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Galois involutions and exceptional buildings

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Abstract. We apply the theory of descent for buildings to give elementary constructions of the exceptional buildings of type A_2, B_2, C_3 and F_4 as the fixed point building of a Galois involution of a building of type E_6, E_7 or E_8 or, in one case, a pseudo-split building of type F_4.

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1. Introduction

In this paper we apply the theory of descent for buildings introduced in [MPW] to give elementary constructions of the exceptional buildings of type A_2, B_2, C_3 and F_4 as the fixed point buildings of a Galois involution of either a building of type E_6, E_7 or E_8 or, in one case, a pseudo-split building of type F_4 (as defined in 15.3). Our main results are 11.21, 12.11, 13.12, 14.11, 15.4 and 17.14.

The notion of a building was introduced by J. Tits in order to give a uniform geometric/combinatorial description of the groups of rational points of an isotropic absolutely simple group. The buildings that arise in this context are spherical. In [Tit2], Tits classified irreducible spherical buildings of rank at least 3 and this classification was extended to the rank 2 case in [TW] under the assumption that the building satisfies the Moufang condition (which is automatic when the rank is at least 3). The classification in the rank 2 case is carried out by studying commutator relations; in [TW, Chapter 40] it is used to give another proof of the classification in rank greater than 2. The question of existence is settled in [TW, Chapter 32] for the rank 2 case and in [TW, 40.56] for the remaining cases using the geometric ideas introduced by Ronan and Tits in [RT]. This replaced the earlier existence proofs for the exceptional buildings in [Tit2, 5.12 and 10.3] and [TW, 42.6], where existence is proved using the theory of Galois descent in algebraic groups (see 5.6).

The result of this classification is that most spherical buildings satisfying the Moufang condition are the spherical buildings associated with absolutely simple algebraic groups. The exceptions are buildings determined by algebraic data involving infinite dimensional structures, defective quadratic or pseudo-quadratic forms, inseparable field extension and/or the square root of a Frobenius endomorphism. Most notable among these exceptions are the indifferent quadrangles, the Moufang quadrangles of type F_4 and the Moufang octagons.

The classification results in [Tit2] and [TW] do not reveal the connection between a spherical building and its ambient split building which is the central concern in the theory of Galois descent. In [MPW, Part 3], this shortcoming was remedied with a theory of descent for buildings. This theory gives, in particular, a combinatorial interpretation of the Tits indices which appear in [Tit1]. It applies, moreover, to buildings of arbitrary type. Some central results of this theory are summarized in §6 below and they are applied to buildings of type E_6, E_7, E_8 and F_4 in subsequent sections.

This paper can thus be seen as a contribution to Tits' larger plan of interpreting the classification of isotropic absolutely simple algebraic group purely in the language of buildings.

The results in this paper provide uniform proofs of [MPW, 34.3–34.9]; see [MPW, 34.12]. These results, in turn, are applied in [MPW, Chapter 36] to the study of exceptional affine buildings. Precursors of the results in this paper can be found in [Mue] and [MM1].

We confine our attention in this paper to those exceptional groups which can be constructed as fixed point buildings of Galois involutions (as defined in 4.15 below). This allows various simplifications in the arguments. In particular, we do not treat the Moufang hexagons (which require the action of a larger Galois group) in this paper. The Moufang octagons can be constructed as fixed point buildings of involutions, but these involutions involve a Tits endomorphism rather than a Galois group; see [dMSW] for more about this case.

All known proper Moufang sets can be described in terms of our theory of descent as fixed point buildings of relative rank 1. The methods used in this paper provide a point of access to these buildings which we are presently pursuing. See, in this context, [CdM] and [MM2].

This paper is organized as follows: In §2–§5, we give background material in the theory of buildings, in §6 we summarize the results about descent we require and in §7 and §8 we make some observations about buildings of type A_n and D_n in terms of linear algebra. The proofs of existence for various forms of buildings of type E_6, E_7 and E_8 begin then in §9, where we describe an anisotropic Galois involution of a building of type D_n. The existence proofs are carried out in §10–§15 by extending this involution (for certain small values of n)

to involutions of various ambient buildings. In §16 and §17, finally, we apply our methods to construct the quadrangles of type F_4.

Notation 1.1. We will follow the conventions used in [TW] that $a^b = b^{-1}ab$ and $[a,b] = a^{-1}b^{-1}ab$ for all elements a, b in some group and we will compose permutations from left to right. (When we are not composing them, however, we will usually write functions on the left.) If $i < j$ are integers, we denote by $[i, j]$ the interval $\{m \in \mathbb{Z} \mid i \leq m \leq j\}$; we only use this notation when i and j are subscripts.

2. Coxeter groups

Let Π be a Coxeter diagram with vertex set S and let (W, S) be the corresponding Coxeter system. An *automorphism* of (W, S) is an automorphism of the group W that stabilizes the generating set S. There is a canonical isomorphism from $\mathrm{Aut}(W, S)$ to $\mathrm{Aut}(\Pi)$ and we will think of these two groups as being the same.

Notation 2.1. Let Σ be the graph with vertex set W in which two vertices x and y are joined by an edge labeled with the element s of S whenever $x^{-1}y = s$. Thus each edge of Σ has a unique label in the set S. We call this label the *type* of the edge. The group W acts on Σ by left multiplication and can, in fact, be identified with the group of type-preserving automorphisms of Σ. See [Weil, 3.10] for the definition of a *root* of Σ.

Lemma 2.2. *The only automorphism of* Σ *stabilizing every root is the identity.*

Proof. If c and d are distinct vertices of Σ, there is a root of Σ containing c but not d (by [Weil, 3.20]). Thus a non-trivial automorphism of Σ cannot stabilize every root of Σ. □

Notation 2.3. Let J be a spherical subset of S (by which we mean that the subgroup $W_J := \langle J \rangle$ is finite) and let w_J denote the longest element of the Coxeter group W_J with respect to the generating set J. By [Weil, 5.11], the map $s \mapsto w_J s w_J$ is an automorphism of the subdiagram of Π spanned by the set J. We denote this subdiagram by Π_J and this automorphism by op_J. The map op_J is called the *opposite map* of Π_J.

Remark 2.4. The map op_J stabilizes every connected component of Π_J and acts non-trivially on a given connected component if and only if it is isomorphic to the Coxeter diagram A_n for some $n \geq 2$, to D_n for some odd $n \geq 5$, to E_6 or to $I_2(n)$ for some odd $n \geq 5$.

Suppose now that (W, S) itself is spherical, equivalently, that the graph Σ is finite.

Notation 2.5. We say that two vertices of Σ are *opposite* if they are at maximal distance in Σ. Let $\xi(x) = xw_S$ for all $x \in W$, where w_S is as in 2.3 with $J = S$. Every vertex of Σ has a unique opposite vertex, and the unique vertex opposite a vertex x is precisely $\xi(x)$.

Notation 2.6. Let $\mathrm{op} = \mathrm{op}_S$ be as in 2.3. By [Weil, 5.11], ξ maps edges of type s to edges of type $\mathrm{op}(s)$. The automorphism op is trivial if and only if w_S is in the center of W and in this case, ξ is given by left multiplication by w_S.

Remark 2.7. The permutation op of $S \subset W$ extends to a unique automorphism π of Σ fixing the vertex 1. The automorphisms π is simply conjugation by w_S. The automorphisms π and ξ commute and their product is left multiplication by w_S.

Proposition 2.8. *The automorphism ξ defined in 2.5 is the unique automorphism of Σ mapping every root to its opposite.*

Proof. By [Weil, 5.1], no root of Σ contains two opposite vertices. In other words, $\xi(\alpha) \subset -\alpha$ for each root α. Since all roots contain the same number of vertices (namely $|W|/2$), we conclude that ξ maps each root to its opposite. Uniqueness holds by 2.2. □

Remark 2.9. Suppose that (W, S) is the spherical Coxeter system associated with a root system Φ, so S is the set of reflections corresponding to the walls of a unique chamber c of Φ. If op is non-trivial, then all the roots of Φ have the same length. Hence there always exists a unique automorphism of Φ fixing c and inducing the permutation op on S. We can thus think of π and ξ in 2.7 as automorphisms of Φ and it follows from 2.8 that ξ is the unique automorphism of Φ mapping every root of Φ to its negative.

Remark 2.10. Let Φ and ξ be as in 2.9. If Φ is of type D_n with $n \geq 4$ even, then by 2.4, 2.6 and 2.9, w_S is the unique automorphism of Φ mapping every root of Φ to its negative.

3. Buildings

Let (W, S) be a spherical Coxeter system and let Δ be a building of type (W, S) as defined in [Weil, 7.1]. (All buildings considered in this paper are assumed to be spherical and thick.) Thus Δ is a graph whose vertices are called chambers and whose edges are labeled by elements of S. The apartments of Δ are the subgraphs isomorphic to the graph Σ defined in 2.1. We assume that Δ is Moufang as defined in [Wei1, 11.2]. This means that Δ is irreducible and of rank $|S|$ at least 2 and that for each root of Δ, the corresponding root group U_α defined in [Wei1, 11.1] acts transitively on the set of apartments containing α.

Notation 3.1. We denote by $G^\dagger$ the subgroup of $G := \mathrm{Aut}(\Delta)$ generated by all the root groups of Δ.

Remark 3.2. Let Σ be an apartment of Δ, let c be a chamber of Σ, let $\alpha_1, \ldots, \alpha_n$ be the roots of Σ containing c but not some chamber of Σ adjacent to c and let D be the subgroup of $G^\dagger$ generated by the $2n$ root groups $U_{\pm\alpha_1}, \ldots, U_{\pm\alpha_n}$. By [Wei1, 11.22], the stabilizer D_Σ induces the group W on Σ and hence D contains U_β for all roots β of Σ. By [Wei1, 11.11(ii)], therefore, D contains U_β for all roots of Δ containing c. Since Δ is connected and D acts transitively on each panel containing c, D acts transitively on the set of chambers of c. Thus $D = G^\dagger$.

Moufang buildings were classified in [Tit2] and [TW]. There is a summary of the classification in [Wei2, Appendix B]. We will use the notation for these buildings given in [Wei2, 30.15].

Notation 3.3. Suppose that (K, L, Q) is a regular quadratic space of finite Witt index $\ell \geq 1$. We denote by $\mathcal{B}(Q)$ the building defined in [MPW, 35.5] whose chambers are the maximal flags of subspaces of L that are totally isotropic with respect to the quadratic form Q.

