*L'Enseignement Mathématique* (2) 62 (2016), 189–198 DOI 10.4171/LEM/62-1/2-11

# Fuchsian groups and compact hyperbolic surfaces

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**Abstract.** We present a topological proof of the following theorem of Benoist-Quint: for a finitely generated non-elementary discrete subgroup $\Gamma_1$ of $\mathrm{PSL}(2,\mathbb{R})$ with no parabolics, and for a cocompact lattice $\Gamma_2$ of $\mathrm{PSL}(2,\mathbb{R})$, any $\Gamma_1$ orbit on $\Gamma_2\backslash\mathrm{PSL}(2,\mathbb{R})$ is either finite or dense.

**Mathematics Subject Classification (2010).** Primary: 11N45, 37F35, 22E40.

**Keywords.** Fuchsian group, orbit closure, unipotent flow, homogeneous dynamics.

## 1. Introduction

Let $\Gamma_1$ be a non-elementary finitely generated discrete subgroup with no parabolic elements of $\mathrm{PSL}(2,\mathbb{R})$. Let $\Gamma_2$ be a cocompact lattice in $\mathrm{PSL}(2,\mathbb{R})$. The following is the first non-trivial case of a theorem of Benoist-Quint [BQ1].

**Theorem 1.1.** *Any $\Gamma_1$-orbit on $\Gamma_2\backslash\mathrm{PSL}(2,\mathbb{R})$ is either finite or dense.*

The proof of Benoist-Quint is quite involved even in the case as simple as above and in particular uses their classification of stationary measures [BQ2]. The aim of this note is to present a short, and rather elementary proof.

We will deduce Theorem 1.1 from the following Theorem 1.2. Let

- $H_1 = H_2 := \mathrm{PSL}(2,\mathbb{R})$ and $G := H_1 \times H_2$;
- $H := \{(h,h) : h \in \mathrm{PSL}_2(\mathbb{R})\}$ and $\Gamma := \Gamma_1 \times \Gamma_2$.

**Theorem 1.2.** *For any $x \in \Gamma\backslash G$, the orbit $xH$ is either closed or dense.*

Our proof of Theorem 1.2 is purely topological, and inspired by the recent work of McMullen, Mohammadi and Oh [MMO] where the orbit closures of the $\mathrm{PSL}(2,\mathbb{R})$ action on $\Gamma_0\backslash\mathrm{PSL}(2,\mathbb{C})$ are classified for certain Kleinian subgroups $\Gamma_0$ of infinite co-volume. While the proof of Theorem 1.2 follows closely the sections 8-9 of [MMO], the arguments in this paper are simpler because of the assumption that $\Gamma_2$ is cocompact. We remark that the approach of [MMO] and hence of this paper is somewhat modeled after Margulis's original proof of Oppenheim's conjecture [Mar]. When $\Gamma_1$ is cocompact as well, Theorem 1.2 also follows from [Rat].

Finally we remark that according to [BQ1], both Theorems 1.1 and 1.2 are still true in presence of parabolic elements, more precisely when $\Gamma_1$ is any non-elementary discrete subgroup and $\Gamma_2$ any lattice in $\mathrm{PSL}(2,\mathbb{R})$. The topological method presented here could also be extended to this case.

## 2. Horocyclic flow on convex cocompact surfaces

In this section we prove a few preliminary facts about unipotent dynamics involving only one factor $H_1$.

The group $\mathrm{PSL}_2(\mathbb{R}) := \mathrm{SL}_2(\mathbb{R})/\{\pm e\}$ is the group of orientation-preserving isometries of the hyperbolic plane $\mathbb{H}^2 := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. The isometry corresponding to the element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R})$ is $z \mapsto \frac{az+b}{cz+d}$. It is implicit in this notation that the matrices $g$ stand for their equivalence class $\pm g$ in $\mathrm{PSL}_2(\mathbb{R})$. This group $\mathrm{PSL}_2(\mathbb{R})$ acts simply transitively on the unit tangent bundle $\mathrm{T}^1(\mathbb{H}^2)$ and we choose an identification of $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{T}^1(\mathbb{H}^2)$ so that the identity element $e$ corresponds to the upward unit vector at $i$. We will also identify the boundary of the hyperbolic plane with the extended real line $\partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$ which is topologically a circle.

