal{B}$ containing the support of μ. Then we show almost sure global well-posedness as well as the invariance of μ via a combination of the methods of Bourgain and Zhidkov [48] in the context of NLS, KdV, mKdV. In implementing this scheme however we need to overcome two main obstacles due to the need to gauge the equation to show local well-posedness (e.g. [40, 26]) and to construct an invariant measure. The symplectic form associated to the periodic gauged derivative nonlinear Schrödinger equation GDNLS in (2.6) does not commute with Fourier modes truncation and so the truncated finite-dimensional systems are not necessarily Hamiltonian. The first (mild) obstacle is to show the conservation of the Lebesgue measure associated to the finite-dimensional approximation to the periodic gauged derivative nonlinear Schrödinger equation FGDNLS, defined in (3.1) by hand, rather than by using the Hamiltonian structure. The second obstacle is much more serious and is at the heart of this work. The energy $\mathcal{E}$ defined in (2.13) associated to the gauged periodic DNLS[1] which we prove to be conserved in time, ceases to be so when computed on solutions of the finite-dimensional approximation equation, that is, $\frac{d}{dt}\mathcal{E}(v^N) \neq 0$, when v^N is a solution to the finite-dimensional gauged DNLS (see (4.9)). In other words the finite-dimensional weighted Wiener measure is not invariant any longer and unlike in Zhidkov's work [48] on KdV we do not have a priori knowledge of global well-posedness. We show however that it is *almost* invariant in the sense that we can control the growth in time of $\mathcal{E}(v^N)(t)$. This idea is reminiscent of the I-method. However, while in the I-method one needs to estimate the variation of the energy of solutions to the infinite-dimensional equation at time t smoothly projected onto frequencies of size up to N, here one needs to control the variation of the energy $\mathcal{E}$ of the solution v^N to the finite-dimensional approximation equation FGDNLS. We note that the loss in energy conservation for solutions to the finite-dimensional equation is principally due to the manner one chooses to approximate the infinite-dimensional gauged equation by using Fourier projections onto the first Nth frequencies. In [3] Bourgain describes an alternative approach that relies on using a discrete system of ODE which seems to preserve the conservation of energy. This approach however entails a number of other difficulties, for one needs to replace the circle $\mathbb{T}$ by the cyclic group $\mathbb{Z}_N$ and carry out the analysis on cyclic groups. We choose not to follow this path here.

We expect the ungauged invariant Wiener measure associated to DNLS (2.1) we obtain in Section 7 to be absolutely continuous with respect to the weighted Wiener measure constructed by Thomann and Tzvetkov [42]. This question is addressed in a forthcoming paper [32].

The paper is organized as follows. In Section 2 we present some general background, notation and results on the derivative nonlinear Schrödinger equation in one dimension. In Section 3 we discuss FGDNLS. In Section 4 we overcome the first two obstacles mentioned above. Namely we prove the invariance of the Lebesgue measure associated to FGDNLS and devote the rest of the section to prove our energy growth estimate Theorem 4.2. In Section 5 we carry out the construction of the weighted Wiener measure associated to the GDNLS. In Section 6 we prove the almost sure global well-posedness result for the GDNLS and the invariance of the measure constructed in section 5. Finally in Section 7 we translate back our results to the ungauged DNLS equation.

Notation. Whenever we write $a+$ for $a \in \mathbb{R}$ we mean $a + \varepsilon$ for some $\varepsilon > 0$; similarly for $a-$. In addition, we write $A \lesssim B$ to mean there exists some absolute constant $C > 0$ such that $A \leq CB$.

[1] We emphasize $\mathcal{E}$ is not the Hamiltonian of the gauged DNLS.

2. The derivative NLS equation in one dimension

The initial value problem for DNLS takes the form

$$\begin{cases} u_t - iu_{xx} = \lambda(|u|^2u)_x, \\ u\big|_{t=0} = u_0, \end{cases} \tag{2.1}$$

where either $(x, t) \in \mathbb{R} \times (-T, T)$ or $(x, t) \in \mathbb{T} \times (-T, T)$ and λ is real. In this paper we will take $\lambda = 1$ for convenience. DNLS is a Hamiltonian PDE whose flow conserves also mass and energy, i.e. the following quantities are conserved in time[2] (cf. [28, 25, 26]):

$$\begin{aligned} \text{mass:} \quad & M(u)(t) = \int |u(x,t)|^2\, dx, \\ \text{energy:} \quad & E(u)(t) = \int |u_x|^2\, dx + \frac{3}{2} \operatorname{Im} \int u^2 \overline{u}\, \overline{u_x}\, dx + \frac{1}{2} \int |u|^6\, dx, \\ \text{hamiltonian:} \quad & H(u)(t) = \operatorname{Im} \int u \overline{u_x}\, dx + \frac{1}{2} \int |u|^4\, dx. \end{aligned}$$

DNLS was introduced as a model for the propagation of circularly polarized Alfvén waves in a magnetized plasma with a constant magnetic field (cf. Sulem–Sulem [39]). The equation is scale invariant for data in L^2, i.e. if $u(x, t)$ is a solution then $u_a(x, t) = a^\alpha u(ax, a^2t)$ is also a solution if and only if $\alpha = 1/2$. Thus *a priori* one expects some form of existence and uniqueness results for (2.1) for data in H^σ, $\sigma \geq 0$. Many results are known for the Cauchy problem with smooth data, including data in H^1, such as those by M. Tsutsumi and I. Fukuda [43], N.Hayashi [23], N. Hayashi and T. Ozawa [24, 25] and T. Ozawa [37] and others (cf. references therein).

In looking for solutions to (2.1) we face a derivative loss arising from the nonlinear term $(|u|^2u)_x = u^2\overline{u}_x + 2|u|^2u_x$ and hence for low regularity data the key is to somehow make up for this loss.

For the nonperiodic case ($x \in \mathbb{R}$) Takaoka [40] proved sharp local well-posedness (LWP) in $H^{1/2}(\mathbb{R})$ relying on the gauge transformation used by Hayashi and Ozawa [24, 25] and the so-called Fourier restriction norm method. Then, Colliander–Keel–Staffilani–Takaoka and Tao [15, 16] established global well-posedness (GWP) for data in $H^\sigma(\mathbb{R})$, $\sigma > 1/2$, of small L^2 norm using the so-called I-method on the gauge equivalent equation (see also [41]). Here, small in L^2 just means less than an appropriate constant $\sqrt{2\pi/\lambda}$ which forces the associated 'energy' to be positive via the Gagliardo–Nirenberg inequality. This result was recently improved by Miao, Wu and Xu to $\sigma \geq 1/2$. The Cauchy initial value problem for DNLS is ill-posed for data in H^σ and $\sigma < 1/2$ (data map fails to be C^3 or uniformly C^0 [40], [2]). In [21] A. Grünrock proved that the nonperiodic DNLS is locally well-posed in the Fourier–Lebesgue spaces $\mathcal{F}L^{s,r}(\mathbb{R})$ which for appropriate choices of (s, r) scale like $H^\sigma(\mathbb{R})$ for any $\sigma > 0$ (cf. (2.2) below).

In the periodic setting, S. Herr [26] showed that the Cauchy problem associated to periodic DNLS is locally well-posed for initial data $u(0) \in H^\sigma(\mathbb{T})$, $\sigma \geq 1/2$, in the sense

[2] In fact, DNLS is completely integrable.

of local existence, uniqueness and continuity of the flow map. Herr's proof is based on an adaptation to the periodic setting of the gauge transformation introduced by Hayashi [23] and Hayashi and Ozawa [24, 25] on $\mathbb{R}$, in conjunction with sharp multilinear estimates for the gauged equivalent equation in periodic Fourier restriction norm spaces $X^{s,b}$ that yield local well-posedness for the gauged equation. Moreover, by use of conservation laws, the problem is also shown to be globally well-posed for $\sigma \geq 1$ and data which is small in L^2 (as in [15, 16]) [26]. More recently, Win [47] applied the I-method to prove GWP in $H^\sigma(\mathbb{T})$ for $\sigma > 1/2$.

A. Grünrock and S. Herr [22] showed that the Cauchy problem associated to DNLS is locally well-posed for initial data $u_0 \in \mathcal{F}L^{s,r}(\mathbb{T})$ with $2 < r < 4$ and $s \geq 1/2$ where

$$\|u_0\|_{\mathcal{F}L^{s,r}(\mathbb{T})} := \|\langle n\rangle^s \widehat{u}_0\|_{\ell^r_n(\mathbb{Z})}. \tag{2.2}$$

These spaces scale like the Sobolev $H^\sigma(\mathbb{T})$ ones where $\sigma = s + 1/r - 1/2$. The proof is based on Herr's adapted periodic gauge transformation and new multilinear estimates for the gauged equivalent equation in an appropriate variant of Fourier restriction norm spaces $X^{s,b}_{r,q}$ introduced by Grünrock–Herr [22].[3]

For $s, b \in \mathbb{R}$, $r, q \geq 1$ we define the space $X^{s,b}_{r,q}$ as the completion of the Schwartz space $\mathcal{S}(\mathbb{T}\times\mathbb{R})$ with respect to the norm

$$\|u\|_{X^{s,b}_{r,q}} := \|\langle n\rangle^s \langle \tau + n^2\rangle^b \widehat{u}(n,\tau)\|_{\ell^r_n L^q_\tau}$$

where first we take the L^q_τ norm and then the ℓ^r_n one. We also define the space

$$\|u\|_{X^{s,b}_{r,q;-}} := \|\langle n\rangle^s \langle \tau - n^2\rangle^b \widehat{u}(n,\tau)\|_{\ell^r_n L^q_\tau},$$

and note that $u \in X^{s,b}_{r,q}$ if and only if $\overline{u} \in X^{s,b}_{r,q;-}$.

For $\delta > 0$ fixed, we define the restriction space $X^{s,b}_{r,q}(\delta)$ of all $v = u|_{[-\delta,\delta]}$ for some $u \in X^{s,b}_{r,q}$ with norm

$$\|v\|_{X^{s,b}_{r,q}(\delta)} := \inf\{\|u\|_{X^{s,b}_{r,q}} : u \in X^{s,b}_{r,q} \text{ and } v = u|_{[-\delta,\delta]}\}.$$

When we take $q = 2$ we simply write $X^{s,b}_{r,2} = X^{s,b}_r$. Note $X^{s,b}_{2,2} = X^{s,b}$. Later we will also use the space

$$Z^s_r(\delta) := X^{s,1/2}_{r,2}(\delta) \cap X^{s,0}_{r,1}(\delta).$$

Some simple embeddings are as follows. For $s, b_1, b_2 \in \mathbb{R}$, $r \geq 1$ and $b_1 > b_2 + 1/2$,

$$X^{s,b_1}_{r,2} \subset X^{s,b_2}_{r,1} \quad \text{and} \quad X^{s,0}_{r,1} \subset C(\mathbb{R}, \mathcal{F}L^{s,r})$$

which follow by Cauchy–Schwarz with respect to the L^1_τ norm and by $\mathcal{F}^{-1}L^1 \subset L^\infty$ respectively. In particular

$$Z^s_r(\delta) \subset C([-\delta,\delta], \mathcal{F}L^{s,r}).$$

[3] Note that in our notation the indices (r, q) are the dual of the corresponding ones in Grünrock–Herr [22].

We finally recall the following estimate[4] heavily used in the proof of Theorem 4.2 below.

Lemma 2.1 ([22, Lemma 5.1]). *Let $1/3 < b < 1/2$ and $s > 3(1/2 - b)$. Then*

$$\|uv\overline{w}\|_{L^2_{xt}} \lesssim \|u\|_{X^{s,b}}\|v\|_{X^{s,b}}\|w\|_{X^{0,b}}.$$

In particular if $b = (1/2)-$, then

$$\|uv\overline{w}\|_{L^2_{xt}} \lesssim \|u\|_{X^{\epsilon,\frac{1}{2}-}}\|v\|_{X^{\epsilon,\frac{1}{2}-}}\|w\|_{X^{0,\frac{1}{2}-}}, \tag{2.3}$$

for small $\epsilon > 0$; while when $b = (1/3)+$,

$$\|uv\overline{w}\|_{L^2_{xt}} \lesssim \|u\|_{X^{\frac{1}{2}-,\frac{1}{3}+}}\|v\|_{X^{\frac{1}{2}-,\frac{1}{3}+}}\|w\|_{X^{0,\frac{1}{3}+}}. \tag{2.4}$$

2.1. The periodic gauged derivative NLS equation

We first recall S. Herr's gauge transformation. For $f \in L^2(\mathbb{T})$, let

$$G(f)(x) := \exp(-iJ(f))f(x)$$

where

$$J(f)(x) := \frac{1}{2\pi}\int_0^{2\pi}\int_\theta^x \left(|f(y)|^2 - \frac{1}{2\pi}\|f\|^2_{L^2(\mathbb{T})}\right) dy\, d\theta. \tag{2.5}$$

Note $G(f)$ is 2π-periodic since its integrand has zero mean value. Then for $u \in C([-T,T]; L^2(\mathbb{T}))$ and $m(u) := \frac{1}{2\pi}\int_{\mathbb{T}} |u(x,0)|^2 dx$ the adapted periodic gauge is defined as[5]

$$\mathcal{G}(u)(t,x) := G(u(t))(x - 2tm(u)).$$

Note the L^2 norm of $\mathcal{G}(u)(t,x)$ is still conserved since the torus is invariant under translation. The map

$$\mathcal{G} : C([-T,T]; H^\sigma(\mathbb{T})) \to C([-T,T]; H^\sigma(\mathbb{T}))$$

is a homeomorphism for any $\sigma \geq 0$ and locally bi-Lipschitz on subsets of $C([-T,T]; H^\sigma(\mathbb{T}))$ with prescribed $\|u(0)\|_{L^2}$ ([26]). Moreover the same is true if we replace $H^\sigma(\mathbb{T})$ by $\mathcal{F}L^{s,r}$ with $s > 1/2 - 1/r$ when $2 < r < \infty$ and $s \geq 0$ when $r = 2$ ([22]).

Then if u is a solution to DNLS (2.1) and $v := \mathcal{G}(u)$ we see that v solves the gauged DNLS equation (GDNLS)

$$v_t - iv_{xx} = -v^2\overline{v}_x + \frac{i}{2}|v|^4 v - i\psi(v)v - im(v)|v|^2 v \tag{2.6}$$

4 This is a trilinear refinement of Bourgain's $L^6(\mathbb{T})$ Strichartz estimate [10].
5 Recall $m(u)(t)$ is conserved under the flow of (2.1).

with initial data $v(0) = \mathcal{G}(u(0))$, where

$$m(v)(t) := \frac{1}{2\pi}\int_{\mathbb{T}} |v(x,t)|^2\,dx, \tag{2.7}$$

$$\psi(v)(t) := -\frac{1}{\pi}\int_{\mathbb{T}} \mathrm{Im}(v\overline{v_x})\,dx + \frac{1}{4\pi}\int_{\mathbb{T}} |v|^4\,dx - m(v)^2. \tag{2.8}$$

Note that $m(v)$ is conserved in time, more precisely $m(v)(t) = \frac{1}{2\pi}\int_{\mathbb{T}} |v(x,0)|^2\,dx = m(u)$, and both $m(v)$ and $\psi(v)$ are real.

The initial value problem associated to (2.6) with data in $\mathcal{F}L^{s,r}(\mathbb{T})$ is locally well-posed in $Z_r^s(\delta)$, $2 < r < 4$, $s \geq 1/2$, for some $\delta > 0$. This was proved in Theorem 7.2 of [22].

Remark 2.2. Local well-posedness for GDNLS (2.6) implies local existence, uniqueness and continuity of the flow map for DNLS (2.1) [26, 22]. One cannot however carry back to solutions to DNLS all the auxiliary estimates coming from the local well-posedness result for GDNLS.

Now we show how the energy $E(u)$ and $H(u)$ transform under the gauge. Let u be the solution to DNLS (2.1) and define

$$w = e^{-iJ(u)}u.$$

Then w solves GDNLS (2.6) with the extra $m(w)w_x$ term in the linear part of the equation [26]. So the gauge transform is, properly speaking, the transformation $w = e^{-iJ(u)}u$ followed by the transformation

$$v(x,t) = w(t, x - 2m(w)t).$$

But all the terms involved in the conserved quantities we considered are invariant under this second transformation $w \mapsto v$ (the torus is invariant under translation). We also notice that $m(u) = m(w) = m(v)$, hence below we will be simply using m for this quantity.

Since

$$u = e^{iJ(w)}w$$

we have

$$u_x = e^{iJ(w)}(w_x + iJ(w)_x w)$$

with $J(w)_x = |w|^2 - m$.

We have

$$\begin{aligned} H(u) &= \mathrm{Im}\int_{\mathbb{T}} u\overline{u_x}\,dx + \frac{1}{2}\int_{\mathbb{T}} |u|^4\,dx \\ &= \mathrm{Im}\int_{\mathbb{T}} w(\overline{w_x} - iJ(w)_x\overline{w})\,dx + \frac{1}{2}\int_{\mathbb{T}} |w|^4\,dx \\ &= \mathrm{Im}\int_{\mathbb{T}} w\overline{w_x}\,dx - \frac{1}{2}\int_{\mathbb{T}} |w|^4\,dx + 2\pi m^2 =: \mathscr{H}(w). \end{aligned}$$

In addition

$$\begin{aligned} u_x\overline{u_x} &= (w_x + iJ(w)_x w)(\overline{w_x} - iJ(w)_x\overline{w}) \\ &= w_x\overline{w_x} + iJ(w)_x(w\overline{w_x} - \overline{w}w_x) + J(w)_x^2|w|^2 \\ &= w_x\overline{w_x} - 2\,\mathrm{Im}\,J(w)_x w\overline{w_x} + (|w|^6 - 2m|w|^4 + m^2|w|^2) \\ &= w_x\overline{w_x} - 2\,\mathrm{Im}\,w^2\overline{w}\,\overline{w_x} + 2m\,\mathrm{Im}\,w\overline{w_x} + (|w|^6 - 2m|w|^4 + m^2|w|^2). \end{aligned} \tag{2.9}$$

By the same calculations we also have

$$u^2\overline{uu_x} = w^2\overline{ww_x} - i|w|^6 + im|w|^4. \tag{2.10}$$

We now recall that

$$E(u)(t) = \int |u_x|^2\,dx + \frac{3}{2}\,\mathrm{Im}\int u^2\overline{u}\,\overline{u_x}\,dx + \frac{1}{2}\int |u|^6\,dx, \tag{2.11}$$

hence by using (2.11), (2.9), (2.10) we find

$$E(u) = \int w_x\overline{w_x}\,dx - \frac{1}{2}\,\mathrm{Im}\int w^2\overline{w}\,\overline{w_x}\,dx + 2m\,\mathrm{Im}\int w\overline{w_x}\,dx - \frac{1}{2}m\int |w|^4\,dx + 2\pi m^3.$$

If we define

$$\mathscr{E}(w) := \int_{\mathbb{T}} |w_x|^2\,dx - \frac{1}{2}\,\mathrm{Im}\int_{\mathbb{T}} w^2\overline{w}\,\overline{w_x}\,dx + \frac{1}{4\pi}\left(\int_{\mathbb{T}} |w(t)|^2\,dx\right)\left(\int_{\mathbb{T}} |w(t)|^4\,dx\right), \tag{2.12}$$

then $E(u)$ can be rewritten as

$$E(u) = \mathscr{E}(w) + 2m\mathscr{H}(w) - 2\pi m^3 =: \mathcal{E}(w). \tag{2.13}$$

Remark 2.3. We observe that $H(u)(t) = \mathscr{H}(w)(t)$ and $\frac{d}{dt}H(u)(t) = 0$ since H is the Hamiltonian for DNLS (2.1), hence $\frac{d}{dt}\mathscr{H}(w)(t) = 0$. On the other hand, we also know that $\frac{d}{dt}E(u)(t) = 0$, hence $\frac{d}{dt}\mathscr{E}(w)(t) = 0$. By the translation invariance of integration over $\mathbb{T}$, we find that (2.13) holds with v in place of w and

$$\frac{d}{dt}\mathscr{H}(v)(t) = 0 = \frac{d}{dt}\mathscr{E}(v)(t).$$

3. Finite-dimensional approximation of GDNLS

We denote by $P_N f = \sum_{|n|\le N}\widehat{f}(n)e^{inx}$ the finite-dimensional projection onto the first $2N+1$ modes and $P_N^{\perp} := I - P_N$. Then the finite-dimensional approximation (FGDNLS) is

$$v_t^N = iv_{xx}^N - P_N((v^N)^2\overline{v_x^N}) + \frac{i}{2}P_N(|v^N|^4v^N) - i\psi(v^N)(t)v^N - im(v^N)P_N(|v^N|^2v^N) \tag{3.1}$$

with initial data

$$v_0^N = P_N v_0, \tag{3.2}$$

where m and ψ are as defined in (2.7) and (2.8) respectively.

Lemma 3.1. *We have*

$$\frac{d}{dt}m(v^N)(t) := \frac{d}{dt}\frac{1}{2\pi}\int_{\mathbb{T}} |v^N(x,t)|^2\,dx = 0.$$

Proof. Indeed for simplicity let us momentarily denote by $w := v^N$ a solution to (3.1); note $P_N w = w$. Then using that for any F, $\int P_N(F(v^N))\overline{v^N}\,dx = \int F(v^N)P_N\overline{v^N}\,dx = \int F(v^N)\overline{v^N}\,dx$, we obtain

$$\begin{aligned}
\frac{d}{dt}(2\pi m(w)) &= 2\,\mathrm{Re}\int w_t\overline{w}\,dx \\
&= 2\,\mathrm{Re}\Big(-i\int |w_x|^2 - \int P_N(w^2\overline{w_x})\overline{w} + \frac{i}{2}\int P_N(|w|^4 w)\overline{w} \\
&\quad - i\psi(w)(t)\int |w|^2 - im(w)(t)\int P_N(|w|^2 w)\overline{w}\Big) \\
&= 2\,\mathrm{Re}\Big(-\int (w^2\overline{w}_x)\overline{w} + \frac{i}{2}\int |w|^6 - i\psi(w)\int |w|^2 - im(w)\int |w|^4\Big) \\
&= -\int w^2\overline{w}\,\overline{w}_x - \int w w_x\overline{w}^2 = -\frac{1}{2}\int \partial_x(|w|^4) = 0.
\end{aligned}$$

□

Theorem 3.2 (Local well-posedness). *Let* $2 < r < 4$ *and* $s \geq 1/2$. *Then for every*

$$v_0^N \in B_R := \{v_0^N \in \mathcal{F}L^{s,r}(\mathbb{T}) : \|v_0^N\|_{\mathcal{F}L^{s,r}(\mathbb{T})} < R\}$$

and $\delta \lesssim R^{-\gamma}$, *for some* $\gamma > 0$, *there exists a unique solution*

$$v^N \in Z_r^s(\delta) \subset C([-\delta,\delta];\mathcal{F}L^{s,r}(\mathbb{T}))$$

of (3.1) *and* (3.2). *Moreover the map*

$$(B_R, \|\cdot\|_{\mathcal{F}L^{s,r}(\mathbb{T})}) \to C([-\delta,\delta];\mathcal{F}L^{s,r}(\mathbb{T})) : v_0^N \mapsto v^N$$

is real analytic.

Proof. The proof follows the argument in [22, Theorem 7.2] since P_N acts on a multilinear nonlinearity and it is a bounded operator in L^p, $1 < p < \infty$, and commutes with D^s. Also, although the proof in [22] is presented for $s = 1/2$, a simple argument of persistence of regularity gives the result for any $s \geq 1/2$. □

The following lemma gives control on how close the finite-dimensional approximations are to the solution of (2.6). Our proof is a variation of Bourgain's Lemma 2.27 in [3] (see also [12]).

Lemma 3.3 (Approximation lemma). *Let* $v_0 \in \mathcal{F}L^{s,r}(\mathbb{T})$, $s > 1/2$, $2 < r < 4$, *be such that* $\|v_0\|_{\mathcal{F}L^{s,r}(\mathbb{T})} < A$, *for some* $A > 0$, *and let* N *be a large integer. Assume the solution* v^N *of* (3.1) *with initial data* $v_0^N(x) := P_N(v_0)$ *satisfies the bound*

$$\|v^N(t)\|_{\mathcal{F}L^{s,r}(\mathbb{T})} \leq A \quad \textit{for all } t \in [-T,T],$$

for some given $T > 0$. Then the IVP GDNLS (2.6) with initial data v_0 is well-posed on $[-T, T]$ and there exist $C_0, C_1 > 0$ such that its solution $v(t)$ satisfies the estimate

$$\|v(t) - v^N(t)\|_{\mathcal{F}L^{s_1,r}(\mathbb{T})} \lesssim \exp[C_0(1 + A)^{C_1} T] N^{s_1-s} \tag{3.3}$$

for all $t \in [-T, T]$ and $1/2 \le s_1 < s$.

