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Isometries of two dimensional Hilbert geometries

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Abstract. We prove that every isometry between two dimensional Hilbert geometries is a projective transformation unless the domains are interiors of triangles.

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Dedicated to Pierre de la Harpe on his seventieth birthday

1. Introduction

The *Hilbert distance* between two points x and y in a bounded convex domain Ω of $\mathbb{R}^n$ is defined as

$$d(x,y) := \ln\big((x,y;\bar{x},\bar{y})\big) := \ln\left(\frac{|\bar{y}-x|}{|\bar{y}-y|} : \frac{|\bar{x}-x|}{|\bar{x}-y|}\right), \tag{1.1}$$

where $|u-v|$ denotes the usual Euclidean length between two points u and v in $\mathbb{R}^n$, and $\bar{x}$ and $\bar{y}$ are as Fig 1. It is well known that the distance function d satisfies the standard requirements of a distance function, the only nontrivial point to check being the triangle inequality, see for example [Hil] or [Har, §1]. This distance was introduced by Hilbert in [Hil] and we refer to [H] for a presentation of both classic and contemporary aspects of Hilbert geometry[1].

Recall that straight lines, convexity, and the cross ratio of four aligned points are invariant under projective transformations, this implies immediately that if $f : \mathbb{RP}^n \to \mathbb{RP}^n$ is a projective transformation, then its restriction to Ω defines an isometry $f : \Omega \to f(\Omega)$ with respect to the Hilbert distances in Ω and $f(\Omega)$. (We consider $\mathbb{R}^n$ as a subset of $\mathbb{RP}^n$ by identifying it with an affine chart, the

[1] More generally, the Hilbert distance is well-defined for a domain Ω in $\mathbb{RP}^n$, that is convex and bounded in an appropriate affine chart of $\mathbb{RP}^n$.

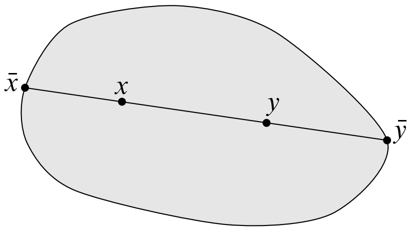

FIGURE 1
The points $\bar{x}$ and $\bar{y}$

Hilbert metric inside Ω does not depend on the choice of the affine chart.) The converse to this statement is not always true: some special Hilbert geometries admit isometries which are not projective transformations. The simplest example is given by the simplex and is discussed in detail in dimension 2 by Pierre de la Harpe in [Har]. This author asked for a full description of all isometries in Hilbert geometry and a complete answer in finite dimension has recently been obtained by Cormac Walsh in [Wal]. Note also that the same author, together with Bas Lemmens, previously described all isometries of polyhedral Hilbert geometries in [LW], while Bas Lemmens, Mark Roelands and Marten Wortel gave some partial results in infinite dimension in [LRW].

Our goal in this paper is to give a short proof of the following two dimensional result:

Theorem 1.1. *Let Ω_1 and Ω_2 be two bounded convex domains in the plane $\mathbb{R}^2$ and d_1, d_2 be the corresponding Hilbert metrics. Suppose that Ω_1 is not the interior of a triangle, then every isometry $f : (\Omega_1, d_1) \to (\Omega_2, d_2)$ is the restriction of a projective transformation of $\mathbb{RP}^2$.*

As mentioned above, this result is false if Ω_1 is the interior of a triangle. In that case (Ω_1, d_1) is isometric to a Minkowski plane whose unit ball is a regular hexagon and its group of isometries is not difficult to describe, see [Har]. Recall also that the above theorem is a special case of the result of C. Walsh [Wal, Theorem 1.3]. For the case of quadrilaterals, the result is also proved by P. de la Harpe in [Har, Proposition 4].

Our proof uses methods completely different from those in Walsh's paper. It is quite direct and only based on the description of metric geodesics in Hilbert geometry, together with a quite old and nontrivial result from line geometry due to Walter Prenowitz.