We recall that $\Gamma_1$ is a non-elementary finitely generated discrete subgroup with no parabolic elements of the group $H_1 = \mathrm{PSL}_2(\mathbb{R})$, that is, $\Gamma_1$ is a convex cocompact subgroup. Let $S_1$ denote the hyperbolic orbifold $\Gamma_1\backslash\mathbb{H}^2$, and let $\Lambda_{\Gamma_1} \subset \partial\mathbb{H}^2$ be the limit set of $\Gamma_1$. Let $A_1$ and $U_1$ be the subgroups of $H_1$ given by

$$A_1 := \{a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t \in \mathbb{R}\} \text{ and } U_1 := \{u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R}\}.$$

Since the subgroup $\Gamma_1$ is convex cocompact, the set

$$\Omega_{\Gamma_1} := \{x \in \Gamma_1\backslash H_1 : xA_1 \text{ is bounded}\} \tag{2.1}$$

is a compact $A_1$-invariant subset and one has the equality

$$\Omega_{\Gamma_1} = \{[h] \in \Gamma_1\backslash H_1 : h(0), h(\infty) \in \Lambda_{\Gamma_1}\}.$$

In geometric words, seen as a subset of the unit tangent bundle of $S_1$, the set $\Omega_{\Gamma_1}$ is the union of all the geodesic lines which stays inside the convex core of $S_1$.

**Definition 2.2.** Let $K > 1$. A subset $T \subset \mathbb{R}$ is called $K$-thick if, for any $t > 0$, $T$ meets $[-Kt, -t] \cup [t, Kt]$.

**Lemma 2.3.** *There exists $K > 1$ such that for any $x \in \Omega_{\Gamma_1}$, the subset $T(x) := \{t \in \mathbb{R} : xu_t \in \Omega_{\Gamma_1}\}$ is $K$-thick.*

*Proof.* Using an isometry, we may assume without loss of generality that $x = [e]$. Since the element $e$ corresponds to the upward unit vector at $i$, and since $x$ belongs to $\Omega_{\Gamma_1}$, both points $0$ and $\infty$ belong to the limit set $\Lambda_{\Gamma_1}$. Since $u_t(\infty) = \infty$ and $u_t(0) = t$, one has the equality

$$T(x) = \{t \in \mathbb{R} : t \in \Lambda_{\Gamma_1}\}.$$

Write $\mathbb{R} - \Lambda_{\Gamma_1}$ as the union $\cup J_\ell$ where $J_\ell$'s are maximal open intervals. Note that the minimum hyperbolic distance between the convex hulls in $\mathbb{H}^2$

$$\delta := \inf_{\ell \neq m} d(\mathrm{hull}(J_\ell), \mathrm{hull}(J_m))$$

is positive, as $2\delta$ is the length of the shortest closed geodesic of the double of the convex core of $S_1$. Choose the constant $K > 1$ so that for $t > 0$, one has

$$d(\mathrm{hull}[-Kt, -t], \mathrm{hull}[t, Kt]) = \delta/2.$$

Note that this choice of $K$ is independent of $t$. If $T(x)$ does not intersect $[-Kt, -t] \cup [t, Kt]$ for some $t > 0$, then the intervals $[-Kt, -t]$ and $[t, Kt]$ must be included in two distinct intervals $J_\ell$ and $J_m$, since $0 \in \Lambda_{\Gamma_1}$. This contradicts the choice of $K$. □

**Lemma 2.4.** *Let $K > 1$ and let $T$ be a $K$-thick subset of $\mathbb{R}$. For any sequence $h_n$ in $H_1 \setminus U_1$ converging to $e$, there exists a sequence $t_n \in T$ such that the sequence $u_{-t_n} h_n u_{t_n}$ has a limit point in $U_1 \setminus \{e\}$.*

*Proof.* Write $h_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$. We compute

$$q_n := u_{-t_n} h_n u_{t_n} = \begin{pmatrix} a_n - c_n t_n & (a_n - d_n - c_n t_n) t_n + b_n \\ c_n & d_n + c_n t_n \end{pmatrix}.$$

