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Line, spiral, dense

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Abstract. Exponential of exponential of almost every line in the complex plane is dense in the plane. On the other hand, for lines through any point, for a set of angles of Hausdorff dimension one, exponential of exponential of a line with angle from that set is not dense in the plane. The third iterate of an oblique line is always dense.

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1. Introduction

In 1914, Harald Bohr and Richard Courant showed that for the Riemann zeta function, if $\sigma \in (\frac{1}{2}, 1]$, then $\overline{\zeta(\sigma + i\mathbb{R})} = \mathbb{C}$, i.e., the image of any vertical line with real part in $(\frac{1}{2}, 1]$ is dense [Boh, §4, p.271]. One hundred years on, we ask what happens under the exponential map.

One may picture the exponential map, $\exp : z \mapsto e^z \in \mathbb{C}$, as mapping Cartesian coordinates onto polar coordinates, since $\exp(x + iy) = e^x e^{iy}$. It maps vertical lines to circles centred on 0 and maps horizontal lines to rays emanating from 0. The map is infinite-to-one and $2\pi i$-periodic; preimages of a point lie along a vertical line. Oblique (slanted) lines get mapped to logarithmic spirals.

Applying exponential a second time, what happens? See Figure 1. Circles are compact, so their images are compact. Rays are subsets of lines, so they get mapped into circles, rays or logarithmic spirals. Intriguingly, the image of a logarithmic spiral under exponential is not obvious, and for good reason.

For $p \in \mathbb{C}$, $\alpha \in \mathbb{R}$, let $L_\alpha(p) := \{p + t(i + \alpha) : t \in \mathbb{R}\}$. Set $\mathcal{L}(p) := \{L_\alpha(p) : \alpha \in \mathbb{R}\}$, the family of non-horizontal lines through a point $p \in \mathbb{C}$, parametrised by $\alpha \in \mathbb{R}$. With this parametrisation, there is a natural one-dimensional Lebesgue measure on the set $\mathcal{L}(p)$. It is equivalent to the measure obtained when parametrising the family by angle (points on the half-circle).

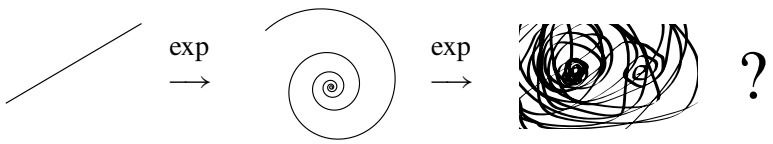

FIGURE 1
Line, spiral, what?

Theorem 1. *Given* $p \in \mathbb{C}$, *for Lebesgue almost every* $\alpha \in \mathbb{R}$,

$$\overline{\exp \circ \exp\left(L_\alpha(p)\right)} = \mathbb{C}.$$

From the topological perspective, a property is *generic* in some space if it holds for all points in a *residual set*, that is, a set which can be written as a countable intersection of open, dense sets.

Theorem 2. *For each* $p \in \mathbb{C}$, *the set* $\{\alpha \in \mathbb{R} : \overline{\exp \circ \exp\left(L_\alpha(p)\right)} = \mathbb{C}\}$ *is residual in* $\mathbb{R}$.

Theorem 3. *The image of an oblique line under* $\exp \circ \exp \circ \exp$ *is dense in* $\mathbb{C}$.

In other words, for each $p \in \mathbb{C}$, for every $\alpha \in \mathbb{R} \setminus \{0\}$,

$$\overline{\exp \circ \exp \circ \exp\left(L_\alpha(p)\right)} = \mathbb{C}.$$

Of course, every subsequent iterate of an oblique line is also dense.

In general, it is hard to determine whether a given line will have dense image or not under $\exp \circ \exp$. Certain ones do, however, and we obtain a concisely defined, explicit, analytic dense curve. Let $a \in (0, 1)$ be the binary Champernowne constant (with binary expansion $0.11011100101\ldots$) or any other number whose binary expansion contains all possible finite strings of zeroes and ones. Let $p_* := \log(2\pi a) + \frac{\pi}{2} i$ and $\alpha_* := \frac{\log 2}{2\pi}$.

Theorem 4. $\overline{\exp \circ \exp\left(L_{\alpha_*}(p_*)\right)} = \mathbb{C}$.

In Theorem 1 we obtained a full-measure set of parameters with dense image. One may be tempted to think that all oblique lines would have dense image under $\exp \circ \exp$. However, this is not true, and to Theorem 1 there is the following complementary statement.

Theorem 5. *For each* $p \in \mathbb{C}$ *and each open set* $X \subset \mathbb{R}$, *the set*

$$\left\{\alpha \in X : \overline{\exp \circ \exp\left(L_\alpha(p)\right)} \neq \mathbb{C}\right\}$$

has Hausdorff dimension 1.

Let Y denote the set of $\theta \in (1,\infty)$ for which $(\theta^k)_{k\geq 0}$ is not dense modulo 1. Kahane [Kah] proved that $Y \subset \mathbb{R}$ has Hausdorff dimension 1; however, in any bounded interval, he only obtained dimension close to 1. In Lemma 8, we establish a connection between intersections of a logarithmic spiral with the imaginary axis and density of the image of the spiral under exponential. This allows us to improve Kahane's result a little.

Corollary 6. *For each open interval $I \subset (1,\infty)$, the set of $\theta \in I$ for which $(\theta^k)_{k\geq 0}$ is not dense modulo 1 has Hausdorff dimension 1.*

Remark: The above-described phenomena are not unique to the exponential map, the most fundamental of transcendental maps. Once one understands exponential, extensions to maps such as $z \mapsto \sin(z), \exp(z^n), \exp\circ\exp(z)$ are not hard to devise, but what of a general statement?

Remark: For a generic entire function of the complex plane, the image of the real line is dense. Indeed, Birkhoff[1] [Bir] showed the existence of an entire function f whose translates $T_n f : x \mapsto f(x-n)$ approximate polynomials in $\mathbb{Q}[x]+i\mathbb{Q}[x]$ arbitrarily well (on compacts). In particular, $(T_n f)_{n\in\mathbb{Z}}$ is dense in the (Fréchet) space of entire functions with the topology of uniform convergence on compacts. Hence, given an open set U of entire functions, there is some $N \in \mathbb{Z}$ with $T_N f \in U$. Since the translation operators T_n are continuous, $\bigcup_{n\in\mathbb{Z}} T_n U$ is an open dense set. Now let $\mathcal{U}$ be a countable basis of open sets for the topology. The set

$$X_{\mathcal{U}} := \bigcap_{U\in\mathcal{U}} \bigcup_{n\in\mathbb{Z}} T_n(U)$$

is residual. Consider $g \in X_{\mathcal{U}}$. One readily checks that the translates $(T_n g)_{n\in\mathbb{Z}}$ enter each set in the basis and hence are dense in the space of entire functions. In particular, the translates approximate all constant functions. Hence $\overline{g(\mathbb{R})} = \mathbb{C}$, as required. The fact that a generic curve has dense image does not tell one what happens for a particular map or for a subfamily (for example, no logarithmic spiral is dense). Besides Birkhoff-style constructions and curves coming from things resembling ζ-functions, we are unaware of other previously-known dense analytic curves.