Proof. We first observe that from the local well-posedness theory ([22] and Theorem 3.2), GDNLS (2.6) with initial data v_0 and FGDNLS (3.1) with initial data v_0^N are both well-posed in $[-\delta, \delta]$ for $\delta \sim (1 + A)^{-\gamma}$. Let $w := v - v^N$; then w satisfies the equation

$$w_t - i w_{xx} = F(v) - P_N F(v^N) = P_N[F(v) - F(v^N)] + (1 - P_N)F(v),$$

where $F(\cdot)$ is the nonlinearity of (2.6). By the Duhamel principle we have

$$w(t) = S(t)[v_0 - v_0^N] + \int_0^t S(t - t')\big(P_N[F(v) - F(v^N)](t') + (1 - P_N)F(v)(t')\big)\, dt',$$

where $S(t) = e^{it\Delta}$, and from the proof of Theorem 7.2 in [22] we have the bound

$$\begin{aligned}\|w\|_{Z_r^{s_1}(\delta)} &\lesssim \|v_0 - v_0^N\|_{\mathcal{F}L^{s_1,r}(\mathbb{T})} + \delta^\gamma (1 + \|v^N\|_{Z_r^{s_1}(\delta)} + \|v\|_{Z_r^{s_1}(\delta)})^4 \|w\|_{Z_r^{s_1}(\delta)} \\ &\quad + \left\|(1 - P_N)\int_0^t S(t - t')F(v)(t')\, dt'\right\|_{Z_r^{s_1}(\delta)} \\ &\lesssim A N^{s_1-s} + \delta^\gamma (1 + \|v^N\|_{Z_r^{s_1}(\delta)} + \|v\|_{Z_r^{s_1}(\delta)})^4 \|w\|_{Z_r^{s_1}(\delta)} \\ &\quad + N^{s_1-s}\delta^\gamma (1 + \|v\|_{Z_r^{s}(\delta)})^5.\end{aligned} \tag{3.4}$$

By choosing a smaller δ if necessary we obtain from (3.4)

$$\|w\|_{Z_r^{s_1}(\delta)} \le C A N^{s_1-s} + \frac{1}{2}\|w\|_{Z_r^{s_1}(\delta)},$$

for some absolute constant $C > 0$,which implies

$$\|v(t) - v^N(t)\|_{\mathcal{F}L^{s_1,r}(\mathbb{T})} \le 2CAN^{s_1-s} \quad \text{for all } t \in [-\delta, \delta] \tag{3.5}$$

and by iteration (3.3) follows. □

4. Analysis of the finite-dimensional equation FGDNLS

Recall that equation DNLS is Hamiltonian and hence its gauge equivalent formulation should stay Hamiltonian (change of coordinates). However, the gauge transformation is not a ‘canonical map’ and the symplectic form in the new coordinates depends on v; that is, we lose the simple expression the symplectic form (namely ∂_x) had in the original coordinates. Two problems arise from the lack of commutativity between the gauged skew-selfadjoint form J and P_N:

(1) The conservation of Lebesgue measure associated to FGDNLS is not obvious as before. We must prove that this is indeed the case; see Subsection 4.1 below.

And more seriously:

(2) We lose the conservation of the energy $\mathcal{E}(v^N)$ for the finite-dimensional approximations, that is, $d\mathcal{E}(v^N)/dt \neq 0$. In particular we lose the invariance of μ_N, the associated finite-dimensional weighted Wiener measure. However we have an estimate controlling its growth, namely Theorem 4.2 below.

4.1. Invariance of the Lebesgue measure

If we rewrite FGDNLS (3.1) as a system of complex ODE's for the Fourier coefficients $c_k \equiv \widehat{v^N}(k)$ we obtain a set of $2N+1$ complex equations of the form $\frac{d}{dt}c_k = F_k(\{c_j, \bar{c}_j\})$, or equivalently $4N+2$ equations $\frac{d}{dt}a_k = \operatorname{Re} F_k(\{c_j, \bar{c}_j\})$ and $\frac{d}{dt}b_k = \operatorname{Im} F_k(\{c_j, \bar{c}_j\})$ for the real functions $a_k = \operatorname{Re} F_k$ and $b_k = \operatorname{Im} F_k$.

To show that this set of equations preserves volume we need to verify that the divergence of the vector field vanishes, i.e.,

$$\sum_k \left(\frac{\partial \operatorname{Re} F_k}{\partial a_k} + \frac{\partial \operatorname{Im} F_k}{\partial b_k} \right) = 0.$$

This is easily shown to be equivalent to

$$\sum_k \left(\frac{\partial F_k}{\partial c_k} + \frac{\partial \bar{F}_k}{\partial \bar{c}_k} \right) = 0.$$

And indeed we have

Lemma 4.1. *The Lebesgue measure $\prod_{|j|\le N} da_j db_j$ is invariant under the flow of the system of ODE's (4.1).*

Proof. The FGDNLS (3.1) as a system of complex ODE's for the Fourier coefficients c_k takes the form

$$\begin{aligned} \frac{d}{dt}c_k &= -ik^2 c_k + i \sum_{n_1,n_2,n_3} n_3 c_{n_1} c_{n_2} \bar{c}_{n_3} \delta_{n_1+n_2-n_3-k} \\ &\quad + \frac{i}{2} \sum_{n_1,n_2,n_3,n_4,n_5} c_{n_1} c_{n_2} c_{n_3} \bar{c}_{n_4} \bar{c}_{n_5} \delta_{n_1+n_2+n_3-n_4-n_5-k} \\ &\quad - i\psi(\{c_j, \bar{c}_j\}) c_k - im(\{c_j, \bar{c}_j\}) \sum_{n_1,n_2,n_3} c_{n_1} c_{n_2} \bar{c}_{n_3} \delta_{n_1+n_2-n_3-k} \end{aligned} \tag{4.1}$$

with $m(\{c_j, \bar{c}_j\}) = \sum_j |c_j|^2$ and

$$\psi(\{c_j, \bar{c}_j\}) = -2\sum_k k|c_k|^2 + \frac{1}{2} \sum_{n_1,n_2,n_3,n_4} c_{n_1} c_{n_2} \bar{c}_{n_3} \bar{c}_{n_4} \delta_{n_1+n_2-n_3-n_4} - \Big(\sum_j |c_j|^2 \Big)^2.$$

To show that this set of equations preserves volume we need to verify

$$\sum_k \left(\frac{\partial F_k}{\partial c_k} + \frac{\partial \bar{F}_k}{\partial \bar{c}_k} \right) = 0.$$

The vector field F_k consists of several terms which we analyze separately.

(1) $F_k^{(1)} = -ik^2 c_k$. Then $\frac{\partial F_k^{(1)}}{\partial c_k} + \frac{\partial \bar{F}_k^{(1)}}{\partial \bar{c}_k} = -ik^2 + ik^2 = 0$.

(2) $F_k^{(2)} = i \sum_{n_1,n_2,n_3} n_3 c_{n_1} c_{n_2} \bar{c}_{n_3} \delta_{n_1+n_2-n_3-k}$. To differentiate we consider the terms with $n_1 = k$ and $n_2 = k$ to obtain

$$\frac{\partial F_k^{(2)}}{\partial c_k} = i2\pi \sum_{n_2,n_3} n_3 c_{n_2} \bar{c}_{n_3} \delta_{n_2-n_3} + i2\pi \sum_{n_1,n_3} n_3 c_{n_1} \bar{c}_{n_3} \delta_{n_1-n_3} = i4\pi \sum_n n|c_n|^2$$

and similarly

$$\frac{\partial \bar{F}_k^{(2)}}{\partial \bar{c}_k} = -i4\pi \sum_n n|c_n|^2,$$

and thus all the contributions of this term to the divergence disappear.

(3) $F_k^{(3)} = \frac{i}{2} \sum_{n_1,n_2,n_3,n_4,n_5} c_{n_1} c_{n_2} c_{n_3} \bar{c}_{n_4} \bar{c}_{n_5} \delta_{n_1+n_2+n_3-n_4-n_5-k}$. This term is treated similarly to (2) and is left to the reader.

(4) $F_k^{(4)} = 2i(\sum_j j|c_j|^2) c_k$. We have

$$\frac{\partial F_k^{(4)}}{\partial c_k} = 2ik|c_k|^2 + 2i \sum_j j|c_j|^2, \qquad \frac{\partial \bar{F}_k^{(4)}}{\partial \bar{c}_k} = -2ik|c_k|^2 - 2i \sum_j j|c_j|^2,$$

and so these terms do not contribute to the divergence.

(5) $F_k^{(5)} = i(\sum_j |c_j|^2)^2 c_k$. We have

$$\frac{\partial F_k^{(5)}}{\partial c_k} = 2i\Big(\sum_j |c_j|^2\Big)|c_k|^2 + i\Big(\sum_j |c_j|^2\Big)^2$$

and again $\frac{\partial F_k^{(5)}}{\partial c_k} + \frac{\partial \bar{F}_k^{(5)}}{\partial \bar{c}_k} = 0$.

(6) $F_k^{(6)} = -\frac{i}{2} \sum_{n_1,n_2,n_3,n_4} c_{n_1} c_{n_2} \bar{c}_{n_3} \bar{c}_{n_4} \delta_{n_1+n_2-n_3-n_4} c_k$. We have

$$\begin{aligned} \frac{\partial F_k^{(6)}}{\partial c_k} &= -\frac{i}{2} \sum_{n_1,n_2,n_3,n_4} c_{n_1} c_{n_2} \bar{c}_{n_3} \bar{c}_{n_4} \delta_{n_1+n_2-n_3-n_4} \\ &\quad - i \sum_{n_2,n_3,n_4} c_k c_{n_2} \bar{c}_{n_3} \bar{c}_{n_4} \delta_{k+n_2-n_3-n_4} \end{aligned} \tag{4.2}$$

and

$$\frac{\partial \bar{F}_k^{(6)}}{\partial \bar{c}_k} = +\frac{i}{2} \sum_{n_1,n_2,n_3,n_4} \bar{c}_{n_1}\bar{c}_{n_2}c_{n_3}c_{n_4}\delta_{n_1+n_2-n_3-n_4} + i \sum_{n_2,n_3,n_4} \bar{c}_k\bar{c}_{n_2}c_{n_3}c_{n_4}\delta_{k+n_2-n_3-n_4}. \tag{4.3}$$

The first terms in (4.2) and (4.3) cancel for each k. By summing the second terms over k, we see that they do not contribute to the divergence.

(7) $F_k^{(7)} = -i\sum_j |c_j|^2 \sum_{n_1,n_2,n_3} c_{n_1}c_{n_2}\bar{c}_{n_3}\delta_{n_1+n_2-n_3-k}$. We have

$$\frac{\partial F_k^{(7)}}{\partial c_k} = -i \sum_{n_1,n_2,n_3} c_{n_1}c_{n_2}\bar{c}_{n_3}\bar{c}_k\delta_{n_1+n_2-n_3-k} - 2i\Big(\sum_j |c_j|^2\Big)^2,$$

$$\frac{\partial \bar{F}_k^{(7)}}{\partial \bar{c}_k} = i \sum_{n_1,n_2,n_3} \bar{c}_{n_1}\bar{c}_{n_2}c_{n_3}c_k\delta_{n_1+n_2-n_3-k} + 2i\Big(\sum_j |c_j|^2\Big)^2.$$

The second terms add to 0 for each k while the first terms cancel if we sum over all k. □

4.2. Energy growth estimate

Theorem 4.2. *Let $v^N(t)$ be a solution to FGDNLS (3.1) in $[-\delta,\delta]$, and let $K > 0$ be such that $\|v^N\|_{X_3^{(2/3)-,1/2}(\delta)} \le K$. Then there exists $\beta > 0$ such that*

$$|\mathcal{E}(v^N(\delta)) - \mathcal{E}(v^N(0))| = \left|\int_0^\delta \frac{d}{dt}\mathcal{E}(v^N)(t)\,dt\right| \lesssim C(\delta)N^{-\beta}\max(K^6, K^8). \tag{4.4}$$

Remark 4.3. It is possible that the estimate (4.4) may still hold for a different choice of $X_r^{s,1/2}(\delta)$ norm, with $s \ge 1/2$, $2 < r < 4$ so that local well-posedness holds. On the other hand the pair (s,r) should also be such that $(s-1)\cdot r < -1$ in order for $\mathcal{F}L^{s,r}$ to contain the support of the Wiener measure (cf. Section 5). Our choice of $s = (2/3)-$ and $r = 3$ allows us to prove (4.4) while satisfying the conditions for local well-posedness and the support of the measure. Note that $\mathcal{F}L^{(2/3)-,3}$ scales like $H^{(1/2)-}$.

4.3. Preparation for the proof of Theorem 4.2

Let v^N denote the solution of FGDNLS (3.1) which we rewrite as

$$v_t^N = \mathcal{L}v^N + P_N^{\perp}((v^N)^2\overline{v_x^N}) - \frac{i}{2}P_N^{\perp}(|v^N|^4v^N) + im(v^N)P_N^{\perp}(|v^N|^2v^N),$$

where

$$\mathcal{L}v^N := iv_{xx}^N - (v^N)^2\overline{v_x^N} + \frac{i}{2}|v^N|^4v^N - i\psi(v^N)v^N - im(v^N)|v^N|^2v^N. \tag{4.5}$$

We first observe that from (2.13) and Lemma 3.1 we have

$$\frac{d}{dt}\mathcal{E}(v^N) = \frac{d}{dt}\mathscr{E}(v^N) + 2m_N \frac{d}{dt}\mathscr{H}(v^N), \tag{4.6}$$

where $m_N := m(v^N)$.

Lemma 4.4. *With the above notation we have*

$$\begin{aligned}\frac{d}{dt}\mathscr{E}(v^N)(t) &= -2\,\mathrm{Im}\int v^N\,\overline{v^N v^N_x}\,P^\perp_N((v^N)^2\overline{v^N_x})\,dx + \mathrm{Re}\int v^N\,\overline{v^N v^N_x}\,P^\perp_N(|v^N|^4 v^N)\,dx\\ &\quad - 2m_N\,\mathrm{Re}\int v^N\,\overline{v^N v^N_x}\,P^\perp_N(|v^N|^2 v^N)\,dx + 2m_N\,\mathrm{Re}\int v^N\,\overline{v^N}^2\,P^\perp_N((v^N)^2\overline{v^N_x})\,dx\\ &\quad + m_N\,\mathrm{Im}\int v^N\,\overline{v^N}^2\,P^\perp_N(|v^N|^4 v^N)\,dx - 2m_N^2\,\mathrm{Im}\int v^N\,\overline{v^N}^2\,P^\perp_N(|v^N|^2 v^N)\,dx,\end{aligned} \tag{4.7}$$

$$\begin{aligned}\frac{d}{dt}\mathscr{H}(v^N)(t) &= -2\,\mathrm{Re}\int_{\mathbb{T}}(\overline{v^N})^2 v^N\,P^\perp_N((v^N)^2\overline{v^N_x})\,dx\\ &\quad + \mathrm{Im}\int v^N(\overline{v^N})^2 P^\perp_N(|v^N|^4 v^N)\,dx - 2m_N\,\mathrm{Im}\int v^N(\overline{v^N})^2 P^\perp_N(|v^N|^2 v^N)\,dx,\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(v^N)(t) &= -2\,\mathrm{Im}\int v^N\,\overline{v^N v^N_x}\,P^\perp_N((v^N)^2\overline{v^N_x})\,dx + \mathrm{Re}\int v^N\,\overline{v^N v^N_x}\,P^\perp_N(|v^N|^4 v^N)\,dx\\ &- 2m_N\,\mathrm{Re}\int v^N\,\overline{v^N v^N_x}\,P^\perp_N(|v^N|^2 v^N)\,dx - 2m_N\,\mathrm{Re}\int_{\mathbb{T}}(\overline{v^N})^2 v^N\,P^\perp_N((v^N)^2\overline{v^N_x})\,dx\\ &+ 3m_N\,\mathrm{Im}\int v^N(\overline{v^N})^2 P^\perp_N(|v^N|^4 v^N)\,dx - 6m_N\,\mathrm{Im}\int v^N(\overline{v^N})^2 P^\perp_N(|v^N|^2 v^N)\,dx.\end{aligned} \tag{4.9}$$

Proof. From (2.12) and integration by parts we have

$$\begin{aligned}\frac{d}{dt}\mathscr{E}(v^N)(t) &= -2\,\mathrm{Re}\int v^N_t\,\overline{v^N_{xx}}\,dx - 2\,\mathrm{Im}\int v^N v^N_t\,\overline{v^N v^N_x}\,dx\\ &\quad + 2m_N\,\mathrm{Re}\int v^N v^N_t\,\overline{v^N}^2\,dx.\end{aligned} \tag{4.10}$$

Due to the energy conservation for the GDNLS (infinite system), one can see that the contribution in (4.10) from $\mathcal{L}v^N$ defined in (4.5) is zero. On the other hand by orthogonality we also have

$$-2\,\mathrm{Re}\int \overline{v^N_{xx}}\Big(P^\perp_N((v^N)^2\overline{v^N_x}) - \frac{i}{2}P^\perp_N(|v^N|^4 v^N) + im(v^N)P^\perp_N(|v^N|^2 v^N)\Big)\,dx = 0.$$

Hence (4.7) follows. By a similar argument we obtain (4.8) as well. The lemma follows by substituting (4.7) and (4.8) into (4.6). □

Remark 4.5. To establish Theorem 4.2 we need to estimate the terms in (4.9). In doing so we will ignore absolute constants and whether we are looking at the real or imaginary parts of the terms.

The first term in (4.9) gives a contribution to (4.4) which is essentially

$$I_1 = \int_0^\delta \int v^N\, \overline{v^N v_x^N}\, P_N^\perp((v^N)^2\, \overline{v_x^N})\, dx\, dt.$$

This term is the hardest to control since it has two derivatives, so we will treat this one first. We start by discussing how to absorb the rough time cut-off. Assume ϕ is any function in $X_3^{(2/3)-,1/2}$ such that

$$\phi|_{[-\delta,\delta]} = v^N.$$

Then we write

$$\begin{aligned} I_1 &= \int_{\mathbb{T}\times\mathbb{R}} \chi_{[0,\delta]}(t) P_N^\perp((v^N)^2\partial_x\overline{v^N})v^N\, \overline{v^N v_x^N}\, dx\, dt \\ &= \int_{\mathbb{T}\times\mathbb{R}} P_N^\perp((\chi_{[0,\delta]}\phi^N)^2\chi_{[0,\delta]}\overline{\phi_x^N})\chi_{[0,\delta]}\phi^N\chi_{[0,\delta]}\overline{\phi^N}\,\chi_{[0,\delta]}\overline{\phi_x^N}\, dx\, dt \end{aligned}$$

and by denoting

$$w := \chi_{[0,\delta]}\phi, \qquad w = P_N(w),$$

we will in fact show that

$$|I_1| = \left|\int_{\mathbb{T}\times\mathbb{R}} P_N^\perp((w)^2\partial_x\overline{w})w\,\overline{w w_x}\, dx\, dt\right| \le C(\delta)N^{-\beta}\|w\|^6_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}. \tag{4.11}$$

To go back to v^N we use the following lemma:

Lemma 4.6 (time cutoff). *Let $b < b_1 < 1/2$. Then the exists $C'(\delta) > 0$ such that*

$$\|w\|_{X_3^{\frac{2}{3}-,b}} \le C'(\delta)\|\phi\|_{X_3^{\frac{2}{3}-,b_1}} \le C'(\delta)\|v^N\|_{X_3^{\frac{2}{3}-,\frac{1}{2}}(\delta)}$$

where w, ϕ and v^N are as above.

Proof. Since the regularity in x does not play any role, without any loss of generality we ignore the power $s = (2/3)-$. Then

$$\begin{aligned} \|w\|_{X_3^{0,b}} &= \left(\sum_n\left(\int |\widehat{\chi_{[0,\delta]}\phi}(n,\tau)|^2\langle\tau+n^2\rangle^{2b}\, d\tau\right)^{3/2}\right)^{1/3} \\ &= \left(\sum_n\left(\int\left|\int_{\tau_1}\widehat{\chi_{[0,\delta]}}(\tau-\tau_1)\widehat{\phi}(n,\tau_1)\, d\tau_1\right|^2\langle\tau+n^2\rangle^{2b}\, d\tau\right)^{3/2}\right)^{1/3}. \end{aligned} \tag{4.12}$$

Writing $\tau + n^2 = (\tau - \tau_1) + (\tau_1 + n^2)$ we bound (4.12) by

$$\lesssim \left(\sum_n\left(\int\left|\int_{\tau_1}\widehat{\chi_{[0,\delta]}}(\tau-\tau_1)\langle\tau-\tau_1\rangle^b\widehat{\phi}(n,\tau_1)\, d\tau_1\right|^2 d\tau\right)^{3/2}\right)^{1/3} \tag{4.13}$$

$$+\left(\sum_n\left(\int\left|\int_{\tau_1}\widehat{\chi_{[0,\delta]}}(\tau-\tau_1)\widehat{\phi}(n,\tau_1)\langle\tau_1+n^2\rangle^b\, d\tau_1\right|^2 d\tau\right)^{3/2}\right)^{1/3}. \tag{4.14}$$

We treat the first sum (4.13), the second one (4.14) being similar. If $\langle\tau-\tau_1\rangle < \langle\tau_1+n^2\rangle$

then by Young's inequality, (4.13) can be bounded by

$$\lesssim \|\widehat{\chi_{[0,\delta]}}(\tau)/\langle\tau\rangle^{\varepsilon}\|_{L^1} \big\| \|\widehat{\phi}(\tau,n)\langle\tau+n^2\rangle^{b+\varepsilon}\|_{L^2}\big\|_{\ell^3} \lesssim \|\chi\|_{H^\beta}\|\phi\|_{X_3^{0,b_1}}$$

by Cauchy–Schwarz on the $\widehat{\chi}$ term provided $\beta+\varepsilon > 1/2$, $\beta < 1/2$ and where $b_1 := b+\varepsilon < 1/2$.

On the other hand if $\langle\tau-\tau_1\rangle \geq \langle\tau_1+n^2\rangle$, then again by Young's inequality, (4.13) can be bounded by

$$\lesssim \|\widehat{\chi_{[0,\delta]}(\tau)}\langle\tau\rangle^{b+\varepsilon}\|_{L^2} \big\| \|\widehat{\phi}(\tau,n)\langle\tau+n^2\rangle^{-\varepsilon}\|_{L^1}\big\|_{\ell^3} \lesssim \|\chi\|_{H^{b+\varepsilon}}\|\phi\|_{X_3^{0,b_1}}$$

by Cauchy–Schwarz on the $\widehat{\phi}$ term provided $b_1+\varepsilon > 1/2$ and $b_1 < 1/2$. Finally by taking the infimum and using the definition of $X_3^{0,1/2}(\delta)$, a bound in terms of $\|v^N\|_{X_3^{0,1/2}(\delta)}$ follows. □

4.4. Proof of Theorem 4.2

Returning to (4.11) we write

$$\begin{aligned}
I_1 &= \int_{\mathbb{T}\times\mathbb{R}} P_N^{\perp}(w^2\partial_x\overline{w})w\overline{w}\overline{w_x}\,dx\,dt \\
&= \int_\tau \sum_{|n|>N} \widehat{(w^2\overline{w_x})}(n,\tau)\overline{\widehat{(\overline{w}ww_x)}}(n,\tau)\,d\tau \\
&= \int \sum_{|n|>N} \Bigg(\int_{\tau=\tau_1+\tau_2-\tau_3} \sum_{\substack{n=n_1+n_2-n_3\\ |n_j|\leq N}} \widehat{w}(n_1,\tau_1)\widehat{w}(n_2,\tau_2)(-in_3)\overline{\widehat{w}}(n_3,\tau_3)\,d\tau_1\,d\tau_2 \Bigg) \\
&\quad \times \Bigg(\int_{-\tau=\tau_4-\tau_5-\tau_6} \sum_{\substack{-n=n_4-n_5-n_6\\ |n_j|\leq N}} \widehat{w}(n_4,\tau_4)\overline{\widehat{w}}(n_5,\tau_5)(-in_6)\overline{\widehat{w}}(n_6,\tau_6)\,d\tau_4\,d\tau_5 \Bigg)\,d\tau \\
&= \int \sum_{N<|n|\leq 3N} \Bigg(\int_{\tau=\tau_1+\tau_2+\tau_3} \sum_{\substack{n=n_1+n_2+n_3\\ |n_j|\leq N}} \widehat{w}(n_1,\tau_1)\widehat{w}(n_2,\tau_2)(in_3)\widehat{\overline{w}}(n_3,\tau_3)\,d\tau_1\,d\tau_2 \Bigg) \\
&\quad \times \Bigg(\int_{-\tau=\tau_4+\tau_5+\tau_6} \sum_{\substack{-n=n_4+n_5+n_6\\ |n_j|\leq N}} \widehat{w}(n_4,\tau_4)\widehat{\overline{w}}(n_5,\tau_5)(in_6)\widehat{\overline{w}}(n_6,\tau_6)\,d\tau_4\,d\tau_5 \Bigg)\,d\tau \\
&= \int \sum_{N<|n|\leq 3N} \Bigg(\int_{\tau=\tau_1+\tau_2+\tau_3} \sum_{\substack{n=n_1+n_2+n_3\\ |n_j|\leq N}} \widehat{w_1}(n_1,\tau_1)\widehat{w_2}(n_2,\tau_2)(in_3)\widehat{\overline{w_3}}(n_3,\tau_3)\,d\tau_1\,d\tau_2 \Bigg) \\
&\quad \times \Bigg(\int_{-\tau=\tau_4+\tau_5+\tau_6} \sum_{\substack{-n=n_4+n_5+n_6\\ |n_j|\leq N}} \widehat{w_4}(n_4,\tau_4)\widehat{\overline{w_5}}(n_5,\tau_5)(in_6)\widehat{\overline{w_6}}(n_6,\tau_6)\,d\tau_4\,d\tau_5 \Bigg)\,d\tau,
\end{aligned}$$

where $w_1 = w_2 = w_4 = w$ and $\overline{w_3} = \overline{w_5} = \overline{w_6} = \overline{w}$.

