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An example of a non-algebraizable singularity of a holomorphic foliation

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Abstract. We say that the the germ of a singular holomorphic foliation on $(\mathbb{C}^2, 0)$ is algebraizable whenever it is holomorphically conjugate to the singularity of a foliation defined globally on a projective algebraic surface. The object of this work is to construct a concrete example of a non-algebraizable singularity. Previously only existential results were known [GT, CS].

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1. Introduction

Let $\mathcal{F}$ be the germ of a holomorphic foliation on $(\mathbb{C}^2, 0)$ with an isolated singularity. We are interested in understanding whether there exists or not a complex projective algebraic surface S and a point p on it, such that $\mathcal{F}$ is holomorphically conjugate to the germ at p of a globally defined foliation on S. Those germs for which such an algebraic foliation exists are called *algebraizable*. The existence of non-algebraizable singularities remained unknown until Genzmer and Teyssier proved in [GT] the existence of countably many classes of saddle-node singularities which are not algebraizable. Their proof however, does not provide us with any concrete examples of such singularities and, as far as the author knows, no concrete examples of non-algebraizable singularities are known. Following Casale [Cas], we split the problem into two parts: first, to give an example of a germ of a non-algebraizable singularity; second, to identify algebraizable singularities. In this paper we address the first question and construct explicitly the germ of a degenerate singularity of order two (i.e. of *algebraic multiplicity* two) on $(\mathbb{C}^2, 0)$ which is not algebraizable.

The strategy we shall follow to construct our example is based on the following observation: Any algebraic singularity depends on finitely many complex parameters – these generate a field extension of $\mathbb{Q}$ of finite transcendence degree. Note, however, that an arbitrary change of coordinates need not preserve this finiteness. In order to keep track of this transcendence degree in a coordinate independent way, we introduce the following definition.

Definition 1. Let $\eta = \alpha(x,y)\,dx + \beta(x,y)\,dy$ be a 1-form on $(\mathbb{C}^2,0)$. Denote by $\mathbb{Q}(\eta)$ the field extension of $\mathbb{Q}$ obtained by adjoining to $\mathbb{Q}$ the coefficients on the power series expansion of α and β. We define the *transcendence degree* of η to be

$$\operatorname{tr.deg}(\eta) = \min\big\{\operatorname{tr.deg}\big(\mathbb{Q}(\tilde{\eta})/\mathbb{Q}\big) \mid \tilde{\eta} \text{ is formally conjugate to } \eta\big\},$$

where $\operatorname{tr.deg}(\mathbb{Q}(\tilde{\eta})/\mathbb{Q})$ denotes the transcendence degree of the field extension $\mathbb{Q}(\tilde{\eta})/\mathbb{Q}$.

Above, and throughout this text, we say that two 1-forms η, ω on $(\mathbb{C}^2,0)$ are *formally conjugate* if there exists a formal change of coordinates Φ and a unit $\mathcal{K} \in \mathbb{C}[[x,y]]$ such that

$$\Phi^*\eta = \mathcal{K}\omega.$$

Remark 2. From the above definition it is clear that any polynomial 1-form has finite transcendence degree. In order to define a non-algebraizable singularity, we need only construct a 1-form of infinite transcendence degree.

Our main result is stated below.

Theorem 3. *Let $\lambda_1, \lambda_2, \lambda_3$ be non-rational numbers satisfying $\lambda_1 + \lambda_2 + \lambda_3 = 1$, and let $f_j = a_j x + b_j y$, $j = 1,2,3$, be arbitrary different linear forms in $\mathbb{C}[x,y]$. Define the homogeneous quadratic 1-form*

$$\omega_0 = f_1 f_2 f_3 \sum_{j=1}^{3} \lambda_j \frac{df_j}{f_j}.$$

Let $\mathcal{B} = \{b_0, b_1, \ldots\}$ be a subset of $\mathbb{C}$ such that the field extension $\mathbb{Q}(\mathcal{B})/\mathbb{Q}$ has infinite transcendence degree and such that the power series

$$b(x) = \sum_{k=0}^{\infty} b_k x^k$$

has a positive radius of convergence. Then the germ of the holomorphic foliation on $(\mathbb{C}^2,0)$ defined by the 1-form

$$\omega = \omega_0 + x^2 b(x)(x\,dy - y\,dx) \tag{1}$$

has infinite transcendence degree and thus, according to Remark 2, it is not algebraizable.