One can also ask (in the spirit of [BJ1, BJ2]) about the distributions of the curves considered, in the following sense. Given α, p, let $\rho : t \mapsto \exp\circ\exp(p + t(i+\alpha))$, so ρ parametrises $\exp\circ\exp$ of the line $L_\alpha(p)$. For every measurable set A and $T > 1$, let $\mu_T(A) := \frac{1}{2T} m(\{t \in [-T,T] : \rho(t) \in A\})$, where m denotes Lebesgue measure. Then μ_T is a probability measure. With

[1] The author thanks P. Gauthier for a helpful conversation in this regard.

the weak*-topology on the space of probability measures on $\overline{\mathbb{C}}$, we obtain the following unilluminating result.

Theorem 7. *For every oblique line $L_\alpha(p)$, the corresponding measures μ_T satisfy*

$$\lim_{T\to\infty} \mu_T = \frac{\delta_0}{4} + \frac{\delta_1}{2} + \frac{\delta_\infty}{4},$$

where δ_z denotes the Dirac mass at the point z.

We shall use $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ to denote the real and imaginary parts of a complex number z. We denote one-dimensional Lebesgue measure by m and denote the length of an interval I by $m(I)$ or by $|I|$. If $\Sigma : t \mapsto \exp(p+t(i+\alpha))$, then $\frac{d}{dt}\Sigma(t) = \Sigma(t)(i+\alpha)$. Therefore the spiral Σ has tangent of slope $-\alpha$ when it intersects the imaginary axis.

The proofs are provided in linear fashion.

2. Dense analytic curves

In this section we prove Theorems 1–3.

Proof of Theorem 1. Let f denote $\exp\circ\exp$. Fix p and write L_α for $L_\alpha(p)$. Let

$$X_U := \{\alpha : f(L_\alpha) \cap U \neq \varnothing\}. \tag{1}$$

Given a sequence $(q_n)_{n=1}^\infty$ dense in $\mathbb{C}$ and a decreasing sequence of positive reals $(\delta_n)_{n=1}^\infty$ with $\delta_n \to 0^+$, let $\mathcal{U} := \{B(q_n,\delta_n) : n \geq 1\}$. Then a set is dense in $\mathbb{C}$ if and only if it has non-empty intersection with each $U \in \mathcal{U}$. Since $\mathcal{U}$ is countable, if for each $U \in \mathcal{U}$, X_U has full measure, then $X_\infty := \bigcap_{U\in\mathcal{U}} X_U$ has full measure as a countable intersection of full-measure sets. Of course, for each $\alpha \in X_\infty$, $f(L_\alpha)$ is dense in $\mathbb{C}$.

Thus proving Theorem 1 reduces to showing that for any open ball U, X_U has full measure. We say a point x is an ε-density point for a set $X \subset \mathbb{R}$ if $\lim_{r\to 0^+} \frac{m(X\cap B(x,r))}{m(B(x,r))} \geq \varepsilon$. By the Lebesgue density point theorem, almost every point of X is a 1-density point for X. On the other hand, if $\varepsilon > 0$ and almost every point in $\mathbb{R}$ is an ε-density point for a set $X \subset \mathbb{R}$, then the set of 1-density points for the complement of X has zero measure, so the complement has zero measure, so X must have full measure. It therefore suffices to prove that, given a ball U, there exists $\varepsilon > 0$ such that each $\alpha_0 \in \mathbb{R}\setminus\{0\}$ is an ε-density point for X_U. So let us do this.

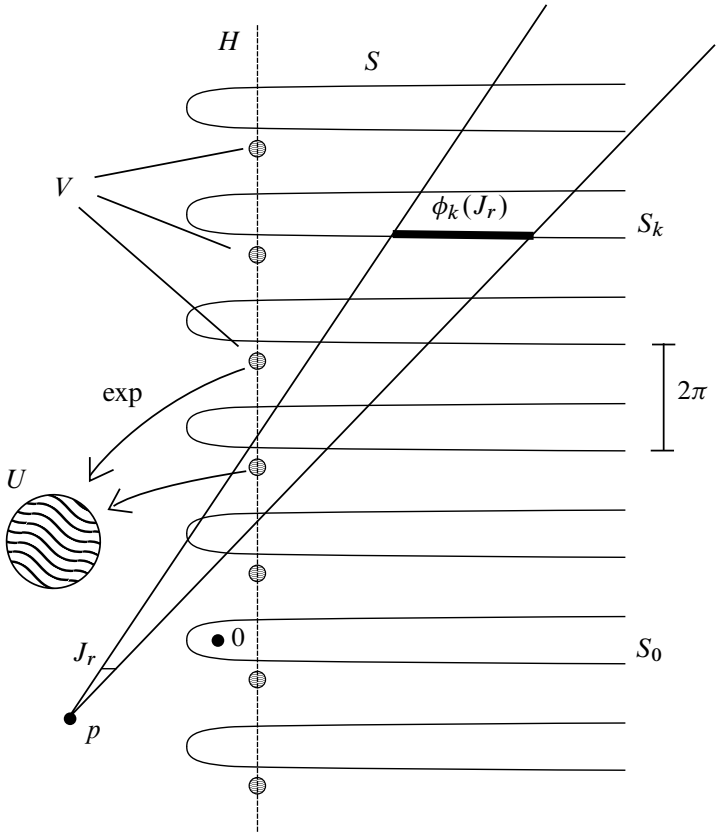

FIGURE 2
An open ball U, $V = \exp^{-1}(U)$, a vertical line H passing through V, $S = \exp^{-1}(H)$ and the projection ϕ_k onto a component S_k of S.