Proposition 3.4. *Let (K, L, Q) be a regular but not hyperbolic quadratic space with finite Witt index $\ell \geq 1$. Then $\mathcal{B}(Q) \cong \mathsf{B}_\ell^Q(\Lambda)$, where Λ is the anisotropic part of (K, L, Q) and $\mathsf{B}_\ell^Q(\Lambda)$ is as in* [Wei2, 30.15].

Proof. By [MPW, 35.6], it suffices to assume that $\ell = 1$. Let $\hat{L} = K \oplus K \oplus L$ and let $\hat{Q}\colon \hat{L} \to K$ be the quadratic form given by $\hat{Q}(x, y, v) = xy + Q(v)$ for all $(x, y, v) \in \hat{L}$. Then $\mathcal{B}(Q)$ is a residue of $\mathcal{B}(\hat{Q})$ and we have $\mathcal{B}(Q) \cong \mathsf{B}_1^Q(\Lambda)$ by [MPW, 3.8 and 3.20] applied to $\hat{Q}$. □

The remaining results in this section will be needed in §13.

Definition 3.5. Let Σ be an apartment and let R be a residue of Δ containing chambers of Σ. We say that a root α of Σ *cuts* R if it contains some but not all chambers of the apartment $\Sigma \cap R$ of R. Equivalently, a root cuts a residue if the residue contains panels in the wall of the root.

Notation 3.6. Let Π be the Coxeter diagram corresponding to (W, S), let J be a subset of S such that the subdiagram Π_J spanned by J is irreducible and $|J| \geq 2$ and suppose that K is a subset of S such that $J \cap K = \varnothing$ and $[J, K] = 1$. Let $L = J \cup K$, let R be a J-residue of Δ, let T be an L-residue containing R, let π be the restriction of the projection map proj_R (as defined in [Weil, 8.23]) to T, let $G_{T,J}$ denote the subgroup G consisting of those elements of the stabilizer G_T which induce an automorphism of the Coxeter diagram Π mapping J to itself and let

$$x^{\xi(g)} = \pi(x^g)$$

for all $g \in G_{T,J}$ and all chambers x of R. By [MPW, 21.40], ξ is a homomorphism from $G_{T,J}$ to $\mathrm{Aut}(R)$.

Notation 3.7. Let R, T, π, etc., be as in 3.6, let Σ be an apartment containing chambers of R, let α be a root of Σ cutting R, let g be an element of $G_{T,J}$ stabilizing Σ, let $R_1 = R$ and let $R_2 = R^g$. By [MPW, 21.38(i)], the residues R_1 and R_2 are parallel as defined in [MPW, 21.7]. By [MPW, 21.19(i)], therefore, α cuts R_2 and by [MPW, 21.8(v)], the restriction $\hat{\pi}$ of π to R_2 is an isomorphism from R_2 to R_1. Let X denote the set of apartments of Δ containing α (so $\Sigma \in X$) and for $i \in [1, 2]$, let Y_i be the set of apartments of R_i containing the root $\alpha \cap R_i$ of R_i. The map $A \mapsto A \cap R_i$ is a bijection from X to Y_i for $i \in [1, 2]$. By [Weil, 8.23], $\pi(A \cap R_2) \subset A \cap R_1$ for all $A \in X$. Since $\hat{\pi}$ is a bijection, it follows that

$$\hat{\pi}(A \cap R_2) = A \cap R_1 \tag{3.8}$$

for all $A \in X$. Hence, in particular, we have

$$\hat{\pi}(\alpha \cap R_2) = \alpha \cap R_1. \tag{3.9}$$

For $i \in [1, 2]$, let φ_i denote the map that sends each element of U_α to its restriction to R_i. By [Weil, 9.3 and 11.10] U_α acts faithfully on X, the root group $U_{\alpha \cap R_i}$ of R_i acts faithfully on Y_i and φ_i is an isomorphism from U_α to $U_{\alpha \cap R_i}$ such that

$$A^a = (A \cap R_i)^{\varphi_i(a)}$$

for all $A \in X$, all $a \in U_\alpha$ and for $i \in [1, 2]$. By (3.8) and (3.9), therefore,

$$\hat{\pi}^{-1} \cdot \varphi_1(a) \cdot \hat{\pi} = \varphi_2(a) \tag{3.10}$$

for all $a \in U_\alpha$. This means that if we identify U_α with $U_{\alpha \cap R_i}$ via φ_i for $i \in [1, 2]$, then $\hat{\pi}$ simply centralizes the root group U_α.

Proposition 3.11. *Let R and ξ be as in 3.6, let Σ, α, g and φ_1 be as in 3.7 and let $\beta = \alpha^g$. Then $\xi(g)$ is an automorphism of R, β is a root of Σ cutting R, $\beta \cap R = (\alpha \cap R)^{\xi(g)}$ and for each $a \in U_\alpha$, the restriction of $a^g \in U_\beta$ to R equals $\varphi_1(a)^{\xi(g)} \in U_{\beta\cap R}$.*

Proof. By 3.6, $\xi(g) \in \mathrm{Aut}(R)$. By [MPW, 21.19(i) and 21.38(i)], β is a root of Σ cutting R. We can thus replace α by β everywhere in 3.7. By (3.9), therefore, $\beta \cap R = (\alpha \cap R)^{\xi(g)}$. The last assertion holds by 3.10. □

Remark 3.12. Let α, ξ, etc., be as in 3.11 and for each root γ of Σ cutting R, let U_γ be identified with the root group $U_{\gamma\cap R}$ of R via the map that sends an element to its restriction to R. Then the last assertion in 3.11 says simply that $a^g = a^{\xi(g)}$ for all $a \in U_\alpha$.

4. Simply laced buildings

We continue to let Δ be a spherical building of type (W, S) satisfying the Moufang condition. In this section we assume that Δ is simply laced and split. This means that there exists a field E such that Δ is isomorphic to $\mathsf{A}_n(E)$ for some $n \geq 1$, to $\mathsf{D}_n(E)$ for some $n \geq 3$, to $\mathsf{E}_6(E)$, to $\mathsf{E}_7(E)$ or to $\mathsf{E}_8(E)$.

Notation 4.1. Let Φ be the corresponding root system of type A_n, D_n, E_6, E_7 or E_8, let $\alpha_1, \dots, \alpha_n$ be the basis of the root system Φ described in [Bou, Plate I or IV–VII] and let d be the unique chamber of Φ which is the intersection of the half-spaces determined by the roots $\alpha_1, \dots, \alpha_n$, let Σ be an apartment of Δ and let c be a chamber of Σ. We denote the reflection associated with a root β of Φ by s_β and we identify W with the Weyl group of Φ in such a way that $S = \{s_{\alpha_1}, \dots, s_{\alpha_n}\}$. There is then a unique W-equivariant bijection θ from the set of chambers of Σ to the set of chambers of Φ mapping c to d. The bijection θ induces a bijection from $\mathrm{Aut}(\Phi)$ into $\mathrm{Aut}(\Sigma)$ that carries the stabilizer of d to the stabilizer of c and it induces a bijection from the set of roots of Σ to the set of half-spaces associated with the roots of Φ and thus to Φ itself. From now on, we identify $\mathrm{Aut}(\Phi)$ with its image in $\mathrm{Aut}(\Sigma)$ under θ and we identify the roots of Σ with the corresponding roots of Φ. In particular, $W \subset \mathrm{Aut}(\Phi)$ is the group of type-preserving automorphisms of Σ and to each root β of Φ, we have a root group U_β of Δ (as defined in [Weil, 11.1]).

Theorem 4.2. *There exists a collection of isomorphisms $x_\beta\colon E \to U_\beta$, one for each root β of Φ, and a mapping $\tau\colon \Phi \times \Phi \to \{1, -1\}$ such that for all ordered pairs (α, β) of roots of Φ such that $\alpha \neq \pm\beta$ and for all $s, t \in E$, the following hold:*

(i) $[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}\big(\tau(\alpha,\beta)st\big)$ *if* $\alpha+\beta \in \Phi$.

(ii) $[x_\alpha(s), x_\beta(t)] = 1$ *if* $\alpha+\beta \notin \Phi$.

(iii) $U_\alpha^{x_{-\alpha}(t)} = U_{-\alpha}^{x_\alpha(t^{-1})}$ *if* $t \neq 0$.

Proof. The building Δ is the building obtained by applying [TW, Prop. 42.3.6] to the root group data associated with the corresponding Chevalley group. The assertions (i) and (ii) hold, therefore, by [Ste, (R2) on p. 30]; see also [Car, Thm. 5.2.2]. Assertion (iii) holds by [Ste, (R7) on p. 30 and Lemma 59 on p. 160]. □

Remark 4.3. Let $\alpha \in \Phi$ and suppose that $U_\alpha^g = U_{-\alpha}^{x_\alpha(t^{-1})}$ for some $g \in U_{-\alpha}$ and some $t \in E^*$. Since the identity is the only element of $U_{-\alpha}$ normalizing U_α, it follows from 4.2(iii) that $g = x_{-\alpha}(t)$.

Notation 4.4. We call a set $\{x_\beta\}_{\beta\in\Phi}$ satisfying the three conditions in 4.2 for some map τ a *coordinate system* for Δ and we call the map τ the *sign function* of $\{x_\beta\}_{\beta\in\Phi}$. This notion depends, of course, on the choice of the apartment Σ and the choice of the identification of Φ with the set of roots of Σ which we made (once and for all) in 4.1.

If $\{x_\beta\}_{\beta\in\Phi}$ is a coordinate system, then we obtain new coordinate system (with a new sign function) by choosing $\beta \in \Phi$ and replacing x_β and $x_{-\beta}$ by x'_β and $x'_{-\beta}$, where $x'_\beta(t) = x_\beta(-t)$ and $x'_{-\beta}(t) = x_{-\beta}(-t)$ for all $t \in E$.

Notation 4.5. We call two coordinate systems $\{x_\beta\}_{\beta\in\Phi}$ and $\{x'_\beta\}_{\beta\in\Phi}$ *equivalent* if there exists a map $\beta \mapsto \varepsilon_\beta$ from the set of positive roots Φ^+ to $\{1,-1\}$ such that $x'_\beta(t) = x_\beta(\varepsilon_\beta t)$ and $x'_{-\beta}(t) = x_{-\beta}(\varepsilon_\beta t)$ for each $t \in E$ and for each $\beta \in \Phi^+$.