2. The case of strictly convex domains

It will be convenient to start with the case of a strictly convex domain. In fact we will prove the following result:

Proposition 2.1. *Assume that Ω_1 and Ω_2 are bounded convex domains in $\mathbb{R}^n$. If Ω_1 is strictly convex, then every isometry $f : (\Omega_1, d_1) \to (\Omega_2, d_2)$ is the restriction of a projective transformation of $\mathbb{RP}^n$.*

This result is proved in [Har, Proposition 3], but we shall give a slightly more direct proof. The result has recently been extended in infinite dimension in [LRW, Theorem 1.2].

The proof is based on the structure of geodesics for the Hilbert distance. It is easy to check from the definition of the Hilbert distance that if three points $x, y, z \in \Omega_1$ are aligned and $z \in [x, y]$, then $d_1(x, y) = d_1(x, z) + d_1(z, y)$. In other words the intersection of Euclidean straight lines with Ω_1 are geodesics for the Hilbert metric. Furthermore, the following fact is classical (see [Har, Proposition 2] or [PT, Theorem 12.5]):

Lemma 2.2. *Let p and q be two points on the boundary of Ω_1, and suppose that at least one of them is an extreme point of Ω_1. Then the open interval (p, q) is the unique geodesic between any pair of its points, that is if $x, y \in (p, q)$ and $z \in \Omega_1$, then $d_1(x, y) = d_1(x, z) + d_1(z, y)$ if and only if $z \in [x, y]$.*

Proof of Proposition 2.1. It is easy to prove the proposition for one dimensional Hilbert geometries; we therefore assume $n \geq 2$. Let $f : \Omega_1 \to \Omega_2$ be an isometry for the Hilbert distances between bounded convex domains in $\mathbb{R}^n$, where $\Omega_1 \subset \mathbb{R}^n$ is strictly convex. From the previous Lemma, it then follows that the affine segment $[x, y]$ between two points $x, y \in \Omega_1$ is the unique geodesic joining these two points. Since f is an isometry, there is also a unique geodesic joining the images $f(x)$ and $f(y)$ in Ω_2 and because the Euclidean segment $[f(x), f(y)] \subset \Omega_2$ is known to be geodesic we conclude that f maps the segment $[x, y] \subset \Omega_1$ to the segment $[f(x), f(y)] \subset \Omega_2$. Since x and y are arbitrary points in Ω_1, we conclude that f is a *local collineation*, that is a mapping sending Euclidean segments to Euclidean segments. The conclusion now follows from the local version of the fundamental theorem of projective geometry (see, e.g., [Shi, Lemma 4]), which states that any local collineation defined in some open connected set of the real projective space $\mathbb{RP}^n$ is the restriction of a projective transformation. $\square$

3. Proof of the main Theorem

The proof of Theorem 1.1 will be based on a 1935 result of Prenowitz [Pre] which generalizes the fundamental theorem of projective geometry in dimension 2. We will need the following definitions.

Definitions 3.1. Let U be a plane domain, that is an open connected nonempty subset of $\mathbb{R}^2$. By a *line in* U we mean a connected component of the intersection of a Euclidean straight line with U. A *family of lines* in U is a partition of U by lines, that is a collection of lines in U such that each point of U lies on exactly one line of the collection. If all lines in a family extend to Euclidean straight lines passing through a common point A, the family is called a *pencil with pole* A. A (linear) n*-web* in U is a set of n families of lines on U such that no two families have a common line.

Figure 2 shows a pencil with pole A in the domain U. By taking the pencils through n pairwise distinct poles $A_1, \dots, A_n \notin U$ we obtain an n-web in any subdomain $U' \subset U$ disjoint from any line through a pair of distinct points A_i, A_j.

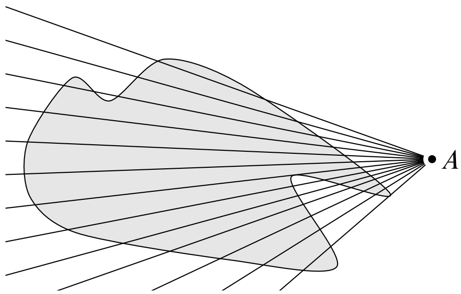

FIGURE 2
A pencil of lines covering a plane domain.