Let $V := \exp^{-1}(U)$. Then V is an open set. Let H be a vertical line, with real part $h \neq 0$, which intersects V, see Figure 2. Since exp is $2\pi i$-periodic, $H \cap V$ contains an open interval I and all $2\pi i$-translates of I. In particular, for any subinterval $T \subset H$ of length at least 2π,

$$m(T \cap V)/m(T) \geq m(I)/4\pi. \tag{2}$$

Now consider $S = \exp^{-1}(H)$. If $h > 0$ then one connected component of S, S_0 say, can be parametrised by

$$\gamma_+ : t \mapsto \frac{1}{2}\log(t^2 + h^2) + i \arctan\frac{t}{h}$$

with $\gamma_+(\mathbb{R}) = S_0$. If $h < 0$ then S_0 can be parametrised by $\gamma_- : t \mapsto \pi i + \gamma_+(t)$. Taking the derivative of γ_+ and γ_-,

$$\gamma'_+(t) = \gamma'_-(t) = \frac{t}{t^2+h^2} + i\frac{h}{t^2+h^2}, \tag{3}$$

so the slope of $\gamma_\pm$ tends to 0 as $|t| \to \infty$. For $k \in \mathbb{Z}$, if $\alpha_0 > 0$ let $S_k := S_0 + 2k\pi i$; otherwise let $S_k := S_0 - 2k\pi i$. Then S_k for $k \in \mathbb{Z}$ are the connected components of S.

Let $W_k := S_k \cap \exp^{-1}(V)$. The absolute value of the derivative of exp on S is bounded below by $|h| > 0$, so any segment of S_k of length at least $2\pi/|h|$ gets mapped onto a segment of H of length at least 2π. The distortion of exp (by *distortion*, we mean the ratio of the absolute value of the derivative at any two points) is bounded by $e^{4\pi/|h|}$ on each vertical strip of width $4\pi/|h|$. By the distortion bound and (2), for any segment B of S_k of length between $2\pi/|h|$ and $4\pi/|h|$,

$$\frac{m(B \cap W_k)}{m(B)} \geq \frac{m\big(\exp(B) \cap V\big)}{m\big(\exp(B)\big)e^{4\pi/|h|}} \geq \frac{m(I)}{4\pi e^{4\pi/|h|}}. \tag{4}$$

Any segment B of S_k of length at least $2\pi/|h|$ can be divided into segments of length between $2\pi/|h|$ and $4\pi/|h|$, so (4) continues to hold for all segments B of S_k of length at least $2\pi/|h|$.

Let $\xi : \alpha \mapsto p + i + \alpha$. Let $\alpha_0 \in \mathbb{R} \setminus \{0\}$ and let $r_0 = |\alpha_0|/2$. For $r \in (0, r_0)$, let $J_r := \xi(B(\alpha_0, r))$ be the open line segment joining the points $p + i + \alpha_0 - r$ and $p + i + \alpha_0 + r$. For some $K \geq 1$, for every $k \geq K$, for each $\alpha \in B(\alpha_0, r_0)$, L_α intersects S_k transversely (twice). For $k \geq K$, let ϕ_k denote the central projection with respect to p from J_{r_0} to S_k (taking the first point of intersection). For some $K_0 > K$ and each $k \geq K_0$, $\phi_k(J_{r_0})$ is almost horizontal and the distortion of ϕ_k on J_{r_0} is close to 1, in particular it is bounded by 2. Now simple geometry entails that $m(\phi_k(J_r))/\pi k r \to 1$ as $k \to \infty$ so, for each $r \in (0, r_0)$, there exists $k_r \geq K_0$ with $m(\phi_{k_r}(J_r)) > 2\pi/|h|$. Let $X_r := J_r \cap \phi_{k_r}^{-1}(W_{k_r})$. From (4) and the distortion bound of 2, we deduce that $m(X_r)/m(J_r) \geq \varepsilon$, for $\varepsilon := m(I)/8\pi e^{4\pi/|h|}$. For $\alpha \in \xi^{-1}(X_r)$, $L_\alpha \cap W_{k_r} \neq \varnothing$ so $f(L_\alpha) \cap U \neq \varnothing$. In particular, $\xi^{-1}(X_r) \subset X_U$ and

$$\frac{m\big(\xi^{-1}(X_r)\big)}{m\big(B(\alpha_0, r)\big)} \geq \varepsilon.$$

Noting that ε depends only on U and h, we have shown that α_0 is an ε-density point for X_U for each $\alpha_0 \in \mathbb{R} \setminus \{0\}$. □

Proof of Theorem 2. Fix $p \in \mathbb{C}$. Let $(q_n)_{n=1}^\infty$ be a dense sequence in $\mathbb{C}$ and let $(\delta_n)_{n=1}^\infty$ be a decreasing sequence of positive reals with $\delta_n \to 0^+$. Let $\mathcal{U} := \{B(q_n, \delta_n) : n \geq 1\}$. As per (1), given an open set U, let

$$X_U := \big\{\alpha : \exp \circ \exp\big(L_\alpha(p)\big) \cap U \neq \varnothing\big\}.$$

Since exp is continuous (so $\exp^{-2}(U)$ is open) and the central projection is an open map, X_U is open. By Theorem 1, X_U has full measure and thus is dense and open for each open set U. Consequently, $X_\infty := \bigcap_{U\in\mathcal{U}} X_U$ is a countable intersection of open, dense sets. As in the proof of Theorem 1, each point $\alpha \in X_\infty$ satisfies $\exp\circ\exp(L_\alpha(p))$ is dense. □

Proof of Theorem 3. We wish to show that the image of an oblique line under $\exp\circ\exp\circ\exp$ is dense. Let us reprise the notation of the preceding proof, so U is an open set, $V = \exp^{-1}(U)$, H a vertical line (not containing 0) intersecting V, $S = \exp^{-1}(H)$ and S_k the connected components of S. Let v_0 be a point in $H \cap V$ and let $v_j := v_0 + 2j\pi i$, so $v_j \in H \cap V$ for all $j \in \mathbb{Z}$. Let w_j^k denote the preimage of v_j in S_k, and write ω_j for the real part of w_j^k, noting that this is independent of k. As $j \to \infty$, ω_j tends to $+\infty$. Therefore the slope of the line segment Z_j^k joining w_j^k to w_{j+1}^k tends to 0 as $j \to \infty$ (cf. (3)). Since H is a vertical line, S_k lies in a horizontal strip of height π, and so

$$\gamma_{j_0}^k := \bigcup_{j\geq j_0} Z_j^k$$

is a curve, contained in a strip of height π, joining $w_{j_0}^k$ to ∞.

