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Pseudoholomorphic simple Harnack curves

Erwan BRUGALLÉ

Abstract. We give a new proof of Mikhalkin's Theorem on the topological classification of simple Harnack curves, which in particular extends Mikhalkin's result to real pseudo-holomorphic curves.

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A non-singular (abstract) *real algebraic curve* is a non-singular complex algebraic curve C equipped with an anti-holomorphic involution $conj_C$. The real part of C, denoted by $\mathbb{R}C$, is by definition the set of fixed points of $conj_C$. If C is compact, then $\mathbb{R}C$ is a disjoint union of at most $g(C)+1$ smooth circles, where $g(C)$ is the genus of C. When $\mathbb{R}C$ has precisely $g(C)+1$ connected components, we say that the real curve C is *maximal*. Equivalently, a real algebraic curve C is maximal if and only if the quotient $C/conj_C$ is a disk with $g(C)$ holes (see for example [Vir84b]).

A *real map* $\phi : C \to \mathbb{C}P^2$ from a real algebraic curve is a map such that $\phi \circ conj_C = conj \circ \phi$, where $conj([x:y:z]) = [\overline{x}:\overline{y}:\overline{z}]$ is the standard complex conjugation on $\mathbb{C}P^2$. Note that $\phi(\mathbb{R}C) \subset \mathbb{R}\phi(C)$ if ϕ is real, however this inclusion might be strict as ϕ may map pairs of $conj_C$-conjugated points to $\mathbb{R}P^2$. Given $\phi : C \to \mathbb{C}P^2$ a real smooth map, a point $p \in \mathbb{R}\phi(C)$ is called a *solitary node* if there exists a neighborhood U of p in $\mathbb{R}P^2$ such that $\phi^{-1}(U) = \phi^{-1}(p)$ which in addition consists of two $conj_C$-conjugated points at which the differential of ϕ is injective (i.e., locally at p, $\phi(C)$ is the transverse intersection of two complex conjugated disks).

1. Introduction

Let L_0, L_1, and L_2 be three distinct real lines in $\mathbb{C}P^2$ with $L_0 \cap L_1 \cap L_2 = \varnothing$. A *simple Harnack curve* is a real algebraic map $\phi : C \to \mathbb{C}P^2$ satisfying the following two conditions:

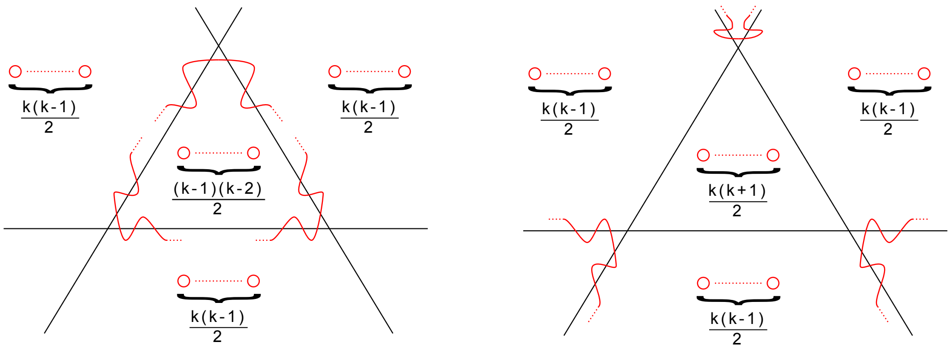
- C is a non-singular maximal real algebraic curve;
- there exist a connected component $\mathcal{O}$ of $\mathbb{R}C$, and three disjoint arcs l_0, l_1, l_2 contained in $\mathcal{O}$ such that $\phi^{-1}(L_i) \subset l_i$.

Note that by Bézout's Theorem, the set $\phi^{-1}(L_i)$ contains finitely many points. We depict in Figure 1 examples of simple Harnack curves with a non-singular image in $\mathbb{C}P^2$ and intersecting transversely all lines L_i. Theorem 1 below says that these are essentially the only simple Harnack curves.

Let $\phi : C \to \mathbb{C}P^2$ be a simple Harnack curve, and choose an orientation of $\mathcal{O}$. This induces an ordering of the intersection points of $\mathcal{O}$ (or C) with L_i, and we denote by s_i the corresponding sequence of intersection multiplicities. Let s be the sequence (s_0, s_1, s_2) considered up to the equivalence relation generated by

$$(s_0, s_1, s_2) \sim (\overline{s}_0, \overline{s}_1, \overline{s}_2), \quad (s_0, s_1, s_2) \sim (s_2, s_0, s_1),$$
$$\text{and} \quad (s_0, s_1, s_2) \sim (s_0, s_2, s_1),$$

where $\overline{(u_i)_{1\le i\le n}} = (u_{n-i})_{1\le i\le n}$. This equivalence relation is such that s does not depend on the chosen orientation on $\mathcal{O}$, nor on the labeling of the three lines L_i.



(a) $d = 2k$ (b) $d = 2k + 1$

FIGURE 1
Simple Harnack curves of degree d and genus $\frac{(d-1)(d-2)}{2}$; in particular three quadrants of $\mathbb{R}P^2 \setminus (\bigcup_{i=0}^{2})\mathbb{R}L_i$ contain $\frac{k(k-1)}{2}$ circles in $\phi(\mathbb{R}C)$, while the fourth one contains either $\frac{(k-1)(k-2)}{2}$ or $\frac{k(k+1)}{2}$ such circles depending on the parity of d.