Since the element $h_n$ does not belong to $U_1$, it follows that the $(1,2)$-entries $P_n(t_n) := (a_n - d_n - c_n t_n) t_n + b_n$ are non-constant polynomial functions of $t_n$ of degree at most $2$ whose coefficients converge to $0$. Hence, by Lemma 2.5 below, we can choose $t_n \in T$ going to $\infty$ so that $k \le |P_n(t_n)| \le 1$, for some constant $k > 0$ depending only on $K$. Since the entry $P_n(t_n)$ is bounded and since $h_n$ converges to $e$, the product $c_n t_n$ must converge to $0$ and the sequence $q_n$ has a limit point in $U_1 - \{e\}$. $\square$

We have used the following basic lemma :

**Lemma 2.5.** *For every $K > 1$ and $d \ge 1$, there exists $k > 0$ such that, for every non-constant polynomial $P$ of degree $d$ with $|P(0)| \le k$, and for every $K$-thick subset $T$ of $\mathbb{R}$, there exists $t$ in $T$ such that $k \le |P(t)| \le 1$.*

*Proof.* Using a suitable homothety in the variable $t$, we can assume with no loss of generality that $P$ belongs to the set $\mathcal{P}_d$ of polynomials of degree at most $d$ such that $P(1) = \max_{[-1,1]} |P(t)| = 1$.

Assume by contradiction that there exists a sequence $P_n$ of polynomials in $\mathcal{P}_d$ and a sequence of $K$-thick subsets $T_n$ of $\mathbb{R}$ such that $\sup_{T_n \cap [-1,1]} |P_n(t)|$ converge to $0$. After extraction, the sequence $T_n$ converges to a $K$-thick subset $T_\infty$ and the sequence $P_n$ converges to a polynomial $P_\infty \in \mathcal{P}_d$ which is equal to $0$ on the set $T_\infty \cap [-1,1]$. This is not possible since this set is infinite. $\square$

We record also, for further use, the following classical lemma :

**Lemma 2.6.** *Let $U_1^+$ be the semigroup $\{u_t : t \ge 0\}$. If the quotient space $X_1 := \Gamma_1 \backslash H_1$ is compact, any $U_1^+$-orbit is dense in $X_1$.*

*Proof.* For $x \in X_1$, set $x_n := x u_n$. We then have $x_n u_{-n} U_1^+ = x U_1^+$. Hence if $z$ is a limit point of the sequence $x_n$ in $X_1$, we have $zU \subset \overline{x U_1^+}$. By Hedlund's theorem [Hed], $zU$ is dense. Hence the orbit $x U_1^+$ is also dense. $\square$

## 3. Proof of Theorems 1.1 and 1.2

In this section, using minimal sets and unipotent dynamics on the product space $\Gamma \backslash G$, we provide a proof of Theorem 1.2.

**3.1. Unipotent dynamics.** We recall the notation $G := \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ and $\Gamma := \Gamma_1 \times \Gamma_2$. Set

- $H_1 = \{(h,e)\}$, $H_2 = \{(e,h)\}$, $H = \{(h,h)\}$;
- $U_1 = \{(u_t,e)\}$, $U_2 = \{(e,u_t)\}$, $U = \{(u_t,u_t)\}$;

- $A_1 = \{(a_t, e)\}$, $A_2 = \{(e, a_t)\}$, $A = \{(a_t, a_t)\}$;
- $X_1 = \Gamma_1 \backslash H_1$, $X_2 = \Gamma_2 \backslash H_2$, $X = \Gamma \backslash G = X_1 \times X_2$.

Recall that $\Gamma_1$ is a non-elementary finitely generated discrete subgroup of $H_1$ with no parabolic elements and that $\Gamma_2$ is a cocompact lattice in $H_2$.

For simplicity, we write $\widetilde{u}_t$ for $(u_t, u_t)$ and $\widetilde{a}_t$ for $(a_t, a_t)$. Note that the normalizer of $U$ in $G$ is $AU_1U_2$.