Remark 4.7. In what follows we always think of N_j, N as dyadic; more precisely $N_j := 2^{K_j}$, $N := 2^K$ where $K_j < K$ since $n_j \in \mathbb{Z}$. By a slight abuse of notation we then denote by N_j both $|n_j|$ and the dyadic interval $[2^{K_j}, 2^{K_j+1})$ to which $|n_j|$ belongs when $n_j \neq 0$. Moreover we denote by w_{N_j} the function such that $\widehat{w_{N_j}}(n_j) = \chi_{\{|n_j| \sim N_j\}} \widehat{w_j}(n_j)$.

From the expression above we then have

$$|n_j| \leq N, \quad N \leq |n| \leq 3N, \quad n = n_1 + n_2 + n_3, \quad -n = n_4 + n_5 + n_6, \tag{4.15}$$

$$N \sim \max(N_1, N_2, N_3) \sim \max(N_4, N_5, N_6), \tag{4.16}$$

$$\tau + n^2 - (\tau_1 + n_1^2) - (\tau_2 + n_2^2) - (\tau_3 - n_3^2) = 2(n - n_1)(n - n_2), \tag{4.17}$$

$$\tau + n^2 + (\tau_4 + n_4^2) + (\tau_5 - n_5^2) + (\tau_6 - n_6^2) = 2(n + n_5)(n + n_6). \tag{4.18}$$

So if we let $\tilde{\sigma}_j := \tau_j \pm n_j^2$ and $\sigma_j := \langle \tau_j \pm n_j^2 \rangle$, by subtracting (4.17) from (4.18) we have

$$\sum_{j=1}^{6} \tilde{\sigma}_j = -2(n(n_1 + n_2 + n_5 + n_6) - n_1 n_2 + n_5 n_6). \tag{4.19}$$

This in turn can also be rewritten using $n_1+n_2+n_3+n_4+n_5+n_6 = 0$ or $n = n_1+n_2+n_3$ and $-n = n_4 + n_5 + n_6$ as

$$\sum_{j=1}^{6} \tilde{\sigma}_j = 2(n(n_3 + n_4) + n_1 n_2 - n_5 n_6). \tag{4.20}$$

In addition, since $\tau_1+\tau_2+\tau_3+\tau_4+\tau_5+\tau_6 = 0$, adding and subtracting n_j^2, $j = 1, \dots, 6$, in the appropriate fashion, we obtain

$$\sum_{j=1}^{6} \tilde{\sigma}_j = (n_3^2 + n_5^2 + n_6^2) - (n_1^2 + n_2^2 + n_4^2). \tag{4.21}$$

Hence we need to estimate

$$\begin{aligned}
|I_1| &= \left| \sum_{N_i \leq N; i=1,\dots 6} \int_{\mathbb{R}} \int_{\mathbb{T}} P_N^{\perp}(w_{N_1} w_{N_2} \partial_x \overline{w_{N_3}}) w_{N_4} \overline{w_{N_5}} \partial_x \overline{w_{N_6}}\, dx\, dt \right| \\
&= \left| \sum_{N_i \leq N; i=1,\dots 6} \sum_{|n| \geq N} \int_{\tau} \left(\int_{\tau=\tau_1+\tau_2+\tau_3} \sum_{n=n_1+n_2+n_3} \widehat{w_{N_1}} \widehat{w_{N_2}} (in_3) \widehat{\overline{w_{N_3}}}\, d\tau_1\, d\tau_2 \right) \right. \\
&\qquad \left. \times \left(\int_{-\tau=\tau_4+\tau_5+\tau_6} \sum_{-n=n_4+n_5+n_6} \widehat{w_{N_4}} \widehat{\overline{w_{N_5}}} (in_6) \widehat{\overline{w_{N_6}}}\, d\tau_4\, d\tau_5 \right) d\tau \right| \\
&\leq \sum_{N \leq |n| \leq 3N} \sum_{N_i \leq N; i=1,\dots 6} \int_{\tau} \left(\int_{\tau=\tau_1+\tau_2+\tau_3} \sum_{n=n_1+n_2+n_3} |\widehat{w_{N_1}}|\, |\widehat{w_{N_2}}|\, |n_3|\, |\widehat{\overline{w_{N_3}}}|\, d\tau_1\, d\tau_2 \right) \\
&\qquad \times \left(\int_{-\tau=\tau_4+\tau_5+\tau_6} \sum_{-n=n_4+n_5+n_6} |\widehat{w_{N_4}}|\, |\widehat{\overline{w_{N_5}}}|\, |n_6|\, |\widehat{\overline{w_{N_6}}}|\, d\tau_4\, d\tau_5 \right) d\tau.
\end{aligned} \tag{4.22}$$

Remark 4.8. This expression (4.22) will be our point of departure in beginning our estimate. In what follows we will abuse notation and write w_{N_j} for $|\widecheck{w_{N_j}}|$ and $\overline{w_{N_k}}$ for $|\widecheck{\overline{w_{N_k}}}|$ since at the end we will estimate all functions in the $X_r^{s,b}$ norms which depend solely on the absolute value of the Fourier transform.

We start by laying out all possible cases and organizing them according to the sizes of the two derivative terms.

Types:

I. $N_3 \sim N$ and $N_6 \sim N$,
II. $N_3 \sim N$ and $N_6 \ll N$,
III. $N_6 \sim N$ and $N_3 \ll N$,
IV. $N_3 \ll N$ and $N_6 \ll N$.

Now we subdivide into all subcases in each situation and group them according to how many low frequencies (i.e. $N_j \ll N$) we have overall, taking into account (4.16).

All cases for each type:

IA. $N_3 \sim N$, $N_6 \sim N$ and 4 lows: $N_1, N_2, N_4, N_5 \ll N$.
IB. $N_3 \sim N$, $N_6 \sim N$ and 3 lows:
  (i) $N_1, N_2, N_4 \ll N$ and $N_5 \sim N$,
  (ii) $N_1, N_2, N_5 \ll N$ and $N_4 \sim N$,
  (iii) $N_1, N_4, N_5 \ll N$ and $N_2 \sim N$,
  (iv) $N_2, N_4, N_5 \ll N$ and $N_1 \sim N$.
IC. $N_3 \sim N$, $N_6 \sim N$ and 2 lows:
  (i) $N_1, N_2 \ll N$ and $N_4, N_5 \sim N$,
  (ii) $N_1, N_4 \ll N$ and $N_2, N_5 \sim N$,
  (iii) $N_1, N_5 \ll N$ and $N_2, N_4 \sim N$.
  (iv) $N_2, N_4 \ll N$ and $N_1, N_5 \sim N$,
  (v) $N_2, N_5 \ll N$ and $N_1, N_4 \sim N$,
  (vi) $N_4, N_5 \ll N$ and $N_1, N_2 \sim N$.
ID. $N_3 \sim N$, $N_6 \sim N$ and 1 low:
  (i) $N_1 \ll N$ and $N_2, N_4, N_5 \sim N$,
  (ii) $N_2 \ll N$ and $N_1, N_4, N_5 \sim N$,
  (iii) $N_4 \ll N$ and $N_1, N_2, N_5 \sim N$,
  (iv) $N_5 \ll N$ and $N_1, N_2, N_4 \sim N$.
IE. $N_3 \sim N$, $N_6 \sim N$ and $N_1, N_2, N_4, N_5 \sim N$.
IIA. $N_3 \sim N$ and $N_6 \ll N$ and 3 lows:
  (i) $N_1, N_2, N_4 \ll N$ and $N_5 \sim N$,
  (ii) $N_1, N_2, N_5 \ll N$ and $N_4 \sim N$,
IIB. $N_3 \sim N$ and $N_6 \ll N$ and 2 lows:
  (i) $N_1, N_2 \ll N$ and $N_4, N_5 \sim N$,
  (ii) $N_1, N_4 \ll N$ and $N_2, N_5 \sim N$,

(iii) $N_1, N_5 \ll N$ and $N_2, N_4 \sim N$,
(iv) $N_2, N_4 \ll N$ and $N_1, N_5 \sim N$.
(v) $N_2, N_5 \ll N$ and $N_1, N_4 \sim N$.

IIC. $N_3 \sim N$ and $N_6 \ll N$ and 1 low:
(i) $N_1 \ll N$ and $N_2, N_4, N_5 \sim N$,
(ii) $N_2 \ll N$ and $N_1, N_4, N_5 \sim N$,
(iii) $N_4 \ll N$ and $N_1, N_2, N_5 \sim N$,
(iv) $N_5 \ll N$ and $N_1, N_2, N_4 \sim N$.

IID. $N_3 \sim N$ and $N_6 \ll N$ and $N_1, N_2, N_4, N_5 \sim N$.

IIIA. $N_6 \sim N$ and $N_3 \ll N$ and 3 lows:
(i) $N_2, N_4, N_5 \ll N$ and $N_1 \sim N$,
(ii) $N_1, N_4, N_5 \ll N$ and $N_2 \sim N$.

IIIB. $N_6 \sim N$ and $N_3 \ll N$ and 2 lows:
(i) $N_4, N_5 \ll N$ and $N_1, N_2 \sim N$,
(ii) $N_1, N_4 \ll N$ and $N_2, N_5 \sim N$,
(iii) $N_1, N_5 \ll N$ and $N_2, N_4 \sim N$,
(iv) $N_2, N_4 \ll N$ and $N_1, N_5 \sim N$,
(v) $N_2, N_5 \ll N$ and $N_1, N_4 \sim N$.

IIIC. $N_6 \sim N$ and $N_3 \ll N$ and 1 low:
(i) $N_1 \ll N$ and $N_2, N_4, N_5 \sim N$,
(ii) $N_2 \ll N$ and $N_1, N_4, N_5 \sim N$,
(iii) $N_4 \ll N$ and $N_1, N_2, N_5 \sim N$,
(iv) $N_5 \ll N$ and $N_1, N_2, N_4 \sim N$.

IIID. $N_6 \sim N$ and $N_3 \ll N$ and $N_1, N_2, N_4, N_5 \sim N$.

IVA. $N_3 \ll N$, $N_6 \ll N$ and 2 lows:
(i) $N_1, N_4 \ll N$ and $N_2, N_5 \sim N$,
(ii) $N_1, N_5 \ll N$ and $N_2, N_4 \sim N$,
(iii) $N_2, N_4 \ll N$ and $N_1, N_5 \sim N$,
(iv) $N_2, N_5 \ll N$ and $N_1, N_4 \sim N$.

IVB. $N_3 \ll N$, $N_6 \ll N$ and 1 low:
(i) $N_1 \ll N$ and $N_2, N_4, N_5 \sim N$,
(ii) $N_2 \ll N$ and $N_1, N_4, N_5 \sim N$,
(iii) $N_4 \ll N$ and $N_1, N_4, N_5 \sim N$,
(iv) $N_5 \ll N$ and $N_1, N_2, N_4 \sim N$.

IVC. $N_3 \ll N$, $N_6 \ll N$ and $N_1, N_2, N_4, N_5 \sim N$.

In what follows we will use the following estimates repeatedly:

Lemma 4.9. *Let w_{N_i} be as above. Then*

$$\|w_{N_i}\|_{X^{0+,\frac{1}{2}-}} \leq N_i^{-\frac{1}{2}+} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}, \tag{4.23}$$

$$\|w_{N_i}\|_{X^{\frac{1}{2}-,\frac{1}{3}+}} \leq \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}. \tag{4.24}$$

We also have

$$\|w_{N_i}\|_{L^8_{xt}} \le \|w_{N_i}\|_{X_3^{\frac{13}{24}+,\frac{3}{8}+}}. \tag{4.25}$$

If we assume that $\sigma_i \lesssim N^\gamma$ *for any* $\gamma > 0$*, then*

$$\|w_{N_i}\|_{L^\infty_{xt}} \le N^{0+}\|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}. \tag{4.26}$$

Proof. The estimates (4.23) and (4.24) are a consequence of frequency localization and Hölder's inequality. The estimate (4.26) is a consequence of Sobolev embedding together with the assumption that $\sigma_i \lesssim N^\gamma$. □

Lemma 4.10. *Let* $0 < \beta < 2$*,* $\rho \ge 0$ *and* $\delta > 0$*. Let* $M > 0$ *and* w_M *be such that* $\operatorname{supp} w_M(\cdot, x) \subset [-\delta, \delta]$*,* $x \in \mathbb{T}$*. Define*

$$\widehat{J_\beta w_M}(\tau, n) := \chi_{\{|n|\sim M\}}\chi_{\{|\tau+n^2|\le M^\beta\}}|\widehat{w_M}(\tau, n)|.$$

Then

$$\|J_\beta w_M\|_{X^{0,\rho}} \lesssim C_\delta A(\beta, M)^{1/6} M^{\rho\beta+}\|w_M\|_{X_3^{0,1/6}},$$

where $A(M,\beta)$ *defined below is bounded by* $1 + M^{\beta-1}$*.*

Proof. We write

$$\begin{aligned}
\|J_\beta w_M\|^2_{X^{0,\rho}} &= \sum_{|n|\sim M}\int_{|\tau+n^2|\le M^\beta}|\widehat{w_M}(\tau,n)|^2\langle\tau+n^2\rangle^{2\rho}\,d\tau\\
&\le M^{2\rho\beta}\int_\tau\Bigg(\sum_{|n|\sim M,\ |\tau+n^2|\le M^\beta}|\widehat{w_M}(\tau,n)|^2\Bigg)d\tau\\
&\le M^{2\rho\beta}\int_\tau\Bigg[\sum_{|n|\sim M,\ |\tau+n^2|\le M^\beta}|\widehat{w_M}(\tau,n)|^3\Bigg]^{2/3}|S(\tau,M,\beta)|^{1/3}\,d\tau, \qquad (4.27)
\end{aligned}$$

where

$$S(\tau, M, \beta) := \{n \in \mathbb{Z} : |n| \sim M \text{ and } |\tau + n^2| \le M^\beta\},$$

and $|S|$ represents the counting measure of the set.

We will show below that

$$A(M, \beta) := \sup_\tau |S(\tau, M, \beta)| \le 1 + M^{\beta-1}. \tag{4.28}$$

Hence (4.27) is less than or equal to

$$\begin{aligned}
&A(M,\beta)^{1/3}M^{2\rho\beta}\int_\tau\Bigg[\sum_n\chi_{\{|n|\sim M\}}(n)\chi_{\{|\tau+n^2|\le M^\beta\}}(\tau,n)|\widehat{w_M}(\tau,n)|^3\Bigg]^{2/3}d\tau\\
&= A(M,\beta)^{1/3}M^{2\rho\beta}\int_\tau\big\|\big\{\chi_{\{|\tau+n^2|\le M^\beta\}}(\tau,n)\widehat{w_M}(\tau,n)\big\}_n\big\|^2_{\ell^3(|n|\sim M)}\,d\tau\\
&\sim A(M,\beta)^{1/3}M^{2\rho\beta}\int_t\big\|\mathcal{F}_\tau^{-1}\big(\{\chi_{\{|\tau+n^2|\le M^\beta\}}(\tau,n)\widehat{w_M}(\tau,n)\}_n\big)(t)\big\|^2_{\ell^3(|n|\sim M)}\,dt\\
&= A(M,\beta)^{1/3}M^{2\rho\beta}\int_t\big\|\big\{\mathcal{F}_\tau^{-1}(\chi_{\{|\tau+n^2|\le M^\beta\}}(\tau,n)) * \mathcal{F}_\tau^{-1}(\widehat{w_M}(\tau,n))\big\}_n(t)\big\|^2_{\ell^3(|n|\sim M)}\,dt.
\end{aligned}$$

Note that $\mathcal{F}_\tau^{-1}(\widehat{w_M}(\cdot,n))(t)$ is still supported on $[-\delta,\delta]$ for all n and

$$\mathcal{F}_\tau^{-1}(\chi_{\{|\tau+n^2|\leq M^\beta\}}(\cdot,n))(t) = 2e^{-itn^2}\frac{\sin(M^\beta t)}{t}.$$

We then continue the above chain of inequalities with

$$\begin{aligned}
&= A(M,\beta)^{1/3}M^{2\rho\beta}\\
&\quad\times\int_t\bigg\|\int_{\mathbb{R}}\chi_{[-\delta,\delta]}(t')\mathcal{F}_n(w_M(t',\cdot))(n)e^{-i(t-t')n^2}\frac{\sin(M^\beta(t-t'))}{t-t'}\,dt'\bigg\|^2_{\ell^3(|n|\sim M)}dt\\
&\leq A(M,\beta)^{1/3}M^{2\rho\beta}\\
&\quad\times\int_{\mathbb{R}}\bigg[\int_{\mathbb{R}}\chi_{[-\delta,\delta]}(t')\|\mathcal{F}_n(w_M(t',\cdot))(n)\|_{\ell^3(|n|\sim M)}\bigg|\frac{\sin(M^\beta(t-t'))}{t-t'}\bigg|\,dt'\bigg]^2dt.
\end{aligned}$$

Let $p=2-$ and $q=1+$; then we compute

$$\begin{aligned}
\bigg\|\frac{\sin(M^\beta(t-t'))}{t-t'}\bigg\|_{L^q_t} &= M^\beta\bigg(\int_{\mathbb{R}}\bigg|\frac{\sin(M^\beta t)}{tM^\beta}\bigg|^q dt\bigg)^{1/q}\\
&= M^\beta M^{-\beta/q}\bigg(\int_{\mathbb{R}}\bigg|\frac{\sin(r)}{r}\bigg|^q dr\bigg)^{1/q}\lesssim M^{0+}. \qquad (4.29)
\end{aligned}$$

On the other hand, for $1/\gamma = 1/p-1/3$,

$$\begin{aligned}
\big\|\chi_{[-\delta,\delta]}(\cdot)\|\mathcal{F}_n(w_M(t,\cdot))(n)\|_{\ell^3(|n|\sim M)}\big\|^2_{L^p_t} &\lesssim \delta^{2/\gamma}\big\|\|\mathcal{F}_n(w_M(t,\cdot))(n)\|_{\ell^3(|n|\sim M)}\big\|^2_{L^3_t}\\
&\lesssim \delta^{2/\gamma}\big\|\|e^{itn^2}\mathcal{F}_n(w_M(t,\cdot))(n)\|_{\ell^3(|n|\sim M)}\big\|^2_{L^3_t}\\
&= \delta^{2/\gamma}\big\|\|e^{itn^2}\mathcal{F}_n(w_M(t,\cdot))(n)\|_{L^3_t}\big\|^2_{\ell^3(|n|\sim M)}\\
&\lesssim \delta^{2/\gamma}\big\|\|e^{itn^2}\mathcal{F}_n(w_M(t,\cdot))(n)\|_{H^{1/6}_t}\big\|^2_{\ell^3(|n|\sim M)} = \delta^{2/\gamma}\,\|w_M\|^2_{X_3^{0,1/6}}, \qquad (4.30)
\end{aligned}$$

where we used the Sobolev theorem and the definition of $X_r^{s,b}$. Finally by Young's inequality, (4.29) and (4.30) we have the desired estimate.

It remains to show (4.28). We use an argument similar to [18]. For fixed τ let $S := S(\tau,M,\beta)\neq\emptyset$. Then there exists $n_0\in S$ and hence

$$\begin{aligned}
|S| &\leq 1+|\{l\in\mathbb{Z}: |n_0+l|\sim M, |\tau+(n_0+l)^2|\leq M^\beta\}|\\
&\leq 1+|\{l\in\mathbb{Z}: |l|\leq M, |2n_0l+l^2|\lesssim M^\beta\}|.
\end{aligned}$$

We have $|2n_0l+l^2| = |(l+n_0)^2-n_0^2|\lesssim M^\beta$ if and only if

$$-CM^\beta+n_0^2\leq (l+n_0)^2\leq n_0^2+CM^\beta.$$

Hence we need $|l|\leq M$ to satisfy

$$\begin{gathered}
-\sqrt{n_0^2+CM^\beta}\leq l+n_0\leq\sqrt{n_0^2+CM^\beta},\\
l+n_0\geq\sqrt{n_0^2-CM^\beta}\quad\text{or}\quad l+n_0\leq-\sqrt{n_0^2-CM^\beta}.
\end{gathered}$$

In other words we need to know the size of

$$\Big[-\sqrt{n_0^2+CM^\beta},-\sqrt{n_0^2-CM^\beta}\Big]\cup\Big[\sqrt{n_0^2-CM^\beta},\sqrt{n_0^2+CM^\beta}\Big]$$

which is of the order of $M^\beta/|n_0|$. Hence since $|n_0|\sim M$ we have

$$|S|\leq 1+M^{\beta-1},$$

which implies (4.28) by taking $\sup_\tau$. □

In what follows we are under the assumption that $\sigma_j\lesssim N^7$ for all $j=1,\dots,6$. Towards the end of the proof we remove this assumption. We begin by treating all cases with at least two high frequencies in the nonderivative terms. All cases in IC, ID, IE, IIB, IIC, IID, IIIB, IIIC, IIID, IVA, IVB, IVC follow from the following lemma applied with the exponent σ set equal to 0.

Lemma 4.11. *Assume there are* $i,j\in\{1,2,4,5\}$ *such that* $N_i\geq N^{1-\sigma}$ *for* $0\leq\sigma<1/6$ *and* $N_j\sim N$. *Then* (4.22) *can be estimated by* $N^{-1/12+\sigma/2}\prod_{i=1}^6\|w_i\|_{X_3^{s,b}}$.