Remark 4. In virtue of the Lindemann-Weierstrass theorem, if $\{a_1, a_2, \dots\}$ is a collection of algebraic numbers spanning an infinite-dimensional vector space over $\mathbb{Q}$, then the set $\{e^{a_1}, e^{a_2}, \dots\}$ generates a field extension of $\mathbb{Q}$ of infinite transcendence degree. This gives us an immense amount of flexibility defining the set $\mathcal{B}$. In particular we can choose $\mathcal{B} \subset \mathbb{R}$, and we can make $b(x)$ as rapidly convergent as desired. For example, defining

$$b_k = e^{-k\sqrt{k}}, \qquad k = 0, 1, 2, \dots,$$

gives rise to an entire function $b(x) = \sum_{k=0}^{\infty} b_k x^k$. Under these choices the 1-form ω defined by (1) is analytic on all $\mathbb{C}^2$.

In order to prove Theorem 3 we make use of the formal classification of non-dicritic degenerate singularities given by Ortiz-Bobadilla, Rosales-González and Voronin in [ORV]. Indeed, we shall show in Lemma 8 that given a 1-form η defined over a subfield $\mathbb{K}$ of $\mathbb{C}$ the formal reduction taking this 1-form to its *formal normal form* is given by a map whose coefficients also belong to $\mathbb{K}$. In particular, the coefficients of the formal normal form belong to $\mathbb{K}$. This immediately implies that the transcendence degree $\operatorname{tr.deg}(\mathbb{Q}(\tilde{\eta})/\mathbb{Q})$ is minimized when $\tilde{\eta}$ is a suitable formal normal form of η (cf. Remark 9). The formal normal form in question is discussed in Section 2.

It is worth mentioning that, on the other hand, we do have a few criteria for deciding algebraizability. It is known since Poincaré and Dulac that non-degenerate planar singularities with spectrum on the so-called *Poincaré domain* are analytically equivalent to foliations given by a polynomial 1-form on $\mathbb{C}^2$. In addition, Casale proved in [Cas] that the class of dicritical foliations on $(\mathbb{C}^2, 0)$ which are regular after a single blow-up and have a unique leaf tangent to the exceptional divisor are algebraizable whenever they admit a meromorphic first integral. More recently, Calsamiglia and Sad [CS] generalized this result to the class of all dicritic foliations which are regular after one blow-up process, thus removing the requirement of a single tangency with the exceptional divisor.

An interesting question is whether or not an algebraizable germ is always conjugate to the singularity of a polynomial foliation on $\mathbb{P}^2$. It is proved in [CS] that simple dicritic singularities with a *generic* meromorphic first integral are not only algebraizable, but in fact, conjugate to polynomial singularities on $\mathbb{P}^2$.

2. Formal classification of non-dicritic singularities

We wish to construct a 1-form ω on $(\mathbb{C}^2,0)$ of infinite transcendence degree. As we will see later, it is enough to construct a 1-form which is on its formal normal form (as defined in [ORV]), and such that its defining coefficients generate an extension of $\mathbb{Q}$ of infinite transcendence degree.

For the sake of briefness, we state the theorem on formal classification (cf. Theorem 1.1 and Corollary 1.4 in [ORV]) only for non-dicritic degenerate singularities of order two. For our own convenience, we state their result in terms of differential forms and not vector fields; the adaptation is straightforward.

Theorem 5 ([ORV]). *A generic non-dicritic 1-form η on $(\mathbb{C}^2,0)$ having a degenerate singularity of order two is formally equivalent to a formal 1-form* H *of the form*

$$\mathrm{H} = \eta_0 + x^2 b(x)(x\,dy - y\,dx), \tag{2}$$

where η_0 is the quadratic homogeneous part of η and $b(x) \in \mathbb{C}[\![x]\!]$. Such normal form is unique up to pull-backs by homotheties and multiplication by a scalar factor.

Let $R(x,y) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ be the radial vector field and let η_0 denote the quadratic homogeneous part of η as above. The *tangent cone* of η is defined to be the polynomial $\eta_0(R)$ and, by definition, η is *non-dicritic* if $\eta_0(R) \not\equiv 0$.

Definition 6. In the above theorem and throughout this text, we will say that a 1-form η having a degenerate singularity of order two is a *generic singularity of order two* if it satisfies:

(i) the tangent cone $T = \eta_0(R)$ is non-zero and has three simple linear factors l_1, l_2, l_3;

(ii) the residues $\alpha_1, \alpha_2, \alpha_3$ of the meromorphic 1-form

$$\frac{\eta_0}{T} = \sum_{j=1}^{3} \alpha_j \frac{dl_j}{l_j}$$

are not rational numbers.