**Lemma 3.1.** *Let $g_n$ be a sequence in $G \setminus AU_1U_2$ converging to $e$, and let $T$ be a $K$-thick subset of $\mathbb{R}$ for some $K > 1$. Then for any neighborhood $G_0$ of $e$ in $G$, there exist sequences $s_n \in T$ and $t_n \in \mathbb{R}$ such that the sequence $\widetilde{u}_{-s_n} g_n \widetilde{u}_{t_n}$ has a limit point $q \neq e$ in $AU_2 \cap G_0$.*

*Proof.* Fix $0 < \varepsilon \leq 1$. Write $g_n = (g_n^{(1)}, g_n^{(2)})$ with $g_n^{(i)} = \begin{pmatrix} a_n^{(i)} & b_n^{(i)} \\ c_n^{(i)} & d_n^{(i)} \end{pmatrix}$. Then the products $q_n := \widetilde{u}_{-s_n} g_n \widetilde{u}_{t_n}$ are given by

$$q_n^{(i)} = u_{-s_n} g_n^{(i)} u_{t_n} = \begin{pmatrix} a_n^{(i)} - c_n^{(i)} s_n & (b_n^{(i)} - d_n^{(i)} s_n) - t_n (c_n^{(i)} s_n - a_n^{(i)}) \\ c_n^{(i)} & d_n^{(i)} + c_n^{(i)} t_n \end{pmatrix}.$$

Set

$$t_n = \frac{b_n^{(1)} - d_n^{(1)} s_n}{c_n^{(1)} s_n - a_n^{(1)}}.$$

The differences $q_n - e$ are now rational functions in $s_n$ of the form

$$q_n - e = \frac{1}{c_n^{(1)} s_n - a_n^{(1)}} P_n(s_n),$$

where $P_n(s)$ is a polynomial function of $s$ of degree at most $2$ with values in $M_2(\mathbb{R}) \times M_2(\mathbb{R})$. Since the elements $g_n$ do not belong to $AU_1U_2$, these polynomials $P_n$ are non-constants. In particular, the real valued polynomial functions $s \mapsto \|P_n(s)\|^2$ are non-constant of degree at most $4$.

Since $\|P_n(0)\| \to 0$ as $n \to \infty$, it follows from Lemma 2.5 that for any $0 < \epsilon$, we can choose $s_n \in T$ going to $\infty$ so that $k\varepsilon \leq \|P_n(s_n)\| \leq \varepsilon$ for some constant $k > 0$ depending only on $K$. Moreover we can deduce $1/2 \leq |c_n^{(1)} s_n - a_n^{(1)}| \leq 2$ from the condition $\|P_n(s_n)\| \leq \varepsilon$ by looking at the $(1,1)$ and $(2,2)$ entries of the first component of $P_n(s_n)$.

Therefore

$$k\varepsilon/2 \leq \|q_n - e\| \leq 2\varepsilon.$$

By construction, when $\varepsilon$ is small enough, the sequence $q_n$ has a limit point $q \neq e$ in $A_1 A_2 U_2 \cap G_0$.

We claim that this limit $q=(q^{(1)},q^{(2)})$ belongs to the group $AU_2$. It suffices to check that the diagonal entries of $q^{(1)}$ and $q^{(2)}$ are equal. If not, the two sequences $c_n^{(i)}s_n$ converge to real numbers $c^{(i)}$ with $c^{(1)}\neq c^{(2)}$, and a simple calculation shows that the $(1,2)$- entries of $q_n^{(2)}$ are comparable to $\frac{c^{(2)}-c^{(1)}}{c^{(1)}-1}s_n$ which tends to $\infty$, yielding a contradiction. Hence $q$ belongs to $AU_2$. □

### 3.2. *H*-minimal and *U*-minimal subsets.

Let

$$\Omega := \Omega_{\Gamma_1}\times X_2$$

where $\Omega_{\Gamma_1}\subset X_1$ is defined in (2.1). Note that, since $\Gamma_2$ is cocompact, one has the equality $\Omega_{\Gamma_2}=X_2$.