Proof. By Plancherel we see that (4.22) is less than or equal to

$$\sum_{N_j\sim N;\;N_i\geq N^{1-\sigma};\;N_k\leq N,1\leq k\leq 6}\int_{\mathbb{R}}\int_{\mathbb{T}}N_3N_6w_{N_1}w_{N_2}\overline{w_{N_3}}w_{N_4}\overline{w_{N_5}w_{N_6}}\,dx\,dt. \tag{4.31}$$

Let $0<\beta<1$ to be determined below. Assume

$$\sigma_3\leq N_3^\beta. \tag{4.32}$$

By Cauchy–Schwarz's inequality, grouping the first three functions in (4.31) in L^2_{xt} and the last three in L^2_{xt} and using (2.3) we see that (4.31) is less than or equal to

$$\sum_{N_j\sim N;\;N_i\geq N^{1-\sigma};N_k\leq N}N_3N_6\prod_{i=1}^6\|w_{N_i}\|_{X^{\epsilon,\frac12-}}. \tag{4.33}$$

Note now that by (4.32), w_{N_3} is equal to $J_\beta w_{N_3}$ as defined in Lemma 4.10. Then we have

$$\|w_{N_3}\|_{X^{\epsilon,\frac12-}}\leq C_\delta N_3^{1/2\beta+}\|w_{N_3}\|_{X_3^{0,\frac16+}}. \tag{4.34}$$

Hence by (4.23), (4.34) we deduce that (4.33) is less than or equal to

$$\begin{aligned}&\sum_{N_j\sim N;\;N_i\geq N^{1-\sigma};\;N_k\leq N}N_3N_6N_1^{-\frac12+}N_2^{-\frac12+}N_3^{\frac12\beta+}N_3^{-\frac23}N_4^{-\frac12+}N_5^{-\frac12+}N_6^{-\frac12+}\\&\qquad\times\Big(\prod_{i=1}^6\|w_{N_i}\|_{X_3^{\frac23-,\frac12-}}\Big).\\&\lesssim\sum_{N_j\sim N;\;N_i\geq N^{1-\sigma};\;N_k\leq N}N_3^{\frac13+\frac\beta2+}N^{\frac12+}N^{-1+\frac\sigma2}\Big(\prod_{i=1}^6\|w_{N_i}\|_{X_3^{\frac23-,\frac12-}}\Big).\end{aligned}$$

Now we apply Hölder's inequality with $r = 3$, $r' = 3/2$ to sum in N_j, N_i, N_k (multiply and divide by $N_j^{-\epsilon}$ with a loss of N^ϵ for each term). For example,

$$\sum_{N_j \le N} \|w_{N_j}\|_{X_3^{s,b}} = \sum_{N_j \le N} \left\| \|\langle n_j\rangle^s \langle \tau + n_j^2\rangle^b \widehat{w}_{N_j}(\tau, n_j)\|_{L^2_\tau} \right\|_{\ell^3}. \tag{4.35}$$

Set $Y_{N_j}(n_j) := \|\langle n_j\rangle^s \langle \tau - n_j^2\rangle^b \widehat{w}_{N_j}(\tau, n_j)\|_{L^2_\tau}$. Then the expression in (4.35) equals

$$\begin{aligned}\sum_{N_j \le N} N_j^\varepsilon N_j^{-\varepsilon} \|Y_{N_j}\|_{\ell^3} &\le N^\varepsilon \Big(\sum_{N_j \le N} N_j^{-\frac{3}{2}\varepsilon}\Big)^{2/3} \Big(\sum_{N_j \le N} \|Y_{N_j}\|^3_{\ell^3}\Big)^{1/3} \\ &\lesssim N^\varepsilon \Big(\sum_{N_j \le N} \sum_{|n_j| \sim N_j} \|\langle n_j\rangle^s \langle \tau + n_j^2\rangle^b \widehat{w}_j(\tau, n_j)\|^3_{L^2_\tau}\Big)^{1/3} \\ &\sim N^\varepsilon \|w_j\|_{X_3^{s,b}}.\end{aligned}$$

Note then that all in all we get at worst a factor of $N^{-\frac{1}{6}+\frac{\beta}{2}+\frac{\sigma}{2}+}$.

Now assume that

$$\sigma_3 \ge N_3^\beta.$$

Then rewrite (4.31) as

$$\sum_{N_j \sim N; N_i \ge N^{1-\sigma}; N_k \le N} \int_{\mathbb{R}} \int_{\mathbb{T}} N_3 N_6 |\sigma_3|^{-\frac{1}{2}+} w_{N_1} w_{N_2} |\sigma_3|^{\frac{1}{2}-} \overline{w_{N_3}} w_{N_4} \overline{w_{N_5}}\, \overline{w_{N_6}}\, dx\, dt. \tag{4.36}$$

We do Hölder by placing $|\sigma_3|^{(1/2)-}\overline{w_{N_3}}$ in L^2_{xt}, the product of $\overline{w_{N_6}}$ with the two largest among $w_{N_1}, w_{N_2}, w_{N_4}, \overline{w_{N_5}}$ in L^2_{xt}, while the remaining ones in L^∞_{xt}. Then by (4.23) and (4.26), we bound (4.36) by

$$\begin{aligned}&\lesssim \sum_{N_j \sim N; N_i \ge N^{1-\sigma}; N_k \le N} N_3 N_3^{-\frac{1}{2}-\frac{\beta}{2}+} N_6 N_6^{-\frac{1}{2}+} N^{-\frac{1}{2}+} N^{-\frac{1}{2}+\frac{\sigma}{2}+} \Big(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\Big) \\ &\lesssim \sum_{N_j \sim N; N_i \ge N^{1-\sigma}; N_k \le N} N_3^{\frac{1}{2}-\frac{\beta}{2}+} N^{-\frac{1}{2}+\frac{\sigma}{2}+} \Big(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\Big).\end{aligned}$$

We want that $\beta > \sigma$ to conclude by Hölder the desired inequality with a decay in N. We now impose that

$$-\frac{1}{6} + \frac{\beta}{2} + \frac{\sigma}{2} = -\frac{\beta}{2} + \frac{\sigma}{2},$$

whence $\beta = 1/6$ and provided $0 < \sigma < 1/6$ the lemma follows. □

It remains then to treat cases IA, IB, IIA and IIIA. Before starting we note the following support condition that will be used throughout what follows.

Support condition

By (4.15) and (4.16) the triplet $(w_{N_1}, w_{N_2}, \overline{w_{N_3}})$ satisfies $n = n_1 + n_2 + n_3$, $|n_j| \le N$, $N \le |n| \le 3N$ and $N \sim \max(N_1, N_2, N_3)$.

Suppose that, say, $\max(N_1, N_2) \le N^\theta$ for some $0 < \theta < 1$. Without any loss of generality assume $n > 0$. Then $N \le n \le (n_1+n_2)+n_3 \le 2N^\theta + N$ and hence $n = N+k$ where $0 \le k \le 2N^\theta$. Next observe that $n_3 = n - (n_1+n_2) = N + k - (n_1+n_2)$ with $|k - (n_1+n_2)| \le 4N^\theta$, whence $n_3 = N + O(N^\theta)$. In other words, whenever $\max(N_1, N_2) \le N^\theta$ the support of $\widehat{\overline{w_{N_3}}}$ is of size $O(N^\theta)$. Note that we could have just as well said that the support of $\widehat{\overline{w_{N_3}}}$ is of size $O(\max(N_1, N_2))$ in lieu of $O(N^\theta)$.

When we are in this situation we say we have the *support condition* on $\overline{w_{N_3}}$. This argument is symmetric with respect to w_{N_1}, w_{N_2} or $\overline{w_{N_3}}$. The exact same analysis holds for $(w_{N_4}, \overline{w_{N_5}}, \overline{w_{N_6}})$. By abuse of notation we still write, for example, $\widehat{\overline{w_{N_3}}}(n_3)$ for $\widehat{\overline{w_{N_3}}}(n_3)\chi_{I_3}(n_3)$, where I_3 is the support of $\widehat{\overline{w_{N_3}}}$ when the support condition holds.

Remark 4.12. As a consequence of the support condition, estimate (4.23) can be improved. For example if we have the support condition on $\widehat{\overline{w_{N_3}}}$ then

$$\|w_{N_3}\|_{X^{0+,\frac{1}{2}-}} \lesssim |I_3|^{1/6}\|w_{N_3}\|_{X_3^{0+,\frac{1}{2}-}}.$$

Case IIIA. Note that (i) and (ii) are symmetric with respect to $j = 1$ and $j = 2$. So we only consider (i). Observe also that a priori there is no help from a large σ_j. Let σ, δ be two positive constants to be determined later but such that $1 - \sigma > \delta$.

Subcase 1. Assume $N_2, N_4, N_5 < N^{1-\sigma}$, $N_3 \lesssim N^\delta$ and $N_1 \sim N \sim N_6$ in (4.22). Then we have the support condition on w_{N_1} and $\overline{w_{N_6}}$. Let us denote by $\sum_*$ the sum over the set of $N_j \le N$, $1 \le j \le 6$, such that $N_1, N_6 \sim N$, $N_j < N^{1-\sigma}$ for $j = 2, 4, 5$ and $N_3 \lesssim N^\delta$. By Cauchy–Schwarz, (2.3), Lemma 4.9 and Remark 4.12 we then conclude that (4.22) is less than or equal to

$$\begin{aligned}
&\sum\nolimits_* N_3 N_6 \max(N_2, N_3)^{1/6} N_1^{-\frac{2}{3}+} N_2^{-\frac{1}{2}+} N_3^{-\frac{1}{2}+} N_4^{-\frac{1}{2}+} N_5^{-\frac{1}{2}+} \\
&\qquad\qquad \times \max(N_4, N_5)^{1/6} N_6^{-\frac{2}{3}+} \Big(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\Big) \\
&\qquad \lesssim \sum\nolimits_* N_3^{\frac{1}{2}+} N_6^{\frac{1}{3}+} \max(N_2, N_3)^{1/6} N_2^{-\frac{1}{2}+} N^{-\frac{2}{3}+} \Big(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\Big)
\end{aligned}$$

since $N_4^{-(1/2)+} N_5^{-(1/2)+} \max(N_4, N_5)^{1/6}$ is bounded. On the other hand the latter expression is worst possible when $\max(N_2, N_3) \sim N_3$; hence if $\delta < 1/2$ we conclude by Hölder as before with a decay of $N^{-1/3} N^{2\delta/3}$.

Subcase 2. Assume $N_2, N_4, N_5 < N^{1-\sigma}$, $N_3 \gtrsim N^\delta$ and $N_1 \sim N \sim N_6$ in (4.22). We further subdivide as follows:

Subcase 2a. Assume $N_2, N_4, N_5 \ll N^\delta$, $N_3 \gtrsim N^\delta$ and $N_1 \sim N \sim N_6$ in (4.22). Then from (4.20) there exists $\sigma_j \gtrsim N^{1+\delta}$. Denote by $\sum_*$ the sum over the set of $N_j \le N$, $1 \le j \le 6$, such that $N_1, N_6 \sim N$, $N_j < N^\delta$ for $j = 2, 4, 5$ and $N_3 \ge N^\delta$.

• Suppose $j = 2, 4$ or 5; $j = 2$ or 4 are symmetric. So we treat first $j = 2$ and then $j = 5$. By Plancherel, (4.22) is less than or equal to

$$\sum\nolimits_* \int_{\mathbb{R}}\int_{\mathbb{T}} N_3 N_6 \sigma_2^{-\frac{1}{2}+} w_{N_1} \sigma_2^{\frac{1}{2}-} w_{N_2} \overline{w_{N_3}} w_{N_4} \overline{w_{N_5}}\, \overline{w_{N_6}}\, dx\, dt$$
$$\lesssim \sum\nolimits_* N_3^{\frac{1}{2}+} N_6^{\frac{1}{2}+} N^{-\frac{1}{2}-\frac{\delta}{2}} N_1^{-\frac{1}{2}+} N_2^{-\frac{1}{2}+} N_4^{0+} N_5^{0+} \Big(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}} \Big)$$

by Cauchy–Schwarz placing $w_{N_1}\overline{w_{N_3} w_{N_6}}$ in L^2, $\sigma_2^{1/2} w_{N_2}$ in L^2 and $w_{N_4}\overline{w_{N_5}}$ in L^∞. From (2.3) and Lemma 4.9 we obtain the desired estimate with decay $N^{-\delta/2}$ so long as $\delta > 0$.

If $j = 5$ we proceed as above with the same grouping in L^2 but exchanging the roles of w_{N_2} and $\overline{w_{N_5}}$ for the other L^2 and one of the L^∞ bounds.

• Suppose $j = 3, 6$ or 1; $j = 3$ or 6 are symmetric. So we treat first $j = 3$ and then $j = 1$. Proceeding as above from (4.22) we now have

$$\sum\nolimits_* \int_{\mathbb{R}}\int_{\mathbb{T}} N_3 N_6 \sigma_3^{-\frac{1}{2}+} w_{N_1} w_{N_2} \sigma_3^{\frac{1}{2}-} \overline{w_{N_3}} w_{N_4} \overline{w_{N_5}}\, \overline{w_{N_6}}\, dx\, dt$$
$$\lesssim \sum\nolimits_* N_3^{\frac{1}{2}+} N_6^{\frac{1}{2}+} N^{-\frac{1}{2}-\frac{\delta}{2}} N_1^{-\frac{1}{2}+} N_2^{0+} N_4^{-\frac{1}{2}+} N_5^{0+} \Big(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}} \Big)$$

by Cauchy–Schwarz placing $w_{N_1} w_{N_4}\overline{w_{N_6}}$ in L^2, $\sigma_3^{(1/2)-}\overline{w_{N_3}}$ in L^2 and $w_{N_2}\overline{w_{N_5}}$ in L^∞. We thus obtain the desired estimate as before with decay $N^{-\delta/2}$ so long as $\delta > 0$.

If $j = 1$ then we group $\overline{w_{N_3}} w_{N_4} \overline{w_{N_6}}$ in L^2, $\sigma_1^{(1/2)-} w_{N_1}$ in L^2 and the other two in L^∞ to reach the same estimate.

Subcase 2b. Suppose there exists $i \in \{2, 4, 5\}$ such that $N_i \gtrsim N^\delta$ and $N_j \ll N^\delta$ for $j \neq i$, and $i, j \in \{2, 4, 5\}$ while still $N_3 \gtrsim N^\delta$ and $N_1 \sim N \sim N_6$ in (4.22).

• Suppose $i = 2$ first. Then we further split the sum over this set into three sums, S_1, S_2 and S_3 according to whether $N^\delta \lesssim N_2 \ll N_3$, $N_2 \sim N_3$ or $N_2 \gg N_3$ respectively. When considering the sums over S_1 or S_3 we deduce from (4.20) that there exists $\sigma_j \gtrsim N^{1+\delta}$ and hence the estimates for S_1 and S_3 follow exactly as those in Subcase 2a.

We then treat S_2. Since $N_2 \sim N_3$ and $N_2 < N^{1-\sigma}$, we also have $N_3 < N^{1-\sigma}$, while $N_4, N_5 \lesssim N^\delta$. Thus we have the support condition on w_{N_1} and $\overline{w_{N_6}}$. Then from (4.22) by Cauchy–Schwarz, (2.3), Lemma 4.9 and Remark 4.12, grouping $w_{N_1} w_{N_2}\overline{w_{N_3}}$ in L^2 and $w_{N_4}\overline{w_{N_5}}\,\overline{w_{N_6}}$ and (4.23) we have

$$\sum_{S_2} N_3 N_6 \max(N_2, N_3)^{1/6} N_1^{-\frac{2}{3}+} N_2^{-\frac{1}{2}+} N_3^{-\frac{1}{2}+} N_4^{-\frac{1}{2}+} N_5^{-\frac{1}{2}+}$$
$$\times \max(N_4, N_5)^{1/6} N_6^{-\frac{2}{3}+} \Big(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}} \Big)$$
$$\lesssim \sum_{S_2} N_6^{\frac{1}{3}+} \max(N_2, N_3)^{1/6} N_1^{-\frac{2}{3}+} \Big(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}} \Big)$$

since $N_4^{-(1/2)+} N_5^{-(1/2)+} \max(N_4, N_5)^{1/6}$ is bounded and $N_2 \sim N_3$. Summing as usual, we get the desired estimate with decay $N^{-(1/6)+}$ regardless of $\sigma > 0$.

• Suppose $i = 4$. Again, we further split the sum over this set into three sums, over S_1, S_2 and S_3, according now to whether $N^\delta \lesssim N_4 \ll N_3$, $N_4 \sim N_3$ or $N_4 \gg N_3$ respectively. For the sums over S_1 or S_3, from (4.20) we have $\sigma_j \gtrsim N^{1+\delta}$ and hence the estimates for S_1 and S_3 follow exactly as those in Subcase 2a.

We then treat S_2. Since $N_4 \sim N_3$, $N_3 < N^{1-\sigma}$ while $N_2, N_5 \lesssim N^\delta$, once again we have the support condition on w_{N_1} and $\overline{w_{N_6}}$. Proceeding as before we have

$$\begin{aligned}\sum_{S_2} & N_3 N_6 \max(N_2, N_3)^{1/6} N_1^{-\frac{2}{3}+} N_2^{-\frac{1}{2}+} N_3^{-\frac{1}{2}+} N_4^{-\frac{1}{2}+} N_5^{-\frac{1}{2}+} \\ &\qquad \times \max(N_4, N_5)^{1/6} N_6^{-\frac{2}{3}+} \Big(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\Big) \\ &\lesssim \sum_{S_2} N_3^{\frac{1}{2}+} N_6 N_3^{1/6} N^{-\frac{2}{3}+} N_2^{-\frac{1}{2}+} N_4^{-\frac{1}{2}+\frac{1}{6}+} N_5^{-\frac{1}{2}+} N^{-\frac{2}{3}+} \Big(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\Big) \\ &\lesssim \sum_{S_2} N_3^{\frac{1}{3}+} N N^{-\frac{4}{3}+} \Big(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\Big).\end{aligned}$$

Since $N_4 \sim N_3$ and $N_3 < N^{1-\sigma}$, summing as before we have the desired estimate with decay $N^{-\sigma/3}$ so long as $\sigma > 0$.

• Suppose $i = 5$. We split the sum over this set into three sums, over S_1, S_2 and S_3, according to whether $N^\delta \lesssim N_5 \ll N_3$, $N_5 \sim N_3$ or $N_5 \gg N_3$ respectively. Again for the sums over S_1 or S_3, from (4.20) we have $\sigma_j \gtrsim N^{1+\delta}$ and hence the estimates for S_1 and S_3 follow exactly as those in Subcase 2a.

We then treat S_2. Since $N_5 \sim N_3$, $N_3 < N^{1-\sigma}$ while $N_2, N_4 \lesssim N^\delta$, we have the support condition on w_{N_1} and $\overline{w_{N_6}}$. Proceeding as before we have

$$\begin{aligned}\sum_{S_2} & N_3 N_6 \max(N_2, N_3)^{1/6} N_1^{-\frac{2}{3}+} N_2^{-\frac{1}{2}+} N_3^{-\frac{1}{2}+} N_4^{-\frac{1}{2}+} N_5^{-\frac{1}{2}+} \\ &\qquad \times \max(N_4, N_5)^{1/6} N_6^{-\frac{2}{3}+} \Big(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\Big) \\ &\lesssim \sum_{S_2} N_3^{\frac{1}{2}+} N_6^{\frac{1}{3}+} N_3^{1/6} N^{-\frac{2}{3}+} N_2^{-\frac{1}{2}+} N_4^{-\frac{1}{2}+} N_5^{-\frac{1}{2}+\frac{1}{6}+} \Big(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\Big) \\ &\lesssim \sum_{S_2} N_3^{\frac{1}{3}+} N^{-\frac{1}{3}+} \Big(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\Big),\end{aligned}$$

which gives the desired estimate with the same $N^{-\sigma/3}$ decay as in the previous case so long as $\sigma > 0$.

Subcase 2c. Suppose that there exist at least $i, j \in \{2, 4, 5\}$ $(i \neq j)$ such that $N_i, N_j \gtrsim N^\delta$ while $N_3 \gtrsim N^\delta$ and $N_1 \sim N \sim N_6$ in (4.22). Note that $N_4, N_5 < N^{1-\sigma}$, which ensures the support condition on $\overline{w_{N_6}}$.

• Suppose $(i, j) = (4, 5)$. Proceeding as above and using similar arguments we have

$$\sum_* N_3 N_6 N_1^{-\frac{1}{2}+} N_2^{-\frac{1}{2}+} N_3^{-\frac{1}{2}+} N_4^{-\frac{1}{2}+} N_5^{-\frac{1}{2}+} \max(N_4, N_5)^{1/6} N_6^{-\frac{2}{3}+} \Big(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\Big)$$
$$= \sum_* N_3^{\frac{1}{2}+} N_6^{\frac{1}{3}+} N^{-\frac{1}{2}+} N_4^{-\frac{1}{2}+} N_5^{-\frac{1}{2}+} \max(N_4, N_5)^{1/6} \Big(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\Big),$$

from which using that $N_4, N_5 \gtrsim N^\delta$ and $N_3 \gtrsim N^\delta$ we get the desired bound with decay $N^{1/3-5\delta/6}$ so long as $\delta > 2/5$.

• Suppose $(i, j) = (2, 5)$. Once again proceeding as before and using similar arguments we have

$$\sum_* N_3 N_6 N_1^{-\frac{1}{2}+} N_2^{-\frac{1}{2}+} N_3^{-\frac{1}{2}+} N_4^{-\frac{1}{2}+} N_5^{-\frac{1}{2}+} \max(N_4, N_5)^{1/6} N_6^{-\frac{2}{3}+} \Big(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\Big)$$
$$\lesssim \sum_* N_3^{\frac{1}{2}+} N_6^{\frac{1}{3}+} N^{-\frac{1}{2}+} N^{-\frac{\delta}{2}} N^{-\frac{\delta}{2}+\frac{\delta}{6}} \Big(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\Big)$$

using that $N_2 \gtrsim N^\delta$ and that $N_4^{-(1/2)+} N_5^{-(1/2)+} \max(N_4, N_5)^{1/6}$ is worse possible when $N_4 \ll N_5$ but $N_5 \gtrsim N^\delta$. Hence we once again obtain the desired estimate with decay $N^{1/3-5\delta/6}$ so long as $\delta > 2/5$.

• Suppose $(i, j) = (2, 4)$. This is exactly as in the previous case by exchanging the roles of 4 and 5.

Subcase 3. Assume there exists at least one $i \in \{2, 4, 5\}$ such that $N_i \gtrsim N^{1-\sigma}$, $N_2, N_4, N_5 \ll N$ while $N_3 \ll N$ and $N_1 \sim N \sim N_6$ in (4.22). This case follows from Lemma 4.11 with $0 < \sigma < 1/6$ as in its statement.

All in all, for Case IIIA we need $2/5 < \delta < 1/2$ and $0 < \sigma < 1/6$.

Remark 4.13. In the proof of the remaining cases, in order to keep the notation lighter, we will ignore the $+\epsilon$ appearing in the exponent of the N_i's in (4.23). For example we simply write $N_i^{-1/2}$ instead of $N_i^{-(1/2)+}$.

Case IA. Assume $N_3 \sim N \sim N_6$ while $N_1, N_2, N_4, N_5 \ll N$ in (4.22) and denote as before by $\sum_*$ the sum over this set. Observe that from (4.17)–(4.21) there exists $\sigma_j \gtrsim N^2$.

Subcase 1. Assume in addition $N_1, N_2 < N^\delta$ for some $\delta > 0$. We then have the support condition on $\overline{w_{N_3}}$.

• Suppose $j = 3$ or 6; say $j = 3$ ($j = 6$ is similar). Then we rewrite (4.22) as follows:

$$\sum_* \int_{\mathbb{R}} \int_{\mathbb{T}} N^2 \sigma_3^{-\frac{1}{2}+} w_{N_1} w_{N_2} \sigma_3^{\frac{1}{2}-} \overline{w_{N_3}} w_{N_4} \overline{w_{N_5}}\, \overline{w_{N_6}}\, dx\, dt$$
$$\lesssim \sum_* N^2 N^{-1} N_1^{0+} N_2^{0+} \max(N_1, N_2)^{1/6} N_3^{-2/3} N_4^{-1/2} N_5^{-1/2} N_6^{-1/2} \Big(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\Big)$$

by placing $\sigma_3^{(1/2)-}\overline{w_{N_3}}$ in L^2_{xt}, $w_{N_4}\overline{w_{N_5}}\,\overline{w_{N_6}}$ in L^2_{xt}, $w_{N_1}w_{N_2}$ in L^∞_{xt} and using the support condition on $\overline{w_{N_3}}$. By Hölder's inequality, summing as above, we get the desired estimate with decay $N^{\delta/6-1/6}$ so long as $\delta < 1$.

• Suppose $j = 1, 2, 4$ or 5. By symmetry (relative to conjugates) $j = 1, 2, 4$ are similar; so suppose $j = 1$. We rewrite (4.22) as

$$\sum_* \int_{\mathbb{R}}\int_{\mathbb{T}} N^2\sigma_1^{-\frac{1}{2}+} w_{N_1}\sigma_1^{\frac{1}{2}-} w_{N_2}\overline{w_{N_3}}\, w_{N_4}\overline{w_{N_5}}\,\overline{w_{N_6}}\,dx\,dt$$

$$\lesssim \sum_* N^2 N^{-1} N_1^{-1/2} N_2^{0+} \max(N_1, N_2)^{1/6} N_3^{-2/3} N_4^{-1/2} N_5^{0+} N_6^{-1/2}\Big(\prod_{i=1}^{6}\|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\Big)$$

by placing $\sigma_1^{(1/2)-} w_{N_1}$ in L^2_{xt}, $\overline{w_{N_3}}w_{N_4}\overline{w_{N_6}}$ in L^2_{xt}, $w_{N_2}\overline{w_{N_5}}$ in L^∞_{xt} and using the support condition on $\overline{w_{N_3}}$. Once again, by Hölder's inequality, summing as before we get the desired estimate with decay $N^{\delta/6-1/6}$ so long as $\delta < 1$.