Proposition 4.6. *Let* $\{x_\beta\}_{\beta\in\Phi}$ *and* $\{x'_\beta\}_{\beta\in\Phi}$ *be two coordinate systems for* Δ *such that* $x_{\alpha_i} = x'_{\alpha_i}$ *for all* $i \in [1,n]$. *Then* $\{x_\beta\}_{\beta\in\Phi}$ *and* $\{x'_\beta\}_{\beta\in\Phi}$ *are equivalent.*

Proof. By [Hum, §10.2, Cor. to Lemma A] and induction, there exists a map $\beta \mapsto \varepsilon_\beta$ from Φ^+ to $\{1,-1\}$ such that $x'_\beta(t) = x_\beta(\varepsilon_\beta t)$ for all $\beta \in \Phi^+$ and all $t \in E$. By 4.3, it follows that $x'_{-\beta}(t) = x_{-\beta}(\varepsilon_\beta t)$ for all $\beta \in \Phi^+$ and all $t \in E$. □

Theorem 4.7. *Let* $\{x_\beta\}_{\beta\in\Phi}$ *be a coordinate system for* Δ, *let* $\lambda_1,\dots,\lambda_n$ *be non-zero elements of* E *and let* $\sigma \in \mathrm{Aut}(E)$. *Then the following hold:*

(i) *There exists a unique automorphism*

$$g = g_{\lambda_1,\dots,\lambda_n,\sigma}$$

of Δ *that fixes the chamber* c *and stabilizes the apartment* Σ *such that*

$$x_{\alpha_i}(t)^g = x_{\alpha_i}(\lambda_i t^\sigma)$$

for all $i \in [1,n]$ *and all* $t \in E$.

(ii) *If*

$$\beta = \sum_{i=1}^{n} c_i\alpha_i \in \Phi,$$

then

$$x_\beta(t)^g = x_\beta(\lambda_\beta t^\sigma),$$

where

$$\lambda_\beta = \prod_{i=1}^{n} \lambda_i^{c_i}.$$

Proof. The existence assertion in (i) holds by [Ste, Lemma 58 on p. 158] and the existence of field automorphisms; uniqueness holds by [Weil, 9.7]. By 4.3, we have $x_{-\alpha_i}(t)^g = x_{-\alpha_i}(\lambda_i^{-1}t^\sigma)$ for all $t \in E$ and each $i \in [1,n]$. By 4.2(i), [Hum, §10.2, Cor. to Lemma A] and induction, it follows that (ii) holds. □

Remark 4.8. Let $\kappa\colon E \to E$ be given by $\kappa(t) = -t$ for all $t \in E$. Suppose that the set $\{\alpha_1,\dots,\alpha_n\}$ is ordered so that for each $j \in [2,n]$, there is at most one $i \in [1, j-1]$ such that $\alpha_i + \alpha_j \in \Phi$. Let $\{x_\beta\}_{\beta\in\Phi}$ be a coordinate system for Δ. Replacing x_{α_i} by $\kappa \cdot x_{\alpha_i}$ for suitable i, we can find an equivalent coordinate system $\{x'_\beta\}_{\beta\in\Phi}$ whose sign function τ' satisfies $\tau'(\alpha_i,\alpha_j) = 1$ for all $i, j \in [1,n]$ such that $i < j$.

In the following display, x_β^φ denotes the map $t \mapsto x_\beta(t)$ followed by the inner automorphism of the root group U_β induced by the automorphism φ of Δ.

Proposition 4.9. *Let* $\{x_\beta\}_{\beta\in\Phi}$ *and* $\{x'_\beta\}_{\beta\in\Phi}$ *be two coordinate systems for* Δ. *Then there exists a unique automorphism* φ *of* Δ *acting trivially on* Σ *such that*

$$\left\{x_\beta^\varphi\right\}_{\beta\in\Phi}$$

is a coordinate system for Δ *which is equivalent to* $\{x'_\beta\}_{\beta\in\Phi}$ *for all* $\beta \in \Phi$.

Proof. Let τ and τ' be the sign functions of $\{x_\beta\}_{\beta\in\Phi}$ and $\{x'_\beta\}_{\beta\in\Phi}$. Since the Coxeter diagram of Δ has no circuits, it follows from 4.8 that after replacing $\{x'_\beta\}_{\beta\in\Phi}$ by an equivalent coordinate system, we can assume that

$$\tau(\alpha_i,\alpha_j) = \tau'(\alpha_i,\alpha_j) \tag{4.10}$$

for all $i, j \in [1,n]$.

Let M be the set of pairs $i, j \in [1,n]$ such that $\alpha_i + \alpha_j \in \Phi$. For each $\{i,j\} \in M$, let R_{ij} be the unique $\{\alpha_i,\alpha_j\}$-residue containing c. By (4.10) and [TW, 7.5], there exists for each $\{i,j\} \in M$ a unique automorphism φ_{ij} of R_{ij} acting trivially on $\Sigma \cap R_{ij}$ such that

$$x_{\alpha_k}^{\varphi_{ij}} = x'_{\alpha_k}$$

for $k = i$ and j. By 4.7(i) applied to each R_{ij} and then to Δ, it follows that there exists a unique automorphism φ of Δ acting trivially on Σ such that

$$x_{\alpha_k}^{\varphi} = x'_{\alpha_k}$$

for all $k \in [1,n]$. By 4.6, we conclude that $\{x_\beta^\varphi\}_{\beta\in\Phi}$ is a coordinate system equivalent to $\{x'_\beta\}_{\phi\in\Phi}$. □

In the following result, we are identifying U_β with the root group $U_{\beta\cap R}$ of the residue R for each $\beta \in \Phi_1$ via the isomorphism which sends each element of U_β to its restriction to R, and hence for each $\beta \in \Phi_1$, x_β is simultaneously an isomorphism from E to U_β and an isomorphism from E to $U_{\beta\cap R}$.

Proposition 4.11. *Let $M \subset [1,n]$, let $X = \{\alpha_i \mid i \in M\}$, let $J = \{s_{\alpha_i} \mid i \in M\}$ and let R be the unique J-residue of Δ containing c. Suppose that R is irreducible and of rank at least 2, let Φ_1 denote the root system $\langle X\rangle \cap \Phi$ and let $\{x'_\beta\}_{\beta\in\Phi_1}$ be a coordinate system for R with respect to the apartment $\Sigma \cap R$. Then there exists a coordinate system $\{x_\beta\}_{\beta\in\Phi}$ for Δ such that $x_\beta = x'_\beta$ for all $\beta \in \Phi_1$.*

Proof. Let $\{x_\beta\}_{\beta\in\Phi}$ be an arbitrary coordinate system for Δ. Since R is irreducible and of rank at least 2, it is Moufang (by [Weil, 11.8]). By 4.9, therefore, there exists an automorphism φ_R of R acting trivially on $\Sigma \cap R$ such that $\{x_\beta^{\varphi_R}\}_{\beta\in\Phi_1}$ is a coordinate system for R equivalent to the coordinate system $\{x'_\beta\}_{\beta\in\Phi_1}$. Thus there exists a coordinate system $\{x''_\beta\}_{\beta\in\Phi}$ equivalent to $\{x_\beta\}_{\beta\in\Phi}$ such that $(x''_\beta)^{\varphi_R} = x'_\beta$ for all $\beta \in \Phi_1$. By 4.7(i), φ_R can be extended to an automorphism φ of Δ acting trivially on Σ. Hence $\{(x''_\beta)^\varphi\}_{\beta\in\Phi}$ is a coordinate system for Δ extending $\{x'_\beta\}_{\beta\in\Phi_1}$. □

Theorem 4.12. *Let $\{x_\beta\}_{\beta\in\Phi}$ be a coordinate system for Δ and let $\gamma \in \mathrm{Aut}(\Phi)$. Then there exists a unique automorphism $\tilde{\gamma}$ of Δ that stabilizes the apartment Σ such that*

$$x_{\alpha_i}(t)^{\tilde{\gamma}} = x_{\gamma(\alpha_i)}(t)$$

for all $t \in E$. Furthermore, there exists a mapping $\rho_\gamma\colon \Phi \to \{1,-1\}$ such that $x_\beta(t)^{\tilde{\gamma}} = x_{\gamma(\beta)}(\rho_\gamma(\beta)t)$ for all $\beta \in \Phi$ and all $t \in E$.

Proof. This holds by [Ste, Thm. 29 on p. 154]. $\square$

Notation 4.13. Let $\{x_\beta\}_{\beta\in\Phi}$ be a coordinate system for Δ. We set

$$g_{\gamma,\lambda_1,\dots,\lambda_n,\sigma} = g_{\lambda_1,\dots,\lambda_n,\sigma}\cdot\tilde{\gamma}$$

for all $\gamma \in \mathrm{Aut}(\Phi)$, all $\lambda_1,\dots,\lambda_n \in E^*$ and all $\sigma \in \mathrm{Aut}(E)$, where $g_{\lambda_1,\dots,\lambda_n,\sigma}$ is as in 4.7(i) and $\tilde{\gamma}$ is as in 4.12.

Proposition 4.14. *Let $\{x_\beta\}_{\beta\in\Phi}$ be a coordinate system for Δ. If $g \in \mathrm{Aut}(\Delta)$ stabilizes Σ, then there exist $\gamma \in \mathrm{Aut}(\Phi)$, $\lambda_1,\dots,\lambda_n \in E^*$ and $\sigma \in \mathrm{Aut}(E)$ such that*

$$g = g_{\gamma,\lambda_1,\dots,\lambda_n,\sigma}.$$

Proof. It suffices to assume that g is an element of $\mathrm{Aut}(\Delta)$ acting trivially on Σ. Thus g stabilizes every irreducible rank 2 residue containing the chamber c. By [TW, 37.13], we can assume that g acts trivially on each of the n panels containing c. The claim holds, therefore, by [Weil, 9.7]. $\square$

Definition 4.15. Let $\{x_\beta\}_{\beta\in\Phi}$ be a coordinate system for Δ. A *Galois involution* of Δ is an element of order 2 in the coset $g_{\gamma,\lambda_1,\dots,\lambda_n,\sigma}G^\dagger$ for some $\gamma,\lambda_1,\dots,\lambda_n,\sigma$ such that $\sigma \neq 1$, where $G^\dagger$ is as in 3.1. This is a special case of the notion of a Galois involution of an arbitrary Moufang building given in [MPW, 31.1]. By 4.9, in particular, it is independent of the choice of the coordinate system $\{x_\beta\}_{\beta\in\Phi}$. By [MPW, 29.24], it is, in fact, independent also of the choice of Σ and the identification of the set of roots of Σ with Φ in 4.1.

