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Note on the distortion of $(2,q)$-torus knots

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Abstract. We show that the distortion of the $(2,q)$-torus knot is not bounded linearly from below.

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1. Introduction

The notion of distortion was introduced by Gromov [GPL]. If γ is a rectifiable simple closed curve in $\mathbf{R}^3$, then its distortion δ is defined as

$$\delta(\gamma) = \sup_{v,w\in\gamma} \frac{d_\gamma(v,w)}{|v-w|},$$

where $d_\gamma(v,w)$ denotes the length of the shortest arc connecting v and w in γ and $|\cdot|$ denotes the euclidean norm on $\mathbf{R}^3$. For a knot K, its distortion $\delta(K)$ is defined as the infimum of $\delta(\gamma)$ over all rectifiable curves γ in the isotopy class K. Gromov [Gro] asked in 1983 if every knot K has distortion $\delta(K) \leq 100$. The question was open for almost three decades until Pardon gave a negative answer. His work [Par] presents a lower bound for the distortion of simple closed curves on closed PL embedded surfaces with positive genus. Pardon showed that the minimal intersection number of such a curve with essential discs of the corresponding surface bounds the distortion of the curve from below. In particular for the (p,q)-torus knot he showed that $\delta(T_{p,q}) \geq \min(p,q)/160$. By considering a standard embedding of $T_{p,p+1}$ into a torus of revolution one obtains $\delta(T_{p,p+1}) \leq const \cdot p$, hence for $q = p+1$ Pardon's result is sharp up to a constant.

An alternative proof for the existence of families with unbounded distortion was given by Gromov and Guth [GG]. In both works the answer to Gromov's question was obtained by estimating the conformal length, which is up to a constant a lower bound for the distortion of rectifiable closed curves. However, the conformal length is in general not a good estimate for the distortion. For example, one finds easily an embedding of the $(2,q)$-torus knot with conformal length ≤ 100 and distortion $\geq q$ by looking at standard embeddings into a torus of revolution with suitable dimensions. In particular, neither Pardon's nor Gromov and Guth's arguments yield lower bounds for $\delta(T_{2,q})$. While Pardon conjectures that $\lim_{q\to\infty}\delta(T_{2,q}) = \infty$ and that there are to his knowledge no known embeddings of $T_{2,q}$ with sublinear distortion [Par, p. 638], Gromov and Guth [GG, p. 2588] write that the distortion of $T_{2,q}$ appears to be approximately q. In this article we show that the growth rate of $\delta(T_{2,q})$ is in fact sublinear in q.

Theorem. *Let $q \geq 50$. Then $\delta(T_{2,q}) \leq 7q/\log q$. In particular the distortion of the $(2,q)$-torus knot is not bounded linearly from below.*

With the same technique as used in this article and somewhat more effort one can give an embedding γ_q of $T_{2,q}$ with $\delta(\gamma_q) \sim \frac{\pi}{2}\frac{q}{\log q}$. Moreover, a more technical proof yields that this asymptotical upper bound for $\delta(T_{2,q})$ is sharp for those embeddings of $T_{2,q}$ that project orthogonally onto a standard knot diagram. This leads to the following question.

Question. *Is $\delta(T_{2,q})$ up to a constant asymptotically equal to $q/\log q$? And if yes, is the constant equal to $\pi/2$?*

2. Proof of the Theorem

In order to prove the Theorem we need to give for every odd integer $q \geq 50$ an embedding γ of the $(2,q)$-torus knot with distortion smaller or equal to $7q/\log q$. The idea is to use a logarithmic spiral. Let S be a logarithmic spiral of unit length starting at its center $0 \in \mathbf{R}^3$ and ending at some $u \in \mathbf{R}^3$. An elementary calculation shows that its distortion is equal to $1/|u|$. For another path $\alpha \subset \mathbf{R}^3$ of unit length and diameter $\leq 2|u|$ with endpoints $\{v,w\} = \partial\alpha$ we get

$$\delta(\alpha) \geq \frac{d_\alpha(v,w)}{|v-w|} = \frac{1}{|v-w|} \geq \frac{1}{2|u|} = \frac{\delta(S)}{2}.$$

Hence up to at most a factor 2 the logarithmic spiral has the smallest distortion among all paths for a prescribed pathlength-pathdiameter-ratio. It seems therefore natural to pack the q windings of the $(2,q)$-torus knot into a logarithmic spiral in order to minimize distortion.