Theorem 1 (Mikhalkin [Mik00], Mikhalkin-Rullgård [MR01]). *Let $\phi : C \to \mathbb{C}P^2$ be a simple Harnack curve of degree d, and suppose that $\phi(C)$ is the limit of images of a sequence of simple Harnack curves of degree d and genus $g(C) = \frac{(d-1)(d-2)}{2}$. Then the curve $\phi(C)$ has solitary nodes as only singularities (if any). Moreover if either $g(C) = 0$ or $g(C) = \frac{(d-1)(d-2)}{2}$, then the topological type of the pair $\left(\mathbb{R}P^2, \mathbb{R}\phi(C) \bigcup_{i=0}^{2} \mathbb{R}L_i\right)$ depends only on d, $g(C)$, and s.*

Mikhalkin actually proved Theorem 1 for simple Harnack curves in any toric surface, nevertheless this a priori more general statement can be deduced from the particular case of $\mathbb{C}P^2$, see Appendix A.2. Existence of simple Harnack curves of maximal genus with any Newton polygon, and intersecting transversely all toric divisors, was first established by Itenberg (see [IV96]). Simple Harnack curves of any degree, genus, and sequence s were first constructed by Kenyon and Okounkov in [KO06]. In addition, when $g = 0$ they could dispense with the hypothesis that $\phi(C)$ must be the limit of images of a sequence of simple Harnack curves of degree d and genus $g(C) = \frac{(d-1)(d-2)}{2}$. In Theorem 2 below, we delete this hypothesis for any g.

Because they are extremal objects, simple Harnack curves play an important role in real algebraic geometry, and Theorem 1 had a deep impact on subsequent developments in this field. However their importance goes beyond real geometry, as shown by their connection to dimers discovered by Kenyon, Okounkov, and Sheffield in [KOS06].

The goal of this note is to give an alternative proof of Theorem 1. Moreover, our proof is also valid for *real pseudoholomorphic curves*, which are also very important objects in real algebraic and symplectic geometry. Note that a real algebraic map $\phi : C \to \mathbb{C}P^2$ is pseudoholomorphic, but that the converse is not true in general. Mikhalkin's original proof of Theorem 1 uses amoebas of algebraic curves, and does not a priori apply to real pseudoholomorphic maps which are not algebraic.

It is nevertheless possible to read our proof of Theorem 1 in the algebraic category, by going directly to Section 2.2, and defining the map $\pi_i : C \to L_i$ as the composition of ϕ with the linear projection $\mathbb{C}P^2 \setminus \left(L_j \cap L_k\right) \to L_i$, with $\{i, j, k\} = \{0, 1, 2\}$.

We consider $\mathbb{C}P^2$ equipped with the standard Fubini-Study symplectic form ω_{FS}. Recall that an almost complex structure J on $\mathbb{C}P^2$ is said to be *tamed* by ω_{FS} if $\omega_{FS}(v, Jv) > 0$ for any non-null vector $v \in T\mathbb{C}P^2$. Such an almost complex structure is called *real* if the standard complex conjugation $conj$ on $\mathbb{C}P^2$ is J-antiholomorphic (i.e. $conj \circ J = J^{-1} \circ conj$). For example, the standard complex structure on $\mathbb{C}P^2$ is a real almost complex structure.

Let (C,ω) be a compact symplectic surface equipped with a complex structure J_C tamed by ω, and a J_C-antiholomorphic involution $conj_C$, and let J be a real almost complex structure on $\mathbb{C}P^2$. A symplectomorphism $\phi : C \to \mathbb{C}P^2$ is a *real J-holomorphic map* if

$$d\phi \circ J_C = J \circ d\phi \quad \text{and} \quad \phi \circ conj_C = conj \circ \phi.$$

It is of degree d if $\phi_*([C]) = d[\mathbb{C}P^1]$ in $H_2(\mathbb{C}P^2;\mathbb{Z})$. Recall that any intersection of two J-holomorphic curves is positive (see [MS12, Appendix E]).

The definition of simple Harnack curves extends immediately to the real J-holomorphic case. Given three distinct real J-holomorphic lines L_0, L_1, and L_2 in $\mathbb{C}P^2$ such that $\bigcap_{i=0}^2 L_i = \varnothing$, a real J-holomorphic curve $\phi : C \to \mathbb{C}P^2$ is a simple Harnack curve if C is maximal, and if there exists a connected component $\mathcal{O}$ of $\mathbb{R}C$, and three disjoint arcs l_0, l_1, l_2 contained in $\mathcal{O}$ such that $\phi^{-1}(L_i) \subset l_i$.

Theorem 2. *Let $\phi : C \to \mathbb{C}P^2$ be a J-holomorphic simple Harnack curve of degree d. Then the curve $\phi(C)$ has solitary nodes as only singularities (if any). Moreover if either $g(C) = 0$ or $g(C) = \frac{(d-1)(d-2)}{2}$, then the topological type of the pair $\left(\mathbb{R}P^2, \mathbb{R}\phi(C) \bigcup_{i=0}^2 \mathbb{R}L_i\right)$ does not depend on J, once d and s are fixed.*

It follows from Theorem 2 that Figure 1 suffices to recover all topological types of pairs $\left(\mathbb{R}P^2, \mathbb{R}\phi(C) \bigcup \cup_{i=0}^2 \mathbb{R}L_i\right)$ where $\phi : C \to \mathbb{C}P^2$ is a simple Harnack curve, see Appendix A.1. As in the case of algebraic curves, one may generalize Theorem 2 to J-holomorphic simple Harnack curves in any toric surface, see Appendix A.2.