Let $x=(x_1,x_2)\in\Gamma\backslash G$ and consider the orbit $xH$. Note that $xH$ intersects $\Omega$ non-trivially. Let $Y$ be an $H$-minimal subset of the closure $\overline{xH}$ with respect to $\Omega$, i.e., $Y$ is a closed $H$-invariant subset of $\overline{xH}$ such that $Y\cap\Omega\neq\varnothing$ and the orbit $yH$ is dense in $Y$ for any $y\in Y\cap\Omega$. Since any $H$ orbit intersects $\Omega$, it follows that $yH$ is dense in $Y$ for any $y\in Y$. Let $Z$ be a $U$-minimal subset of $Y$ with respect to $\Omega$. Since $\Omega$ is compact, such minimal sets $Y$ and $Z$ exist. Set

$$Y^* = Y\cap\Omega \quad\text{and}\quad Z^*=Z\cap\Omega.$$

In the following, we assume that

the orbit $xH$ is not closed

and aim to show that $xH$ is dense in $X$.

**Lemma 3.2.** *For any $y\in Y$, the identity element $e$ is an accumulation point of the set $\{g\in G\smallsetminus H : yg\in\overline{xH}\}$.*

*Proof.* If $y$ does not belong to $xH$, there exists a sequence $h_n\in H$ such that $xh_n$ converges to $y$. Hence there exists a sequence $g_n\in G$ converging to $e$ such that $xh_n=yg_n$. These elements $g_n$ do not belong to $H$; hence proving the claim.

Suppose now that $y$ belongs to $xH$. If the claim does not hold, then for a sufficiently small neighborhood $G_0$ of $e$ in $G$, the set $yG_0\cap Y$ is included in the orbit $yH$. This implies that the orbit $yH$ is an open subset of $Y$. The minimality of $Y$ implies that $Y=yH$, contradicting the assumption that the orbit $yH=xH$ is not closed. □

**Lemma 3.3.** *There exists an element* $v \in U_2 \smallsetminus \{e\}$ *such that* $Zv \subset \overline{xH}$.

*Proof.* Choose a point $z = (z_1, z_2) \in Z^*$. By Lemma 3.2, there exists a sequence $g_n$ in $G \smallsetminus H$ converging to $e$ such that $zg_n \in \overline{xH}$. We may assume without loss of generality that $g_n$ belongs to $H_2$.

Suppose first that at least one $g_n$ belongs to $U_2$. Set $v = g_n$ be one of those belonging to $U_2$, so that the point $zv$ belongs to $\overline{xH}$. Since $v$ commutes with $U$ and $Z$ is $U$-minimal with respect to $\Omega$, one has the equality $Zv = \overline{zvU}$, hence the set $Zv$ is included in $\overline{xH}$.

Now suppose that $g_n$ does not belong to $U_2$. Then, since the set $T(z_1)$ is $K$-thick for some $K > 1$ by Lemma 2.3, it follows from Lemma 2.4 that there exists a sequence $t_n \to \infty$ in $T(z_1)$ such that, after extraction, the products $\widetilde{u}_{-t_n} g_n \widetilde{u}_{t_n}$ converge to an element $v \in U_2 \smallsetminus \{e\}$.

Since the points $z\widetilde{u}_{t_n}$ belong to $\Omega$, this sequence has a limit point $z' \in Z^*$. Since one has the equality

$$z'v = \lim_{n\to\infty} z\widetilde{u}_{t_n}(\widetilde{u}_{-t_n} g_n \widetilde{u}_{t_n}) = \lim_{n\to\infty} (zg_n)\widetilde{u}_{t_n},$$

the point $z'v$ belongs to $\overline{xH}$. We conclude as in the first case that the set $Zv = \overline{z'vU}$ is included in $\overline{xH}$. □

**Lemma 3.4.** *For any* $z \in Z^*$, *there exists a sequence* $g_n$ *in* $G \smallsetminus U$ *converging to* $e$ *such that* $zg_n \in Z$ *for all* $n$.

*Proof.* Since the group $\Gamma_2$ is cocompact, it does not contain unipotent elements and hence the orbit $zU$ is not compact. By Lemma 2.3, the orbit $zU$ is recurrent in $Z^*$, hence the set $Z^* \smallsetminus zU$ contains at least one point. Call it $z'$. Since the orbit $z'U$ is dense in $Z$, there exists a sequence $\widetilde{u}_{t_n} \in U$ such that $z = \lim z'\widetilde{u}_{t_n}$. Hence one can write $z'\widetilde{u}_{t_n} = zg_n$ with $g_n$ in $G \smallsetminus U$ converging to $e$. □