If $j = 5$, then

$$\sum_* \int_{\mathbb{R}}\int_{\mathbb{T}} N^2\sigma_5^{-\frac{1}{2}+} w_{N_1}w_{N_2}\overline{w_{N_3}}w_{N_4}\sigma_5^{\frac{1}{2}-}\overline{w_{N_5}}\,\overline{w_{N_6}}\,dx\,dt$$

$$\lesssim \sum_* N^2 N^{-1} N_1^{0+} N_2^{0+} \max(N_1, N_2)^{1/6} N_3^{-\frac{2}{3}} N_4^{-\frac{1}{2}} N_5^{-\frac{1}{2}} N_6^{-\frac{1}{2}}\Big(\prod_{i=1}^{6}\|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\Big)$$

by placing $\sigma_5^{(1/2)-}\overline{w_{N_5}}$ in L^2_{xt}, $\overline{w_{N_3}}w_{N_4}\overline{w_{N_6}}$ in L^2_{xt}, $w_{N_1}w_{N_2}$ in L^∞_{xt} and using the support condition on $\overline{w_{N_3}}$. Once again, by Hölder's inequality, summing as before we get the desired estimate with decay $N^{\delta/6-1/6}$ so long as $0 < \delta < 1$.

Subcase 2. Assume either N_1 or N_2 is $> N^\delta$. Suppose $N_1 > N^\delta$; otherwise exchange the roles of w_{N_1} and w_{N_2} below. We no longer rely on the support condition but on the lower bound on N_1 as follows.

• Suppose $j = 3$ or 6; say $j = 3$ ($j = 6$ is similar). Then proceeding as before we rewrite (4.22) as

$$\sum_* \int_{\mathbb{R}}\int_{\mathbb{T}} N^2\sigma_3^{-\frac{1}{2}+} w_{N_1}w_{N_2}\sigma_3^{\frac{1}{2}-}\overline{w_{N_3}}w_{N_4}\overline{w_{N_5}}\,\overline{w_{N_6}}\,dx\,dt$$

$$\lesssim \sum_* N^2 N^{-1} N_1^{-1/2} N_2^{0+} N_3^{-1/2} N_4^{0+} N_5^{-1/2} N_6^{-1/2}\Big(\prod_{i=1}^{6}\|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\Big)$$

by placing $\sigma_3^{(1/2)-}\overline{w_{N_3}}$ in L^2_{xt}, $w_{N_1}\overline{w_{N_5}}\,\overline{w_{N_6}}$ in L^2_{xt}, $w_{N_2}w_{N_4}$ in L^∞_{xt}. By Hölder's inequality, summing as above, we get the desired estimate with decay $N^{-\delta/2}$ so long as $\delta > 0$.

• Suppose $j = 1$ or 2; say $j = 1$ ($j = 2$ is similar). We now write

$$\sum_* \int_{\mathbb{R}}\int_{\mathbb{T}} N^2\sigma_1^{-\frac{1}{2}+} w_{N_1}\sigma_1^{\frac{1}{2}-} w_{N_2}\overline{w_{N_3}}w_{N_4}\overline{w_{N_5}}\,\overline{w_{N_6}}\,dx\,dt$$

$$\lesssim \sum_* N^2 N^{-1} N_1^{-1/2} N_2^{-1/2} N_3^{-1/2} N_4^{0+} N_5^{0+} N_6^{-1/2}\Big(\prod_{i=1}^{6}\|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\Big)$$

by placing $\sigma_1^{(1/2)-} w_{N_1}$ in L^2_{xt}, $w_{N_2}\overline{w_{N_3}}\,\overline{w_{N_6}}$ in L^2_{xt}, $w_{N_4}\overline{w_{N_5}}$ in L^∞_{xt}. Once again, by Hölder's inequality and summing as above, we get the desired estimate with decay $N^{-\delta/2}$ so long as $\delta > 0$.

• Suppose $j = 4$. Then proceed as above but place $\sigma_4^{(1/2)-} w_{N_4}$ in L^2_{xt}, $w_{N_1}\overline{w_{N_3} w_{N_6}}$ in L^2_{xt}, and $w_{N_2}\overline{w_{N_5}}$ in L^∞_{xt}.

• Suppose $j = 5$. Then once again we proceed as above but now place $\sigma_5^{(1/2)-}\overline{w_{N_5}}$ in L^2_{xt}, $w_{N_1}\overline{w_{N_3}}\,\overline{w_{N_6}}$ in L^2_{xt}, and $w_{N_2} w_{N_4}$ in L^∞_{xt}.

Remark 4.14. Matching Subcases 1 and 2 above means $-\delta/2 = \delta/6 - 1/6$, which requires $\delta = 1/4$ and yields a decay of $N^{-(1/8)+}$.

Case IIA. Part (i) will follow similarly to Case IA while part (ii) to Case IIIA.

Part (i). We are under the assumptions $N_3 \sim N \sim N_5$ while $N_1, N_2, N_4, N_6 \ll N$. It follows from (4.21) that there exists $\sigma_j \gtrsim N^2$. We proceed exactly as in IA exchanging in each instance the roles of $\overline{w_{N_6}}$ and $\overline{w_{N_5}}$.

Part (ii). We are under the assumptions $N_3 \sim N \sim N_4$ while $N_1, N_2, N_5, N_6 \ll N$. We have a priori no help from a large σ_j at our disposal. We then proceed as in IIIA above with the role of $(N_3; \overline{w_{N_3}})$ switched with that of $(N_6; \overline{w_{N_6}})$, and $(N_1; w_{N_1})$ with $(N_4; w_{N_4})$. Hence for $\sigma, \delta > 0$ to be determined, in Subcase 1 we are under the assumption $N_1, N_2, N_5 < N^{1-\sigma}$, $N_6 \lesssim N^\delta$ and $N_3 \sim N \sim N_4$. In Subcase 2 we assume that $N_1, N_2, N_5 < N^{1-\sigma}$ while $N_6 \gtrsim N^\delta$ and $N_3 \sim N \sim N_4$, and further subdivide just as before into Subcase 2a: $N_1, N_2, N_5 \ll N^\delta$ while $N_6 \gtrsim N^\delta$ which implies from (4.19) the existence of a $\sigma_j \gtrsim N^{1+\delta}$; Subcase 2b: there exists $i \in \{1, 2, 5\}$ such that $N_i \gg N^\delta$ and $N_j \lesssim N^\delta$ for $j \neq i$ and $i, j \in \{1, 2, 5\}$ while still $N_6 \gtrsim N^\delta$ and $N_3 \sim N \sim N_4$ in (4.22), and Subcase 2c: there exist at least $i, j \in \{1, 2, 5\}$ $(i \neq j)$ such that $N_i, N_j \gg N^\delta$ while $N_6 \gtrsim N^\delta$ and $N_3 \sim N \sim N_4$ in (4.22). Note that $N_1, N_2 < N^{1-\sigma}$, which ensures the support condition on $\overline{w_{N_3}}$. Subcase 3: Assume there exists at least one $i \in \{1, 2, 5\}$ such that $N_i \gtrsim N^{1-\sigma}$, $N_2, N_1, N_5 \ll N$ while $N_6 \ll N$ and $N_3 \sim N \sim N_4$ in (4.22). This case follows from Lemma 4.11 with $0 < \sigma < 1/6$ as in its statement.

Proceeding then just as in IIIA we deduce the desired estimate with the same decay in N as in IIIA as long as $2/5 < \delta < 1/2$ and $0 < \sigma < 1/6$ as before.

Case IB. We first note that parts (ii), (iii) and (iv) are all symmetric relative to conjugation; so we only consider (i) and (ii).

Part (i). We are under the assumptions $N_3 \sim N_5 \sim N_6 \sim N$ while $N_1, N_2, N_4 \ll N$. It follows from (4.21) that there exists $\sigma_j \gtrsim N^2$.

• Suppose $j = 1, 2$ or 4. By symmetry it is enough to consider $j = 1$ and $j = 4$. To obtain decay we need to use the support condition. Thus we further subdivide into two cases.

Subcase 1. Assume in addition $N_1, N_2 < N^\delta$ for some $\delta > 0$. We then have the support condition on $\overline{w_{N_3}}$. For $j = 1$ we have

$$\sum\nolimits_* \int_{\mathbb{R}}\int_{\mathbb{T}} N^2 \sigma_1^{-\frac{1}{2}+} w_{N_1} \sigma_1^{\frac{1}{2}-} w_{N_2} \overline{w_{N_3}} w_{N_4} \overline{w_{N_5}}\, \overline{w_{N_6}}\, dx\, dt$$
$$\lesssim \sum\nolimits_* N^2 N^{-1} N_1^{-1/2} N_2^{-1/2} N_3^{-2/3} N^{\delta/6} N_4^{0+} N_5^{0+} N_6^{-1/2} \Big(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}} \Big)$$

by placing $\sigma_1^{(1/2)-} w_{N_1}$ in L^2_{xt}, $w_{N_2}\overline{w_{N_3}}\, \overline{w_{N_6}}$ in L^2_{xt}, $w_{N_4}\overline{w_{N_5}}$ in L^∞_{xt}. By Hölder's inequality, summing as above, we get the desired estimate with decay $N^{-1/6+\delta/6}$ so long as $0 < \delta < 1$.

For $j = 4$, we place $\sigma_4^{1/2} w_{N_4}$ in L^2_{xt}, $w_{N_1}\overline{w_{N_3} w_{N_6}}$ in L^2_{xt} and $w_{N_2}\overline{w_{N_5}}$ in L^∞_{xt} and proceed similarly.

Subcase 2. Assume either N_1 or N_2 is $> N^\delta$. By symmetry suppose $N_1 > N^\delta$; otherwise exchange the roles of w_{N_1} and w_{N_2} below. We use then the lower bound on N_1 as follows. For $j = 1$,

$$\sum\nolimits_* \int_{\mathbb{R}}\int_{\mathbb{T}} N^2 \sigma_1^{-\frac{1}{2}+} w_{N_1} \sigma_1^{\frac{1}{2}-} w_{N_2} \overline{w_{N_3}} w_{N_4} \overline{w_{N_5}}\, \overline{w_{N_6}}\, dx\, dt$$
$$\lesssim \sum\nolimits_* N^2 N^{-1} N_1^{-1/2} N_2^{-1/2} N_3^{-1/2} N_4^{0+} N_5^{0+} N_6^{-1/2} \Big(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}} \Big)$$

by placing $\sigma_1^{(1/2)-} w_{N_1}$ in L^2_{xt}, $w_{N_2}\overline{w_{N_3}}\, \overline{w_{N_6}}$ in L^2_{xt}, $w_{N_4}\overline{w_{N_5}}$ in L^∞_{xt}. Hence, by Hölder's inequality and summing as usual we get the desired estimate with decay $N^{-\delta/2}$ so long as $\delta > 0$.

For $j = 4$, we place $\sigma_4^{(1/2)-} w_{N_4}$ in L^2_{xt}, $w_{N_1}\overline{w_{N_3}}\, \overline{w_{N_6}}$ in L^2_{xt} and $w_{N_2}\overline{w_{N_5}}$ in L^∞_{xt} and proceed similarly.

Remark 4.15. Note that once again, matching Subcases 1 and 2 above means $-\delta/2 = \delta/6 - 1/6$, which requires $\delta = 1/4$ and yields a decay of $N^{-(1/8)+}$.

• Suppose $j = 3$, 6 or 5. By symmetry relative to conjugation it is enough to consider $j = 3$. We have

$$\sum\nolimits_* \int_{\mathbb{R}}\int_{\mathbb{T}} N^2 \sigma_3^{-\frac{1}{2}+} w_{N_1} w_{N_2} \sigma_3^{\frac{1}{2}-} \overline{w_{N_3}} w_{N_4} \overline{w_{N_5}}\, \overline{w_{N_6}}\, dx\, dt$$
$$\lesssim \sum\nolimits_* N^2 N^{-1} N_1^{0+} N_2^{0+} N_3^{-1/2} N_4^{-1/2} N_5^{-1/2} N_6^{-1/2} \Big(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}} \Big)$$

by placing $\sigma_3^{(1/2)-}\overline{w_{N_3}}$ in L^2_{xt}, $w_{N_4}\overline{w_{N_5}}\, \overline{w_{N_6}}$ in L^2_{xt}, $w_{N_1} w_{N_2}$ in L^∞_{xt}. Hence, by Hölder's inequality, summing as usual we get the desired estimate with decay $N^{-(1/2)+}$.

Part (ii). We are under the assumptions $N_3 \sim N_4 \sim N_6 \sim N$ while $N_1, N_2, N_5 \ll N$. It follows from (4.19) that there exists $\sigma_j \gtrsim N^2$.

• Suppose $j = 1, 2$ or 5. If $j = 1$ then

$$\sum_* \int_{\mathbb{R}} \int_{\mathbb{T}} N^2 \sigma_1^{-\frac{1}{2}+} \sigma_1^{\frac{1}{2}-} w_{N_1} w_{N_2} \overline{w_{N_3}} w_{N_4} \overline{w_{N_5}}\, \overline{w_{N_6}}\, dx\, dt$$
$$\lesssim \sum_* N^2 N^{-1} N_1^{-1/2} N_2^{0+} N_3^{-1/2} N_4^{-1/2} N_5^{0+} N_6^{-1/2} \Big(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}} \Big)$$

by placing $\sigma_1^{(1/2)-} w_{N_1}$ in L^2_{xt}, $w_{N_4} \overline{w_{N_3}}\, \overline{w_{N_6}}$ in L^2_{xt}, $w_{N_2} \overline{w_{N_5}}$ in L^∞_{xt}. Hence, by Hölder's inequality and summing as usual we get the desired estimate with decay $N^{-(1/2)+}$.

If $j = 2, 5$ we proceed similarly, keeping $w_{N_4} \overline{w_{N_3}}\, \overline{w_{N_6}}$ in L^2_{xt} and exchanging the roles of either w_{N_2} or $\overline{w_{N_5}}$ with that of w_{N_1} above.

• Suppose $j = 3, 6$ or 4. If $j = 3$ then

$$\sum_* \int_{\mathbb{R}} \int_{\mathbb{T}} N^2 \sigma_3^{-\frac{1}{2}+} w_{N_1} w_{N_2} \sigma_3^{\frac{1}{2}-} \overline{w_{N_3}} w_{N_4} \overline{w_{N_5}}\, \overline{w_{N_6}}\, dx\, dt$$
$$\lesssim \sum_* N^2 N^{-1} N_1^{0+} N_2^{-1/2} N_3^{-1/2} N_4^{-1/2} N_5^{0+} N_6^{-1/2} \Big(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}} \Big)$$

by placing $\sigma_3^{(1/2)-} \overline{w_{N_3}}$ in L^2_{xt}, $w_{N_2} w_{N_4} \overline{w_{N_6}}$ in L^2_{xt}, $w_{N_1} \overline{w_{N_5}}$ in L^∞_{xt}. Hence, by Hölder's and summing as usual we get the desired estimate with decay $N^{-(1/2)+}$.

If $j = 6$ we proceed similarly exchanging the roles of $\overline{w_{N_3}}$ and $\overline{w_{N_6}}$ above.

If $j = 4$ we place $\sigma_4^{(1/2)-} w_{N_4}$ in L^2_{xt} and group $w_{N_2} \overline{w_{N_3}}\, \overline{w_{N_6}}$ in L^2_{xt} to derive the same conclusion.

We now remove the assumption we made at the beginning of the proof. Suppose that there is at least one $\sigma_j > N^7$. It follows from (4.19) and (4.20) that there are two indices $1 \le i_1 \neq i_2 \le 6$ such that $\sigma_{i_1}, \sigma_{i_2} \gtrsim N^7$. Then, by (2.4) and (4.24), we have

$$\begin{aligned}
|I_1| &\lesssim \sum_{N \le |n| \le 3N} \sum_{N_i \le N; i=1,\dots 6} \int_\tau \Big(\int_{\tau=\tau_1+\tau_2+\tau_3} \sum_{n=n_1+n_2+n_3} |\widehat{w_{N_1}}|\, |\widehat{w_{N_2}}|\, |n_3|\, |\widehat{\overline{w_{N_3}}}|\, d\tau_1\, d\tau_2 \Big) \\
&\qquad \times \Big(\int_{-\tau=\tau_4+\tau_5+\tau_6} \sum_{-n=n_4+n_5+n_6} |\widehat{w_{N_4}}|\, |\widehat{\overline{w_{N_5}}}|\, |n_6|\, |\widehat{\overline{w_{N_6}}}|\, d\tau_4\, d\tau_5 \Big)\, d\tau \\
&\lesssim \sum_{N \le |n| \le 3N} \sum_{N_i \le N;\, i=1,\dots 6} N^2 \|w_{N_1} w_{N_2} \overline{w_{N_3}}\|_{L^2_{xt}} \|w_{N_4} \overline{w_{N_5} w_{N_6}}\|_{L^2_{xt}} \\
&\lesssim \sum_{N \le |n| \le 3N} \sum_{N_i \le N;\, i=1,\dots 6} N^{-\frac{1}{3}+} \|w_{N_{i_1}}\|_{X^{\frac{1}{2}-,\frac{1}{2}-}} \|w_{N_{i_2}}\|_{X^{\frac{1}{2}-,\frac{1}{2}-}} \prod_{j \neq i_1, i_2} \|w_j\|_{X^{\frac{1}{2}-,\frac{1}{3}+}} \\
&\lesssim N^{-\frac{1}{3}+} \prod_{j=1}^{6} \|w_j\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}.
\end{aligned} \tag{4.37}$$

To treat the remaining terms in (4.9) we first note that these are either higher order with no derivatives or the same order as the first but with only one derivative term. We again start by assuming that $\sigma_j \lesssim N^9$ for all j. Under this assumption the estimate follows from the following lemma.

Lemma 4.16 (Remaining terms). *There exists $\beta > 0$ such that*

$$\sum_{N\leq|n|\leq 3N}\sum_{N_i\leq N;\, i=1,\dots 6}\int_\tau\Big(\int_{\tau=\tau_1+\tau_2+\tau_3}\sum_{n=n_1+n_2+n_3}|\widehat{w_{N_1}}|\,|\widehat{w_{N_2}}|\,|\widehat{\overline{w_{N_3}}}|\Big)$$
$$\times\Big(\int_{-\tau=\tau_4+\tau_5+\tau_6}\sum_{-n=n_4+n_5+n_6}|\widehat{w_{N_4}}|\,|\widehat{\overline{w_{N_5}}}|\,|m(n_6)|\,|\widehat{\overline{w_{N_6}}}|\Big)\,d\tau \lesssim N^{-\beta}\prod_{i=1}^{6}\|w_i\|_{X_3^{\frac23-,\frac12-}}, \tag{4.38}$$

$$\sum_{N\leq|n|\leq 3N}\sum_{N_i\leq N; i=1,\dots 8}\int_\tau\Big(\int_{\tau=\sum_{i=1}^5\tau_i}\sum_{n=\sum_{i=1}^5 n_i}|\widehat{w_{N_1}}|\,|\widehat{w_{N_2}}|\,|\widehat{\overline{w_{N_3}}}|\,|\widehat{w_{N_4}}|\,|\widehat{\overline{w_{N_5}}}|\Big)$$
$$\times\Big(\int_{-\tau=\tau_6+\tau_7+\tau_8}\sum_{-n=n_6+n_7+n_8}|\widehat{w_{N_6}}|\,|\widehat{\overline{w_{N_7}}}|\,|m(n_8)|\,|\widehat{\overline{w_{N_8}}}|\Big)\,d\tau \lesssim N^{-\beta}\prod_{i=1}^{8}\|w_i\|_{X_3^{\frac23-,\frac12-}}, \tag{4.39}$$

where the multiplier m satisfies $|m(\xi)| \leq \langle\xi\rangle$.

Proof. Here we will only prove (4.39) since (4.38) is similar but simpler. Without loss of generality we can assume that $N_1 \sim N \sim N_8$. Fix any $0 < \sigma < 1$ and consider the following cases.

Case 1. Assume that $N_i \lesssim N^\sigma$, $i \neq 1, 8$. Then we have the support condition on w_{N_1} and $\overline{w_{N_8}}$. By Plancherel, (4.39) is less than or equal to

$$\sum_{N_1,N_8\sim N;\, N_i\leq N^\sigma,\, i\neq 1,8}\int_{\mathbb{R}}\int_{\mathbb{T}} N_8 w_{N_1} w_{N_2}\overline{w_{N_3}}w_{N_4}\overline{w_{N_5}}w_{N_6}\overline{w_{N_7}w_{N_8}}\,dx\,dt$$
$$\lesssim \sum_{N_1,N_8\sim N;\, N_i\leq N^\sigma,\, i\neq 1,8} N\|w_{N_1}w_{N_2}\overline{w_{N_3}}\|_{L^2_{xt}}\|w_{N_4}\overline{w_{N_5}}\|_{L^\infty_{xt}}\|w_{N_6}\overline{w_{N_7}w_{N_8}}\|_{L^2_{xt}}$$
$$\lesssim \sum_{N_1,N_8\sim N;\, N_i\leq N^\sigma,\, i\neq 1,8} NN_1^{-2/3}\max(N_2,N_3,N_4,N_5)^{1/6}N_4^{0+}N_5^{0+}N_6^{-1/2}N_7^{-1/2}N_8^{-2/3}$$
$$\times\max(N_6,N_7)^{1/6}\Big(\prod_{i=1}^{8}\|w_{N_i}\|_{X_3^{\frac23-,\frac12-}}\Big) \lesssim N^{-\frac13+\frac{\sigma}{6}+}\Big(\prod_{i=1}^{8}\|w_i\|_{X_3^{\frac23-,\frac12-}}\Big). \tag{4.40}$$

Case 2. Assume there exists $k \neq 1, 8$ such that $N_k > N^\sigma$. Without loss of generality $k = 4$. Then we bound (4.40) as follows:

$$\sum_{N_1,N_8\sim N;\ N_4>N^\sigma, N_i\le N;\ i\neq 1,4,8} N\|w_{N_1}w_{N_4}\overline{w_{N_3}}\|_{L^2_{xt}}\|w_{N_2}\overline{w_{N_5}}\|_{L^\infty_{xt}}\|w_{N_6}\overline{w_{N_7}w_{N_8}}\|_{L^2_{xt}}$$
$$\lesssim \sum_{N_1,N_8\sim N;\ N_4>N^\sigma,\ N_i\le N;\ i\neq 1,4,8} NN_1^{-1/2}N_3^{-1/2}N_4^{-1/2}N_2^{0+}N_5^{0+}N_6^{-1/2}N_7^{-1/2}N_8^{-1/2}$$
$$\times\Big(\prod_{i=1}^8\|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\Big)\lesssim N^{-\frac{\sigma}{2}+}\Big(\prod_{i=1}^8\|w_i\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\Big). \qquad \square$$

We now remove the assumption we made before the lemma above. Suppose that there is at least one $\sigma_j > N^9$. The term with six factors is handled just as in (4.37). To estimate the term with eight factors we first observe that as before there are at least two indices $1\le i_1\neq i_2\le 8$ such that $\sigma_i, \sigma_j \gtrsim N^9$. Next we use Hölder's inequality to bound the left hand side of (4.39) by

$$\sum_{N\le|n|\le 3N}\ \sum_{N_i\le N;\ i=1,\dots 8} N\prod_{i=1}^8\|w_{N_i}\|_{L^8_{tx}} \lesssim \sum_{N\le|n|\le 3N}\ \sum_{N_i\le N;\ i=1,\dots 8} N\prod_{i=1}^8\|w_{N_i}\|_{X_3^{\frac{13}{24}+,\frac{3}{8}+}}$$

by (4.25). Using $\sigma_{i_1}, \sigma_{i_2} > N^9$ we conclude that the above is

$$\lesssim \sum_{N\le|n|\le 3N}\ \sum_{N_i\le N;\ i=1,\dots 8} N^{-\frac{1}{8}+}\|w_{N_{i_1}}\|_{X_3^{\frac{13}{24}+,\frac{1}{2}-}}\|w_{N_{i_2}}\|_{X_3^{\frac{13}{24}+,\frac{1}{2}-}}\prod_{i\neq i_1,i_2}\|w_{N_i}\|_{X_3^{\frac{13}{24}+,\frac{3}{8}+}}$$
$$\lesssim N^{-\frac{1}{8}+}\prod_{i=1}^8\|w_i\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}.$$

5. Construction of weighted Wiener measures

In this section we construct weighted Wiener measures and associated probability spaces on which we establish well-posedness. To construct these measures we make use of the conserved quantities $\mathcal{E}(v)$ (given in (2.13)) and the L^2-norm. As a motivation we recall a well known fact in finite-dimensional spaces. Suppose we have a well-posed ODE $y_t = F(y)$, where F is a divergence-free vector field. Assume $G(y)$ is a constant of motion such that for reasonable f, $f(G(y)) \in L^1(dy)$. Then by Liouville's Theorem, $d\mu(y) = Z^{-1}f(G(y))dy$ is, for a suitable normalization constant Z, an invariant probability measure for the flow map for the ODE.