Theorem 3.2 (Prenowitz 1935). *A one to one continuous map defined in a plane domain that carries a 4-web into a 4-web is the restriction of a projective transformation.*

Recall that, by Brouwer's theorem, an injective continuous map defined in a domain of $\mathbb{R}^n$ is a homeomorphism onto its image. The above result is proved in [Pre]; a much simpler proof is given in [Kas] assuming the map is a diffeomorphism. Some generalizations in higher dimensions are given in [AAS].

The following corollary will be useful in the proof of Theorem 1.1:

Corollary 3.3. *Let $f : U \to \mathbb{R}^2$ be a one to one continuous map defined in a domain $U \subset \mathbb{R}^2$ and let $A_1, \cdots, A_5 \in \mathbb{R}^2$ be five pairwise distinct points. Assume that f maps the intersection of every line through A_j with U to a straight line $(1 \le j \le 5)$. Then f is the restriction of a projective transformation.*

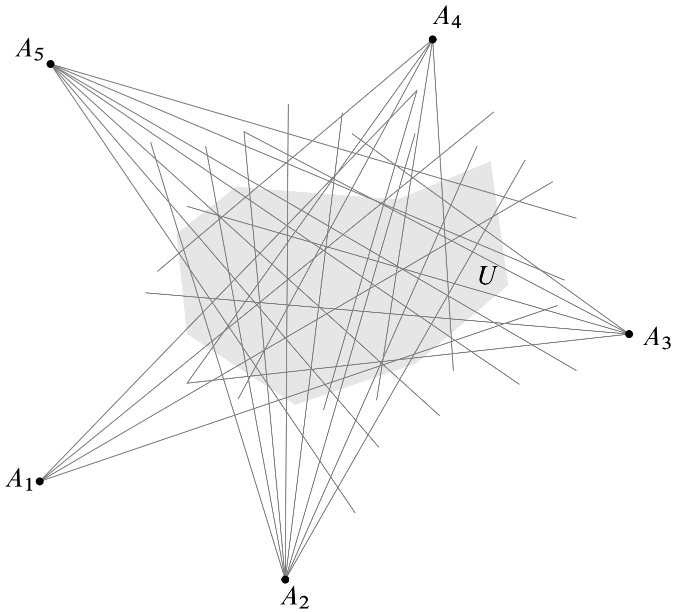

Figure 3
A polygonal region U covered by 5 pencils. Corollary 3.3 states that a homeomorphism defined in U carrying all those lines into lines is a projective transformation.

Proof. There are 10 lines through any pair of the points A_j and the pairwise intersections of those 10 lines determine (at most) 20 points[2]. Let us denote this set by $\mathcal{I}$ and call it the set of *intersection points*. For any point $X \in U \setminus \mathcal{I}$, at least four of the directions $\overrightarrow{XA_j}$ are mutually distinct and this property holds in a neighborhood V of X. The pencils with the corresponding four points A_j as poles form a 4-web in V, see Figure 3, which is mapped by f to a 4-web in $f(V)$. By Theorem 3.2, we know that the restriction of f to V is the restriction

[2] 10 distinct lines in a projective plane define $\binom{10}{2} = 45$ intersection points counted with multiplicity, the 5 points A_j have multiplicity 6.

of a projective transformation. By real analyticity, two projective transformations that coincide on an open subset coincide everywhere. Since $U \setminus \mathcal{I}$ is connected the restriction of f to $U \setminus \mathcal{I}$ is a projective transformation and since $\mathcal{I}$ is finite, f is a projective transformation on the whole domain U by continuity. $\square$

Proof of Theorem 1.1. Recall that we assumed that the bounded convex domain $\Omega_1 \in \mathbb{R}^2$ is not the interior of a triangle. We first assume that Ω_1 is also not a quadrilateral. Then, $\overline{\Omega}_1$ has at least five distinct extreme points $A_1, A_2, A_3, A_4, A_5 \in \partial\Omega_1$. Because the points A_j are extreme points of $\overline{\Omega}_1$, Lemma 2.2 implies that each line through one of the points A_j intersects Ω_1 on a unique geodesic (for the Hilbert distance) between any of its pairs of points. Since f is an isometry, it sends each line from the five pencils into a straight line in Ω_2 and it follows from Corollary 3.3 that f is the restriction of a projective transformation.