Proposition 4.16. *Let $\{x_\beta\}_{\beta\in\Phi}$ be a coordinate system for Δ, let g be an element of $\mathrm{Aut}(\Delta)$ acting trivially on Σ and let $\gamma,\lambda_1,\dots,\lambda_n,\sigma$ be as in 4.14. If $\{x'_\beta\}_{\beta\in\Phi}$ is another coordinate system for Δ, then there exists a map $i \mapsto \varepsilon_i$ from $[1,n]$ to $\{1,-1\}$ such that $\varepsilon_i = 1$ if $w(\alpha_i) = \pm\alpha_i$ and*

$$g = g'_{\gamma,\lambda'_1,\dots,\lambda'_n,\sigma},$$

where $\lambda'_i = \varepsilon_i\lambda_i$ for all $i \in [1,n]$ and $g'_{\gamma,\lambda'_1,\dots,\lambda'_n,\sigma}$ is as defined in 4.13 with $\{x_\beta\}_{\beta\in\Phi}$ replaced by $\{x'_\beta\}_{\beta\in\Phi}$.

Proof. This holds by 4.9. $\square$

5. The exceptional Moufang quadrangles

A *Moufang quadrangle* is a building of type B_2 satisfying the Moufang condition. The *exceptional* Moufang quadrangles are the Moufang quadrangles defined in [TW, 16.6–16.7]. These are the Moufang quadrangles denoted by $B_2^{\mathcal{E}}(\Lambda)$ and $B_2^{\mathcal{F}}(\Lambda)$ in [Wei2, 30.15], where Λ is a quadratic space of type E_6, E_7 or E_8 in the first case and Λ is a quadratic space of type F_4 in the second.

Definition 5.1. A quadratic space (K, V, q) is *of type* E_k for $k = 6$, 7 or 8 if it is anisotropic and for some $\eta_1, \dots, \eta_d \in K$, where $d = 2 + 2^{k-6}$, and some separable quadratic extension E/K with norm N, the quadratic form q is equivalent to the quadratic form Q on E^d given by

$$Q(u_1, \dots, u_d) = \eta_1 N(u_1) + \cdots + \eta_d N(u_d) \tag{5.2}$$

for all $(u_1, \dots, u_d) \in E^d$ with the additional conditions that

$$\eta_1\eta_2\eta_3\eta_4 \notin N(E) \tag{5.3}$$

if $k = 7$ and

$$-\eta_1\eta_2 \cdots \eta_6 \in N(E) \tag{5.4}$$

if $k = 8$.

Remark 5.5. Let (K, V, q) be a quadratic space of type E_k for $k = 6$, 7 or 8. If E is as in 5.1, then $N \otimes_K E$ is hyperbolic and hence $q_E := q \otimes_K E$ is also hyperbolic. By [dMed, Lemma 4.2] and [MPW, 8.5], if E/K is an arbitrary separable quadratic extension such that q_E is hyperbolic, then there exist $\eta_1, \dots, \eta_d \in K$ satisfying (5.3) if $k = 7$ and (5.4) if $k = 8$ such that q is equivalent to the quadratic form $Q\colon E^d \to K$ given by (5.2).

Remark 5.6. In [dMed, Thm. 5.3], it is shown that for each $\ell \in \{6, 7, 8\}$, an anisotropic quadratic form is of type E_ℓ if and only if its even Clifford algebra has a certain structure. In the paragraphs entitled "Type (2)", "Type (3)" and "Type (4)" in [TW, 42.6], it is shown (given [dMed, Thm. 5.3]) that a quadratic form of type E_6, E_7, respectively, E_8 is precisely the ingredient needed to construct a form of type ${}^2E_{6,2}^{16'}$, $E_{7,2}^{31}$, respectively, $E_{8,2}^{66}$ (in the notation of [Tit1]). See also [Tit3, §5].

The following notion was introduced in [TW, 14.1].

Definition 5.7. A quadratic space (K, V, q) is *of type* F_4 if it is anisotropic, $\mathrm{char}(K) = 2$ and for some separable quadratic extension E/K with norm N, some extension F/K (of arbitrary dimension, possibly infinite) such that $F^2 \subset K$ and some $\eta_1, \eta_2 \in K$ such that

$$\eta_1\eta_2 \in F^2,$$

the quadratic form q is similar to the quadratic form Q on $E \oplus E \oplus F$ given by

$$Q(u_1, u_2, t) = \eta_1 N(u_1) + \eta_2 N(u_2) + t^2 \tag{5.8}$$

for all $(u_1, u_2, t) \in E \oplus E \oplus F$. (Here F^2 denotes $\{t^2 \mid t \in F\}$, not $F \oplus F$.)

Remark 5.9. Let (K, V, q) be a quadratic space of type F_4, let F be as in 5.7 and let D denote the radical of the bilinear form ∂q. Then $F^2 = q(D)/q(v)$ for every non-zero $v \in D$. Thus the extension F/K is an invariant of the similarity class of q.

Remark 5.10. If $\Delta = \mathsf{B}_2^{\mathcal{E}}(\Lambda)$ for some quadratic space Λ of type E_6, E_7 or E_8, then by [TW, 35.11], Λ is an invariant of Δ up to similarity. If $\Delta = \mathsf{B}_2^{\mathcal{F}}(\Lambda)$ for some quadratic space $\Lambda = (K, V, q)$ of type F_4 and F is as in 5.9, then by [TW, 35.12], the similarity class of Λ determines a second similarity class of quadratic spaces over F of type F_4 and this pair of similarity classes is an invariant of Δ.

Definition 5.11. We call a quadratic space (K, V, q) *pseudo-split* if it is the orthogonal sum of a finite dimensional hyperbolic space and an anisotropic totally singular space (of arbitrary dimension). See [MPW, 2.31–2.33].

Remark 5.12. Let (K, V, q) be a quadratic space of type F_4, let $f = \partial q$ and let E/K be as in 5.7. Since $N \otimes_K E$ is hyperbolic, the quadratic form q_E is pseudo-split as defined in 5.11. Suppose that E/K is an arbitrary separable quadratic extension such that q_E is pseudo-split. Let v, v' be two elements of V such that $v \otimes 1$ and $v' \otimes 1$ span a hyperbolic pair in $V \otimes_K E$ and $f(v, v') = 1$. The restriction of q to $\langle v, v' \rangle$ is similar to N. Let $\eta_1 = q(v)$. By [MPW, 9.7], there exists $\eta_2 \in K$ such that $\eta_1\eta_2 \in F^2$ and q is similar to the quadratic form $Q\colon E \oplus E \oplus F \to K$ given by (5.8).

Remark 5.13. In [CP, D.2.7], forms of relative rank 2 of a pseudo-split group of type F_4 are classified in terms of quadratic forms of type F_4. The quadratic forms which appear in this context are those where at least one of the two extensions K/F or F/K^2 in 5.9 is finite.

Proposition 5.14. *Let $\Lambda = (K, V, q)$ be an anisotropic quadratic space. Suppose that either Λ is a quadratic space of type E_6, E_7 or E_8 or that the bilinear form ∂q is degenerate but not identically zero. Then q is not similar to the norm of a composition algebra.*

Proof. Let Q be the norm of a composition algebra (as defined in [Wei2, 30.17]). Then the bilinear form ∂Q is either non-degenerate or identically zero. If ∂Q is non-degenerate, then $\dim(Q)$ divides 8 and if $\dim(Q) = 8$, its Hasse invariant is trivial. If Λ is of type E_6, E_7 or E_8, then ∂q is non-degenerate, but its dimension divides 8 only if Λ is of type E_7 and in this case the Hasse invariant is non-trivial (by [MPW, 8.3]). $\square$

In the following, $\mathsf{A}_1(D)$ and $\mathsf{B}_1^{\mathcal{Q}}(\Lambda)$ are as defined in [MPW, 3.8]. Thus $\mathsf{A}_1(D)$ is the Moufang set (as defined in [MPW, 1.5]) associated with the projective line $D \cup \{\infty\}$ and $\mathsf{B}_1^{\mathcal{Q}}(\Lambda)$ is the Moufang set associated with an anisotropic quadratic space $\Lambda = (K, V, \varphi)$ on the "projective line" $V \cup \{\infty\}$.

Proposition 5.15. *Let Λ be as in 5.14. Then there is no field or skew field D such that $\mathsf{B}_1^{\mathcal{Q}}(\Lambda) \cong \mathsf{A}_1(D)$.*

Proof. Let D be a field or skew field and let F be its center. By [Wei3, 31.21], $\mathsf{B}_1^{\mathcal{Q}}(K, V, q) \cong \mathsf{A}_1(D)$ for some anisotropic quadratic space (K, V, q) if and only if (D, F) is a composition algebra, $F \cong K$ and q is similar to the norm of (D, F). The claim holds, therefore, by 5.14. $\square$

We will use the following result, which depends on the classification of Moufang polygons, to identify the fixed point buildings that we construct. Alternatively, we could have used [MPW, 24.32] to identify these buildings by calculating their commutator relations. This is what is done, for instance, in [MM1].