To construct measures on infinite-dimensional spaces we will consider conserved quantities of the form $\exp(-\frac{\beta}{2}\mathcal{E}(v))$. But there is a priori little hope of constructing a finite measure using this quantity since (a) the nonlinear part of $\mathcal{E}(v)$ is not bounded below and (b) the linear part is only nonnegative but not positive definite. To resolve this we use the conservation of L^2-norm and consider instead the conserved quantity

$$\chi_{\{\|v\|_{L^2}\le B\}}e^{-\frac{\beta}{2}\mathcal{N}(v)}e^{-\frac{\beta}{2}\int(|v|^2+|v_x|^2)\,dx}$$

where $\mathcal{N}(v)$ is the nonlinear part of the energy, i.e.

$$\mathcal{N}(v) = -\frac{1}{2}\operatorname{Im}\int_{\mathbb{T}} v^2\overline{v}\overline{v}_x\,dx - \frac{1}{4\pi}\Big(\int_{\mathbb{T}}|v|^2\,dx\Big)\Big(\int_{\mathbb{T}}|v|^4\,dx\Big) + \frac{1}{\pi}\Big(\int_{\mathbb{T}}|v|^2\,dx\Big)\Big(\operatorname{Im}\int_{\mathbb{T}} v\overline{v}_x\,dx\Big) + \frac{1}{4\pi^2}\Big(\int_{\mathbb{T}}|v|^2\,dx\Big)^3, \tag{5.1}$$

and B is a (suitably small) constant.

By analogy with the finite-dimensional case we would like to construct the measure (with $v(x) = u(x) + iw(x)$)

$$\text{“ } d\mu_\beta = Z^{-1}\chi_{\{\|v\|_{L^2}\le B\}}e^{-\frac{\beta}{2}\mathcal{N}(v)}e^{-\frac{\beta}{2}\int(|v|^2+|v_x|^2)\,dx}\prod_{x\in\mathbb{T}}du(x)dw(x)\text{ ”.}$$

This is a purely formal, although suggestive, expression since it is impossible to define the Lebesgue measure on an infinite-dimensional space as a countably additive measure. Moreover, it will turn out that $\int|u_x|^2 = \infty$, μ-almost surely.

One uses instead a Gaussian measure as reference measure and the measure μ is constructed in two steps. First one constructs a Gaussian measure ρ as the limit of the finite-dimensional measures on $\mathbb{R}^{4N+2}$ given by

$$d\rho_N = Z_{0,N}^{-1}\exp\Big(-\frac{\beta}{2}\sum_{|n|\le N}(1+|n|^2)|\widehat{v}_n|^2\Big)\prod_{|n|\le N}da_n db_n \tag{5.2}$$

where $\widehat{v}_n = a_n + ib_n$. The construction of such Gaussian measures is a classical subject (see e.g. Gross [20] and Kuo [29]). For our purpose we will need to realize this measure as a measure supported on a suitable Banach space. Once this measure ρ has been constructed one constructs the measure μ as a measure which is absolutely continuous with respect to ρ and whose Radon–Nikodym derivative is

$$\frac{d\mu}{d\rho} = \tilde{Z}^{-1}\chi_{\{\|v\|_L^2\le B\}}e^{-\frac{\beta}{2}\mathcal{N}(v)}.$$

For this measure to be normalizable it turns out that one needs B to be sufficiently small. Also the constant β in the measure does not play any role in the analysis (although the cutoff B depends on β) and thus in the sequel we will set $\beta = 1$. But note that the measures for different β are all invariant and they are all mutually singular [20, 29].

First let us recall some facts on Gaussian measures in Hilbert spaces and Banach spaces. For details see Zhidkov [48], Gross [20] and Kuo [29]. Let $n \in \mathbb{N}$ and $\mathcal{T}$ be a symmetric positive $n\times n$ matrix with real entries. The Borel measure ρ in $\mathbb{R}^n$ given by

$$d\rho(x) = \frac{1}{\sqrt{(2\pi)^n\det(\mathcal{T})}}\exp(-\tfrac{1}{2}\langle\mathcal{T}^{-1}x, x\rangle_{\mathbb{R}^n})\,dx$$

is called a (nondegenerate centered) *Gaussian measure* in $\mathbb{R}^n$. Note that $\rho(\mathbb{R}^n) = 1$.

Now, we consider the analogous definition of the infinite-dimensional (centered) Gaussian measures. Let H be a real separable Hilbert space and $\mathcal{T} : H \to H$ be a linear positive self-adjoint operator (generally unbounded) with eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ and the corresponding eigenvectors $\{e_n\}_{n\in\mathbb{N}}$ forming an orthonormal basis of H. We call a set $M \subset H$ *cylindrical* if there exists an integer $n \geq 1$ and a Borel set $F \subset \mathbb{R}^n$ such that

$$M = \{x \in H : (\langle x, e_1\rangle_H, \dots, \langle x, e_n\rangle_H) \in F\}. \tag{5.3}$$

Given the operator $\mathcal{T}$, we denote by $\mathcal{A}$ the set of all cylindrical subsets of H; one can easily verify that $\mathcal{A}$ is a field. The *centered Gaussian measure* in H with correlation operator $\mathcal{T}$ is defined as the additive (but not countably additive in general) measure ρ defined on the field $\mathcal{A}$ via

$$\rho(M) = (2\pi)^{-n/2}\prod_{j=1}^{n}\lambda_j^{-1/2}\int_F e^{-\frac{1}{2}\sum_{j=1}^{n}\lambda_j^{-1}x_j^2}dx_1\cdots dx_n \quad \text{for } M \in \mathcal{A} \text{ as in (5.3).} \tag{5.4}$$

The following proposition tells us when this Gaussian measure ρ is countably additive.

Proposition 5.1. *The Gaussian measure ρ defined in (5.4) is countably additive on the field $\mathcal{A}$ if and only if $\mathcal{T}$ is an operator of trace class, i.e., $\sum_{n=1}^{\infty}\lambda_n < \infty$. If the latter holds, then the minimal σ-field $\mathcal{M}$ containing the field $\mathcal{A}$ of all cylindrical sets is the Borel σ-field on H.*

Consider a sequence of finite-dimensional Gaussian measures $\{\rho_n\}_{n\in\mathbb{N}}$ defined as follows. For fixed $n \in \mathbb{N}$, let $\mathcal{M}_n$ be the set of all cylindrical sets in H of the form (5.3) with this fixed n and arbitrary Borel sets $F \subset \mathbb{R}^n$. Clearly, $\mathcal{M}_n$ is a σ-field, and setting

$$\rho_n(M) = (2\pi)^{-n/2}\prod_{j=1}^{n}\lambda_j^{-1/2}\int_F e^{-\frac{1}{2}\sum_{j=1}^{n}\lambda_j^{-1}x_j^2}dx_1\cdots dx_n$$

for $M \in \mathcal{M}_n$, we obtain a countably additive measure ρ_n defined on $\mathcal{M}_n$. Then, one can extend the measure ρ_n onto the whole Borel σ-field $\mathcal{M}$ of H by setting $\rho_n(A) := \rho_n(A \cap \operatorname{span}\{e_1, \dots, e_n\})$ for $A \in \mathcal{M}$.[6] Then we have

Proposition 5.2. *Let ρ in (5.4) be countably additive. Then $\{\rho_n\}_{n\in\mathbb{N}}$ constructed above converges weakly to ρ as $n \to \infty$.*

For our problem we consider the Gaussian measure ρ which is the weak limit of the finite-dimensional Gaussian measures

$$d\rho_N = Z_{0,N}^{-1}\exp\Big(-\frac{1}{2}\sum_{|n|\leq N}(1+|n|^2)|\widehat{v}_n|^2\Big)\prod_{|n|\leq N}da_n db_n. \tag{5.5}$$

Let $J_s := (1-\Delta)^{s-1}$. Then we have

$$\sum_n (1+|n|^2)|\widehat{v}_n|^2 = \langle v, v\rangle_{H^1} = \langle J_s^{-1}v, v\rangle_{H_s}.$$

[6] Note a slight abuse of notation. We use ρ_n to denote a Gaussian measure on $\operatorname{span}\{e_1, \dots, e_n\}$ as well as its extension on H. A similar comment applies in the following.

The operator $J_s : H_s \to H_s$ has the set of eigenvalues $\{(1+|n|^2)^{s-1}\}_{n\in\mathbb{Z}}$ and the corresponding eigenvectors $\{(1+|n|^2)^{-s/2}e^{inx}\}_{n\in\mathbb{Z}}$ form an orthonormal basis of H^s. Since J_s is of trace class if and only if $s < 1/2$, by Proposition 5.1, ρ is a countably additive measure on H^s for any $s < 1/2$ (but not for $s \geq 1/2$).

Unfortunately, (2.6) is locally well-posed in $H^s(\mathbb{T})$ only for $s \geq 1/2$ [26]. Instead, we propose to work in the Fourier–Lebesgue space $\mathcal{F}L^{s,r}(\mathbb{T})$ defined in (2.2) in view of the local well-posedness result by Grünrock–Herr [22]. Since $\mathcal{F}L^{s,r}$ is not a Hilbert space, we need to construct ρ as a measure supported on a Banach space.

5.1. *General Banach space setting*

Let us recall the basic theory of abstract Wiener spaces [29]. Given a real separable Hilbert space H with norm $\|\cdot\|$, let $\mathcal{F}$ denote the set of finite-dimensional orthogonal projections $\mathbb{P}$ of H. Then define a cylinder set E by $E = \{x \in H : \mathbb{P}x \in F\}$ where $\mathbb{P} \in \mathcal{F}$ and F is a Borel subset of $\mathbb{P}H$, and let $\mathcal{R}$ denote the collection of such cylinder sets. Note that $\mathcal{R}$ is a field but not a σ-field. The Gaussian measure ρ on H is defined by

$$\rho(E) = (2\pi)^{-n/2}\int_F e^{-\|x\|^2/2}\,dx$$

for $E \in \mathcal{R}$, where $n = \dim \mathbb{P}H$ and dx is the Lebesgue measure on $\mathbb{P}H$. It is known that ρ is finitely additive but not countably additive in $\mathcal{R}$.

Definition 5.3 (Gross [20]). A seminorm $|||\cdot|||$ in H is called *measurable* if for every $\varepsilon > 0$, there exists $\mathbb{P}_\varepsilon \in \mathcal{F}$ such that

$$\rho(|||\mathbb{P}x||| > \varepsilon) < \varepsilon$$

for $\mathbb{P} \in \mathcal{F}$ orthogonal to $\mathbb{P}_\varepsilon$.

Any measurable seminorm is weaker than the norm of H, and H is not complete with respect to $|||\cdot|||$ unless H is finite-dimensional. Let $\mathcal{B}$ be the completion of H with respect to $|||\cdot|||$ and denote by i the inclusion map of H into $\mathcal{B}$. The triple $(i, H, \mathcal{B})$ is called an *abstract Wiener space*.

Now, regarding $y \in \mathcal{B}^*$ as an element of $H^* \equiv H$ by restriction, we embed $\mathcal{B}^*$ in H. Define the extension of ρ onto $\mathcal{B}$ (still denoted by ρ) as follows. For a Borel set $F \subset \mathbb{R}^n$, set

$$\rho(\{x \in \mathcal{B} : ((x, y_1), \dots, (x, y_n)) \in F\}) := \rho(\{x \in H : (\langle x, y_1\rangle_H, \dots, \langle x, y_n\rangle_H) \in F\}),$$

where y_j's are in $\mathcal{B}^*$ and $(\cdot,\cdot)$ denotes the natural pairing between $\mathcal{B}$ and $\mathcal{B}^*$. Let $\mathcal{R}_\mathcal{B}$ denote the collection of cylinder sets $\{x \in \mathcal{B} : ((x, y_1), \dots, (x, y_n)) \in F\}$ in $\mathcal{B}$.

Proposition 5.4 (Gross [20]). *ρ is countably additive on the σ-field generated by $\mathcal{R}_\mathcal{B}$.*

5.2. *Back to our setting*

In the present context, we will let $H = H^1(\mathbb{T})$ and $\mathcal{B} = \mathcal{F}L^{s,r}(\mathbb{T})$ with $2 \leq r < \infty$ and $(s-1)r < -1$. First we prove the following result.

Proposition 5.5. *Let $2 \le r < \infty$ and assume $(s-1)r < -1$. Then the seminorm $\|\cdot\|_{\mathcal{F}L^{s,r}}$ is measurable. Moreover, we have the following exponential tail estimate: there exist $C > 0$ and $c > 0$ (which both depend on (s, r)) such that, for $K > 0$,*

$$\rho(\|v\|_{\mathcal{F}L^{s,r}} > K) \le Ce^{-cK^2}. \tag{5.6}$$

This shows that $(i, H, \mathcal{B}) = (i, H^1, \mathcal{F}L^{s,r})$ $(2 \le r < \infty)$ is an abstract Wiener space if $(s-1)r < -1$ and thus the Wiener measure ρ can be realized as a countably additive measure supported on $\mathcal{F}L^{s,r}$ for $(s-1)r < -1$. This is hardly surprising since this is equivalent to $\sigma \equiv s + 1/r - 1/2 < 1/2$, and $\mathcal{F}L^{s,r}$ scale as H^σ.

The second part of Proposition 5.5 is a consequence of Fernique's theorem [19] (cf. Theorem 3.1 of Chapter III in [29]).

Remark 5.6. Proposition 5.5 was essentially proved in [35] in the context of white noise for the KdV equation. We include here a proof in our DNLS context for completeness.[7]

It is useful to note that the measure ρ_N given in (5.5) can be regarded as the induced probability measure on $\mathbb{C}^{2N+1} \cong \mathbb{R}^{4N+2}$ under the map

$$\omega \mapsto \left\{g_n/\sqrt{1+|n|^2}\right\}_{|n|\le N}, \tag{5.7}$$

where $g_n(\omega)$, $|n| \le N$, are independent standard complex Gaussian random variables on a probability space $(\Omega, \mathcal{F}, P)$ (i.e. $\widehat{v}_n = g_n/\sqrt{1+|n|^2}$). In a similar manner, we can view ρ as the induced probability measure under the map $\omega \mapsto \{g_n/\sqrt{1+|n|^2}\}_{n\in\mathbb{Z}}$, where $g_n(\omega)$ are independent standard complex Gaussian random variables.

For the proof of Proposition 5.5, we first recall the following result.

Lemma 5.7 ([36, Lemma 4.7]). *Let $\{g_n\}$ be a sequence of independent standard complex-valued Gaussian random variables. Then, for M dyadic and $\delta < 1/2$, we have*

$$\lim_{M\to\infty} M^{2\delta} \frac{\max_{|n|\sim M} |g_n|^2}{\sum_{|n|\sim M} |g_n|^2} = 0 \quad a.s.$$

Proof of Proposition 5.5. Let $2 \le r < \infty$ and $(s-1)r < -1$. In view of Definition 5.3, it suffices to show that for given $\varepsilon > 0$, there exists a large M_0 such that

$$\rho(\|P_{M_0}^\perp v\|_{\mathcal{F}L^{s,r}} > \varepsilon) < \varepsilon,$$

where $P_{M_0}^\perp$ is the projection onto the frequencies $|n| > M_0$. Note that if $\mathbb{P}$ is a finite-dimensional projection such that $\mathbb{P} \perp P_{M_0}$ then $\|\mathbb{P}v\|_{\mathcal{F}L^{s,r}} \le \|P_{M_0}^\perp v\|_{\mathcal{F}L^{s,r}}$.

In view of (5.7), we assume that v is of the form

$$v(x) = \sum_n \frac{g_n}{\sqrt{1+|n|^2}} e^{inx}, \tag{5.8}$$

where $\{g_n\}$ is as in (5.7).

[7] Proposition 5.5 also holds for $r < 2$ and $(s-1)r < -1$, albeit with a different proof (see [1] for details). For our purposes $2 \le r < \infty$ suffices and so we restrict ourselves to that case.

Let $\delta < 1/2$ to be chosen later. Then, by Lemma 5.7 and Egoroff's theorem, there exists a set E such that $\rho(E^c) < \frac{1}{2}\varepsilon$ and the convergence in Lemma 5.7 is uniform on E, i.e. we can choose dyadic M_0 large enough such that

$$\frac{\|\{g_n(\omega)\}_{|n|\sim M}\|_{L_n^\infty}}{\|\{g_n(\omega)\}_{|n|\sim M}\|_{L_n^2}} \leq M^{-\delta} \tag{5.9}$$

for all $\omega \in E$ and dyadic $M > M_0$. In the following, we will work only on E and drop '$\cap E$' for notational simplicity. However, it should be understood that all the events are under the intersection with E so that (5.9) holds.

Let $\{\sigma_j\}_{j\geq 1}$ be a sequence of positive numbers such that $\sum \sigma_j = 1$, and let $M_j = M_0 2^j$ dyadic. Note that $\sigma_j = C2^{-\lambda j} = CM_0^\lambda M_j^{-\lambda}$ for some small $\lambda > 0$ (to be determined later). Then, from (5.8), we have

$$\rho(\|P_{M_0}^\perp v(\omega)\|_{\mathcal{F}L^{s,r}} > \varepsilon) \leq \sum_{j=1}^\infty \rho\big(\|\{\langle n\rangle^{s-1} g_n(\omega)\}_{|n|\sim M_j}\|_{L_n^r} > \sigma_j\varepsilon\big). \tag{5.10}$$

By interpolation and (5.9),

$$\begin{aligned}
&\|\{\langle n\rangle^{s-1} g_n\}_{|n|\sim M_j}\|_{L_n^r} \\
&\quad\sim M_j^{s-1}\|\{g_n\}_{|n|\sim M_j}\|_{L_n^r} \leq M_j^{s-1}\|\{g_n\}_{|n|\sim M_j}\|_{L_n^2}^{2/r}\|\{g_n\}_{|n|\sim M_j}\|_{L_n^\infty}^{(r-2)/r} \\
&\quad\leq M_j^{s-1}\|\{g_n\}_{|n|\sim M}\|_{L_n^2}\bigg(\frac{\|\{g_n\}_{|n|\sim M_j}\|_{L_n^\infty}}{\|\{g_n\}_{|n|\sim M_j}\|_{L_n^2}}\bigg)^{(r-2)/r} \leq M_j^{s-1-\delta(r-2)/r}\|\{g_n\}_{|n|\sim M_j}\|_{L_n^2}.
\end{aligned}$$

Thus, if $\|\{\langle n\rangle^{s-1} g_n\}_{|n|\sim M_j}\|_{L_n^r} > \sigma_j\varepsilon$, then $\|\{g_n\}_{|n|\sim M_j}\|_{L_n^2} \gtrsim R_j$ where $R_j := \sigma_j\varepsilon M_j^\alpha$ with $\alpha := -s+1+\delta(r-2)/r$. With $r = 2+\theta$, we have

$$\alpha = \frac{-(s-1)r+\delta\theta}{2+\theta} > \frac{1}{2}$$

by taking δ sufficiently close to $1/2$ since $-(s-1)r > 1$. Then, by taking $\lambda > 0$ sufficiently small, $R_j = \sigma_j\varepsilon M_j^\alpha = C\varepsilon M_0^\lambda M_j^{\alpha-\lambda} \gtrsim C\varepsilon M_0^\lambda M_j^{(1/2)+}$. By a direct computation in polar coordinates, we have

$$\rho(\|\{g_n\}_{|n|\sim M_j}\|_{L_n^2} \gtrsim R_j) \sim \int_{B^c(0,R_j)} e^{-\frac{1}{2}|g_n|^2}\prod_{|n|\sim M_j} dg_n \lesssim \int_{R_j}^\infty e^{-\frac{1}{2}s^2} s^{2\#\{|n|\sim M_j\}-1}\,ds.$$

Note that, in the inequality, we have dropped the implicit constant $\sigma(S^{2\#\{|n|\sim M_j\}-1})$, a surface measure of the $2\#\{|n| \sim M_j\}-1$-dimensional unit sphere, since $\sigma(S^n) = 2\pi^{n/2}/\Gamma(n/2) \lesssim 1$. By the change of variable $t = M_j^{-1/2}s$, we have $s^{2\#\{|n|\sim M_j\}-2} \lesssim s^{4M_j} \sim M_j^{2M_j}t^{4M_j}$. Since $t > M_j^{-1/2}R_j = C\varepsilon M_0^\lambda M_j^{0+}$, we have $M_j^{2M_j} = e^{2M_j\ln M_j} <$

$e^{\frac{1}{8}M_j t^2}$ and $t^{4M_j} < e^{\frac{1}{8}M_j t^2}$ for M_0 sufficiently large. Thus, $s^{2\#\{|n|\sim M_j\}-2} < e^{\frac{1}{4}M_j t^2} = e^{\frac{1}{4}s^2}$ for $s > R_j$. Hence,

$$\rho(\|\{g_n\}_{|n|\sim M_j}\|_{L_n^2} \gtrsim R_j) \leq C\int_{R_j}^{\infty} e^{-\frac{1}{4}s^2} s\,ds \leq e^{-cR_j^2} = e^{-cC^2 M_0^{2\lambda} M_j^{1+\varepsilon^2}}. \tag{5.11}$$

From (5.10) and (5.11), we have

$$\rho(\|P_{M_0}^{\perp} v\|_{\mathcal{F}L^{s,r}} > \varepsilon) \leq \sum_{j=1}^{\infty} e^{-cC^2 M_0^{1+2\lambda+}(2^j)^{1+\varepsilon^2}} \leq \tfrac{1}{2}\varepsilon$$

by choosing M_0 sufficiently large as long as $(s-1)r < -1$. Hence, the seminorm $\|\cdot\|_{\mathcal{F}L^{s,r}}$ is measurable for $(s-1)r < -1$.

The tail estimate (5.6) is a direct consequence of Fernique's theorem [29, Theorem 3.1]. □

To construct the weighted Wiener measure μ let us define

$$R(v) := \chi_{\{\|v\|_{L^2}\leq B\}} e^{-\frac{1}{2}\mathcal{N}(v)}, \qquad R_N(v) := R(v^N), \tag{5.12}$$

where $\mathcal{N}(v)$ is the nonlinear part of the energy defined in (5.1) and at this stage and for the remainder of this section $v^N = P_N(v)$ for some generic function v. In the next section v^N will denote the solution to the FGDNLS (3.1) as in Section 3. We write

$$\mathcal{N}_N(v) := \mathcal{N}(v^N) = F_N(v) + G_N(v) + K_N(v),$$

where

$$\begin{aligned}
F_N(v) &= -\frac{1}{2}\operatorname{Im}\int_{\mathbb{T}} (v^N)^2\overline{v^N}\,\overline{v_x^N}\,dx,\\
G_N(v) &= -\frac{1}{4\pi}\Big(\int_{\mathbb{T}}|v^N|^2\,dx\Big)\Big(\int_{\mathbb{T}}|v^N|^4\,dx\Big),\\
K_N(v) &= \frac{1}{\pi}\Big(\int_{\mathbb{T}}|v^N|^2\,dx\Big)\Big(\operatorname{Im}\int_{\mathbb{T}} v^N\overline{v_x^N}\,dx\Big) + \frac{1}{4\pi^2}\Big(\int_{\mathbb{T}}|v^N|^2\,dx\Big)^3.
\end{aligned}$$

We will construct the measure

$$d\mu = Z^{-1}R(v)d\rho,$$

for sufficiently small B, as the weak limit of the finite-dimensional weighted Wiener measures μ_N on $\mathbb{R}^{4N+2}$ given by

$$d\mu_N = Z_N^{-1}R_N(v)d\rho_N = Z_N^{-1}\chi_{\{\|v^N\|_{L^2}\leq B\}} e^{-\frac{1}{2}\mathcal{N}(v^N)} d\rho_N \tag{5.13}$$

for a suitable normalization Z_N.

Lemma 5.8. (a) *The sequence F_N converges in $L^2(d\rho)$ to*

$$F(v) = -\frac{1}{2}\,\mathrm{Im}\int_{\mathbb{T}} v^2\overline{v}\,\overline{v_x}\,dx.$$

Moreover, for $\alpha < 3/4$, there are $C, \delta > 0$ such that for all $M \geq N \geq 1$ and $\lambda > 0$,

$$\rho(|F_M(v) - F_N(v)| > \lambda) \leq Ce^{-\delta N^{\alpha}\lambda^{1/2}}$$

(b) *Let $p \in [2, \infty)$. Then there exist α, C such that for all $M \geq N \geq 1$ and $\lambda > 0$,*

$$\rho(\|P_N v\|_{L^p(\mathbb{T})} > \lambda) < Ce^{-c\lambda^2}, \tag{5.14}$$

$$\rho(\|P_M v - P_N v\|_{L^p(\mathbb{T})} > \lambda) < Ce^{-cN^{2\alpha}\lambda^2}. \tag{5.15}$$

Proof. Part (a) was proved by Thomann and Tzvetkov in [42, Proposition 3.1] using Proposition 5.10 below. Note that their proof only uses the fact that v is in the support of the measure and is independent of the function space v is in.