The proof of Theorem 2 proceeds along the following lines: the three projections from $\mathbb{C}P^2 \setminus \left(L_j \cap L_k\right)$ to L_i induce three ramified coverings $\pi_i : C \to L_i$; by considering the arrangement of the real Dessins d'enfants $\pi_i^{-1}(\mathbb{R}L_i)$ on $C/conj_C$, we deduce the number of connected components of $\mathbb{R}\phi(C)$ in each quadrant of $\mathbb{R}P^2 \setminus \left(\cup_{i=0}^2 \mathbb{R}L_i\right)$, as well as its complex orientation; the mutual position of all these connected components is then deduced from Rokhlin's complex orientation formula.

2. Proof of Theorem 2

Let $\phi : C \to \mathbb{C}P^2$ be a J-holomorphic simple Harnack curve in $\mathbb{C}P^2$ of degree d and genus g. We define $p_{i,j} = L_i \cap L_j$.

2.1. Construction of the maps $\pi_i : C \to L_i$. Gromov proved in [Gro85] that there exists a unique J-holomorphic line passing through two distinct points in $\mathbb{C}P^2$. By uniqueness, this line is real if the two points are in $\mathbb{R}P^2$, hence there exists a real pencil of J-holomorphic lines through any point of $\mathbb{R}P^2$. In particular if $\{i, j, k\} = \{0, 1, 2\}$, the map $\mathbb{C}P^2 \setminus \{p_{j,k}\} \to L_i$, which associates to each point p the unique intersection point of L_i with the J-holomorphic line passing through p and $p_{j,k}$, is a real smooth map. We define $\pi_i : C \to L_i$ as the composition of ϕ with this projection. By positivity of intersections of J-holomorphic curves, the map π_i is a real ramified covering.

2.2. Dessins d'enfants on C. We denote by $\widetilde{C}$ the quotient of C by $conj_C$. Since C is maximal, the surface $\widetilde{C}$ is a disk with g holes.

Let $\Gamma_i \subset \widetilde{C}$ be the graph $\pi_i^{-1}(\mathbb{R}L_i)/conj_C$. Note that $\Gamma_j \cap \Gamma_k = \phi^{-1}(\mathbb{R}P^2)$ if $j \neq k$, in particular $\Gamma_j \cap \Gamma_k = \bigcap_{i=0}^2 \Gamma_i$. We call a *triple point* an isolated point in $\bigcap_{i=0}^2 \Gamma_i$. By construction, a triple point corresponds to a singular point of $\phi(C)$ in $\mathbb{R}P^2$, where at least two complex conjugated non-real branches intersect. By the adjunction formula (see [MS12, Chapter 2] in the case of J-holomorphic curves), the graph $\bigcup_{i=0}^2 \Gamma_i$ has no more than $\frac{(d-1)(d-2)}{2} - g$ triple points, and $\phi(C)$ is nodal with only solitary nodes in case of equality.

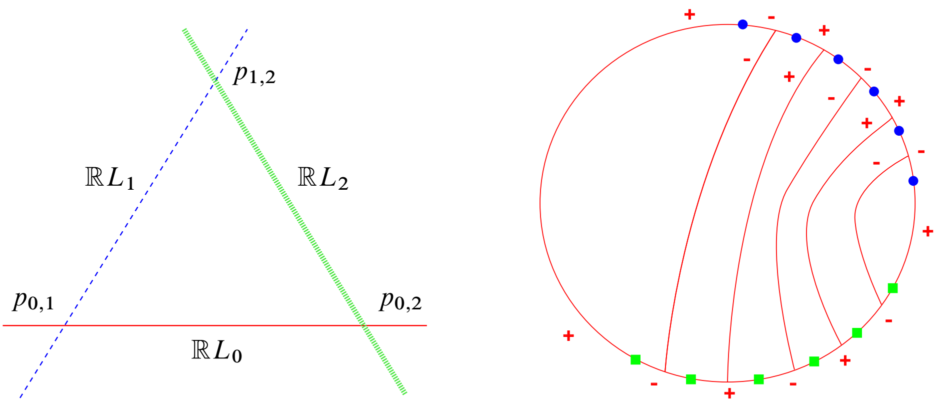
Let $\{i, j, k\} = \{0, 1, 2\}$. We label by $+$ (resp. $-$) the connected component of $\mathbb{R}L_i \setminus \{p_{i,j}, p_{i,k}\}$ containing (resp. disjoint from) $\phi(\mathcal{O}) \cap L_i$. We endow each connected component of $\Gamma_i \setminus \pi_i^{-1}\left(\{p_{i,j}, p_{i,k}\}\right)$ with the sign of the corresponding component of $\mathbb{R}L_i \setminus \{p_{i,j}, p_{i,k}\}$. We also denote by $(\varepsilon_0, \varepsilon_1) \in \{+,-\}^2$ the connected component of $\mathbb{R}P^2 \setminus \left(\bigcup_{i=0}^2 \mathbb{R}L_i\right)$ which project to the components labeled by ε_0 and ε_1 of $\mathbb{R}L_0$ and $\mathbb{R}L_1$ under the projections of center $p_{1,2}$ and $p_{0,2}$ respectively.