Proposition 5.16. *Let Δ be a Moufang quadrangle, let $G = \mathrm{Aut}(\Delta)$, let c be a chamber, let R_1 and R_2 be the two panels containing c and for $i = 1$ and 2, let $\mathbb{M}_i$ be the Moufang set induced by the stabilizer G_{R_i} on R_i. Suppose that $\mathbb{M}_1 \cong \mathsf{B}_1^{\mathcal{Q}}(\Lambda)$ for some quadratic space $\Lambda = (K, V, q)$ of type E_6, E_7, E_8 or F_4 and that either*

(a) *$\mathbb{M}_2$ has non-abelian root groups or*

(b) *$\mathbb{M}_2 \cong \mathsf{B}_1^{\mathcal{Q}}(\Theta)$ for some anisotropic quadratic space $\Theta = (F, L, Q)$ such that ∂Q is degenerate but not identically zero.*

Then Λ is of type E_6, E_7 or E_8 and $\Delta \cong \mathsf{B}_2^{\mathcal{E}}(\Lambda)$ if (a) *holds and Λ is of type F_4 and $\Delta \cong \mathsf{B}_2^{\mathcal{F}}(\Lambda)$ if* (b) *holds.*

Proof. By [TW, 38.9], Δ is in one of the six cases described in [MPW, 4.2], where the quadrangles are described in terms of root group sequences as defined in [TW, 8.7]. The root groups of $\mathbb{M}_1$ are abelian and if (b) holds, then by [MPW, 4.8(iii)], the tori of $\mathbb{M}_2$ (as defined in [MPW, 1.6]) are non-abelian. If Δ were as in [MPW, 4.2(iii)], then the root groups and (by [MPW, 4.8(iv)]) the tori of $\mathbb{M}_i$ for both $i = 1$ and 2 would have to be abelian. Hence Δ is not as in [MPW, 4.2(iii)]. If Δ were as [MPW, 4.2(i), (ii) or (iv)], then there would exist a field or a skew field D such that $\mathcal{M}_i \cong \mathsf{A}_1(D)$ for $i = 1$ or 2. This is impossible by 5.15. Only the cases (v) and (vi) of [MPW, 4.8] remain. Thus $\Delta \cong B_2^{\mathcal{E}}(\Lambda')$ for some quadratic space Λ' of type E_6, E_7 or E_8 if (a) holds and $\Delta \cong \mathsf{B}_2^{\mathcal{F}}(\Lambda')$ for some quadratic space Λ' of type F_4 if (b) holds. Suppose that (a) holds. Then $\mathbb{M}_1 \cong \mathsf{B}_1^{\mathcal{Q}}(\Lambda')$ and hence by [MPW, 6.10], Λ' is similar to Λ. Thus $\Delta \cong \mathsf{B}_2^{\mathcal{E}}(\Lambda)$ (by [TW, 35.11]). Suppose that (b) holds and let Λ'' denote the dual of Λ' as defined in [MPW, 9.5]. By [TW, 28.45], there is a non-type-preserving isomorphism from $\mathsf{B}_2^{\mathcal{F}}(\Lambda')$ to $\mathsf{B}_2^{\mathcal{F}}(\Lambda'')$. Thus $\mathbb{M}_1$ is isomorphic to $\mathsf{B}_1^{\mathcal{Q}}(\Lambda')$ to $\mathsf{B}_1^{\mathcal{Q}}(\Lambda'')$. By [MPW, 6.10] again, Λ is similar to Λ' or Λ''. Hence $\Delta \cong \mathsf{B}_2^{\mathcal{F}}(\Lambda)$ (by [TW, 35.12]). □

6. Descent

In this section we assemble the results in [MPW] on descent in buildings that we will require.

Definition 6.1. Let Δ be a building and let Γ be a subgroup of $\mathrm{Aut}(\Delta)$. A Γ*-residue* is a residue of Δ stabilized by Γ. A Γ*-chamber* is a Γ-residue which is minimal with respect to inclusion. A Γ*-panel* is a Γ-residue P such that for some Γ-chamber C, P is minimal in the set of all Γ-residues containing C properly.

Definition 6.2. Let Δ and Γ be as in 6.1. The group Γ is *anisotropic* if Δ itself is the unique Γ-chamber and *isotropic* if this is not the case. Thus Γ is isotropic if and only if there exist Γ-panels (equivalently, if there exist Γ-residues other than Δ itself).

Notation 6.3. Let Δ be a building and let Γ be an isotropic subgroup of $\mathrm{Aut}(\Delta)$. We denote by Δ^Γ the graph with vertex set the set of all Γ-chambers, where two Γ-chambers are joined by an edge of Δ^Γ if and only if there is a Γ-panel containing them both.

Definition 6.4. Let Δ be a building. A *descent group* of Δ is an isotropic subgroup Γ of $\mathrm{Aut}(\Delta)$ such that each Γ-panel contains at least three Γ-chambers.

Theorem 6.5. *Let Δ be a simply laced spherical building which is Moufang and split. If Ω is an isotropic Galois involution of Δ as defined in 4.15 and 6.2, then $\Gamma := \langle\Omega\rangle$ is a descent group of Δ.*

Proof. By [MPW, 28.16], Δ satisfies [MPW, 30.1(i)]. The claim holds, therefore, by [MPW, 32.27]. □

Proposition 6.6. *Suppose that R is a residue of a Moufang building Δ. Let Σ be an apartment containing chambers of R and let U_R denote the subgroup generated by the root groups U_α for all roots α of Σ containing $R \cap \Sigma$. Then U_R is independent of the choice of Σ.*

Proof. This holds by [MPW, 24.17]. □

Definition 6.7. The group U_R in 6.6 is called the *unipotent radical* of the residue R.

Definition 6.8. A *Tits index* is a triple (Π, Θ, A) where Π is a Coxeter diagram, Θ is a subgroup of $\mathrm{Aut}(\Pi)$ and A is a Θ-invariant subset of the vertex set S of Π such that for each $s \in S\backslash A$, the subset $A \cup \Theta(s)$ of S is spherical (i.e., the subgroup $\langle A \cup \Theta(s)\rangle$ of W is finite) and A is stabilized by the opposite map $\mathrm{op}_{A\cup\Theta(s)}$ defined in 2.3. Here $\Theta(s)$ denotes the Θ-orbit containing s.

Definition 6.9. Let $T = (\Pi, \Theta, A)$ be a Tits index. For each $s \in S\backslash A$, let $\tilde{s} = w_A w_{A\cup\Theta(s)}$, where w_J for $J = A$ and $J = A \cup \Theta(s)$ is as in 2.3. Thus there is one element $\tilde{s}$ for each Θ-orbit in $S\backslash A$. Let $\tilde{S}$ be the set of all these elements $\tilde{s}$. By [MPW, 20.32], $(\tilde{W}, \tilde{S})$ is a Coxeter system. Let $\tilde{\Pi}$ be the corresponding Coxeter diagram. We call Π the *absolute Coxeter diagram* of T and $\tilde{\Pi}$ the *relative Coxeter diagram* of T. An algorithm for calculating the relative Coxeter diagram of a Tits index is described in [TW, 42.3.5(c)].

Conventions 6.10. Our notion of a Tits index generalizes the usual notion of a Tits index as defined, for example, in [TW, 42.3.4], where it is called a Witt index. We use Tits' conventions for indicating a Tits index (Π, A, Θ), drawing the Coxeter diagram Π with a circle around each Θ-orbit disjoint from A and with vertices in the same Θ-orbit brought near to one another. See [MPW, 34.2] for a more precise description of these conventions.

Examples 6.11. There are Tits indices (drawn using the conventions in 6.10) in all of our main results. Using [TW, 42.3.5(c)], we can check that the relative type of the indices in 11.21, 13.12, 14.11 and 17.14 is B_2, the relative type of the index in 12.11 is A_2, the relative type of the first three indices in 15.4 is F_4 and the relative type of the last index in 15.4 is C_3. We observe, too, that the Tits index in 17.14 does not appear in [Tit1].

The following is a special case of the main results of [MPW, Part 3].

Theorem 6.12. *Let Γ be a descent group of a spherical building Δ. Let Π be the Coxeter diagram of Δ, let S denote the vertex set of Π and let Θ denote the subgroup of* $\mathrm{Aut}(\Pi)$ *induced by Γ. Then the following hold:*

(i) *The graph Δ^Γ is a building with respect to a canonical coloring of its edges.*

(ii) *All Γ-chambers are residues of Δ of the same type $A \subset S$, the set A is Θ-invariant and the rank k of Δ^Γ is the number of Θ-orbits in S disjoint from A.*

(iii) *The triple $T := (\Pi, \Theta, A)$ is a Tits index and Δ^Γ is a building of type $\tilde{\Pi}$, where $\tilde{\Pi}$ is the relative Coxeter diagram of T.*

(iv) *If Δ is Moufang and $k \geq 2$, then Δ^Γ is also Moufang.*

(v) *Suppose that Δ is Moufang and that $k = 1$ and let X denote the set of all Γ-chambers. For each $R \in X$, let $\tilde{U}_R$ denote the subgroup of* $\mathrm{Sym}(X)$ *induced by the centralizer $C_{U_R}(\Gamma)$ of Γ in the unipotent radical U_R. Then*

$$\big(X, \{\tilde{U}_R \mid R \in X\}\big)$$

is a Moufang set.

Proof. Assertions (i) and (ii) hold by [MPW, 22.20(v) and (viii)], assertion (iii) holds by [MPW, 22.20(iv) and (viii)] and the remaining two assertions hold by [MPW, 24.31]. □

Definition 6.13. Let Γ and Δ be as in 6.12. We refer to the triple T in 6.12(iii) as the *Tits index of* Γ. (In fact, the Tits index of a descent group Γ is defined also when Δ is not assumed to be spherical; see [MPW, 22.20 and 22.22].)

Definition 6.14. A *fixed point building* is a building of the form Δ^Γ for some pair Δ, Γ as in 6.12. If the rank of Δ^Γ is 1 and Δ is Moufang, we interpret Δ^Γ to mean the Moufang set described in 6.12(v).

Remark 6.15. Let Δ, Γ, Θ, A, etc., be as in 6.12 and suppose that Δ is Moufang. Let $\tilde{\Delta} = \Delta^\Gamma$ and let $\tilde{G} = \mathrm{Aut}(\tilde{\Delta})$. By 6.9, we can identify the vertex set of the relative Coxeter diagram $\tilde{\Pi}$ with the set of Θ-orbits disjoint from A. Let $I = \Theta(s)$ be one of these orbits, let $J = A \cup I$, let R be a Γ-residue of type J and let Γ_R denote the restriction of Γ to R. By [MPW, 22.39], $P := R^{\Gamma_R}$ is an I-panel of $\tilde{\Delta}$ and by [MPW, 24.30], R^{Γ_R} is isomorphic as a Moufang set (see 6.14) to the Moufang set induced on P by the stabilizer of P in $\tilde{G}$.