To prove (b) we first note that for any $2 \leq p < \infty$ and $N \leq M$,

$$\|P_N v\|_{L^p(\mathbb{T})} \leq C\|P_N v\|_{\mathcal{F}L^{\frac{2}{3}-,3}(\mathbb{T})}, \tag{5.16}$$

$$\|P_N v - P_M v\|_{L^p(\mathbb{T})} \leq C\frac{1}{N^{\alpha}}\|P_M v\|_{\mathcal{F}L^{\frac{2}{3}-,3}(\mathbb{T})}, \tag{5.17}$$

where $\alpha = (1/p)-$. Then use (5.16) and (5.17) in conjunction with (5.6) to conclude the proof. □

Lemma 5.9. *$K_N(v)$ is Cauchy in measure, i.e. for every $\gamma > 0$ and $N \leq M$,*

$$\lim_{N,M\to\infty} \rho(|K_M(v) - K_N(v)| > 2\gamma) = 0,$$

and hence K_N converges in measure to

$$K(v) = \frac{1}{\pi}\left(\int_{\mathbb{T}} |v|^2\,dx\right)\left(\mathrm{Im}\int_{\mathbb{T}} v\overline{v_x}\,dx\right) + \frac{1}{4\pi^2}\left(\int_{\mathbb{T}} |v|^2\,dx\right)^3.$$

Before the proof we need the following Proposition 5.10 (see Thomann and Tzevtkov [42] for a proof) and Lemma 5.11 which we prove below.

Proposition 5.10. *Let $d \geq 1$ and $c(n_1, \ldots, n_k) \in \mathbb{C}$. Let $\{g_n\}_{1\leq n\leq d} \in \mathcal{N}_{\mathbb{C}}(0,1)$ be complex L^2 normalized independent Gaussians. For $k \geq 1$ set $A(k,d) := \{(n_1, \ldots, n_k) \in \{1, \ldots, d\}^k : n_1 \leq \cdots \leq n_k\}$ and*

$$S_k(\omega) = \sum_{A(k,d)} c(n_1, \ldots, n_k) g_{n_1}(\omega) \ldots g_{n_k}(\omega). \tag{5.18}$$

Then for all $d \geq 1$ and $p \geq 2$,

$$\|S_k\|_{L^p(\Omega)} \leq \sqrt{k+1}\,(p-1)^{k/2}\|S_k\|_{L^2(\Omega)}.$$

Let $X_N(v) = \int_{\mathbb{T}} v^N \overline{v_x^N}$.

Lemma 5.11. *For any $N \le M$ and $\varepsilon > 0$ we have*

$$|X_N(v)| \lesssim N^{2\varepsilon} \|v^N\|^2_{\mathcal{F}L^{2/3-\varepsilon,3}}, \tag{5.19}$$

$$\|X_M(v) - X_N(v)\|_{L^4} \lesssim \frac{1}{N^{1/2}}, \tag{5.20}$$

$$\|X_M(v) - X_N(v)\|_{L^q} \lesssim c(q-1)\frac{1}{N^{1/2}} \quad \text{for any } q \ge 2. \tag{5.21}$$

Proof. To prove (5.19) we use Plancherel and Hölder's inequality to obtain

$$\begin{aligned} |X_N(v)| &\le \sum_{|n|\le N} |n|\, |\widehat{v^N}(n)|^2 \\ &\le \Big(\sum_{|n|\le N} |n|^{-1+6\varepsilon}\Big)^{1/3} \Big(\sum_{|n|\le N} (|n|^{2/3-\varepsilon}|\widehat{v^N}(n)|)^3\Big)^{2/3} \le N^{2\varepsilon}\|v^N\|^2_{\mathcal{F}L^{2/3-\varepsilon,3}}. \end{aligned}$$

To prove (5.20) we start by recalling that $v^N(\omega, x) := \sum_{|n|\le N} \frac{g_n(\omega)}{\langle n\rangle} e^{inx}$. Then by Plancherel,

$$X_N(v) = -i \sum_{|n|\le N} n \frac{|g_n(\omega)|^2}{\langle n\rangle^2} \quad \text{and} \quad X_M(v) - X_N(v) = -i \sum_{N\le|n|<M} n \frac{|g_n(\omega)|^2}{\langle n\rangle^2},$$

and

$$|X_M(v) - X_N(v)|^2 = \sum_{N\le|n_1|,|n_2|<M} n_1 n_2 \frac{|g_{n_1}(\omega)|^2 |g_{n_2}(\omega)|^2}{\langle n_1\rangle^2\langle n_2\rangle^2} =: Y^1_{N,M} + Y^2_{N,M} + Y^3_{N,M}, \tag{5.22}$$

where

$$\begin{aligned} Y^1_{N,M} &:= \sum_{N\le|n_1|,|n_2|<M} n_1 n_2 \frac{(|g_{n_1}(\omega)|^2 - 1)(|g_{n_2}(\omega)|^2 - 1)}{\langle n_1\rangle^2\langle n_2\rangle^2}, \\ Y^2_{N,M} &:= \sum_{N\le|n_1|,|n_2|<M} n_1 n_2 \frac{(|g_{n_1}(\omega)|^2 - 1) + (|g_{n_2}(\omega)|^2 - 1)}{\langle n_1\rangle^2\langle n_2\rangle^2}, \\ Y^3_{N,M} &:= \sum_{N\le|n_1|,|n_2|<M} \frac{n_1 n_2}{\langle n_1\rangle^2\langle n_2\rangle^2}. \end{aligned}$$

By symmetry $Y^3_{N,M} = 0$. We now observe that

$$\|X_M(v) - X_N(v)\|^4_{L^4} \lesssim \|Y^1_{N,M}\|^2_{L^2} + \|Y^2_{N,M}\|^2_{L^2}. \tag{5.23}$$

We now proceed as in [42]. Set $G_n(\omega) := |g_n(\omega)|^2 - 1$ and note that by the independence of $g_n(\omega)$ (cf. (5.7)),

$$\mathbb{E}[G_n(\omega)G_m(\omega)] = 0 \quad \text{for } n \ne m. \tag{5.24}$$

Since

$$|Y^1_{N,M}|^2 = \sum_{N\le|n_1|,|n_2|,|n_3|,|n_4|<M} n_1n_2n_3n_4 \frac{G_{n_1}G_{n_2}G_{n_3}G_{n_4}}{\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2\langle n_4\rangle^2}.$$

We compute $\mathbb{E}[|Y^1_{N,M}|^2]$ and by (5.24) the only contributions come from ($n_1 = n_3$ and $n_2 = n_4$), ($n_1 = n_2$ and $n_3 = n_4$) and ($n_2 = n_3$ and $n_1 = n_4$) . Hence by symmetry and using that the fourth moments of the Gaussians $g_n(\omega)$ are bounded we have

$$\|Y^1_{N,M}\|^2_{L^2} = \mathbb{E}[|Y^1_{N,M}|^2] \le C \sum_{N\le|n_1|,|n_2|<M} \frac{n_1^2n_2^2}{\langle n_1\rangle^4\langle n_2\rangle^4} \lesssim \frac{1}{N^2}. \tag{5.25}$$

On the other hand, since

$$|Y^2_{N,M}|^2 = \sum_{N\le|n_1|,|n_2|,|n_3|,|n_4|<M} n_1n_2n_3n_4 \frac{(G_{n_1}+G_{n_2})(G_{n_3}+G_{n_4})}{\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2\langle n_4\rangle^2},$$

by symmetry it is enough to consider a single term of the form

$$\sum_{N\le|n_1|,|n_2|,|n_3|,|n_4|<M} n_1n_2n_3n_4 \frac{G_{n_j}G_{n_k}}{\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2\langle n_4\rangle^2},$$

with $1 \le j \ne k \le 4$, which we set without any loss of generality to be $j = 1, k = 3$. We then have

$$\|Y^2_{N,M}\|^2_{L^2} = \mathbb{E}[|Y^2_{N,M}|^2] \le C \sum_{N\le|n_1|,|n_2|,|n_4|\le M} \frac{n_1^2n_2n_4}{\langle n_1\rangle^4\langle n_2\rangle^2\langle n_4\rangle^2} = 0$$

by symmetry. From (5.23) and (5.25) we obtain (5.20) as desired.

To prove (5.21) we use (5.22) to define

$$S_{M,N}(v) := |X_M(v) - X_N(v)|^2 = \sum_{N\le|n_1|,|n_2|<M} n_1n_2 \frac{|g_{n_1}(\omega)|^2|g_{n_2}(\omega)|^2}{\langle n_1\rangle^2\langle n_2\rangle^2} \tag{5.26}$$

which fits the framework of (5.18) in Proposition 5.10 with $k = 4$. Then it follows that for any $p \ge 2$,

$$\|S_{M,N}(v)\|_{L^p} \lesssim (p-1)^2\|S_{M,N}(v)\|_{L^2} = (p-1)^2\|X_M(v) - X_N(v\|^2_{L^4} \lesssim (p-1)^2\frac{1}{N}. \tag{5.27}$$

On the other hand if we set $q = 2p$, then by (5.27) we have

$$\|X_M(v) - X_N(v)\|_{L^q} = \|S_{M,N}(v)\|^{1/2}_{L^p} \lesssim (q-1)\frac{1}{N^{1/2}},$$

hence (5.21) for $q \ge 4$. Finally, Hölder's inequality gives (5.21) for $2 \le q \le 4$. □

Proof of Lemma 5.9. Let us denote $M_N(v) := \fint_{\mathbb{T}} |v_N|^2\, dx$. Up to absolute constants we write

$$\rho(|K_M(v) - K_N(v)| > 2\gamma) \leq \rho(|X_M(v)M_M(v) - X_N(v)M_N(v)| > \gamma) + \rho(|M_M(v)^3 - M_N(v)^3| > \gamma). \tag{5.28}$$

Then

$$\rho(|X_M(v)M_M(v) - X_N(v)M_N(v)| > \gamma)$$
$$\leq \rho(|X_M(v) - X_N(v)|M_M(v) > \gamma/2) + \rho(|M_M(v) - M_N(v)||X_N(v)| > \gamma/2) = I_1 + I_2.$$

Let $\lambda > 0$ to be determined. Then by (5.19), (5.6) and (5.17) with $p = 2$, $\alpha = (1/2)-$, we have

$$I_2 \leq \rho(|X_N(v)| > \lambda) + \rho\Big(|M_M(v) - M_N(v)| > \frac{\gamma}{2}\lambda^{-1}\Big)$$
$$\leq e^{-c\lambda N^{-2\varepsilon}} + \rho\Big(\|v^N - v^M\|_{L^2} > \frac{\gamma}{4B}\lambda^{-1}\Big) \leq e^{-c\lambda N^{-2\varepsilon}} + e^{-c_{\gamma,B}N^{1-}\lambda^{-2}}.$$

By setting $\lambda = N^{1/3+(2\varepsilon/3)-}$ we have $I_2 \lesssim e^{-c_{\gamma,B}N^{1/3-(4\varepsilon/3)-}}$.

To estimate I_1 we first note that

$$M_M(v) \leq \|v\|_{L^2}^2 \leq B^2. \tag{5.29}$$

Then by (5.21) and Chebyshev's inequality[8] we have

$$I_1 \leq \rho\Big(|X_M(v) - X_N(v)| > \frac{\gamma}{2B^2}\Big) \lesssim e^{-C_B N^{\frac{1}{2}}\gamma}. \tag{5.30}$$

To estimate the second term of (5.28), we use (5.29) to obtain

$$\rho(|M_M(v)^3 - M_N(v)^3| > \gamma) \leq \rho(|M_M(v) - M_N(v)| > c_B\gamma) \leq e^{-C_B\gamma^2 N^{1-}}$$

by arguing as in the estimate for I_2 above. □

Lemma 5.12. *$R_N(v)$ converges in measure to $R(v)$.*

Proof. If $\|P_N v\|_{L^2} \leq B$ for all $N \in \mathbb{N}$, then $\|v\|_{L^2} \leq B$. Hence, by continuity from above, we have, for $\delta \in (0, 1)$,

$$\lim_{N\to\infty} \rho\big(\{v : |\chi_{\{\|v^N\|_{L^2} \leq B\}} - \chi_{\{\|v\|_{L^2} \leq B\}}| > \delta\}\big)$$
$$= \lim_{N\to\infty} \rho(\|v^N\|_{L^2} \leq B) - \rho(\|v\|_{L^2} \leq B)$$
$$= \rho\Big(\bigcap_{N=1}^{\infty} \{\|v^N\|_{L^2} \leq B\}\Big) - \rho(\|v\|_{L^2} \leq B) = 0.$$

Thus, $\chi_{\{\|v^N\|_{L^2} \leq B\}}$ converges to $\chi_{\{\|v\|_{L^2} \leq B\}}$ in measure. By Lemma 5.8(a), F_N converges to F in measure, and by Lemma 5.9, K_N converges to K in measure.

[8] Cf. Lemma 4.5 in [46].

Lastly, we consider $G_N(v)$ and show it is Cauchy in measure provided $\|v\|_{L^2} \leq B$. Assume $N \leq M$. Then

$$
\begin{aligned}
&4\pi G_N(v) - 4\pi G_M(v) \\
&= \left(\int_{\mathbb{T}} (|v^M|^2 - |v^N|^2)\, dx\right)\left(\int_{\mathbb{T}} |v^M|^4\, dx\right) + \left(\int_{\mathbb{T}} |v^N|^2\, dx\right)\left(\int_{\mathbb{T}} (|v^M|^4 - |v^N|^4)\, dx\right) \\
&\leq c_B \|v^M - v^N\|_{L^2} \|v^M\|_{L^4}^4 + \|v^N\|_{L^2}^2 \left| \|v^M\|_{L^4}^4 - \|v^N\|_{L^4}^4 \right| \\
&\leq C_B [\|v^M - v^N\|_{L^2} \|v^M\|_{L^4}^4 + 3(\|v^M\|_{L^4}^3 + \|v^N\|_{L^4}^3) \|v^M - v^N\|_{L^4}].
\end{aligned}
$$

Fix any $\gamma > 0$; then

$$
\begin{aligned}
\rho(|4\pi G_M(v) - 4\pi G_N(v)| > \gamma) \leq{}& \rho\left(\|v^M - v^N\|_{L^2}^2 \|v^M\|_{L^4}^4 > \frac{\gamma}{2C_B}\right) \\
&+ \rho\left((\|v^M\|_{L^4}^3 + \|v^N\|_{L^4}^3) \|v^M - v^N\|_{L^4} > \frac{\gamma}{6C_B}\right).
\end{aligned}
$$

To treat the first term we write

$$
\rho\left(\|v^M - v^N\|_{L^2} \|v^M\|_{L^4}^4 > \frac{\gamma}{2C_B}\right) \leq \rho\left(\|v^M - v^N\|_{L^2} > \lambda^{-1} \frac{\gamma}{2C_B}\right) + \rho(\|v^M\|_{L^4}^4 > \lambda)
$$

for some $\lambda > 0$ to be determined. We use (5.15) with $\alpha = (1/2)-$ corresponding to $p = 2$ and (5.14) to get

$$
\rho(\|v^M - v^N\|_{L^2} > c_B \gamma \lambda^{-1}) \leq e^{-c_B' \gamma^2 N^{1-} \lambda^{-2}}
$$

and

$$
\rho(\|v^M\|_{L^4} > \lambda^{1/4}) \leq e^{-c\lambda^{1/2}}.
$$

A decay of $e^{-C_B N^{(1/5)-} \gamma^{2/5}}$ follows by setting $\lambda = N^{(2/5)-} \gamma^{4/5}$.

For the second term write

$$
\begin{aligned}
&\rho\left(\|v^M - v^N\|_{L^4} (\|v^M\|_{L^4}^3 + \|v^N\|_{L^4}^3) > \frac{\gamma}{6C_B}\right) \\
&\qquad \leq \rho(\|v^M - v^N\|_{L^4} > c_B \gamma \lambda^{-1}) + \rho(\|v^M\|_{L^4} > c_1 \lambda^{1/3}) + \rho(\|v^N\|_{L^4} > c_2 \lambda^{1/3}) \\
&\qquad \leq e^{-c_B' \gamma^2 N^{\frac{1}{2}-} \lambda^{-2}} + 2e^{-c\lambda^{2/3}},
\end{aligned}
$$

since $\alpha = (1/4)-$ when $p = 4$ in (5.15). A decay of $e^{-C_B N^{(1/8)-} \gamma^{1/2}}$ follows by setting $\lambda = N^{(3/16)-} \gamma^{3/4}$.

Thus, $G_N(v)$ converges to $G(v)$ in measure and hence, by composition and multiplication of continuous functions, $R_N(v)$ converges to $R(v)$ in measure. □

The following proposition shows that the weight $R(v)$ is indeed integrable with respect to the Wiener measure ρ.

Proposition 5.13. (a) *For sufficiently small $B > 0$, we have $R(v) \in L^2(d\rho)$. In particular, the weighted Wiener measure μ is a probability measure, absolutely continuous with respect to the Wiener measure ρ.*

(b) *We have the following tail estimate: Let $2 \leq r < \infty$ and $(s-1)r < -1$; then there exists a constant c such that*

$$\mu(\|v\|_{\mathcal{F}L^{s,r}} > K) \leq e^{-cK^2} \tag{5.31}$$

for sufficiently large $K > 0$.

(c) *The finite-dimensional weighted Wiener measure μ_N in (5.13) converges weakly to μ.*

Proof. (a) By Hölder's inequality, we have

$$\begin{aligned}\int R_N^2(v)\,d\rho(v) \leq C_B &\bigg(\int \chi_{\{\|v^N\|_{L^2}\leq B\}} e^{-3\,\mathrm{Im}\int (v^N)^2 \overline{v^N}\,\overline{v_x^N}\,dx}\,d\rho(v)\bigg)^{1/3}\\ &\times \bigg(\int \chi_{\{\|v^N\|_{L^2}\leq B\}} e^{\frac{3B^2}{2\pi}(\int |v^N|^4\,dx)}\,d\rho(v)\bigg)^{1/3}\\ &\times \bigg(\int \chi_{\{\|v^N\|_{L^2}\leq B\}} e^{-\frac{6}{\pi}M_N(v)\,\mathrm{Im}\int v^N\overline{v_x^N}\,dx}\,d\rho(v)\bigg)^{1/3}.\end{aligned}$$

It follows from Lemma 3.10 in [3] (see also [30]) that the second factor is finite for any $B > 0$, whereas it was shown in [42, Proposition 4.2] that the first factor is finite for sufficiently small $B > 0$. For the third factor we proceed as in the proof of [42, Proposition 4.2]. In what follows we always implicitly assume that $\|v_N\|_{L^2} \leq B$. If we define

$$A_{\gamma,N} = \{\chi_{\{\|v^N\|_{L^2}\leq B\}} e^{-\frac{6}{\pi}M_N(v)\,\mathrm{Im}\int v^N\overline{v_x^N}\,dx} > \gamma\},$$

then we need to show that

$$\int_0^\infty \gamma^2 \rho(A_{\gamma,N})\,d\gamma \tag{5.32}$$

is convergent uniformly with respect to N for $B > 0$ small enough. Let $N_0 = \ln\gamma$ and assume first that $N \leq N_0 \leq (C/B^2)\ln\gamma$ for B small enough. We first observe that

$$\bigg|M_N(v)\,\mathrm{Im}\int v^N\overline{v_x^N}\,dx\bigg| \leq CB^2\|\partial_x(v^N)^2\|_{L^\infty(\mathbb{T})}.$$

We also note that

$$\rho(A_{\gamma,N}) \leq \rho\bigg(\bigg|M_N(v)\,\mathrm{Im}\int v^N\overline{v_x^N}\,dx\bigg| > C\ln\gamma\bigg); \tag{5.33}$$

combining (5.33) and (5.32) with Proposition 4.1 in [42], we can continue with

$$\rho(A_{\gamma,N}) \leq \rho(\|\partial_x(v^N)^2\|_{L^\infty(\mathbb{T})} > CB^{-2}\ln\gamma) \lesssim e^{-\frac{C}{B^2}\ln\gamma} = \gamma^{-C/B^2},$$

and the convergence of (5.32) follows by taking B small enough.

Assume now that $N > N_0 = \ln\gamma$. Then we observe that $A_{\gamma,N} \subset B_{\gamma,N} \cup C_{\gamma,N}$ where

$$B_{\gamma,N} := \left\{ |X_{N_0}(v)| > \frac{\pi}{12B^2}\ln\gamma \right\}, \quad C_{\gamma,N} := \left\{ |X_N - X_{N_0}(v)| > \frac{\pi}{12B^2}\ln\gamma \right\}.$$

We first observe that from the argument above

$$\rho(B_{\gamma,N}) \le \rho(\|\partial_x (v^{N_0})^2\|_{L^\infty(\mathbb{T})} > CB^{-2}\ln\gamma) \lesssim \gamma^{-C/B^2}.$$

On the other hand from (5.30) and the fact that $N > \ln\gamma$ we have

$$\rho(C_{\gamma,N}) \lesssim e^{-C_B N^{1/2}\ln\gamma} \le e^{-C_B(\ln\gamma)^{1+1/2}} \le C_{B,L}\gamma^L,$$

for any $L \ge 1$ and an appropriate constant $C_{B,L}$ depending on B and L. From this again the convergence of (5.32) follows.

Hence we see that $R_N(v) \in L^2(d\rho)$ for sufficiently small $B > 0$, independent of N. Then, by Lemma 5.12 and Fatou's lemma, we obtain $R(v) \in L^2(d\rho)$.

(b) By the Cauchy–Schwarz inequality, we have

$$\int \chi_{\{\|v\|_{\mathcal{F}L^{s,r}} > K\}}\, d\mu \le \|R(v)\|_{L^2(d\rho)}\{\rho(\|v\|_{\mathcal{F}L^{s,r}} > K)\}^{1/2}.$$

Then (5.31) follows from (5.6).

(c) Let us define

$$\mathcal{H} := \bigcup_M \{F : F = G(\widehat{v}_{-M}, \dots, \widehat{v}_M),\, G \text{ bounded and continuous}\}. \tag{5.34}$$

Note this is a dense set in $L^1(\mathcal{F}L^{s,r}, \mu)$ with $2 \le r < \infty$ and $(s-1)r < -1$. Fix $F \in \mathcal{H}$; then F depends on M, finitely many modes, for some M. Fix $\varepsilon > 0$. Then, for $N > M$, we have

$$\begin{aligned}
\left|\int F(v)\, d\mu_N - \int F(v)\, d\mu\right| &= \left|\int F(v)(R_N(v) - R(v))\, d\rho\right| \\
&\le \left|\int_{\{|R_N(v)-R(v)|<\varepsilon\}} F(v)(R_N(v) - R(v))\, d\rho\right| \\
&\quad + \left|\int_{\{|R_N(v)-R(v)|\ge\varepsilon\}} F(v)(R_N(v) - R(v))\, d\rho\right| \\
&\le \varepsilon \sup|F| + \sup|F|\|R_N(v) - R(v)\|_{L^2(d\rho)}\{\rho(|R_N(v) - R(v)| \ge \varepsilon)\}^{1/2}.
\end{aligned}$$

From the proof of Proposition 5.13, we have $\|R_N(v) - R(v)\|_{L^2(d\rho)} \le \|R_N(v)\|_{L^2(d\rho)} + \|R(v)\|_{L^2(d\rho)} < C < \infty$ for all N. By Lemma 5.12, $\rho(|R_N(v) - R(v)| \ge \varepsilon) \to 0$ as $n \to \infty$.