For some $r \in (0,1)$, $B(v_j, r) \subset V$. Estimating via the derivative of exp, we obtain $B(w_j^k, \frac{r}{2|v_j|}) \subset \exp^{-1}(V)$ for all large j, and similarly that $|w_{j+1}^k - w_j^k| < 2\pi/|v_j| < 1$. Setting $\delta := r/4\pi$, we deduce that

$$B_j^k := B\big(w_j^k, \delta|w_{j+1}^k - w_j^k|\big) \subset \exp^{-1}(V).$$

Simple geometry then entails that if ρ is a smooth curve with slope bounded in absolute value by $\delta/2$ which intersects the line segment Z_j^k and whose projection onto the real line contains (ω_j, ω_{j+1}), then ρ intersects B_j^k. This holds for all $j \geq j_0$, for some large j_0, independent of k.

Now any curve in the half-plane $\{\mathrm{Re}(z) > \omega_{j_0}\}$ whose imaginary part has range at least 3π long must intersect a curve $\gamma_{j_0}^k$ for some k. From this we deduce that if ρ' is a smooth curve contained in $\{\mathrm{Re}(z) > \omega_{j_0}\}$, with slope lying in $(\delta/4, \delta/2)$ and of horizontal length at least $2 + 12\pi/\delta$, then ρ' must intersect some B_j^k. Indeed, there is a subcurve whose projection is $(\omega_{j_1}, \omega_{j_2})$ (say) and has horizontal length at least $12\pi/\delta$. By the slope estimate, the range of its imaginary part is at least 3π long, so it intersects some $\gamma_{j_0}^k$, so it intersects some Z_j^k, with $j_1 \leq j < j_2$, and so it intersects B_j^k.

Given an oblique line, under exponential it gets mapped to a spiral Σ, say. Every revolution, the spiral has two stretches where the slope lies in $(\delta/4, \delta/2)$, one in the right half-plane, one in the left half-plane. Let $(\Sigma_n)_{n\in\mathbb{Z}}$ denote the sequence of those stretches lying in the right half-plane, ordered so that the

distance of Σ_n from 0 increases with n. For n large enough, $\mathrm{Re}(\Sigma_n) \subset (\omega_{j_0}, \infty)$ and the horizontal length of Σ_n is arbitrarily large, in particular it can be taken bigger than $2 + 12\pi/\delta$. Therefore it intersects some B_j^k.

Since exp of the line intersects B_j^k, $\exp \circ \exp \circ \exp$ of the line intersects U. This holds for every open set U so the theorem is proven. □

3. An explicit dense curve

Given $R > 1$, let A_R denote the annulus $B(0, R) \setminus B(0, 1/R)$, the image of the vertical strip $H_R := \{z : \mathrm{Re}(z) \in [-\log R, \log R)\}$ under exp.

Lemma 8. *Let Σ be a logarithmic spiral whose intersections with the imaginary axis occur at points $(w_k)_{k\in\mathbb{Z}}$, ordered by distance from 0. Then $\exp(\Sigma)$ is dense in $\mathbb{C}$ if and only if $(w_k/2\pi i)_{k\geq 0}$ are dense modulo 1.*

Proof. Denote by Σ_k the connected component of the $\Sigma \cap H_R$ containing w_k. There exists k_0 for which, for all $k \leq k_0$, $\Sigma_k = \Sigma_{k_0}$, which spirals all the way in to 0. The set $\exp(\Sigma_{k_0})$ has finite length and is not dense anywhere. Of course this then holds for $\exp(\Sigma_k)$ for each k, so we only need to consider positive k.

If $\Sigma = \exp(L_\alpha(p))$ say, denote by Z_k the intersection of H_R and the line which passes through w_k with slope $-\alpha$. Then the Hausdorff distance of Σ_k to Z_k decreases to 0 as $k \to +\infty$.

Taken sequentially, the following statements are (clearly) equivalent.

- $(w_k/2\pi i)_{k\geq 0}$ is dense modulo 1.
- the union of all $2\pi i$-translates of $\{Z_k\}_{k\geq 0}$ is dense in H_R.
- the union of all $2\pi i$-translates of $\{\Sigma_k\}_{k\geq 0}$ is dense in H_R.
- $\bigcup_{k\geq 0} \exp(\Sigma_k)$ is dense in A_R.
- $\exp(\Sigma)$ is dense in $\mathbb{C}$.

This completes the proof of the lemma. □

Proof of Theorem 4. Let $a \in (0, 1)$ have a binary expansion containing all possible strings of zeroes and ones; let $p := \log(2\pi a) + \frac{\pi}{2} i$ and $\alpha := \log 2/2\pi$ as per Theorem 4. By choice of α, each time the imaginary part of the line $L_\alpha(p)$ increases by 2π, the real part increases by $\log 2$. Thus the intersections of the spiral $\Sigma := \exp(L_\alpha(p))$ with the positive imaginary axis $(i\mathbb{R}^+)$ occur at values $2\pi a 2^k i$, $k \in \mathbb{Z}$.

By choice of a, for all k_0 the set $\{2^k a\}_{k\geq k_0}$ is dense modulo 1. Now apply Lemma 8. □

4. Hausdorff dimension of the complementary set of parameters

In this section we prove Theorem 5 and Corollary 6. The Mass Distribution Principle is a standard source of lower bounds for the Hausdorff dimension. It is infused into the following lemma.

Lemma 9. *Let J be a non-degenerate interval, let $Y \subset J$ and let μ be a measure with $\mu(Y) > 0$. For each $n \geq 1$, let $\mathcal{P}_n$ be a finite partition of J into intervals, each of length at most 2^{-n}. Let $\varepsilon \in (0,1)$, let $\beta > 1$ and suppose*

$$\mu(P) \leq \beta(1+\varepsilon)^n |P| \tag{5}$$

for every $P \in \mathcal{P}_n$. Then the Hausdorff dimension of Y is at least $1-2\varepsilon$.