7. Linear groups

Let V be an $(n+1)$-dimensional vector space over a field E (by which we mean a commutative field) for some $n \geq 1$ and let

$$\mathcal{B} = (e_1, \dots, e_{n+1})$$

be an ordered basis of V. For each ordered pair (i, j) of distinct integers i, j in the interval $[1, n+1]$ and each $t \in E$, let $x_{ij}(t)$ denote element of $\mathrm{SL}(V)$ that maps e_j to $e_j + te_i$ and fixes e_k for $k \neq j$.

Let Φ be the root system of type A_n and let $\varepsilon_1, \dots, \varepsilon_{n+1}$, $\alpha_1, \dots, \alpha_n$ and $\tilde{\alpha}$ be as in [Bou, Plate I]. Thus, in particular, $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for each $i \in [1, n]$ and $\tilde{\alpha} = \varepsilon_1 - \varepsilon_{n+1}$. For each $\beta \in \Phi$, we set set $x_\beta = x_{ij}$ if $\beta = \varepsilon_i - \varepsilon_j$. Let Δ be the building of type A_n associated with V. Thus the chambers of Δ are the maximal flags of subspaces of V, and $\Delta \cong \mathsf{A}_n(E)$ in the notation in [Wei2, 30.15]. The groups $x_\beta(E)$ act faithfully on Δ and we will simply identify them with their images in $\mathrm{Aut}(\Delta)$. Let Σ the apartment of Δ whose chambers are maximal flags involving only subspaces spanned by subsets of the basis $\mathcal{B}$, let c denote the chamber

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, e_2, \dots, e_n \rangle \tag{7.1}$$

of Σ and let Φ be identified with the set of roots of Σ and $\mathrm{Aut}(\Phi)$ with a subgroup of $\mathrm{Aut}(\Sigma)$ as in 4.1. Thus $\alpha_1, \dots, \alpha_n$ are the roots of Σ containing c but not some chamber of Σ adjacent to c and $\{x_\beta\}_{\beta \in \Phi}$ is a coordinate system for Δ. By [Tit2, Prop. 6.6], there is a natural homomorphism from $\mathrm{Aut}(\mathrm{SL}(V))$ to $\mathrm{Aut}(\Delta)$.

The following observation will be used in §14.

Lemma 7.2. *There exists a unique automorphism Ω of Δ stabilizing Σ such that $x_{\alpha_1}(t) \mapsto x_{\tilde{\alpha}-\alpha_1}(-t)$, $x_{\alpha_n}(t) \mapsto x_{\tilde{\alpha}-\alpha_n}(-t)$ and $x_{\alpha_i}(t) \mapsto x_{-\alpha_i}(-t)$ for all $i \in [2, n-1]$. The automorphism Ω has order 2.*

Proof. Let T denote the linear automorphism of V that interchanges e_1 and e_{n+1} and fixes e_i for all $i \in [2,n]$, let $\Omega \in \mathrm{Aut}(\mathrm{SL}(V))$ denote the composition of the automorphism $A \mapsto (A^t)^{-1}$ followed by conjugation by T. The automorphism of Δ induced by Ω has the desired properties. Uniqueness holds by 4.7(i). $\square$

The following observation will be used in the proof of 15.4.

Lemma 7.3. *There exists a unique automorphism Ω of Δ stabilizing Σ such that $x_{\alpha_i}(t)^\Omega = x_{\alpha_{n+1-i}}(-t)$ for all $i \in [1,n]$ and all $t \in E$. The automorphism Ω has order 2.*

Proof. Let T denote the linear automorphism of V that interchanges e_i and e_{n+2-i} for all $i \in [1,n+1]$ and let $\Omega \in \mathrm{Aut}(\mathrm{SL}(V))$ denote the composition of the automorphism $A \mapsto (A^t)^{-1}$ of $\mathrm{SL}(V)$ followed by conjugation by T. The automorphism of Δ induced by Ω has the desired properties. Uniqueness holds by 4.7(i). $\square$

Remark 7.4. Let Ω be as in 7.3 and let c be the flag in (7.1). Then c is the unique chamber of the apartment Σ stabilized by the root group U_{α_i} for all $i \in [1,n]$. Since Ω stabilizes Σ and interchanges these root groups, it fixes c.

Remark 7.5. The automorphisms Ω of Δ in 7.2 and 7.3 are not type-preserving.

8. Orthogonal groups

Notation 8.1. Let E be a field, let V be a vector space over E of dimension $2n$ for some $n \geq 3$, let

$$\mathcal{B} = \{e_1, \dots, e_n, f_1, \dots, f_n\}$$

be a basis of V, let $q\colon V \mapsto E$ be the quadratic form given by

$$q\Big(\sum_{i=1}^{n}(x_i e_i + y_i f_i)\Big) = \sum_{i=1}^{n} x_i y_i$$

for all $x_1, \dots, y_n \in E$ and let $\mathrm{O}(q)$ denote the corresponding orthogonal group.

Notation 8.2. For distinct $i,j \in [1,n]$ and all $t \in E$, we denote by $x_{ij}(t)$ the element of $\mathrm{O}(q)$ fixing e_k and f_m for all $k \neq j$ and all $m \neq i$ that maps e_j to $e_j + te_i$ and f_i to $f_i - tf_j$.

For i,j such that $1 \leq i < j \leq n$ and all $t \in E$, we denote by $y_{ij}(t)$ the element of $\mathrm{O}(q)$ fixing e_k and f_m for all k and all $m \notin \{i,j\}$ that maps f_i to $f_i - te_j$ and f_j to $f_j + te_i$.

For i, j such that $1 \le i < j \le n$ and all $t \in E$, we denote by $z_{ij}(t)$ the element of $\mathrm{O}(q)$ fixing e_k and f_m for all $k \notin \{i, j\}$ and all m that maps e_i to $e_i + tf_j$ and e_j to $e_j - tf_i$.

Notation 8.3. Let $\Delta = \mathsf{D}_n(E)$ denote the building of type D_n associated with q. The chambers of Δ are the maximal elements of the set $\mathcal{F}(q)$ described in [MPW, 35.9], where q is the quadratic form in 8.1. We will call these maximal elements *oriflammes*. Thus an oriflamme is a set of n subspaces $Z_1, \ldots, Z_n$ of V each of which is totally isotropic with respect to q such that $\dim_E Z_i = i$ for all $i \in [1, n-2]$, $\dim_E Z_{n-1} = \dim_E Z_n = n$, $\dim_E(Z_{n-1} \cap Z_n) = n - 1$ and $Z_i \subset Z_j$ for all $i \in [1, n-2]$ and all $j \in [1, n]$ whenever $i \le j$. Let c denote the oriflamme consisting of the subspaces

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, e_2, \ldots, e_{n-2} \rangle$$

together with $\langle e_1, e_2, \ldots, e_{n-1}, e_n \rangle$ and $\langle e_1, e_2, \ldots, e_{n-1}, f_n \rangle$.

Notation 8.4. Let Φ be the root system of type D_n and let $\varepsilon_1, \ldots, \varepsilon_n$, $\alpha_1, \ldots, \alpha_n$ and $\tilde{\alpha}$ be as in [Bou, Plate IV]. Thus $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i \in [1, n-1]$, $\alpha_n = \varepsilon_{n-1} + \varepsilon_n$ and $\tilde{\alpha} = \varepsilon_1 + \varepsilon_2$. For each $\beta \in \Phi$, we set $x_\beta = x_{ij}$ if $\beta = \varepsilon_i - \varepsilon_j$, we set $x_\beta = y_{ij}$ if $\beta = \varepsilon_i + \varepsilon_j$ and we set $x_\beta = z_{ij}$ if $\beta = -\varepsilon_i - \varepsilon_j$. The groups $x_\beta(E)$ for $\beta \in \Phi$ act faithfully on Δ and we will simply identify them with their images in $\mathrm{Aut}(\Delta)$. Let S denote the set of reflections $\{s_{\alpha_1}, \ldots, s_{\alpha_n}\}$ and let $W = \langle S \rangle \subset \mathrm{Aut}(\Phi)$ be the Weyl group of Φ. Let Σ be the apartment of Δ whose chambers are the oriflammes containing only subspaces spanned by a subset of $\mathcal{B}$ and let Φ be identified with the set of roots of Σ and $\mathrm{Aut}(\Phi)$ (and hence, in particular, W) with a subgroup of $\mathrm{Aut}(\Sigma)$ as in 4.1. Thus $\alpha_1, \ldots, \alpha_n$ are the roots of Σ containing c but not some chamber of Σ adjacent to c. For each $\beta \in \Phi$, the group $x_\beta(E)$ is the root group of Δ corresponding to the root β of Σ, and there exists a map τ such $\{x_\beta\}_{\beta \in \Phi}$ is a coordinate system for Δ as defined in 4.4.

Notation 8.5. The symbol $\Omega(q)$ denotes the subgroup of $\mathrm{O}(q)$ generated by all its root groups. The group $\Omega(q)$ is the kernel of the spinor norm from $\mathrm{O}(q)$ to $E^*/(E^*)^2$. In particular, the quotient $\mathrm{O}(q)/\Omega(q)$ is an elementary abelian 2-group; see, for example, [Die, II, §6.4 and §10.4].

We will apply 8.6–8.13 in §13.

Notation 8.6. Let n be even and at least 6 and let $\Phi_1 = \langle \alpha_3, \ldots, \alpha_n \rangle \cap \Phi$. Thus Φ_1 is a root system of type D_{n-2}. Let J be the set of reflections $\{s_{\alpha_i} \mid i \in [3, n]\}$, let w_1 be the longest element in the Coxeter group $W_J = \langle J \rangle$ with respect to the set of generators J and let $w_0 = s_{\alpha_1} w_1$. The roots α_1 and $\tilde{\alpha}$ are perpendicular to Φ_1 and hence fixed by w_1, and $w_1(\alpha_i) = -\alpha_i$ for all $i \in [3, n]$ by 2.10. Since

$$\tilde{\alpha} = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n, \tag{8.7}$$

it follows that $\alpha_1 + \alpha_2 + w_1(\alpha_2) = \tilde{\alpha}$. Thus

$$w_1(\alpha_2) = \varepsilon_2 + \varepsilon_3, \tag{8.8}$$

so $w_1(\alpha_2)$ is the highest root of the root system $\langle \alpha_2, \ldots, \alpha_n \rangle \cap \Phi$ of type D_{n-1} (by 8.4). It also follows from (8.8) that

$$w_0(\alpha_2) = \varepsilon_1 + \varepsilon_3 = \tilde{\alpha} - \alpha_2. \tag{8.9}$$

Finally, we have

$$w_0(\alpha_i) = -\alpha_i \tag{8.10}$$

for all $i \in [1, n]$ other than 2.