Now, let F be a general bounded continuous function on $\mathcal{F}L^{s,r}$ with $2 \leq r < \infty$ and $(s-1)r < -1$. Let F_M denote its restriction to E_M, i.e. $F_M(v) = F(v^M)$ where $v^M = P_M v$. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\left| \int F(v)\, d\mu - \int F_M(v)\, d\mu \right| &= \left| \int (F(v) - F(v^M)) R(v)\, d\rho \right| \\ &\leq \|R(v)\|_{L^2(d\rho)} \left(\int |F(v) - F(v^M)|^2\, d\rho \right)^{1/2}. \end{aligned} \tag{5.35}$$

By continuity of F, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|P_M^{\perp} v\|_{\mathcal{F}L^{s,r}} = \|v - v^M\|_{\mathcal{F}L^{s,r}} < \delta \;\Rightarrow\; |F(v) - F(v^M)| < \varepsilon.$$

Then the contribution to (5.35) from $\{v : \|P_M^{\perp} v\|_{\mathcal{F}L^{s,r}} < \delta\}$ is at most $\varepsilon \|R(v)\|_{L^2(d\rho)}$. Without loss of generality, assume $\delta \leq \varepsilon^2$. By the measurability of the $\mathcal{F}L^{s,r}$-norm (see Definition 5.3), the contribution to (5.35) from $\{v : \|P_M^{\perp} v\|_{\mathcal{F}L^{s,r}} \geq \delta\}$ is at most

$$\begin{aligned} 2 \sup |F| \cdot \|R(v)\|_{L^2(d\rho)} &\{\rho(\|P_M^{\perp} v\|_{\mathcal{F}L^{s,r}} \geq \delta)\}^{1/2} \\ &\leq 2 \sup |F| \cdot \|R(v)\|_{L^2(d\rho)} \delta^{1/2} \leq 2 \sup |F| \cdot \|R(v)\|_{L^2(d\rho)} \varepsilon \end{aligned}$$

for sufficiently large M. A similar argument can be used to show $|\int F(v)\, d\mu_N - \int F_M(v)\, d\mu_N| \leq C(f, R)\varepsilon$, independent of N. Hence, μ_N converges weakly to μ. □

Remark 5.14. A tail estimate similar to (5.31) holds for the finite-dimensional weighted Wiener measure μ_N, i.e. we have

$$\mu_N(\|v^N\|_{\mathcal{F}L^{s,r}} > K) \leq e^{-cK^2}, \tag{5.36}$$

where the constant is independent of N.

Remark 5.15. The measure ρ_N is not absolutely continuous with respect to μ_N but its restriction to $\{\|v^N\|_{L^2} \leq B\}$, i.e., $\widetilde{\rho}_N = \widehat{Z}_N^{-1} \chi_{\{\|v^N\|_{L^2} \leq B\}} \rho_N$ is absolutely continuous with respect to μ_N, and from (5.13) we have

$$\frac{d\widetilde{\rho}_N}{d\mu_N} := \tilde{R}_N = \tilde{Z}_N^{-1} \chi_{\{\|v^N\|_{L^2} \leq B\}} e^{\frac{1}{2}\mathcal{N}(v^N)}$$

for a suitable renormalization $\tilde{Z}_N$. Since $\mathcal{N}(v^N)$ does not have a definite sign, Lemma 5.8, Lemma 5.12 and Proposition 5.13(a) hold for $\tilde{R}_N$ and its corresponding limit $\tilde{R}$. In particular, for sufficiently small B, $\tilde{R}_N \in L^2(d\rho)$ for all N with bound independent of N. The latter fact will be used in the proof of Proposition 6.2 below.

Remark 5.16. Given any $p < \infty$, one can prove $R(v) \in L^p(d\rho)$ for sufficiently small $B \leq B(p)$. However, $B(p) \to 0$ as $p \to \infty$, i.e. there is no uniform lower bound on the size of the L^2-cutoff. For our purpose, the integrability with $p = 2$ suffices.

6. Almost sure well-posedness of FGDNLS and invariance of the measure

In order to prove the global well-posedness of μ-almost all solutions of FGDNLS (3.1) we fix once again $s = (2/3)-$ and $r = 3$ so that we have at our disposal the local well-posedness result in $\mathcal{F}L^{s,r}$, that the measure is supported on $\mathcal{F}L^{s,r}$, and also the energy growth estimates in Theorem 4.2 as explained in Remark 4.3.

We first use the almost invariance of the finite-dimensional measure μ_N under the flow of the truncated equation (3.1) to control the growth of solutions.

Lemma 6.1. *For any given $T > 0$ and $\varepsilon > 0$ there exists an integer $N_0 = N_0(T, \varepsilon)$ and sets $\widetilde{\Omega}_N = \widetilde{\Omega}_N(\varepsilon, T) \subset \mathbb{R}^{4N+2}$ such that for $N > N_0$:*

(a) $\mu_N(\widetilde{\Omega}_N) \geq 1 - \varepsilon$.
(b) *For any initial condition $v_0^N \in \widetilde{\Omega}_N$, FGDNLS (3.1) is well-posed on $[-T, T]$ and its solution $v^N(t)$ satisfies the bound*

$$\sup_{|t|\leq T} \|v^N(t)\|_{\mathcal{F}L^{\frac{2}{3}-,3}} \lesssim \left(\log \frac{T}{\varepsilon}\right)^{1/2}.$$

Proof. It is enough to consider $t \in [0, T]$; the argument for $t \in [-T, 0]$ is similar. We set

$$C_N(K, B) := \{w^N \in \mathbb{R}^{4N+2} : \|w^N\|_{\mathcal{F}L^{\frac{2}{3}-,3}} \leq K,\ \|w^N\|_{L^2} \leq B\}.$$

If the initial condition v_0^N is in $C_N(K, B)$ then FGDNLS (3.1) is locally well-posed on the time interval of length $\delta \sim K^{-\gamma}$ by Theorem 3.2, where $\gamma > 0$ is independent of N. Furthermore, if μ_N is given by (5.13), then for sufficiently large K we have $\mu_N(C_N(K, B)^c) \leq e^{-cK^2}$ for some constant c which is independent of N by (5.36).

Let $\Phi_N(t)$ be the flow map of (3.1). We define

$$\widetilde{\Omega}_N := \{v_0^N : \Phi_N(j\delta)(v_0^N) \in C_N(K, B),\ j = 0, 1, \ldots, [T/\delta]\}.$$

Note that $\widetilde{\Omega}_N^c = \bigcup_{k=0}^{[T/\delta]} D_k$, where

$$\begin{aligned} D_k &= \{v_0^N; k = \min\{j : \Phi_N(j\delta)(v_0^N) \in C_N(K, B)^c\}\}, \\ &= \left[\bigcap_{j=0}^{k-1} \Phi_N(-j\delta)(C_N(K, B))\right] \cap \Phi_N(-k\delta)(C_N(B, K)^c). \end{aligned} \tag{6.1}$$

One verifies easily that the sets D_k satisfy

$$D_0 = C_N(K, B)^c, \qquad D_k = C_N(K, B) \cap \Phi_N(-\delta)(D_{k-1}). \tag{6.2}$$

By Lemma 4.1, the Lebesgue measure $d\mu_N^0 \equiv \prod_{|n|\leq N} da_n db_n$ is invariant under the flow $\Phi_N(t)$ (i.e. for any $f \in L^1(d\mu_N^0)$ we have $\int f \circ \Phi_N(t)\, d\mu_N^0 = \int f\, d\mu_N^0$).

Using the energy growth estimate[9] in Theorem 4.2 and the invariance of the L^2 norm $m(v) = \frac{1}{2\pi}\|v\|_{L^2}$ under $\Phi_N(t)$ (i.e. $m \circ \Phi_N(t) = m$ for all t; see Remark 3.1) we have, for any set $A \subset \mathbb{R}^{4N+2}$,

$$\begin{aligned}
\mu_N(C_N(K,B)\cap A) &= Z_N^{-1}\int \chi_{\{C_N(K,B)\cap A\}}\chi_{\{m\le 2\pi B^2\}}e^{-\frac{1}{2}\mathcal{E}-\pi m}\,d\mu_N^0\\
&= Z_N^{-1}\int \chi_{\{C_N(K,B)\cap A\}}\circ\Phi_N(-\delta)\chi_{\{m\le 2\pi B^2\}}e^{-\mathcal{E}\circ\Phi_N(-\delta)-\pi m}\,d\mu_N^0\\
&= \int \chi_{\{\Phi_N(\delta)(C_N(K,B)\cap A)\}}e^{-\frac{1}{2}(\mathcal{E}\circ\Phi_N(-\delta)-\mathcal{E})}\,d\mu_N\\
&\le e^{c(\delta)N^{-\beta}K^8}\mu_N\left(\Phi_N(\delta)(C_N(K,B)\cap A)\right)\\
&\le e^{c(\delta)N^{-\beta}K^8}\mu_N\left(\Phi_N(\delta)(A)\right). \qquad (6.3)
\end{aligned}$$

Applying (6.3) to (6.2) with $A = \Phi_N(-\delta)(D_{k-1})$ and iterating in $k \in \{0,\dots,[T/\delta]\}$, we obtain

$$\mu_N(D_K) \le e^{c(\delta)N^{-\beta}K^8}\mu_N(D_{K-1}) \le e^{kc(\delta)N^{-\beta}K^8}e^{-cK^2}$$

and thus

$$\mu_N(\widetilde{\Omega}_N^c) \le \sum_{k=0}^{[T/\delta]} e^{kc(\delta)N^{-\beta}K^8}e^{-cK^2} \lesssim \left[\frac{T}{\delta}\right]e^{-cK^2} \sim TK^\gamma e^{-cK^2}$$

for $N \ge N_0(T,K)$. By choosing $K \sim (\log(T/\varepsilon))^{1/2}$, we have $\mu_N(\widetilde{\Omega}_N^c) < \varepsilon$ as desired.

Finally, by construction, we have $\|v^N(j\delta)\|_{\mathcal{F}L^{(2/3)-,3}} \le K$ for $j = 0,\dots,[T/\delta]$ and by the local theory, we have

$$\sup_{0\le t\le T}\|v^N(t)\|_{\mathcal{F}L^{\frac{2}{3}-,3}} \le 2K \sim \left(\log\frac{T}{\varepsilon}\right)^{1/2}. \qquad \square$$

Combining Lemma 6.1 with the approximation Lemma 3.3 we can now prove a similar result for the solution of the initial value problem GDNLS (2.6).

Proposition 6.2. *For any given $T > 0$ and $\varepsilon > 0$ there exists a set $\Omega(\varepsilon, T)$ such that:*

(a) $\mu\left(\Omega(\varepsilon,T)\right) \ge 1-\varepsilon$.
(b) *For any initial condition $v_0 \in \Omega(\varepsilon,T)$ the initial value problem GDNLS (2.6) is well-posed on $[-T,T]$ with the bound*

$$\sup_{|t|\le T}\|v(t)\|_{\mathcal{F}L^{\frac{2}{3}-,3}} \lesssim \left(\log\frac{T}{\varepsilon}\right)^{1/2}.$$

[9] Without loss of generality we assume $\max(K^6, K^8) = K^8$ in Theorem 4.2.

Proof. Let $\widetilde{\Omega}_N = \widetilde{\Omega}_N(\varepsilon, T)$ be the set given in Lemma 6.1 for $N \geq N_0(\varepsilon, T)$. This set is defined in terms of $K \sim (\log(T/\varepsilon))^{1/2}$ and for that same K we define

$$\Omega_N := \Omega_N(\varepsilon, T) := \{v_0 \in \mathcal{F}L^{\frac{2}{3}-,3} : \|v_0\|_{\mathcal{F}L^{\frac{2}{3}-,3}} \leq K,\ P_N v_0 \in \widetilde{\Omega}_N\}$$

If $v_0 \in \Omega_N$ then by Lemma 6.1 we have

$$\sup_{t \leq T} \|\Phi_N(t)(P_N v_0)\|_{\mathcal{F}L^{\frac{2}{3}-,3}} \leq 2K. \tag{6.4}$$

On the other hand for $v_0 \in \Omega_N$ the local well-posedness theorem in [22] gives a $\delta > 0$ and a solution $v(t)$ of GDNLS (2.6) for $|t| \leq \delta$.

By (3.5) in the proof of Lemma 3.3, with K in place of A, we find that for every $s_1 < (2/3)-$,

$$\|v(\delta) - v^N(\delta)\|_{\mathcal{F}L^{s_1,3}} \lesssim K N^{s_1 - \frac{2}{3}+}.$$

By choosing a larger N_0 if necessary, so that $[T/\delta]\, K N^{s_1-(2/3)+} \ll 1$ for $N > N_0$ we can repeat this argument $[T/\delta]$ times over the intervals $[j\delta, (j+1)\delta]$, $j = 0, 1, \ldots, [T/\delta]-1$, to obtain

$$\|v(j\delta) - v^N(j\delta)\|_{\mathcal{F}L^{s_1,3}} < 1. \tag{6.5}$$

Then from (6.4) and (6.5) we conclude

$$\|v(t)\|_{\mathcal{F}L^{s_1,3}} \lesssim 2K + 1 \sim \left(\log \frac{T}{\varepsilon}\right)^{1/2},$$

and since the right hand side is independent of $s_1 < (2/3)-$, we obtained the desired estimate.

To estimate $\mu(\Omega_N)$ note first that

$$\Omega_N^c \subset \{v_0 \in \mathcal{F}L^{\frac{2}{3}-,3} : \|v_0\|_{\mathcal{F}L^{\frac{2}{3}-,3}} \geq K\} \cup \{v_0 \in \mathcal{F}L^{\frac{2}{3}-,3} : P_N v_0 \in \widetilde{\Omega}_N^c\} \tag{6.6}$$

The first set on the right hand side of (6.6) has μ measure less than ε by the tail bound in Proposition 5.13. The set $F_N \equiv \{v_0 \in \mathcal{F}L^{(2/3)-,3} : P_N v_0 \in \widetilde{\Omega}_N^c\}$ is a cylinder set and we have $F_N \cap E_N = \widetilde{\Omega}_N^c$ (recall $E_N = \text{span}\{e^{inx}\}_{|n| \leq N}$). Thus $\rho(F_N) = \rho_N(F_N) = \rho_N(\widetilde{\Omega}_N^c)$. On the other hand, recall that $\mu \ll \rho$ and that $\widetilde{\rho}_N$, the restriction of ρ_N to the ball $\{\|v^N\|_{L^2} \leq B\}$, is absolutely continuous with respect to μ_N (see Remark 5.15). Then using Cauchy–Schwarz repeatedly we obtain

$$\begin{aligned}
\mu(F_N) &\leq \left(\int R^2\, d\rho\right)^{1/2} \left(\int_{\widetilde{\Omega}_N^c} \chi_{\{\|v^N\|_{L^2} \leq B\}}\, d\rho_N\right)^{1/2} \\
&\leq \left(\int R^2\, d\rho\right)^{1/2} \left(\int \tilde{R}_N^2\, d\mu_N\right)^{1/4} \mu_N(\widetilde{\Omega}_N^c)^{1/4} \\
&\leq \left(\int R^2\, d\rho\right)^{1/2} \left(\int \tilde{R}_N\, d\rho_N\right)^{1/4} \mu_N(\widetilde{\Omega}_N^c)^{1/4},
\end{aligned} \tag{6.7}$$

where $\tilde{R}_N$ is as defined in Remark 5.15 and where in the last inequality we have used that by definition $\tilde{R}_N^2 R_N = \tilde{R}_N$.

By relying on Lemma 5.12, Proposition 5.13 and Remark 5.15 we can bound the first two terms in (6.7) by a constant independent of N. This combined with Lemma 6.1 allows us to conclude that there exist a constant $d > 0$ and $N_1(\varepsilon, T)$ such that $\mu(F_N) \le d\varepsilon$ for $N \ge N_1$. So for $N \ge \max(N_0, N_1)$, any set $\Omega(\varepsilon, T) := \Omega_N(\varepsilon, T)$ satisfies the desired hypothesis. □

Theorem 6.3 (Almost sure global well-posedness). *There exists a subset Ω of the space $\mathcal{F}L^{(2/3)-,3}$ with $\mu(\Omega^c) = 0$ such that for every $v_0 \in \Omega$ the initial value problem GDNLS (2.6) with initial data v_0 is globally well-posed.*

Proof. Fix an arbitrary T and let $\varepsilon = 2^{-i}$. Using the sets given in Proposition 6.2 we set

$$\Omega(T) := \bigcup_i \Omega(2^{-i}, T).$$

If $v_0 \in \Omega(T)$ then the initial value problem GDNLS (2.6) is well-posed up to time T. Since $\mu(\Omega(T)) \ge 1 - 2^{-i}$ for any $i \in \mathbb{N}$, the set $\Omega(T)$ has full measure.

Finally by taking $T := 2^j$ the set

$$\Omega = \bigcap_j \Omega(2^j) \tag{6.8}$$

also has full measure and if $v_0 \in \Omega$ then the initial value problem GDNLS (2.6) is globally well-posed. □

Remark 6.4. We note that by slightly modifying the proof of Theorem 6.3 above we could also derive a logarithmic bound in time on solutions similar to the one in [3] and [12].

Now that we have a well-defined flow on the measure space $(\mathcal{F}L^{(2/3)-,3}, \mu)$, we show that μ is invariant under the flow $\Phi(t)$, following the argument in [38].

Theorem 6.5. *The measure μ is invariant under the flow $\Phi(t)$.*

Proof. Let us consider the measure space $(\mathcal{F}L^{(2/3)-,3}, \mu)$. We need to show that for any measurable A we have $\mu(A) = \mu(\Phi(-t)(A))$ for all $t \in \mathbb{R}$. Note that by the group property of the flow without loss of generality we can assume that $|t| \le \delta$. An equivalent characterization of invariance is that for all $F \in L^1(\mathcal{F}L^{(2/3)-,3}, \mu)$ we have

$$\int F(\Phi(t)(v))\, d\mu = \int F(v)\, d\mu. \tag{6.9}$$

By an elementary approximation argument it is enough to show (6.9) for F in a dense set in $L^1(\mathcal{F}L^{(2/3)-,3}, \mu)$ which we choose as in (5.34) to be

$$\mathcal{H} := \bigcup_M \{F : F = G(\widehat{v}_{-M}, \dots, \widehat{v}_M),\ G \text{ bounded and continuous}\}.$$

For $F \in \mathcal{H}$ choose an arbitrary $\epsilon > 0$ and assume $N \geq M$. By Proposition 5.13, μ_N converges weakly to μ and thus

$$\left|\int F\,d\mu - \int F\,d\mu_N\right| + \left|\int F\circ\Phi(t)\,d\mu - \int F\circ\Phi(t)\,d\mu_N\right| \leq \epsilon. \tag{6.10}$$

Let $\Phi_N(t)$ be the flow map for FGDNLS (3.1). For $s_1 < (2/3)-$, by Lemma 3.3, we deduce that $\|\Phi(t)(v) - \Phi_N(t)(v)\|_{\mathcal{F}L^{s_1,3}}$ converges to 0 uniformly on $\{v : \|v\|_{\mathcal{F}L^{(2/3)-,3}} \leq K\}$. Using the tail estimate $\mu_N(\|v^N\|_{\mathcal{F}L^{(2/3)-,3}} > K) \leq e^{-cK^2}$ (uniformly in N) and the continuity of F in $\mathcal{F}L^{s_1,3}$ we obtain

$$\left|\int F\circ\Phi(t)\,d\mu_N - \int F\circ\Phi_N(t)\,d\mu_N\right| \leq 2\|F\|_\infty e^{-cK^2} + \epsilon \leq 3\epsilon \tag{6.11}$$

for large enough K and N.

Finally using again the tail estimate for μ_N, the invariance of Lebesgue measure under $\Phi_N(t)$ and the energy estimate given in Theorem 4.2 we obtain

$$\begin{aligned}
&\left|\int F\circ\Phi_N(t)\,d\mu_N - \int F\,d\mu_N\right| \\
&\quad\leq 2\|F\|_{L^\infty} e^{-cK^2} + \left|\int_{\{\|v\|_{\mathcal{F}L^{\frac{2}{3}-,3}}\leq K\}} F[e^{-\frac{1}{2}(\mathcal{E}\circ\Phi_N(-t)-\mathcal{E})} - 1]\,d\mu_N\right| \\
&\quad\leq 2\epsilon + \|F\|_{L^\infty}(e^{c(\delta)N^{-\beta}K^8} - 1) \leq 3\epsilon,
\end{aligned} \tag{6.12}$$

for sufficiently large N. By combining (6.10)–(6.12) we obtain invariance. □

7. The ungauged DNLS equation

Recall that if $u(t,x)$ is a solution of DNLS (2.1) then $w(t,x) = G(u(t,x))$ where $G(f)(x) = \exp(-iJ(f))f(x)$ (see (2.5)) is a solution of

$$w_t - iw_{xx} - 2m(w)w_x = -w^2\overline{w_x} + \frac{i}{2}|w|^4 w - i\psi(w)w - im(w)|w|^2 w \tag{7.1}$$

with initial data $w(0) = G(u(0))$. Furthermore $v(t,x) = w(t, x - 2tm(w))$ is a solution of (2.6) with initial condition $v(0) = w(0)$. If $\Phi(t)$ denotes the flow map for GDNLS (2.6), let $\widetilde{\Phi}(t)$ denote the flow map of (7.1) and let $\Psi(t)$ denote the flow map of (2.1).

Clearly we have the relation

$$\Psi(t) = G^{-1}\circ\widetilde{\Phi}(t)\circ G. \tag{7.2}$$

To elucidate the relation between $\Phi(t)$ and $\widetilde{\Phi}(t)$ let $\tau_\alpha(s)$ denote the action of the group of spatial translations on functions, i.e., $(\tau_\alpha(s)w)(x) := w(x - \alpha s)$. We define a state dependent translation

$$(\Gamma(s)w)(x) := (\tau_{2m(w)}(s)w)(x) = w(x - 2sm(w)).$$

Note that the H^s, L^p and $\mathcal{F}L^{s,r}$ norms are all invariant under this transformation. Furthermore we have

$$v(t,x) := (\Gamma(t)w)(t,x).$$

Since m is preserved under G, $\Gamma(s)$ and both flows $\Psi(t)$ and $\widetilde{\Phi}(t)$, we have the relation

$$\Phi(t) = \Gamma(t)\widetilde{\Phi}(t) = \widetilde{\Phi}(t)\Gamma(t), \tag{7.3}$$

in particular $\widetilde{\Phi}(t)$ and $\Gamma(t)$ commute.

Finally if μ is a measure on Ω as in Theorem 6.3 and $\varphi : \Omega \to \Omega$ is a measurable map then we define the measure $\nu = \mu \circ \varphi^{-1}$ by

$$\nu(A) := \mu(\varphi^{-1}(A)) = \mu(\{x : \varphi(x) \in A\}).$$

for all measurable sets A or equivalently by

$$\int F\, d\nu = \int F \circ \varphi\, d\mu$$

for integrable F.

Consider the measure defined by

$$\nu := \mu \circ G. \tag{7.4}$$

Since the measure μ constructed in Proposition 5.13 is invariant under the flow $\Phi(t)$ we show that the flow $\Psi(t)$ for DNLS is well defined ν almost surely and that ν is invariant under the flow $\Psi(t)$.

Theorem 7.1 (Almost sure global well-posedness for DNLS). *There exists a subset Σ of the space $\mathcal{F}L^{(2/3)-,3}$ with $\nu(\Sigma^c) = 0$ such that for every $u_0 \in \Sigma$ the IVP DNLS (2.1) with initial data u_0 is globally well-posed.*

Proof. Let Ω be the set of full μ measure given in Theorem 6.3 and let $\Sigma = G^{-1}(\Omega)$. Note that Σ is a set of full ν measure by (7.4). For $v_0 \in \Omega$ the IVP GDNLS (2.6) with initial data v_0 is globally well-posed. Hence since the map $\mathcal{G} : C([-T,T]; \mathcal{F}L^{s,r}) \to C([-T,T]; \mathcal{F}L^{s,r})$ is a homeomorphism if $s > 1/2 - 1/r$ when $2 < r < \infty$, the IVP (DNLS) (2.1) with initial data $u_0 = G^{-1}(v_0)$ is also globally well-posed. □

Finally we show that the measure ν is invariant under the flow map of DNLS (2.1).

Theorem 7.2. *The measure $\nu = \mu \circ G$ is invariant under the flow $\Psi(t)$.*

Proof. First we note that the measure μ is invariant under $\Gamma(t)$. The density of μ with respect to ρ is $R(v)$ (see (5.12)), and it is verified easily that $R \circ \Gamma(t) = R$. Furthermore one also verifies easily that the finite-dimensional measures ρ_N are also invariant under $\Gamma(t)$. As a consequence, since μ is invariant under $\Phi(t)$ by Theorem 6.5, μ is also invariant under $\widetilde{\Phi}(t)$ because of (7.3). Finally ν is invariant under $\Psi(t)$ since by (7.2),

$$\begin{aligned}\int F \circ \Psi(t)\, d\nu &= \int F \circ G^{-1} \circ \widetilde{\Phi}(t) \circ G\, d\mu \circ G = \int F \circ G^{-1} \circ \widetilde{\Phi}(t)\, d\mu \\ &= \int F \circ G^{-1}\, d\mu = \int F\, d\nu.\end{aligned}$$

□

Acknowledgments. Andrea R. Nahmod and Gigliola Staffilani would like to warmly thank the Radcliffe Institute for Advanced Study at Harvard University for its wonderful hospitality while part of this work was being carried out. They also thank their fellow Fellows for the stimulating environment they created.

The first author is funded in part by NSF DMS 0803160 and a 2009-2010 Radcliffe Institute for Advanced Study Fellowship.

The third author is funded in part by NSF DMS 0605058.

The fourth author is funded in part by NSF DMS 0602678 and a 2009-2010 Radcliffe Institute for Advance Study Fellowship.

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