**Proposition 3.5.** *There exists a one-parameter semi-group* $L^+ \subset AU_2$ *such that* $ZL^+ \subset Z$.

*Proof.* It suffices to find, for any neighborhood $G_0$ of $e$, an element $q \neq e$ in $AU_2 \cap G_0$ such that the set $Zq$ is included in $Z$; then writing $q = \exp w$ for an element $w$ of the Lie algebra of $G$, we can take $L^+$ to be the semigroup $\{\exp(sw_\infty) : s \geq 0\}$ where $w_\infty$ is a limit point of the elements $\frac{w}{\|w\|}$ when the diameter of $G_0$ shrinks to $0$.

Fix a point $z = (z_1, z_2) \in Z^*$. According to Lemma 3.4 there exists a sequence $g_n \in G \smallsetminus U$ converging to $e$ such that $zg_n \in Z$.

Suppose first that $g_n$ belongs to $AU_1U_2$ for infinitely many $n$; then one can find $\widetilde{u}_{t_n} \in U$ such that the product $q_n := g_n\widetilde{u}_{t_n}$ belongs to $AU_2 \smallsetminus \{e\}$ and $zq_n$

belongs to $Z$. Since $q_n$ normalizes $U$ and since $Z$ is $U$-minimal with respect to $\Omega$, one has the equality $Zq_n = \overline{zU}q_n = \overline{zq_nU}$, hence the set $Zq_n$ is included in $Z$.

Now suppose that $g_n$ is not in $AU_1U_2$. By Lemmas 2.3 and 3.1, there exist sequences $s_n \in T(z_1)$ and $t_n \in \mathbb{R}$ such that, after passing to a subsequence, the products $\widetilde{u}_{-s_n} g_n \widetilde{u}_{t_n}$ converge to an element $q \neq e$ in $AU_2 \cap G_0$. Since the elements $z\widetilde{u}_{s_n}$ belong to $Z^*$, they have a limit point $z' \in Z^*$. Since we have

$$z'q = \lim_{n\to\infty} z\widetilde{u}_{s_n}(\widetilde{u}_{-s_n} g_n \widetilde{u}_{t_n}) = \lim_{n\to\infty} (zg_n)\widetilde{u}_{t_n},$$

the element $z'q$ belongs to $Z$. We conclude as in the first case that the set $Zq = \overline{z'qU}$ is included in $Z$. □

**Proposition 3.6.** *There exist an element $z \in \overline{xH}$ and a one-parameter semi-group $U_2^+ \subset U_2$ such that $zU_2^+ \subset \overline{xH}$.*

*Proof.* By Proposition 3.5 there exists a one-parameter semigroup $L^+ \subset AU_2$ such that $ZL^+ \subset Z$. This semigroup $L^+$ is equal to one of the following: $U_2^+$, $A^+$ or $v_0^{-1}A^+v_0$ for some element $v_0 \in U_2 \smallsetminus \{e\}$, where $U_2^+$ and $A^+$ are one-parameter semigroups of $U_2$ and $A$ respectively.

When $L^+ = U_2^+$, our claim is proved.

Suppose now $L^+ = A^+$. By Lemma 3.3 there exists an element $v \in U_2 \smallsetminus \{e\}$ such that $Zv \subset \overline{xH}$. Then one has the inclusions

$$ZA^+vA \subset ZvA \subset \overline{xH}A \subset \overline{xH}.$$

Choose a point $z' \in Z^*$ and a sequence $\widetilde{a}_{t_n} \in A^+$ going to $\infty$. Since $z'\widetilde{a}_{t_n}$ belong to $\Omega$, after passing to a subsequence, the sequence $z'\widetilde{a}_{t_n}$ converges to a point $z \in \overline{xH} \cap \Omega$. Moreover, since the Hausdorff limit of the sets $\widetilde{a}_{-t_n}A^+$ is $A$, one has the inclusions

$$zAvA \subset \lim_{n\to\infty} z'\widetilde{a}_{t_n}(\widetilde{a}_{-t_n}A^+)vA = z'A^+vA \subset \overline{xH}.$$

Now by a simple computation, we can check that the set $AvA$ contains a one-parameter semigroup $U_2^+$ of $U_2$, and hence the orbit $zU_2^+$ is included in $\overline{xH}$ as desired.