Lemma 8.11. *Let n be even and at least 6 and let w_0 be as in 8.6. There exists a unique automorphism Ω of Δ mapping the basis $\mathcal{B}$ to itself such that $x_{\alpha_1}(t) \mapsto x_{w_0(\alpha_1)}(t)$, $x_{\alpha_2}(t) \mapsto x_{w_0(\alpha_2)}(t)$ and $x_{\alpha_i}(t) \mapsto x_{w_0(\alpha_i)}(-t)$ for each $i \in [3, n]$. The automorphism Ω has order 2 and interchanges the residues of Δ corresponding to $\langle e_1 \rangle$ and $\langle e_2 \rangle$.*

Proof. It follows from 8.2, (8.9) and (8.10) that conjugation by the automorphism of V that interchanges e_1 with e_2, f_1 with f_2 and e_i with f_i for each $i \in [3, n]$ induces an automorphism of Δ with the desired properties. Uniqueness holds by 4.7(i). □

Remark 8.12. Let V_1 be a totally isotropic subspace of V of dimension $k \leq n-3$ contained in an oriflamme c_1, let R_1 be the residue of Δ containing all oriflammes that agree with c_1 in all dimensions at least k, let R_2 be the residue of Δ containing all oriflammes that agree with c_1 in all dimensions at most k and let $\pi_i = \mathrm{proj}_{R_i}$ for $i = 1$ and 2 (as defined in [Weil, 8.23]). Let d be an arbitrary oriflamme containing V_1. Then $\pi_1(d)$ is the oriflamme that agrees with c_1 in all dimensions at least k and with d in all dimensions at most k, and $\pi_2(d)$ is the oriflamme that agrees with c_1 in all dimensions at most k and with c_1 in all dimensions at least k.

Remark 8.13. Let Ω be the automorphism of Δ in 8.11, let c_1 be an oriflamme (i.e. a chamber of Δ) containing $\langle e_1\rangle$ and $\langle e_1, e_2\rangle$ and contained in the apartment Σ, let d be the oriflamme containing $\langle e_2\rangle$ that agrees with c_1 in all dimensions greater than 1 and let P be the panel of Δ containing c_1 and d. Thus d is the other chamber in $P \cap \Sigma$. By 8.12, the composition $\Omega \cdot \mathrm{proj}_P$ (that is, Ω followed by proj_P) interchanges c_1 and d and maps the image of d under $x_{\alpha_1}(t)$ to the image of d under $x_{\alpha_1}(t^{-1})$ for all $t \in E^*$.

The following will be applied in §12.

Lemma 8.14. *There exists a unique automorphism Ω of Δ stabilizing Σ such that $x_{\alpha_1}(t) \mapsto x_{\tilde{\alpha}}(t)$ and $x_{\alpha_i}(t) \mapsto x_{-\alpha_i}(-t)$ for each $i \in [2, n]$. The automorphism Ω has order 2.*

Proof. The automorphism of Δ induced by the element of $\mathrm{O}(q)$ that fixes e_1 and f_1 and interchanges e_i and f_i for each $i \in [2, n]$ has the desired properties. Uniqueness holds by 4.7(i). □

Notation 8.15. Let σ be an involution in $\mathrm{Aut}(E)$ and let $K = \mathrm{Fix}_E(\sigma)$. We will usually write $\overline{x}$ in place of x^σ for elements $x \in E$. Let N be the norm of the quadratic extension E/K.

The last two results of this section will be applied in the proof of 15.4. For the definition of the quaternion algebra $(E/K, \kappa)$ that appears in the next result, see, for example, [TW, 9.3].

Lemma 8.16. *Suppose that n is even and that κ is an element of K not in $N(E)$. Let R be the residue of Δ whose chambers are the oriflammes containing the subspaces $\langle e_1, e_2, \dots, e_k\rangle$ for all even $k \in [1, n]$ and let R_1 denote the residue whose chambers are the oriflammes containing the subspace $\langle e_1, e_2, \dots, e_n\rangle$. Then there exists a type-preserving Galois involution Ω on Δ that stabilizes Σ, R and R_1 such that Ω does not stabilize any proper residues of R and*

$$R_1^{\langle \Omega_1\rangle} \cong \mathsf{A}_m(D),$$

where Ω_1 denotes the restriction of Ω to R_1, $m = (n/2) - 1$ and D denotes the quaternion division algebra $(E/K, \kappa)$.

Proof. Let T denote the unique σ-linear automorphism of V that extends the maps $te_i \mapsto \overline{t}e_{i+1}$ and $tf_i \mapsto \kappa\overline{t} f_{i+1}$ for all odd $i \in [1, n]$ and $te_i \mapsto \kappa\overline{t}e_{i-1}$ and $tf_i \mapsto \overline{t} f_{i-1}$ for all even $i \in [1, n]$. Then $q(T(v)) = \kappa \cdot \overline{q(v)}$ for all $v \in V$ and T stabilizes the subspaces $\langle e_1, \dots, e_k\rangle$ for all even $k \in [1, n]$. Let Ω denote the automorphism of Δ induced by T. Then $\Omega^2 = 1$ and Ω stabilizes both R and R_1. Let $\Gamma = \langle\Omega\rangle$ and let Γ_1 denote the restriction of Γ to R_1.

Every subspace of $\tilde{V} := \langle e_1, \dots, e_n\rangle$ of dimension $n-1$ is contained in exactly two totally isotropic subspaces of V of dimension n. It follows that the residue R_1 is isomorphic to the building of type A_{n-1} whose chambers are the maximal flags of subspaces of $\tilde{V} := \langle e_1, \dots, e_n\rangle$.

We have

$$D = \{x + uy \mid x, y \in E\},$$

where $uy \cdot uz = \kappa\overline{y}z$, $uy \cdot z = u(yz)$ and $y \cdot uz = u(\overline{y}z)$ for all $y, z \in E$. The vector space $\tilde{V}$ has a unique structure as a right vector space over D of dimension $n/2$ such that

$$(se_i + te_{i+1})(x + uy) = (xs + \kappa y\overline{t})e_i + (xt + y\overline{s})e_{i+1}$$

for all odd $i \in [1, n]$ and all $s, t, x, y \in E$. We have $T(v) = v \cdot u$ for all $v \in \tilde{V}$. It follows that the T-invariant subspaces of $\tilde{V}$ as a vector space over E are precisely the subspaces of $\tilde{V}$ as a right vector space over D. Thus R is a Γ-chamber and $R_1^{\Gamma_1} \cong \mathsf{A}_m(D)$. □

Lemma 8.17. *If $n = 3$, then there exists a unique automorphism Ω of Δ stabilizing Σ such that $x_{\alpha_1}(t)^\Omega = x_{\alpha_1}(-\overline{t})$ and $x_{\alpha_2}(t)^\Omega = x_{\alpha_3}(-\overline{t})$ for all $t \in E$. The automorphism Ω is a non-type-preserving Galois involution and $\Delta^{\langle\Omega\rangle} \cong \mathsf{B}_2^{\mathcal{Q}}(K, E, N)$.*

Proof. Let T be the unique σ-linear automorphism of V that fixes e_1 and f_1, maps e_2 to $-e_2$ and f_2 to $-f_2$ and interchanges e_3 with f_3. Then $T^2 = 1$ and $q(T(v)) = \overline{q(v)}$ for all $v \in V$ and by 8.2, $x_{\alpha_1}(t)^T = x_{\alpha_1}(-\overline{t})$ and $x_{\alpha_2}(t)^T = x_{\alpha_3}(-\overline{t})$ for all $t \in E$. Let Ω denote the Galois involution of Δ induced by T. Then Ω is non-type-preserving and stabilizes Σ. By 4.7(i), Ω is unique. Since c is the unique chamber of Σ contained in α_i for all $i \in [1, 3]$, Ω fixes c. Thus, in particular, Ω is isotropic.

Let τ be a non-zero element of trace 0 in E, let ω be an element of E not in K and let $V_0 = \mathrm{Fix}_V(T)$, let V_1 denote the subspace over K (rather than E) spanned by the set

$$\mathcal{B}_1 := \{e_1,\ f_1,\ \tau e_2,\ \tau^{-1} f_2,\ e_3 + f_3,\ \omega e_3 + \overline{\omega} f_3\}.$$

Then $V_1 \subset V_0$, so $q(V_1) \subset K$ and by [MPW, 2.40(i)], $V_1 = V_0$. Let $Q\colon V_1 \to K$ denote the restriction of q to V_1. By 6.5, $\Gamma := \langle\Omega\rangle$ is a descent group of Δ. By [MPW, 2.40(ii)], Δ^Γ is isomorphic to the building $\mathcal{B}(Q)$ defined in 3.3. The restriction of Q to $\langle e_1, f_1, \tau e_2, \tau^{-1} f_2\rangle$ is hyperbolic and the map

$$s(e_3 + f_3) + t(\omega e_3 + \overline{\omega} f_3) \mapsto s + t\omega$$

is an isometry from the restriction of Q to the subspace $\langle e_3 + f_3, \omega e_3 + \overline{\omega} f_3\rangle$ of V_1 to the norm N viewed as a quadratic form over K. Thus N is the anisotropic part of Q. By 3.4, we conclude that $\mathcal{B}(Q) \cong \mathsf{B}_2^Q(K, E, N)$. □

9. An anisotropic Galois involution of $\mathsf{D}_n(q)$

We continue with all the notation and assumptions from the previous section. In particular, Δ is the building $\mathsf{D}_n(E)$ whose chambers are the oriflammes of V with respect to the quadratic form q as defined in 8.3.

Notation 9.1. Let σ, K, $x \mapsto \overline{x}$ and N be as in 8.15, let ω be an element of E not in K and let

$$x^2 - ax + b = (x - \omega)(x - \overline{\omega})$$

be the minimal polynomial of ω over K. Thus

$$N(x + y\omega) = x^2 + axy + by^2 \tag{9.2}$$

for all $x, y \in K$.