Proof. For $r \in (0,1)$, let $n := \lceil -\log_2 r \rceil$. Let $x \in J$. If $P \in \mathcal{P}_n$ then $|P| \leq 2^{-n} \leq r$, so if $P \cap B(x,r) \neq \varnothing$ then $P \subset B(x,2r)$. The total length of elements of $\mathcal{P}_n$ intersecting $B(x,r)$ is thus at most $4r$. Summing (5) over such elements, we deduce that

$$\mu\big(B(x,r)\big)/\beta \leq 4r(1+\varepsilon)^n \leq 4r(1+\varepsilon)e^{-\log(1+\varepsilon)\log r/\log 2} \leq 8r^{1-\log(1+\varepsilon)/\log 2}.$$

Now $\log 2 > \frac{1}{2}$ and $\log(1+\varepsilon) < \varepsilon$, so

$$\mu\big(B(x,r)\big)/8\beta \leq r^{1-2\varepsilon}.$$

If $U_1, U_2, \ldots$ is any countable cover of Y by balls of radius at most 1, then

$$\sum_{j\geq 1} |U_j|^{1-2\varepsilon} \geq \sum_{j\geq 1} \mu(U_j)/8\beta \geq \mu(Y)/8\beta > 0.$$

Since this positive lower bound does not depend on the cover, the Hausdorff dimension of Y is at least $1-2\varepsilon$, as required. □

Together with the following lemma, one can glean an insight into the means of proving Theorem 5.

Lemma 10. *Let I be an open subinterval of the imaginary axis and let $\hat{I} := \bigcup_{k\in\mathbb{Z}}(2k\pi i + I)$ be the union of all $2\pi i$-translates of $\hat{I}$. Suppose $\hat{I}$ is disjoint from $B(0,1)$. Let $p \in \mathbb{C}$. Let Y be a compact subset of $\mathbb{R}$ and suppose that $\exp(L_\alpha(p)) \cap \hat{I} = \varnothing$ for every $\alpha \in Y$. Then there is an open set U with $\exp \circ \exp(L_\alpha(p)) \cap U = \varnothing$ for each $\alpha \in Y$.*

Proof. Differentiating $t \mapsto \exp(p + t(i+\alpha))$ gives $(i+\alpha)\exp(p+t(i+\alpha))$. Thus $\exp(L_\alpha(p))$ has slope $-\alpha$ at each intersection with the imaginary axis. Moreover,

since Y is bounded, there is a constant $C > 1$ such that the slope of $\exp(L_\alpha(p))$ is bounded in absolute value by C in the region

$$\left\{z : |\mathrm{Re}(z)| < \frac{1}{2}, |\mathrm{Im}(z)| > \frac{1}{2}\right\}.$$

Let D denote the body of the rhombus with diagonal I and sides of slope $\pm C$, and $\widehat{D}$ the union of all $2\pi i$-translates of D. Then $\exp(L_\alpha(p)) \cap \widehat{D} = \varnothing$ for each $\alpha \in Y$. Let x be the midpoint of I and denote by U the open set $\exp(B(x, |I|/4C)$. By construction, $B(x, |I|/4C) \subset D$ so $\exp^{-1}(U) \subset \widehat{D}$. Thus $\exp \circ \exp(L_\alpha(p)) \cap U = \varnothing$ for each $\alpha \in Y$, as required. □

Now we can prove Theorem 5, which states that for each $p \in \mathbb{C}$ and each open set $X \subset \mathbb{R}$, the set $\{\alpha \in X : \overline{\exp \circ \exp(L_\alpha(p))} \neq \mathbb{C}\}$ has Hausdorff dimension 1.

Proof of Theorem 5. We can assume $0 \notin X$. Writing σ for the map sending points to their complex conjugates, $\exp \circ \sigma = \sigma \circ \exp$ and $\sigma(L_\alpha(p)) = L_{-\alpha}(\sigma(p))$ so, without loss of generality (replacing p by $\sigma(p)$ and X by $-X$, if necessary), one can assume $X \subset \mathbb{R}^+$.

Given X and p, let $X' = (\alpha_0, \alpha_1)$ be a non-degenerate subinterval of X with $0 < \alpha_0 < \alpha_1$. Let $\xi : \alpha \mapsto p + i + \alpha$ and let J be the line segment $\xi(X')$. For $k \in \mathbb{Z}$, let $S_k := (k + \frac{1}{2})\pi i + \mathbb{R}$. Then $\exp(S_k)$ is a vertical ray leaving 0, heading up if k is even and down if k is odd. Let ϕ_k be the central projection with respect to p from J to S_k, so

$$\phi_k(p + i + \alpha) = \mathrm{Re}(p) + \left(\left(k + \frac{1}{2}\right)\pi - \mathrm{Im}(p)\right)\alpha + i\left(\left(k + \frac{1}{2}\right)\pi - \mathrm{Im}(p)\right).$$

In particular, as a map from J to S_k, ϕ_k is affine with derivative

$$D\phi_k(z) = \left(k + \frac{1}{2}\right)\pi - \mathrm{Im}(p)$$

for every $z \in J$. There exists a (possibly negative) $k_0 \in \mathbb{Z}$ such that, for all $k \leq k_0$, $\phi_k(J) \subset \{z : \mathrm{Re}(z) < 0\}$, and thus, for $k \leq k_0$, $\exp \circ \phi_k(J) \subset B(0, 1)$. Writing $\psi_k := \exp \circ \phi_k$ on J, ψ_k maps J onto a subinterval of the imaginary axis, see Figure 3. As ϕ_k is affine, the distortion of ψ_k on an interval $W \subset J$ is bounded by $\exp(|\phi_k(W)|)$.

We have $|D\psi_k| = |D\exp(\phi_k)||D\phi_k| = |D\phi_k|\exp(\mathrm{Re}(\phi_k))$, so

$$|D\psi_k(p + i + \alpha)| = \left|\left(k + \frac{1}{2}\right)\pi - \mathrm{Im}(p)\right| e^{\mathrm{Re}(p)} e^{(\frac{\pi}{2} - \mathrm{Im}(p))\alpha} e^{k\pi\alpha}. \tag{6}$$

Thus for $k > |\mathrm{Im}(p)|/\pi$,

$$\left|D\psi_{k+1}(p + i + \alpha)/D\psi_k(p + i + \alpha)\right| > e^{\alpha\pi}, \tag{7}$$