Remark 7. A generic tangent cone as above decomposes into three linear factors which, by a linear change of coordinates, can be normalized to be y, x and $x-y$. From now on we will assume without loss of generality that all foliations under consideration have $T(x,y) = xy(x-y)$ as tangent cone.

3. Proof of Theorem 3

We shall prove first that the 1-form defined in Theorem 3 is not formally equivalent to a polynomial 1-form on $\mathbb{C}^2$. Once we do this it will follow easily that such a singularity cannot be holomorphically conjugate to a singularity of a foliation on a projective algebraic surface.

Lemma 8. *Let $\mathbb{K}$ be a subfield of $\mathbb{C}$ and let $P, Q \in \mathbb{K}[\![x, y]\!]$ be formal power series. Assume $\eta = P\,dx + Q\,dy$ defines a generic singularity of order two. The formal reduction taking η to its formal normal form (2) is given by a formal map defined over the field $\mathbb{K}$. In particular, η has a formal normal form defined over $\mathbb{K}$ – that is, defined by a 1-form whose coefficients belong to $\mathbb{K}[\![x, y]\!]$.*

Remark 9. The above lemma implies that the transcendence degree of $\mathbb{Q}(\tilde{\eta})$, taken over all $\tilde{\eta}$ formally conjugate to η, is minimized at a formal normal form of η. Indeed, such a minimum is guaranteed to exist, say at $\mathbb{K} = \mathbb{Q}(\tilde{\eta})$. By Lemma 8, $\tilde{\eta}$ is formally equivalent to a formal normal form H defined over $\mathbb{K}$. We conclude that $\mathbb{Q}(\mathrm{H})$ is a subfield of $\mathbb{Q}(\tilde{\eta})$ and so

$$\mathrm{tr.deg}\,(\mathbb{Q}(\mathrm{H})/\mathbb{Q}) \leq \mathrm{tr.deg}\,(\mathbb{Q}(\tilde{\eta})/\mathbb{Q}).$$

Proof of Lemma 8. The lemma follows almost immediately from the proof of the formal classification theorem provided in [ORV] where a *pre-normalized* foliation (i.e., a foliation whose separatrix tangent to the line $x = 0$ has been rectified) is reduced to its formal normal form.

Let us first show that, given the 1-form $\eta = P\,dx + Q\,dy$ as above, we can rectify the separatrix tangent to $x = 0$ by a formal change of coordinates defined over $\mathbb{K}$. We proceed recursively assuming the separatrix tangent to $x = 0$ has been rectified up to jets of order k (i.e. $\eta \wedge dx|_{x=0} = O(y^{k+1})$), and defining a formal change of coordinates ϕ_k that will rectify the separatrix up to jets of order $k + 1$. In fact, it is enough to define a polynomial change of coordinates of the form

$$\phi_k(x, y) = (x + c_k\, y^k, y). \tag{3}$$

A short computation shows that if the separatrix tangent to $x = 0$ was indeed rectified up to jets of order k, then the above polynomial change of coordinates rectifies the separatrix up to jets of order $k+1$ for a suitable coefficient $c_k \in \mathbb{C}$. The condition that $\phi_k^*\eta$ has a straight separatrix up to jets of order $k+1$ is given by the equation

$$(\phi_k^*\,\eta) \wedge dx|_{x=0} = O(y^{k+2}),$$

which reduces to a linear equation on c_k defined over $\mathbb{K}$. In particular, $c_k \in \mathbb{K}$ and the map ϕ_k is defined over $\mathbb{K}$. In this way, we can fully rectify the separatrix by a sequence of maps of the form (3). To be more precise, any finite jet of the sequence of polynomial maps

$$\Phi_N = \phi_N \circ \ldots \circ \phi_2, \quad N = 2, 3, \ldots,$$

eventually stabilizes and thus we obtain a well defined formal map $\Phi = \lim_{N\to\infty} \Phi_N$ whose Taylor coefficients belong to the field $\mathbb{K}$.