Suppose now that Ω_1 is a quadrilateral with vertices $ABCD$. The vertices are extreme points of $\overline{\Omega}_1$, therefore, by Lemma 2.2, any line through a vertex defines a unique geodesic for the Hilbert distance and it is thus mapped on a line by the isometry f. The pencils with the four vertices as poles form a 4-web in each connected component of the complement of the diagonals. These connected components are the interior of the triangles ABM, BCM, CDM, DAM, where M is the intersection of the diagonals, and from Prenowitz' Theorem 3.2, we conclude that the restriction of f to each of those triangles is a projective transformation.

Consider two adjacent such triangles, and consider the f-image of their union, see Fig 4. Since the restriction of f to each of these triangles is a projective transformation, the image of its union is the union of two triangles. By continuity they have a common edge. Since the image of the line AC is a straight line, the

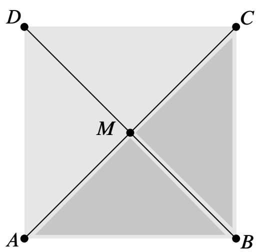

FIGURE 4
The restriction of f to the "dark-gray" triangles ABM and BCM is a projective transformation, and f sends AC to a straight line. Then, the image of ABC is a triangle and the restriction of f to it is a projective transformation.

closure of the image of the union of these triangles is a triangle. Furthermore, the map f sends any line through A or B to a line, we thus conclude that f restricted to the triangle ABC is a projective transformation (see also the Corollary in [Pre] page 567). Similarly, the restrictions of f to BCD, ABD and to CDA are projective transformations, which implies that the map f on the whole quadrilateral $ABCD$ is the restriction of a projective transformation as desired. □

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References

[AAS] S. Artstein-Avidan and B. Slomka, The Fundamental Theorems of Affine and Projective Geometry Revisited. arXiv:1604.01762.

[Bus1] H. Busemann, *Local metric geometry.* Trans. Amer. Math. Soc. **56** (1944) 200–274. Zbl 0061.37403 MR 0010976

[Bus2] H. Busemann, *The geometry of geodesics*, Academic Press (1955), reprinted by Dover in 2005. Zbl 1141.53001

[BK] H. Busemann and P.J. Kelly, *Projective geometry and projective metrics*, Academic Press (1953). Zbl 0052.37305 MR 0054980

[Har] P. de la Harpe, On Hilbert's metric for simplices. *Geometric group theory*, Vol. **1** (Sussex, 1991), 97–119, *London Math. Soc. Lecture Note Ser.* **181**, Cambridge Univ. Press, Cambridge, 1993. Zbl 0832.52002 MR 1238518

[H] *Handbook of Hilbert geometry*, (ed. A. Papadopoulos and M. Troyanov), European Mathematical Society, Zürich, 2014.

[Hil] D. Hilbert, *Ueber die gerade Linie als kürzeste Verbindung zweier Punkte.* Math. Ann. **46** (1895) 91–96. JFM 26.0540.02

[Kas] E. Kasner, *The characterization of collineations.* Bull. Amer. Math. Soc. **9** (1903), 545–546. JFM 34.0716.03 MR 1558030

[LRW] B. Lemmens, M. Roelands and M. Wortel, *Isometries of infinite dimensional Hilbert geometries.* arXiv:1405.4147

[LW] B. Lemmens and C. Walsh, Isometries of polyhedral Hilbert geometries. *J. Topol. Anal.* **3** (2011), 213–241. Zbl 1220.53090 MR 2819195

[PT] A. Papadopoulos and M. Troyanov, *From Funk to Hilbert Geometry.* In *Handbook of Hilbert geometry*, (A. Papadopoulos and M. Troyanov, eds.), European Mathematical Society, Zürich 2014. MR 3329876

[Pre] W. Prenowitz, *The characterization of plane collineations in terms of homologous families of lines.* Trans. Amer. Math. Soc. **38** (1935), 564–599. JFM 61.0626.04 MR 1501829

[Shi] B. SHIFFMAN, Synthetic projective geometry and Poincaré theorem on automorphisms of the ball. *Enseign. Math.* (2) **41** (1995), 201–215. Zbl 0863.51003 MR 1365845

[Wal] C. WALSH, Gauge-reversing maps on cones, and Hilbert and Thompson isometries. arXiv:1312.7871

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