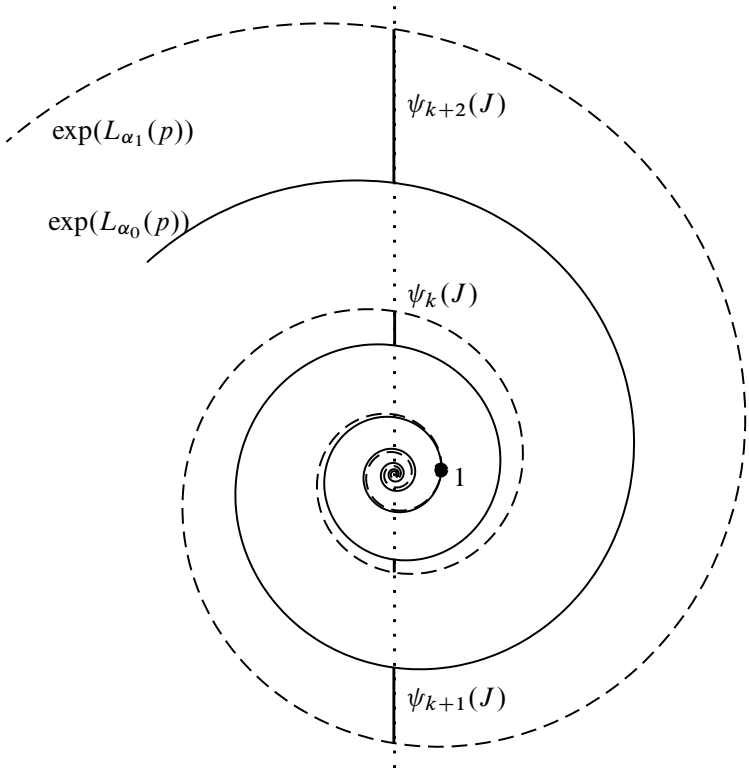

FIGURE 3
Two logarithmic spirals $(\exp(L_{\alpha_0}(p))$ and $\exp(L_{\alpha_1}(p))$, drawn with $p = 0)$ and the increasing (in length) subintervals $\psi_k(J), \psi_{k+1}(J), \psi_{k+2}(J)$ of the imaginary axis.

so the derivatives grow exponentially. Moreover, there exists $C \in (0, 1)$ such that, for each $k \geq k_0$ with $p \notin S_k$,

$$|D\psi_k| > C. \tag{8}$$

Remark: Choice of the constant N: If $\alpha > 0$ is small, then there is not much expansion at each revolution. We shall consider blocks of N (half-) revolutions at a time, for large integers N. Let I be small open sub-interval of the imaginary axis and let $\hat{I} := \bigcup_{k\in\mathbb{Z}}(2k\pi i + I)$. Let $\varepsilon > 0$ and suppose that V is a subinterval of J, that $\psi_j(V) \cap \hat{I} = \varnothing$ for $j \leq nN$ and that $\psi_{nN}(V) \geq \varepsilon$. We shall obtain estimates for the points in V not meeting $\hat{I}$ for $j \leq (n + 1)N$. To continue by induction, we will need to regain the starting condition length $\geq \varepsilon$. By (7), $|\psi_{nN+j}(V)| \geq e^{j\alpha_0\pi}\varepsilon$, and for $j \leq N$ the length is bounded by $L := |\psi_{(n+1)N}(V)|$. Note $L \geq e^{N\alpha_0\pi}\varepsilon$. The number of connected components of the set of points $z \in V$ with $\psi_{nN+j}(z) \notin \hat{I}$ for $j = 1, \ldots, N$ is bounded

by $N(L+2)$ [if $L > 2\pi$, one can improve the bound to $NL/2\pi + 1$]. The proportion of points $z \in V$ with $\psi_{nN+j}(z) \notin \hat{I}$ for $j = 1, \dots, N$ is at least $1 - 2N|I|/\varepsilon$, if one assumes bounded distortion giving a factor of 2. If we remove all connected components whose image under $\phi_{(n+1)N}$ is less than ε, the remaining proportion is at least $1 - 2N|I|/\varepsilon - 2\varepsilon N(L+2)/L$. If one takes $\varepsilon = N^{-2}$, $|I| = N^{-4}$ and N large, then $L > 1$ and the proportion is at least $1 - 8/N$, which can be made as close to 1 as we desire. To get good starting conditions for a forthcoming induction argument, N may need to be taken larger again, and $|I|$ slightly smaller.

Let an integer $N > 2|k_0| + 8\pi$ be large enough that

- $N\pi > 2|\mathrm{Im}(p)|$;
- $e^{N\pi\alpha_0/2} > N^4$;
- $Ne^{\mathrm{Re}(p)} > 1$;
- $1/N^2 < |J|$.

By (6) and choice of N, for all $z = p + i + \alpha \in J$,

$$|D\psi_N(z)| > (N\pi/2)e^{\mathrm{Re}(p)}e^{N\pi\alpha_0/2} > N^4. \tag{9}$$

From (7) and choice of N, we obtain

$$\left|D\psi_{(n+1)N}(z)/D\psi_{nN}(z)\right| > N^4 \tag{10}$$

for each $n \geq 1$ and $z \in J$.

Let $M := \sup_{z\in J} |D\psi_N(z)|$, so for any subinterval $J' \subset J$, $|\psi_N(J')| \leq M|J'|$. Let I be an open subinterval of the imaginary axis of length $N^{-4}C/M$ whose $2\pi i$-translates are disjoint from $\exp(p)$ and from $B(0,1)$. Let $\hat{I} := \bigcup_{k\in\mathbb{Z}}(2k\pi i + I)$. For $k \leq k_0$, $\psi_k(J) \subset B(0,1)$, so $\psi_k(J) \cap \hat{I} = \varnothing$.

Let J' be a subinterval of J of length $1/N^2$. Let J_n be the set of points $z \in J'$ for which $\psi_k(z) \notin \hat{I}$ for all $k \leq n$. Note that $J_{k_0} = J'$.

We now deal with the steps from k_0 to N, to get a good starting interval. We shall later use induction to pass from nN to $(n+1)N$. For $k = k_0 + 1, \dots, N$,

$$|\phi_k(J')| < |J'|\left(\left(N + \frac{1}{2}\right)\pi - \mathrm{Im}(p)\right) < N^{-2}\big((N+1)\pi/2\big) < \pi/N.$$

Hence the distortion of ψ_k is bounded by $e^{\pi/N} < 2$. For $k = k_0 + 1, \dots, N$, $|\psi_k(J')| \leq |\psi_N(J')|$ and by (9), $|\psi_N(J')| > N^4/N^2 = N^2$. For $k \leq N$, the number of connected components of $\hat{I}$ intersecting $\psi_k(J')$ is bounded by $|\psi_N(J')|$; it follows that $m(\hat{I} \cap \psi_k(J')) \leq |\psi_N(J')||I|$. Using (8) and then choice of M and I,

$$\begin{aligned} m(J_N) &= |J'| - m\Big(J' \cap \bigcup_{k=k_0+1}^{N} \psi_k^{-1}(\hat{I})\Big) \\ &\geq |J'| - (N-k_0)|\psi_N(J')||I|/C \\ &\geq |J'| - (N-k_0)|J'|N^{-4} \\ &> |J'|/2, \end{aligned}$$

say. Meanwhile, J_N has at most $(N-k_0)|\psi_N(J')|$ connected components. Therefore, at least one connected component V of J_N must satisfy

$$|V| > |J'|/3(N-k_0)|\psi_N(J')|$$

and, more importantly (by the distortion bound of 2),

$$|\psi_N(V)| > 1/2(N-k_0) > 1/N^2.$$

Let $\mathcal{W}_1 := \{V\}$.