The map $\pi_i : C \to L_i$ is a ramified covering of degree d, so by the Riemann-Hurwitz formula it has exactly $2(d + g - 1)$ ramification points (counted with multiplicity). Given $j \neq i$, a subarc of l_j connecting two consecutive points in $l_j \cap \phi^{-1}(L_j)$ has to contain a ramification point of π_i in its interior, and a point of contact of order c of l_j with $\mathbb{R}L_j$ is a ramification point of multiplicity $c - 1$ of π_i. Alltogether, the set $l_j \cup l_k$ with $\{i, j, k\} = \{0, 1, 2\}$ contains at least $2(d - 1)$ ramification points of π_i (counted with multiplicity). Moreover a connected component of $\mathbb{R}C$ distinct from $\mathcal{O}$ contains at least two ramification points of π_i. Since C has $g + 1$ connected components, if follows that these two previous lower bounds are in fact equalities, in particular all ramification points of π_i are real. This implies that each connected component of $\widetilde{C} \setminus \Gamma_i$ is a disk, and that the restriction of π_i on this disk is a homeomorphism to one the two hemispheres of $L_i \setminus \mathbb{R}L_i$.

Lemma 3. *If $g = 0$, then the arrangement of $\bigcup_{i=0}^{2} \Gamma_i$ in $\widetilde{C}$ depends only, up to orientation preserving homeomorphism, on d and s. In particular it has exactly $\frac{(d-1)(d-2)}{2}$ triple points.*

Proof. Since π_i has no ramification point outside $\mathcal{O}$, the graph Γ_i decomposes $\widetilde{C}$ into a chain of disks, where two adjacent disks intersect along (the closure of) a connected component of $\Gamma_i \setminus \mathcal{O}$. See Figure 2 in the case when $d = 6$ and $\phi^{-1}(L_i)$ consists of 6 distinct points. By definition, the points of Γ_i in l_i are endowed with the sign $+$.

By the adjunction formula, the number of intersection points of the graphs Γ_i and Γ_j, with $i \neq j$, is not more than $\frac{(d-1)(d-2)}{2} = 1 + 2 + \ldots + d - 2$. However, this number is clearly the minimal number of intersection point of Γ_i and Γ_j, and there exists a unique mutual position of those graphs that achieves this lower bound (see Figure 3a). The lemma follows immediately by symmetry (see Figure 3b). □

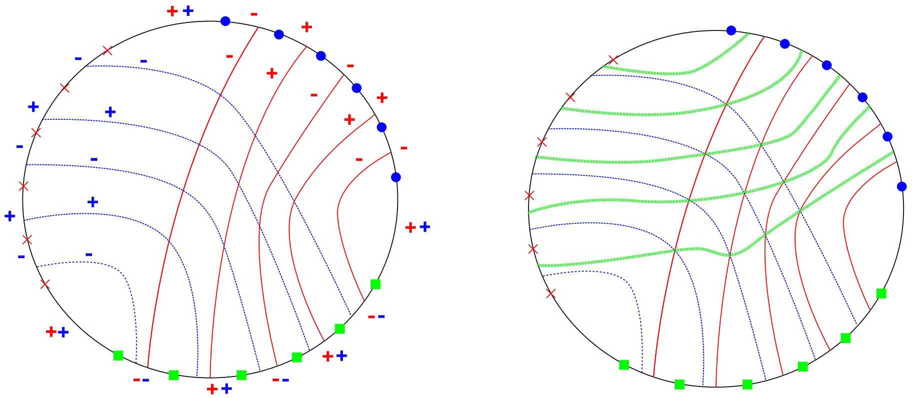


(a) $\mathbb{R}L_0 \cup \mathbb{R}L_1 \cup \mathbb{R}L_2$ in $\mathbb{R}P^2$

(b) The graph $\Gamma_0 = \pi_0^{-1}(L_0)$, dots and squares being points in $\phi^{-1}(L_1)$ and $\phi^{-1}(L_2)$ respectively

FIGURE 2

Simple Harnack curves of degree d and genus $\frac{(d-1)(d-2)}{2}$; in particular three quadrants of $\mathbb{R}P^2 \setminus (\bigcup_{i=0}^{2})\mathbb{R}L_i$ contain $\frac{k(k-1)}{2}$ circles in $\phi(\mathbb{R}C)$, while the fourth one contains either $\frac{(k-1)(k-2)}{2}$ or $\frac{k(k+1)}{2}$ such circles depending on the parity of d.



(a) $\Gamma_0 \cup \Gamma_1$, crosses being points in $\phi^{-1}(L_0)$

(b) $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2$

FIGURE 3

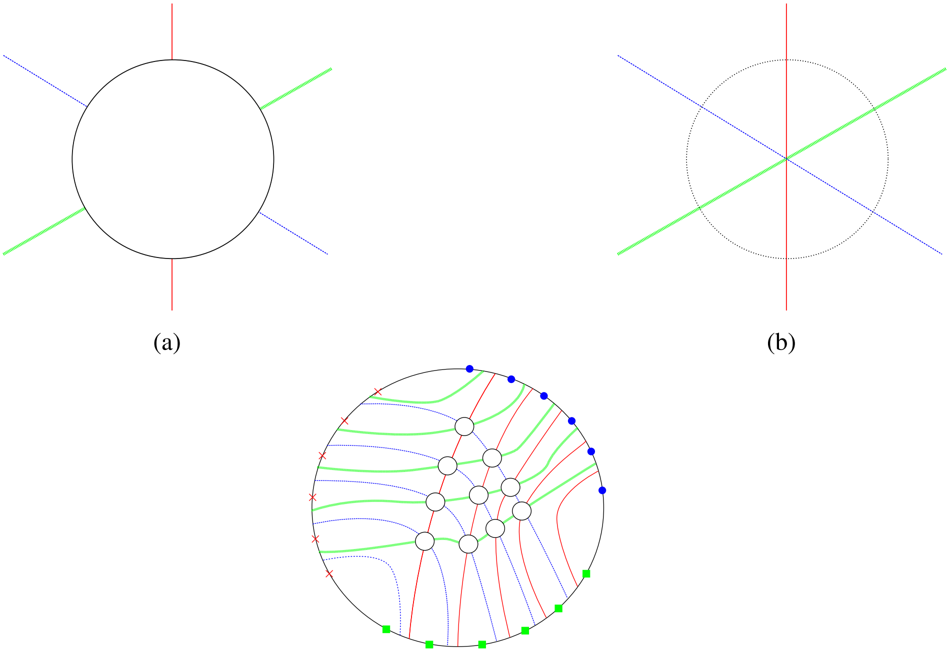
In case of positive genus, we have the following lemma.