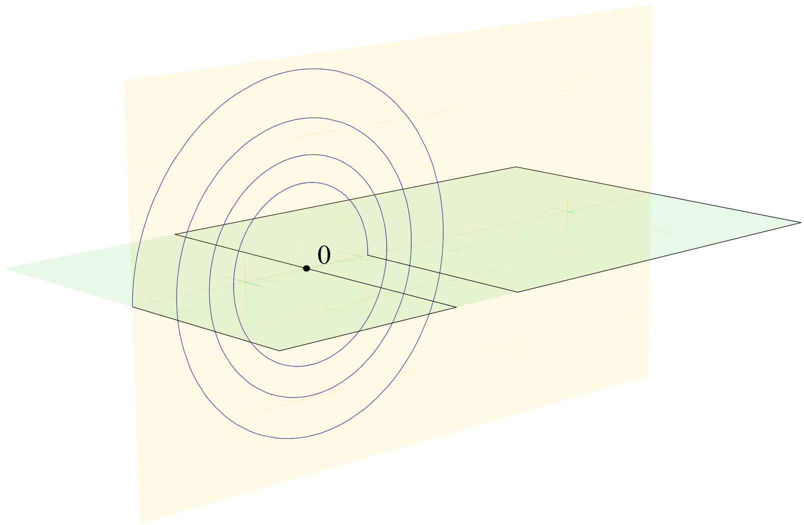

FIGURE 1
The embedding γ for $q = 7$

We define the embedding γ as the union of a segment of the logarithmic spiral S with slope $k = \log(q)/2\pi q$ and a piecewise linear part L, see Figure 1. The segment of the logarithmic spiral S is contained in the vertical (x,z) plane and parametrized by

$$\varphi : [0, \pi q] \to \mathbf{R}^2, \qquad \varphi(s) = e^{ks} \cdot \begin{pmatrix} \cos(s) \\ \sin(s) \end{pmatrix},$$

see Figures 1 and 2. The segment of the piecewise linear part L is in the horizontal (x,y) plane, see Figures 1 and 3. Note that

$$|\varphi(\pi q)| = e^{k\pi q} = \sqrt{q} \quad \text{and} \quad |\varphi(0)| = 1,$$

hence the lengths defining L in Figure 3 are chosen such that the union γ of S and L is the simple closed curve illustrated in Figure 1. The linear segments L_1 and L_2 indicated in Figure 3 are named because of their special role in the following computations.

To see that the obtained curve is an embedded $(2,q)$-torus knot, we perturb γ, see Figure 4. This simple closed curve is ambient isotopic in $\mathbf{R}^3$ to γ and if we project it onto the (x,y) plane, we see a well known diagram of the $(2,q)$-torus knot, see Figure 5.

We now estimate the distortion of γ. One has to show that

$$\frac{d_\gamma(v,w)}{|v-w|} \le \frac{7q}{\log q}$$

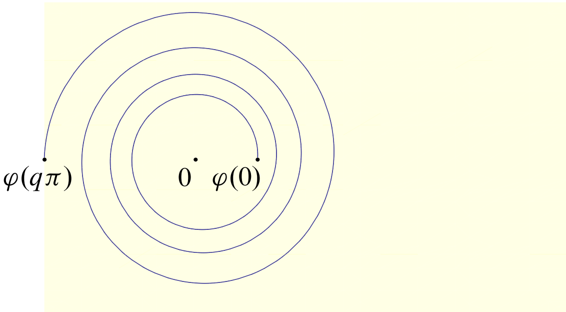

FIGURE 2
The logarithmic spiral S in the (x,z) plane

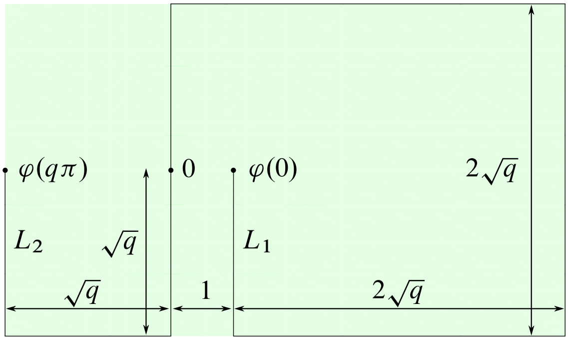

FIGURE 3
The linear part L in the (x,y) plane

for all pairs of points $v,w \in \gamma$. An easy computation shows that

$$\frac{1}{k} \cdot \sqrt{2k^2+1} = \frac{2\pi q}{\log q} \cdot \sqrt{2(\log q/2\pi q)^2+1} \leq \frac{7q}{\log q}$$

for all integers ≥ 2. Therefore, it suffices to show that

$$\frac{d_\gamma(v,w)}{|v-w|} \leq \frac{\sqrt{2k^2+1}}{k}.$$

In order to do this, we distinguish four cases.