Now we repeat the argument for general intervals. Let us define $\mathcal{W}_n$ inductively as follows. For $W \in \mathcal{W}_n$, let $\mathcal{A}_W$ denote the (finite) collection of connected components A of $J_{(n+1)N} \cap W$ for which $|\psi_{(n+1)N}(A)| \geq 1/N^2$. Let

$$\mathcal{W}_{n+1} := \cup_{W \in \mathcal{W}_n} \mathcal{A}_W.$$

Note $\mathcal{W}_1 = \{V\}$ is non-empty. The set

$$\Lambda := \bigcap_{n \geq 1} \bigcup_{W \in \mathcal{W}_n} W$$

is a closed subset of J, as a countable intersection of finite unions of closed sets. For $z \in \Lambda$, $z \in J_k$ for all k, so the image of the line passing through p and z is a spiral which avoids $\hat{I}$. We shall show that Λ is non-empty and has dimension at least $1 - 10/N$.

For $W \in \mathcal{W}_n$, let $W^+ := \cup_{A \in \mathcal{A}_W} A$. In order to apply Lemma 9, we will need to show that $m(W^+)/m(W)$ is close to 1; in particular it will be at least $1 - 4/N$.

Since $W \in \mathcal{W}_n$, $|\psi_{nN}(W)| \geq 1/N^2$. Let k satisfy $nN \leq k(n+1)N$. By (7), $\psi_k(W)$ has length at least $1/N^2$. Hence

$$m\big(\hat{I} \cap \psi_k(W)\big)/|\psi_k(W)| \leq N^2|I|. \tag{11}$$

Now

$$|D\phi_k|/|D\phi_{nN}| = \left(\left(k + \frac{1}{2}\right)\pi - \mathrm{Im}(p)\right) / \left(\left(nN + \frac{1}{2}\right)\pi - \mathrm{Im}(p)\right) < 4.$$

Since $\psi_{nN}(W) \cap \hat{I} = \varnothing$, one obtains $|\psi_{nN}(W)| \leq 2\pi$ and $\phi_{nN}(W)$ has length (crudely) bounded by $1/8$. Hence $|\phi_k(W)|$ is bounded by $1/2$. Therefore the

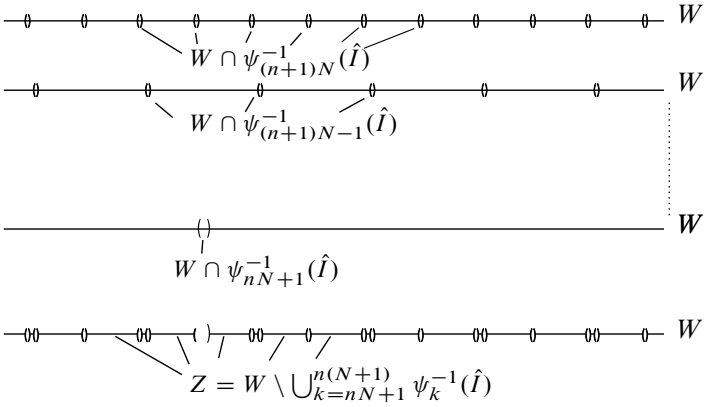

FIGURE 4
A schematic drawing of $Z = W \setminus \bigcup_{k=nN+1}^{(n+1)N} \psi_k^{-1}(\hat{I})$ showing multiple copies of W. Connected components of $W \cap \psi_k^{-1}(\hat{I})$ are tiny, so most of Z will consist of relatively large connected components.

distortion of ψ_k on W is bounded by $e^{1/2} < 2$. We deduce from this and (11) (N times, for $k = nN+1, \ldots, (n+1)N$) that the set $Z := J_{(n+1)N} \cap W$ satisfies $m(Z)/|W| \geq 1 - 2N^3|I|$. Meanwhile, by (10),

$$|\psi_{(n+1)N}(W)| \geq N^4 |\psi_{nN}(W)| \geq N^4/N^2 = N^2.$$

[As an aside, note that the image is long and therefore contains many components of $\hat{I}$, so elements of $\mathcal{A}_W$ will have length much less than $|W|/2$.] The set Z (see Figure 4) has at most $N|\psi_{(n+1)N}(W)|$ connected components. Those of length at least $2|W|/|\psi_{(n+1)N}(W)|N^2$ get mapped by $\psi_{(n+1)N}$ onto an interval of length at least $1/N^2$, by bounded distortion, so they are contained in $\mathcal{A}_W$. Knowing a bound for the number of connected components, we deduce that those of length at most $2|W|/|\psi_{(n+1)N}(W)|N^2$ have measure bounded by $2|W|/N$. Consequently,

$$\begin{aligned} m(W^+)/m(W) &\geq 1 - 2N^3|I| - 2/N \\ &> 1 - 4/N, \end{aligned}$$

noting $|I| \leq 1/N^4$.

Since $|J'| = 1/N^2$, $|V| < 1/2$ for (the unique interval) $V \in \mathcal{W}_1$. It follows that for each $W \in \mathcal{W}_n$, $|W| \leq 2^{-n}$.