Lemma 4. *The arrangement of* $\bigcup_{i=0}^{2} \Gamma_i$ *has exactly* $\frac{(d-1)(d-2)}{2} - g$ *triple points. Moreover if* $d = 2k$ *(resp.* $d = 2k+1$*), then* $\mathbb{R}\phi(C)$ *has exactly* $\frac{(k-1)(k-2)}{2}$ *(resp.* $\frac{k(k+1)}{2}$*) connected components in the quadrant* $(+,+)$ *(resp.* $(-,-)$*), and* $\frac{k(k-1)}{2}$ *connected components in each of the other quadrants.*

Proof. Locally around each boundary component of $\widetilde{C}$ distinct from $\mathcal{O}$, the graph $\bigcup_{i=0}^{2} \Gamma_i$ looks like in Figure 4a. In particular, we may glue a disk as depicted in Figure 4b. Performing this operation to each boundary component of $\widetilde{C}$ distinct from $\mathcal{O}$, the lemma is proved with the same arguments as Lemma 3. □

Even if this will eventually follows from Theorem 2, we do not claim that the disk gluing in the proof of Lemma 4 has any interpretation in terms of degenerations of $\phi(C)$. Note that when $g = \frac{(d-1)(d-2)}{2}$, the arrangement $\bigcup_{i=0}^{2} \Gamma_i$ depends only, up to orientation preserving homeomorphism, on d and s. See Figure 4c in the case $d = 6$.

2.3. Application of Rokhlin's complex orientation formula. To end the proof of Theorem 2 in the case $d = 2k$, it remains to prove the following lemma. The case of curves of odd degree is entirely similar, and is left to the reader.



(a) (b)

(c)

FIGURE 4

Lemma 5. *The following hold:*

(1) *$\phi(\gamma)$ bounds a disk in $\mathbb{R}P^2$ disjoint from $\mathbb{R}\phi(C \setminus \gamma)$ for any connected component γ of $\mathbb{R}C \setminus \mathcal{O}$;*

(2) *a connected component of $\mathbb{R}\phi(C \setminus \mathcal{O})$ is contained in the disk bounded by $\phi(\mathcal{O})$ in $\mathbb{R}P^2$ if and only if it is contained in the quadrant $(+,+)$.*

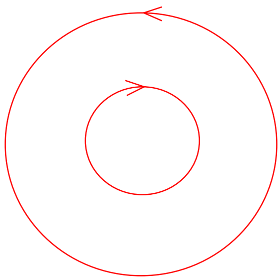
Proof. These two facts will be a consequence of Rokhlin's complex orientation formula ([Rok74] see also [Vir84b]). Since there exists a smoothing $\phi'(C')$ of $\phi(C)$ where $\phi' : C \to \mathbb{C}P^2$ is a real J'-holomorphic curve of degree d and genus $\frac{(d-1)(d-2)}{2}$, we may assume[1] from now on that C has genus $\frac{(d-1)(d-2)}{2}$. Analogously, we may further assume for simplicity that $\phi(C)$ intersects transversely the three J-holomorphic lines L_i.

Recall that since C is maximal, the set $C \setminus \mathbb{R}C$ has two connected components. Moreover the choice of one of these components induces an orientation of $\mathbb{R}C$

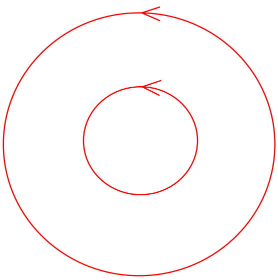
[1] This assumption is intended to simplify the exposition, and is not formally needed for our purposes. Indeed, there exists a generalization of Rokhlin's formula for nodal curves that we could also have used here ([Zvo83] see also [Vir96])

(as boundary). The effect of choosing the other component of $C \setminus \mathbb{R}C$ is to reverse the orientation of $\mathbb{R}C$. Hence there is a canonical orientation, up to a global change of orientation of $\mathbb{R}C$, of all connected components of $\mathbb{R}C$. This orientation is called the *complex orientation of* $\mathbb{R}C$.