Case 1: $v,w \in S$. Let $0 \leq s \leq t \leq \pi q$, $v = \varphi(s), w = \varphi(t)$. From

$$|\varphi'(r)| = \left| \begin{pmatrix} \cos(r) & -\sin(r) \\ \sin(r) & \cos(r) \end{pmatrix} \begin{pmatrix} ke^{kr} \\ e^{kr} \end{pmatrix} \right| = \left| \begin{pmatrix} ke^{kr} \\ e^{kr} \end{pmatrix} \right| = \sqrt{k^2+1} \cdot e^{kr},$$

we get

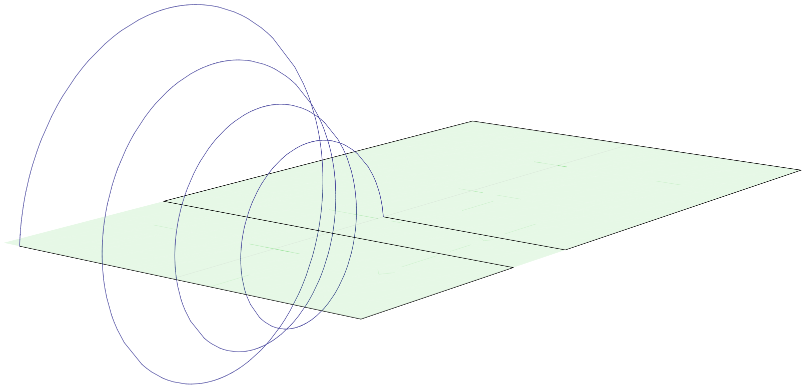

FIGURE 4
Perturbation of γ

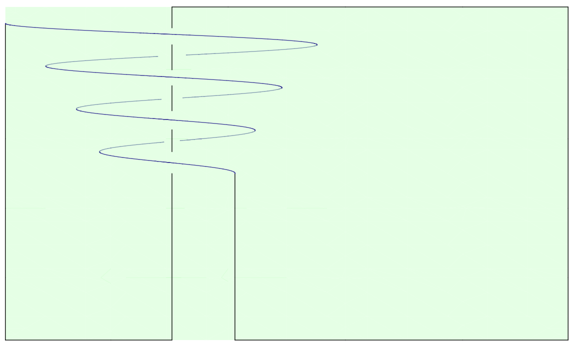

FIGURE 5
Projection onto the (x, y) plane

$$\begin{aligned} d_\gamma(v,w) &\le d_S(v,w) \\ &= \int_s^t |\varphi'(r)| dr \\ &= \sqrt{k^2+1} \int_s^t e^{kr} dr \\ &= \tfrac{\sqrt{k^2+1}}{k} \cdot (e^{kt} - e^{ks}) \\ &= \tfrac{\sqrt{k^2+1}}{k} \cdot \big(|\varphi(t)| - |\varphi(s)|\big) \\ &= \tfrac{\sqrt{k^2+1}}{k} \cdot (|w| - |v|). \end{aligned}$$

Since $|w - v| \ge |w| - |v|$, we conclude that

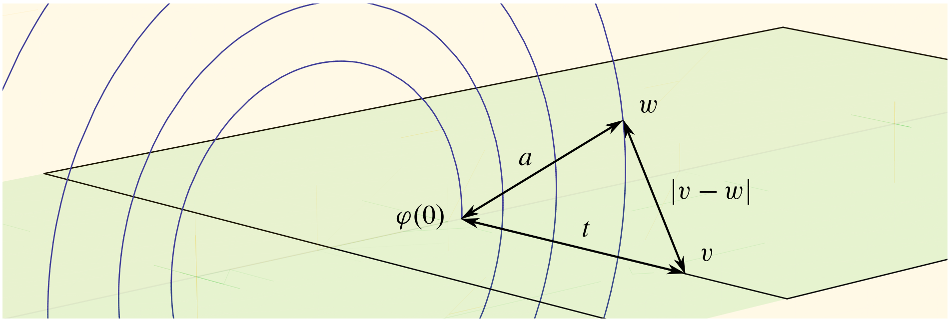

FIGURE 6
Visualization of Case 2

$$\frac{d_\gamma(v,w)}{|v-w|} \le \frac{\sqrt{k^2+1}}{k} \cdot \frac{(|w|-|v|)}{(|w|-|v|)} = \frac{\sqrt{k^2+1}}{k}.$$