Because of the above paragraph we may assume without loss of generality that the 1-form η in Lemma 8 has a straight separatrix given by $x = 0$. We can thus proceed with the formal reduction process given in [ORV]. In the aforementioned paper, the form η is brought to its formal normal form by a sequence of maps $H_k(x, y)$ followed by multiplication by functions $\mathcal{K}_k(x, y)$ of the form

$$H_k(x, y) = \big(x + \alpha_k(x, y)\,,\ y + \beta_k(x, y)\big), \qquad \mathcal{K}_k(x, y) = 1 - \delta_k(x, y),$$

where α_k, β_k are homogeneous polynomials of degree k and δ is a homogeneous polynomial of degree $k - 1$. The coefficients of the polynomials $\alpha_k, \beta_k, \delta_k$ are obtained by solving a linear system of equations which are evidently defined over $\mathbb{K}$ (see Sections 2.2 and 2.3 in [ORV]). This shows that the formal reduction process obtained in [ORV] is given by a formal map with coefficients in $\mathbb{K}$. □

Proof of Theorem 3. Let us prove Theorem 3 by contradiction. Consider the 1-form ω defined by equation (1). Suppose ω is locally holomorphically equivalent to a polynomial 1-form $\eta = P\,dx + Q\,dy$. Let $\mathbb{K}$ be the field generated over $\mathbb{Q}$ by the (finitely many) coefficients of $P, Q \in \mathbb{C}[x, y]$. This field necessarily has finite transcendence degree over $\mathbb{Q}$. By Lemma 8 there exists a formal normal form H of η defined over the field $\mathbb{K}$. Since ω and H are formally equivalent and are in their formal normal form, Theorem 5 implies that H and ω differ at most by a linear change of coordinates followed by multiplication by a scalar. Namely, there exits a linear map $A \in \mathrm{GL}(2, \mathbb{C})$ and a complex number $\lambda \in \mathbb{C}$ such that

$$A^*\mathrm{H} = \lambda\,\omega. \tag{4}$$

This, however, is impossible since the left hand side of (4) is given by power series with coefficients over a field of finite transcendence degree, and the right hand side of (4) is given by power series whose coefficients generate a field of infinite transcendence degree. We conclude that the 1-form ω cannot be holomorphically conjugate to a polynomial 1-form on $\mathbb{C}^2$.

Suppose now that there exists a foliation $\mathcal{F}$ on an algebraic surface $S \subset \mathbb{P}^N$ such that the germ of the singularity defined by ω is holomorphically conjugate to the germ of $\mathcal{F}$ at a point $p \in S$. This conjugacy implies that the point p is a smooth point of S, hence we can find a linear projection $f\colon \mathbb{P}^N \to \mathbb{P}^2$ such that the restriction $f|_S\colon S \to \mathbb{P}^2$ is a branched covering map and p a regular point of f. We use the local inverse $f^{-1}\colon (\mathbb{P}^2, f(p)) \to (S, p)$ of the local biholomorphism $f\colon (S, p) \to (\mathbb{P}^2, f(p))$ to define the germ of a singularity

$$\widetilde{\mathcal{F}} = (f^{-1})^*\mathcal{F},$$

around $f(p)$, locally given by a holomorphic (not necessarily polynomial) 1-form η. Without loss of generality, we may consider η to be a 1-form on $(\mathbb{C}^2, 0)$. Note that the map f, the foliation $\mathcal{F}$ and the surface S are all defined by finitely many rational functions in $\mathbb{C}(\mathbb{P}^N)$. These rational functions are each defined by finitely many complex numbers, thus they are simultaneously defined over a subfield $\mathbb{K}$ of $\mathbb{C}$ of finite transcendence degree over $\mathbb{Q}$. Note that if $f\colon S \to \mathbb{P}^2$ is defined over the field $\mathbb{K}$, then the germ $f^{-1}\colon (\mathbb{P}^2, f(p)) \to (S, p)$ is also defined over $\mathbb{K}$, since we have a formal inverse function theorem for the ring $\mathbb{K}[\![x, y]\!]$. Pulling back $\mathcal{F}$ by the map f^{-1} will result in a foliation, induced by the 1-form η, whose defining coefficients belong to $\mathbb{K}$. This implies that η is defined over a field $\mathbb{K} \subset \mathbb{C}$ of finite transcendence degree over $\mathbb{Q}$ and is holomorphically equivalent to ω, a contradiction. □

Note that the last argument actually proves that if S is any projective surface, $p \in S$ a smooth point, and $\mathcal{F}$ a holomorphic foliation on S with an isolated singularity at p, then the singularity of $\mathcal{F}$ at p has finite transcendence degree.

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