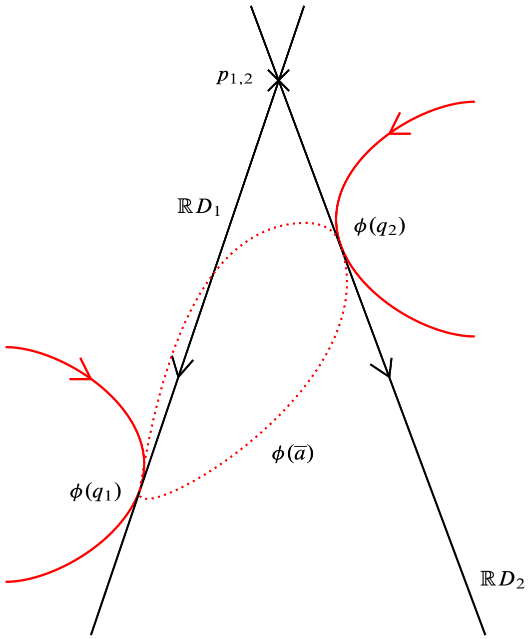
Recall also that a disjoint pair of embedded circles in $\mathbb{R}P^2$ is said to be *injective* if their union bounds an annulus A. If the two circles are oriented and form an injective pair, this latter is said to be *positive* if the two orientations is induced by some orientation of A, and is said to be *negative* otherwise, see Figure 5a and b.



(a) A positive pair



(b) A negative pair



(c) Fiedler's orientation rule

FIGURE 5

We denote respectively by Π_+ and Π_- the number of positive and negative injective pairs of connected components of $\phi(\mathbb{R}C)$ equipped with their complex orientation. Rokhlin's complex orientation formula reduces in our case to

$$\Pi_+ - \Pi_- = \frac{(k-1)(k-2)}{2}. \tag{1}$$

Now we apply Fiedler's orientation rule ([Fie83] see also [Vir84b]) to estimate the quantities Π_+ and Π_-. Consider the projection $\pi_0 : C \to L_0$, and choose an arc a of $\Gamma_0 \setminus \mathbb{R}C$. The arc a lifts to a pair of $conj_C$-conjugated arcs in C, whose topological closure in C, denoted by $\overline{a}$, is homeomorphic to S^1. The set $\overline{a} \cap \mathbb{R}C$ consists of two ramification points q_1 and q_2 of π_0. By construction, each of these two points q_i corresponds to a tangency of $\phi(C)$ with a real J-holomorphic line D_i passing through $p_{1,2}$. Choose a complex orientation of $\mathbb{R}C$, and orient $\mathbb{R}D_1$ in a way compatible with the complex orientation of $\mathbb{R}\phi(C)$ at $\phi(q_1)$, see Figure 5c. Transport this orientation to $\mathbb{R}D_2$ via the portion of the pencil of J-holomorphic lines through $p_{1,2}$ that intersect $\phi(\overline{a})$. Fiedler's orientation rule states that this orientation of $\mathbb{R}D_2$ is still compatible with the complex orientation of $\mathbb{R}\phi(C)$ at $\phi(q_2)$, see Figure 5c.

It follows from Lemmas 3 and 4 that $\phi(q_1)$ is contained in the quadrant $(\varepsilon_1, \varepsilon_2)$ if and only if $\phi(q_2)$ is contained in the quadrant $(\varepsilon_1, -\varepsilon_2)$, see Figures 3 and 4. Hence Fiedler's orientation rule implies that the complex orientation of the curve $\phi(C)$ is as depicted in Figure 6. In particular if γ_1 and γ_2 are two distinct connected components of $\phi(\mathbb{R}C)$ which form an injective pair, we see that this pair contributes to Π_+ if and only if $\gamma_i = \phi(\mathcal{O})$ and γ_{3-i} is in the quadrant $(+,+)$. Hence we deduce from Lemma 4 that

$$\Pi_+ \leq \frac{(k-1)(k-2)}{2} \quad \text{and} \quad \Pi_- \geq 0,$$

with equality if and only if the conclusion of the lemma holds. Now the result follows from Equation (1). □

Remark 6. It is proved in [Mik00] that the index map defined in [FPT00] provides a pairing between connected components of $\mathbb{R}\phi(C \setminus \mathcal{O})$ and points with integer coordinates in the interior of the triangle Δ_d with vertices $(0,0)$, $(d,0)$, and $(0,d)$. It is interesting that this pairing is also visible from the arrangements $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2$, see Figures 3 and 4. In addition to the pairing, a triangulation of Δ_d (dual to a honeycomb tropical curve) is also visible in these pictures. I do not know whether this subdivision has any interpretation.

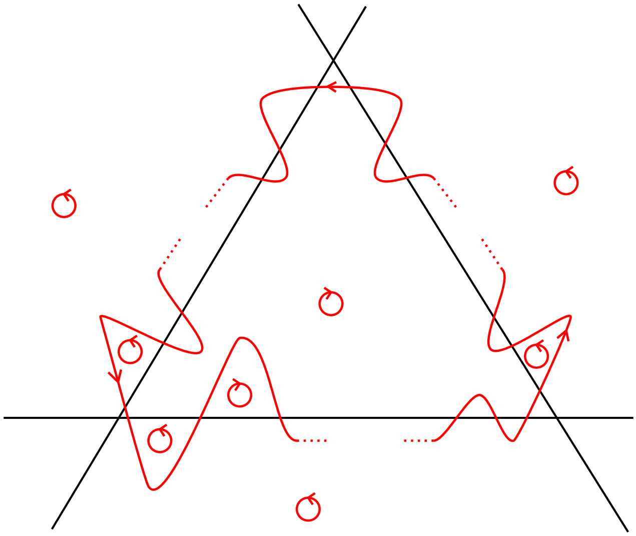


FIGURE 6

Appendix A

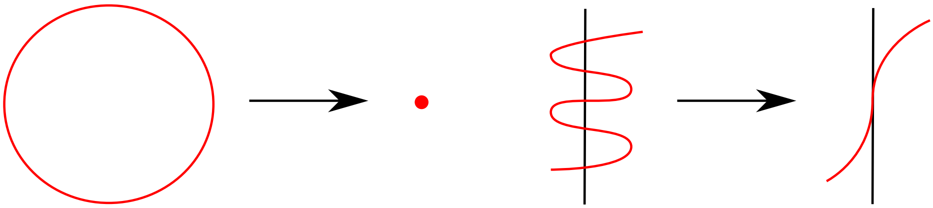
As consequences of Theorem 1, we generalize to simple J-holomorphic Harnack curves some facts that are well known for simple algebraic Harnack curves.