Case 2: $v \in L_1 \cup L_2$, $w \in S$. We consider the case where $v \in L_1$. The idea is to find the maximum of $d_\gamma(v,w)/|v-w|$ for fixed w and varying v. Let $t = |v - \varphi(0)|$, $a = |\varphi(0) - w|$, and $b = d_S(\varphi(0), w)$, see Figure 6. Note that $|v-w| = \sqrt{t^2+a^2}$ and $d_\gamma(v, \varphi(0)) = |v - \varphi(0)| = t$. We get

$$\frac{d_\gamma(v,w)}{|v-w|} \le \frac{d_\gamma\big(v,\varphi(0)\big) + d_S\big(\varphi(0),w\big)}{|v-w|} = \frac{t+b}{\sqrt{t^2+a^2}} =: f(t).$$

Deriving f with respect to t yields a unique critical point at $t = a^2/b$:

$$0 = f'(t) = \frac{a^2 - bt}{(a^2+t^2)^{3/2}} \qquad \Longleftrightarrow \qquad t = a^2/b.$$

Since a^2/b is the only critical point, $f(\infty) = 1 \le b/a = f(0)$ and

$$f(0) = \frac{b}{a} \le \frac{\sqrt{a^2+b^2}}{a} = \frac{\frac{a^2}{b} + b}{\sqrt{(\frac{a^2}{b})^2 + a^2}} = f(a^2/b),$$

a^2/b must be a global maximum. Consequently we get

$$\begin{aligned} \frac{d_\gamma(v,w)}{|v-w|} &\le \frac{\sqrt{a^2+b^2}}{a} \\ &= \sqrt{1 + \left(\frac{b}{a}\right)^2} \\ &= \sqrt{1 + \left(\frac{d_S(\varphi(0),w)}{|\varphi(0)-w|}\right)^2} \end{aligned}$$

$$\stackrel{\text{Case1}}{\leq} \sqrt{1 + \left(\tfrac{\sqrt{k^2+1}}{k}\right)^2}$$
$$= \frac{\sqrt{2k^2+1}}{k}.$$

In the case where $v \in L_2$, we make the estimate with the path that connects v with w through $\varphi(\pi q)$. It works exactly the same and yields the same estimate.

Case 3: $v, w \in L$. Consider Figure 3 and note that all pairs of points $v, w \in L$ of euclidean distance < 1 are either on the same linear segment or on neighboring linear segments of L. It is easy to see that such pairs of points cannot cause distortion $> \sqrt{2}$. For the pairs of points $v, w \in L$ of euclidean distance ≥ 1 we get

$$\frac{d_\gamma(v, w)}{|v - w|} \leq d_L\big(\varphi(0), \varphi(\pi q)\big) = 11\sqrt{q} + 1.$$

A direct calculation shows that

$$11\sqrt{q} + 1 \leq \frac{2\pi q}{\log q} = \frac{1}{k}$$

for $q \geq 50$.

Case 4: $v \in L \setminus (L_1 \cup L_2), w \in S$. Note that for these pairs of points we have $|v - w| \geq |w|$. We estimate $d_\gamma(v, w)$ using results of Cases 1 and 3:

$$\begin{aligned} d_\gamma(v, w) &\leq d_L\big(v, \varphi(0)\big) + d_S\big(\varphi(0), w\big) \\ &\leq \tfrac{1}{k} + \tfrac{\sqrt{k^2+1}}{k} \cdot (|w| - 1) \\ &\leq \tfrac{\sqrt{k^2+1}}{k} \cdot |w|. \end{aligned}$$

We conclude that

$$\frac{d_\gamma(v, w)}{|v - w|} \leq \frac{\frac{\sqrt{k^2+1}}{k} \cdot |w|}{|w|} = \frac{\sqrt{k^2+1}}{k},$$

which finishes the proof.

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