Suppose finally $L^+ = v_0^{-1}A^+v_0$ for some $v_0$ in $U_2 \smallsetminus \{e\}$. We can write $A^+ = \{\widetilde{a}_{\varepsilon t} : t \geq 0\}$ with $\varepsilon = \pm 1$ and $v_0 = (e, u_s)$ with $s \neq 0$. A simple computation shows that the set $U_2' := \{(e, u_{\varepsilon st}) : 0 \leq t \leq 1\}$ is included in $v_0^{-1}A^+v_0A$. Hence one has the inclusions

$$ZU_2' \subset Zv_0^{-1}A^+v_0A \subset ZA \subset \overline{xH}.$$

Choose a point $z' \in Z^*$ and let $z \in \overline{xH}$ be a limit of a sequence $z'\widetilde{a}_{-t_n}$ with $t_n$ going to $+\infty$. Since the Hausdorff limit of the sets $\widetilde{a}_{t_n} U_2' \widetilde{a}_{-t_n}$ is the semigroup $U_2^+ := \{(e, u_{\varepsilon st}) : t \geq 0\}$, one has the inclusions

$$zU_2^+ \subset \lim_{n\to\infty} (z'\widetilde{a}_{-t_n})\widetilde{a}_{t_n} U_2' \widetilde{a}_{-t_n} \subset \overline{ZU_2'A} \subset \overline{xH}.$$

□

### 3.3. Conclusion.

*Proof of Theorem* 1.2. Suppose that the orbit $xH$ is not closed. By Proposition 3.6, the orbit closure $\overline{xH}$ contains an orbit $zU_2^+$ of a one-parameter subsemigroup of $U_2$. Since $\Gamma_2$ is cocompact in $H_2$, by Lemma 2.6, this orbit $zU_2^+$ is dense in $zH_2$. Hence we have the inclusions

$$X = zG = zH_2H \subset \overline{zU_2^+H} \subset \overline{xH}.$$

This proves the claim. □

*Proof of Theorem* 1.1. Let $x = [g]$ be a point of $X_2 = \Gamma_2 \backslash H_2$. By replacing $\Gamma_1$ by $g^{-1}\Gamma_1 g$, we may assume without loss of generality that $g = e$. One deduces Theorem 1.1 from Theorem 1.2 thanks to the following equivalences:
The orbit $[e]H$ is closed (resp. dense) in $\Gamma\backslash G \iff$
The orbit $\Gamma[e]$ is closed (resp. dense) in $G/H \iff$
The product $\Gamma_2\Gamma_1$ is closed (resp. dense) in $\mathrm{PSL}_2(\mathbb{R}) \iff$
The orbit $[e]\Gamma_1$ is closed (resp. dense) in $\Gamma_2\backslash \mathrm{PSL}_2(\mathbb{R})$. □

**Acknowledgment.** Oh was supported in part by an NSF Grant.

## References

[BQ1] Y. Benoist and J. F. Quint, Stationary measures and invariant subsets of homogeneous spaces I. *Annals of Math* **174** (2011), 1111–1162. Zbl 1241.22007 MR 2831114

[BQ2] Y. Benoist and J. F. Quint, Stationary measures and invariant subsets of homogeneous spaces III. *Annals of Math* **178** (2013), 1017–1059. Zbl 1279.22013 MR 3092475

[Hed] G. Hedlund, Fuchsian groups and transitive horocycles. *Duke Math. J.* **2** (1936), 530–542. Zbl 0015.10201 MR 1545946

[Mar] G. Margulis, Indefinite quadratic forms and unipotent flows on homogeneous spaces. In *Dynamical Systems and Ergodic Theory* (Warsaw, 1986), volume 23. Banach Center Publ., 1989. Zbl 0689.10026 MR 1102736

[MMO] C. MCMULLEN, A. MOHAMMADI and H. OH, Geodesic planes in hyperbolic 3-manifolds. Preprint, 2015

[Rat] M. RATNER, Raghunathan's topological conjecture and distributions of unipotent flows. Duke Math. J. **63** (1991), 235–280. Zbl 0733.22007 MR 1106945

(*Reçu le 11 juin 2015*)

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