A.1. Topological types of simple Harnack curves. Here we deduce from Theorem 2 all topological types of pairs $(\mathbb{R}P^2, \mathbb{R}\phi(C) \bigcup \cup_{i=0}^2 \mathbb{R}L_i)$, where $\phi : C \to \mathbb{C}P^2$ is a simple Harnack curve.

Proposition 7. *Let $\phi : C \to \mathbb{C}P^2$ be a simple J-holomorphic Harnack curve of degree d. Then the topological type of the pair $(\mathbb{R}P^2, \mathbb{R}\phi(C) \bigcup \cup_{i=0}^2 \mathbb{R}L_i)$ is obtained from Figure 1 by performing finitely many of the two following operations:*

- *the contraction of a circle disjoint from $\cup_{i=0}^2 \mathbb{R}L_i$ to a point, see Figure 7a;*
- *the replacement of u_j consecutive intersection points with $\mathbb{R}L_i$ by a point of order of contact u_j, see Figure 7b.*

Conversely, any such topological type is realized by a J-holomorphic Harnack curve of degree d.



(a) (b)

FIGURE 7

Proof. Indeed, let $\phi' : C' \to \mathbb{C}P^2$ be a simple J'-holomorphic Harnack curve of degree d and genus $\frac{(d-1)(d-2)}{2}$ such that $\phi'(C')$ is a smoothing of $\phi(C)$, and $\phi'(C')$ intersects transversely a J'-holomorphic perturbation L_i' of L_i for $i = 0, 1, 2$. According to the proof of Theorem 2, the topological type of the pair $(\mathbb{R}P^2, \mathbb{R}\phi'(C') \bigcup \cup_{i=0}^2 \mathbb{R}L_i')$ is given Figure 1. This proves that the topological type of the pair $(\mathbb{R}P^2, \mathbb{R}\phi(C) \bigcup \cup_{i=0}^2 \mathbb{R}L_i)$ is as stated in the proposition.

Analogously, to prove the second statement, it is enough to exhibit a rational Harnack curve of degree d intersecting each lines L_i in a single point of order of contact d. According to Theorem 2, the map

$$\begin{array}{rccc} \phi : & \mathbb{C}P^1 & \longrightarrow & \mathbb{C}P^2 \\ & [x:y] & \longmapsto & [x^d : y^d : (x-y)^d] \end{array}$$

is such a rational Harnack curve. □

A.2. Simple Harnack curves in other toric surfaces. Here we deduce the classification of simple Harnack curves in any toric surface from the classification of simple Harnack curves in $\mathbb{C}P^2$. Theorem 10 below can be proved along the same lines as Theorem 2. The reason why we restricted to $\mathbb{C}P^2$ in Theorem 2 is that, thanks to symmetries, the proof in this particular case is much more transparent and avoids purely technical complications. Furthermore Theorem 10 can be deduced from Theorem 2 thanks to Viro's patchworking. We briefly indicate below how to perform this reduction. We refer to [Vir84a, Vir89, Shu05] for references to patchworking, and to [IS02] for its J-holomorphic version.

Let $\Delta \subset \mathbb{R}^2$ be a convex polygon with vertices in $\mathbb{Z}^2$, and let X_Δ be the complex algebraic toric surface associated to Δ, see [GKZ94]. The complement of the maximal toric orbit of X_Δ is denoted by ∂X_Δ, and is called the *toric boundary* of X_Δ. There is a natural correspondence $e \leftrightarrow X_e$ between edges of Δ and irreducible components of ∂X_Δ, which satisfies $e \cap e' \neq \emptyset$ if and only if $X_e \cap X_{e'} \neq \emptyset$. Note that X_Δ might have isolated singularities located at intersections $X_e \cap X_{e'}$ of irreducible components of ∂X_Δ. Recall that Δ induces an embedding of X_Δ into some projective space $\mathbb{C}P^N$, and we equip X_Δ with the restriction, still denoted by ω_{FS}, of the corresponding Fubini-Study symplectic form. An almost complex structure J on X_Δ tamed by ω_{FS} is said to be *compatible* if it coincides with the toric complex structure on X_Δ in a neighborhood of ∂X_Δ, and *real* if the standard complex conjugation on $(\mathbb{C}^*)^2 = X_\Delta \setminus \partial X_\Delta$ is J-antiholomorphic.

Let (C, ω) be a compact symplectic surface equipped with a complex structure J_C tamed by ω, and a J_C-antiholomorphic involution $conj_C$, and let J be a real compatible almost complex structure on X_Δ. A real J-holomorphic map $\phi : C \to X_\Delta$ is said to have degree Δ if $\phi_*([C])$ is equal, in $H_2(X_\Delta; \mathbb{Z})$, to the class realized by a hyperplane section of X_Δ for the embedding induced by Δ. By the adjunction formula, a J-holomorphic map $\phi : C \to X_\Delta$ of degree Δ which does not factorize through a non-trivial ramified covering has genus at most the number of integer points in the interior $\overset{\circ}{\Delta}$ of Δ. Furthermore $\phi(C)$ is non-singular in case of equality.