Recall we wish to construct a measure on $\Lambda = \cap_{n \geq 1} \cup_{W \in \mathcal{W}_n} W$, in order to estimate its dimension using Lemma 9. For each $n \geq 1$, let us introduce a measure μ_n on $\bigcup_{W \in \mathcal{W}_n} W$. Let μ_1 be Lebesgue measure restricted to the unique interval $V \in \mathcal{W}_1$. Define inductively μ_n, for $n \geq 2$, as follows. For each $W \in \mathcal{W}_{n-1}$, set

$$\mu_n := \frac{m(W)}{m(W^+)}\mu_{n-1} \tag{12}$$

on W^+, and $\mu_n := 0$ on $W \setminus W^+$. As defined, $\mu_n(W^+) = \mu_{n-1}(W)$ for each $W \in \mathcal{W}_{n-1}$, whence $\mu_k(W) = \mu_n(W)$ for all $k \geq n$ and each $W \in \mathcal{W}_n$. Since also $\max_{W\in\mathcal{W}_n} |W| \leq 2^{-n}$, there exists a unique (weak) limit measure

$$\mu := \lim_{n\to\infty} \mu_n$$

and μ is supported on Λ with $\mu(\Lambda) = \mu_n(J') = |V|$. We need to check the limit measure is well-behaved. In particular, it should not have atoms. By induction using (12),

$$\mu_n(W) \leq |W|(1 - 4/N)^{-n+1}$$

for $W \in \mathcal{W}_n$. Thus for $z \in \Lambda$ and $n \geq 1$, there are at most two elements $W_1, W_2 \in \mathcal{W}_n$ intersecting all tiny neighbourhoods of z, and

$$\mu_n(W_i) \leq |W_i|(1 - 4/N)^{-n+1} \leq 2^{-n/2+1}$$

for $i = 1, 2$. Hence $\mu_k(W_i) \leq 2^{-n/2+1}$ for all $k \geq n$, and so $\mu(\{z\}) \leq 2^{-n/2+2}$ for each n; therefore μ is continuous (i.e. it has no atoms). Since μ is continuous, $\mu(W) = \mu_k(W)$ for each $W \in \mathcal{W}_n$ and $k \geq n$.

We are nearly at a stage where we can apply Lemma 9. For each n, let $\mathcal{Q}_n$ denote a finite partition of $J' \setminus \bigcup_{W\in\mathcal{W}_n} W$ into intervals such that each $Q \in \mathcal{Q}_n$ has $|Q| < 2^{-n}$. For each $Q \in \mathcal{Q}_n$, $\mu_k(Q) = 0$ for all $k \geq n$, hence $\mu(Q) = 0$ (using continuity of μ). Let

$$\mathcal{P}_n := \mathcal{Q}_n \cup \mathcal{W}_n,$$

so $\mathcal{P}_n$ is a partition of J'. From the construction,

$$\mu(P) \leq |P|(1 - 4/N)^{-n} \leq |P|(1 + 5/N)^n$$

for each $n \geq 1$ and $P \in \mathcal{P}_n$. By Lemma 9, the Hausdorff dimension of Λ is at least $1 - 10/N$. Recalling $\Lambda \subset J'$, set $Y := \xi^{-1}(\Lambda) \subset X'$. Applying Lemma 10 to Y, we obtain that for each $\alpha \in Y$, $\exp \circ \exp(L_\alpha(p))$ is not dense. As ξ is a translation it preserves Hausdorff dimension, and the dimension of Y is at least $1 - 10/N$. But N could be taken arbitrarily large (of course, I and therefore Y depend on choice of N). Noting that any set with subsets of dimension arbitrarily close to 1 has dimension at least 1, the proof of Theorem 5 is complete. □

Proof of Corollary 6. Taking $p = 2\pi i$, the intersections of the spiral with the positive imaginary axis occur at points $\exp(2\pi\alpha k)2\pi i$, $k \in \mathbb{Z}$. From Theorem 5 and Lemma 8, we deduce that the set of α in any open interval X for which $\exp(2\pi\alpha k)$ is not dense modulo 1 has dimension 1, from which the result follows (taking $X = (\log I)/2\pi$). □

Remark: One could use Lemma 8 to prove Theorem 1 (using Koksma's theorem [Kok]), however the lemma cannot be used to prove Theorem 3, neither does Theorem 3 provide information about distributions of sequences modulo 1.

5. Distribution

Given α, p and the corresponding spiral $\Sigma : t \mapsto \exp(p + t(i + \alpha))$, we set $\rho := \exp \circ \Sigma$, a parametrisation of $\exp \circ \exp$ of the line $L_\alpha(p)$. We now study the distribution of $\rho(t)$. For every measurable set A and $T > 1$, let

$$\mu_T(A) := \frac{1}{2T} m\big(\{t \in [-T, T] : \rho(t) \in A\}\big),$$

where m denotes Lebesgue measure. Then μ_T is a probability measure.

Proof of Theorem 7. We can assume without loss of generality that $\alpha > 0$. Since $\lim_{t\to-\infty} |\Sigma(t)| = 0$,

$$\lim_{t\to-\infty} \rho(t) = 1.$$

Let us define intervals

$$I_n^+ := 2n\pi + \big[-\pi/2 + 1/n - \mathrm{Im}(p), \pi/2 + 1/n - \mathrm{Im}(p)\big].$$

The intervals are chosen so that for $t \in I_n^+$ and n large,

$$\mathrm{Re}\big(\Sigma(t)\big) \geq \sin(1/n) \exp\big(\mathrm{Re}(p) + 2n\pi\alpha - \pi/2 - \mathrm{Im}(p)\big) \gg 1,$$

so

$$\lim_{n\to\infty} |\rho(I_n^+)| = +\infty.$$

Setting $I_n^- := I_n^+ + \pi$, we similarly obtain that

$$\lim_{n\to\infty} |\rho(I_n^-)| = 0.$$

Noting that the intervals $I_n^\pm$ have length approaching π, and the spaces between them have length $\approx 2/n$, it follows that

$$\lim_{T\to\infty} \mu_T = \delta_1/2 + \frac{\delta_0 + \delta_\infty}{4},$$

as required. □

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References

[Bir] G. D. Birkhoff, Démonstration d'un théoreme élémentaire sur les fonctions entières. *C. R. Acad. Sci. Paris*, 189(14):473–475, 1929. Zbl 55.0192.07

[Boh] H. Bohr and R. Courant, Neue Anwendungen der Theorie der Diophantischen Approximationen auf die Riemannsche Zetafunktion. *Journal für die reine und angewandte Mathematik* **144** (1914), 249–274. Zbl 45.0718.02 MR 1580903

[BJ1] H. Bohr and B. Jessen, Über die Werteverteilung der Riemannschen Zetafunktion. *Acta Math.* **54** (1930), 1–35. JFM 56.0287.01 MR 1555301

[BJ2] H. Bohr and B. Jessen, Über die Werteverteilung der Riemannschen Zetafunktion. *Acta Math.* **58** (1932), 1–55. Zbl 0003.38901 MR 1555343

[Kah] J.-P. Kahane, Sur la répartition des puissances modulo 1. *C. R. Math. Acad. Sci. Paris* **352** (2014), 383–385. Zbl 1287.11093 MR 3194243

[Kok] J. F. Koksma, Ein mengentheoretischer Satz über die Gleichverteilung modulo Eins. *Compositio Math.* **2** (1935), 250–258. Zbl 0012.01401 MR 1556918

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