Definition 8. *Let $\Delta \subset \mathbb{R}^2$ be a convex polygon with vertices in $\mathbb{Z}^2$, and let $[e_1, \ldots, e_k]$ be the natural cyclic ordering on the edges of Δ. A* ***simple Harnack curve*** *of degree Δ is a real J-holomorphic map $\phi : C \to X_\Delta$ of degree Δ, for some real compatible almost complex structure J on X_Δ, satisfying the following three conditions:*

- *C is a non-singular maximal real curve;*
- *there exist a connected component $\mathcal{O}$ of $\mathbb{R}C$, and k disjoint arcs $l_1, \ldots, l_k$ contained in $\mathcal{O}$ such that $\phi^{-1}(X_{e_i}) \subset l_i$;*
- *the cyclic orientation on the arcs l_i induced by $\mathcal{O}$ is precisely $[l_1, \ldots, l_k]$.*

Note that the last condition is non-empty only when $k \geq 4$.

Example 9. For Δ_d the triangle with vertices $(0,0)$, $(d,0)$, and $(0,d)$, the surface X_{Δ_d} is the projective plane equipped with a homogeneous coordinate system, and ∂X_{Δ_d} is the union of the three coordinate lines. A simple Harnack curve of degree Δ_d is a simple Harnack curve of degree d in the sense of Section 1. Note however that a J-holomorphic simple Harnack curve of degree d might not be a simple Harnack curve of degree Δ_d, since J is not required to be integrable in a neighborhood of the coordinate axis. This additional requirement is necessary when one wants to consider more general toric surfaces.

As in Section 1, given $\phi : C \to X_\Delta$ a simple Harnack curve, we encode in a sequence the intersections of $\phi(\mathcal{O})$ with the components of ∂X_Δ. The choice of an orientation of $\mathcal{O}$ induces an ordering of the intersection points of $\mathcal{O}$ with X_{e_i}, and we denote by s_i the corresponding sequence of intersection multiplicities. Let s be the sequence $(s_1, \dots, s_k)$ considered up to the equivalence relation generated by

$$(s_1, \dots, s_k) \sim (\overline{s}_1, \dots, \overline{s}_k), \quad (s_1, \dots, s_k) \sim (s_k, s_1, \dots, s_{k-1}),$$

and

$$(s_1, \dots, s_k) \sim (s_k, s_{k-1}, \dots, s_1).$$

Recall that $\overline{(u_i)_{1 \leq i \leq n}} = (u_{n-i})_{1 \leq i \leq n}$.

Theorem 10. *Let $\Delta \subset \mathbb{R}^2$ be a convex polygon with vertices in $\mathbb{Z}^2$, and let $\phi : C \to X_\Delta$ be a simple Harnack curve of degree Δ. Then the curve $\phi(C)$ has solitary nodes as only singularities (if any). Moreover if either $g(C) = 0$ or $g(C) = |\mathbb{Z}^2 \cap \mathring{\Delta}|$, then the topological type of the pair $\big((\mathbb{R}^*)^2, \mathbb{R}\phi(C) \cap (\mathbb{R}^*)^2\big)$ depends only on Δ, $g(C)$, and s.*

Proof. Let us assume for simplicity that $\phi(C)$ intersects ∂X_Δ transversely, and suppose for a moment that we have proved the following:

Claim: for any edge e of Δ, the cyclic orders on the finite set $\mathcal{O} \cap \mathbb{R}X_e$ induced by $\mathcal{O}$ and $\mathbb{R}X_e$ coincide.

Assuming this claim, one constructs exactly as in the proof of [KRS01, Theorem 2(1)] a simple Harnack curve in $\mathbb{C}P^2$ by patchworking $\phi(C)$ with finitely many simple algebraic Harnack curves constructed in [IV96]. Theorem 10 now follows from Theorem 2.

Hence it remains to prove the claim. Let e be an edge of Δ, and define $\mathring{X}_e$ to be X_e from which we remove its two intersection points with the other irreducible components of ∂X_Δ. Since the almost complex structure on Δ is integrable in a neighborhood of ∂X_Δ, there exists a J-holomorphic compactification of $(\mathbb{C}^*)^2 \cup \mathring{X}_e$ into $\mathbb{C}P^2 = (\mathbb{C}^*)^2 \cup L_0 \cup L_1 \cup L_2$ where L_i is a J-holomorphic line in $\mathbb{C}P^2$, and L_0 is a compactification of $\mathring{X}_e$. The map ϕ induces a J-holomorphic map $\phi' : C \to \mathbb{C}P^2$, and exactly as in the beginning of Section 2.2, one proves that the map $\pi_0 : C \to L_0$ has no ramification point on the connected component of $\mathcal{O} \setminus \phi'^{-1}(L_1 \cup L_2)$ containing $\phi'^{-1}(L_0)$. This says precisely that the cyclic orders on the set $\mathcal{O} \cap \mathbb{R}X_e$ induced by $\mathcal{O}$ and $\mathbb{R}X_e$ coincide. □

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Erwan BRUGALLÉ, École Polytechnique, Centre Mathématiques Laurent Schwartz, 91128 Palaiseau Cedex, France

e-mail: erwan.brugalle@math.cnrs.fr