Lemma 9.3. *Let ω, a, b, $x \mapsto \overline{x}$ and N be as in 9.1. Let $i \in [1, n]$, let $e = e_i$, let $f = f_i$, let $\eta \in E$ and let φ be the quadratic form on $\langle e, f\rangle$ given by*

$$\varphi(xe + yf) = xy$$

for all $x, y \in E$. Let $b_1 = \eta e + f$ and let $b_2 = \eta\omega e + \overline{\omega} f$. Then the following hold:

(i) $e = \eta^{-1}(\overline{\omega} - \omega)^{-1}(\overline{\omega} b_1 - b_2)$ *and* $f = -(\overline{\omega} - \omega)^{-1}(\omega b_1 - b_2)$.

(ii) $\varphi(xb_1 + yb_2) = \eta(x^2 + axy + by^2)$ *for all* $x, y \in E$.

(iii) $\varphi \cong N \otimes_K E$.

Proof. It can be verified with a few calculations that (i) and (ii) hold; (iii) follows from (ii) and (9.2). □

Notation 9.4. Let $\eta_1, \ldots, \eta_n$ be non-zero elements of K and let $Q\colon E^n \to K$ denote the quadratic from over K given by

$$Q(y_1, \ldots, y_n) = \sum_{i=1}^{n} \eta_i N(y_i)$$

for all $(y_1, \ldots, y_n) \in E^n$.

Proposition 9.5. *Let $q\colon V \to E$ be as in 8.1, let $x \mapsto \overline{x}$ and K be as in 9.1, let $\eta_1, \dots, \eta_n$ and Q be as in 9.4 and let $\Omega = \Omega_{\eta_1,\dots,\eta_n}$ be the σ-linear automorphism of V given by*

$$\Omega\Big(\sum_{i=1}^{n}(x_i e_i + y_i f_i)\Big) = \sum_{i=1}^{n}(\eta_i \overline{y_i} e_i + \eta_i^{-1}\overline{x_i} f_i) \tag{9.6}$$

for all $x_1, \dots, y_n \in E$. Then the following hold:

(i) *$q(\Omega(v)) = \overline{q(v)}$ for all $v \in V$ and $\Omega^2 = 1$.*

(ii) *$q \cong Q \otimes_K E$.*

(iii) *If the quadratic form Q is anisotropic, then there are no non-zero Ω-invariant subspaces of V that are totally isotropic with respect to q.*

Proof. Assertion (i) is clear and assertion (ii) follows from 9.3(iii). Suppose that V_0 is a non-zero totally isotropic Ω-invariant subspace of V. Thus $q(v) = 0$ for all $v \in V_0$. Let u be a non-zero element of V_0. The sum $v := u + \Omega(u)$ is fixed by Ω. Replacing u by tu for some $t \in E\backslash K$ if necessary, we can assume that v is non-zero. Hence

$$v = \sum_{i=1}^{n}(x_i e_i + y_i f_i)$$

for some $x_1, \dots, y_n \in E$ not all zero. Since v is fixed by Ω, we have $x_i = \eta_i \overline{y_i}$ for each $i \in [1, n]$. Therefore the elements $y_1, \dots, y_n$ are not all zero and

$$Q(y_1, \dots, y_n) = \sum_{i=1}^{n} \eta_i y_i \overline{y_i} = q(v) = 0.$$

Thus (iii) holds. □

Proposition 9.7. *Let $\alpha_1, \dots, \alpha_n$ and x_β for $\beta \in \Phi$ be as in 8.4 and let Ω be as in (9.6). Then*

$$x_{\alpha_i}(t)^{\Omega} = x_{-\alpha_i}(-\eta_i^{-1}\eta_{i+1}\overline{t})$$

for all $i \in [1, n-1]$ and all $t \in E$ and

$$x_{\alpha_n}(t)^{\Omega} = x_{-\alpha_n}(-\eta_{n-1}^{-1}\eta_n^{-1}\overline{t})$$

for all $t \in E$.

Proof. This holds by 8.2, (9.6) and some computation. □

Notation 9.8. Let W be the Weyl group of Φ, let w_1 be the longest element in W with respect to the set of generators $\{s_{\alpha_i} \mid i \in [1,n]\}$ and let $\Omega_1 := \Omega_{1,\dots,1}$ be the involution obtained by setting $\eta_1 = \cdots = \eta_n = 1$ in 9.5. We use the same letters $\Omega = \Omega_{\eta_1,\dots,\eta_n}$ and Ω_1 to denote the automorphisms of Δ induced by these two involutions of V; this convention should not cause any confusion. Since $\eta_1,\dots,\eta_n \in K$, we have

$$\Omega = g_{\lambda_1,\dots,\lambda_n,\mathrm{id}} \cdot \Omega_1 = g_{w_1,-\lambda_1,\dots,-\lambda_n,\sigma} \tag{9.9}$$

if n is even by 2.10, 8.15 and 9.7, where $\lambda_i = \eta_i^{-1}\eta_{i+1}$ for all $i \in [1, n-1]$ and $\lambda_n = \eta_{n-1}^{-1}\eta_n^{-1}$, $g_{\lambda_1,\dots,\lambda_n,\mathrm{id}}$ is as in 4.7(i) and $g_{w_1,-\lambda_1,\dots,-\lambda_n,\sigma}$ is as in 4.13.

Notation 9.10. Let ι be the automorphism of V given by

$$\iota\Big(\sum_{i=1}^{n}(x_ie_i + y_if_i)\Big) = \sum_{i=1}^{n}(\overline{x_i}e_i + \overline{y_i}f_i)$$

for all $x_1,\dots,y_n \in E$. Then $\iota(q(v)) = \overline{q(v)}$ for all $v \in V$, ι commutes with the element Ω_1 in 9.8, the composition $\iota \cdot \Omega_1$ is contained in $\mathrm{O}(q)$ and

$$x_\beta(t)^\iota = x_\beta(\overline{t})$$

for all $\beta \in \Phi$ and all $t \in E$.

Proposition 9.11. *Let n be even and let Ω_1 and ι be as in 9.8 and 9.10. Then the product $\iota \cdot \Omega_1$ induces an automorphism of Δ contained in the group $G^\dagger$ defined in 3.1.*

Proof. Since n is even, there is a unique element of $\mathrm{O}(q)$ that maps e_i to e_{i+1} and f_i to f_{i+1} for all odd $i \in [1,n]$ and e_i to f_{i-1} and f_i to e_{i-1} for all even $[1,n]$, and the square of this element equals $\iota \cdot \Omega_1$. By 8.5, it follows that $\iota \cdot \Omega_1 \in \Omega(q)$. The claim holds, therefore, by 8.5. $\square$

10. An extension from $\mathsf{D}_n(E)$ to $\mathsf{D}_{n+1}(E)$

Let V, E, $\Omega = \Omega_{\eta_1,\dots,\eta_n}$, q, $\mathcal{B}$, Φ, etc., be as in the previous two sections.

Notation 10.1. Let V_0 be a vector space over E containing V as a subspace of co-dimension 2, let

$$\mathcal{B}_0 = \{e_0,\dots,e_n, f_0,\dots,f_n\}$$

be an extension of the basis $\mathcal{B}$ to a basis of V_0, let $q_0\colon V_0 \to E$ be the quadratic form given by

$$q_0\Big(\sum_{i=0}^{n}(x_i e_i + y_i f_i)\Big) = \sum_{i=0}^{n} x_i y_i$$

and let $\alpha_0, \alpha_1, \dots, \alpha_n$ be an extension of the basis $\alpha_1, \dots, \alpha_n$ of Φ to a basis of a root system Φ_0 of type D_{n+1} containing Φ. We extend Ω to a σ-linear automorphism Ω_0 of V_0 by setting

$$\Omega_0(x_0 e_0 + y_0 f_0 + v) = \overline{x_0} e_0 + \overline{y_0} f_0 + \Omega(v) \tag{10.2}$$

for all $x_0, y_0 \in E$ and all $v \in V$. Since Ω is an involution, so is Ω_0.

Notation 10.3. Let Δ_0 denote the building of type D_{n+1} whose chambers are the oriflammes with respect to q_0. We identify the building $\Delta = \mathsf{D}_n(E)$ in §9 with the residue of Δ_0 consisting of all oriflammes containing the subspace $\langle e_0 \rangle$ and we denote the automorphism of Δ_0 induced by Ω_0 also by Ω_0. Thus Δ is a $\langle \Omega_0 \rangle$-residue and Ω_0 is a Galois involution of Δ_0 extending Ω.

Proposition 10.4. *Suppose that the quadratic form Q in 9.4 is anisotropic. Then Δ is a $\langle \Omega_0 \rangle$-chamber and the fixed point building $\Delta_0^{\langle \Omega_0 \rangle}$ is isomorphic to*

$$\mathsf{B}_1^{\mathcal{Q}}(K, E^n, Q),$$

where $\mathsf{B}_1^{\mathcal{Q}}(K, E^n, Q)$ is as defined in [Wei2, 30.15].

Proof. It follows from 9.5(iii) that Δ is a $\langle \Omega_0 \rangle$-chamber. Let

$$Q_0\colon K \oplus K \oplus E^n \to K$$

be the quadratic form over K given by

$$Q_0(x_0 e_0 + y_0 f_0 + v) = x_0 y_0 + Q(v)$$

for all $x_0, y_0 \in K$ and all $v \in E^n$. Thus Q is the anisotropic part of Q_0. Let $\hat{V} = \mathrm{Fix}_{V_0}(\Omega_0)$. By [MPW, 2.40(i)], there is a canonical isomorphism from $\hat{V} \otimes_K E$ to V_0 mapping $\hat{v} \otimes t$ to $t\hat{v}$ for all $\hat{v} \in \hat{V}$ and all $t \in E$. By [MPW, 2.40(ii)], the map $W \mapsto W \cap \hat{V}$ is an inclusion- and dimension-preserving bijection from the set of Ω_0-invariant subspaces of V_0 to the set of all subspaces of $\hat{V}$. For each $i \in [1, n]$, the elements b_1 and b_2 defined in 9.3 are fixed by Ω_0. The set of these elements together with e_0 and f_0 is thus a basis for $\hat{V}$ over K. By 9.3(ii), it follows that Q_0 is the restriction of q_0 to $\hat{V}$. Thus by 9.5(ii), an Ω_0-invariant subspace W of V_0 is totally isotropic with respect to q_0 if and only if $W \cap \hat{V}$ is totally isotropic with respect to Q_0. By 3.4, we conclude that $\Delta_0^{\langle \Omega_0 \rangle} \cong \mathsf{B}_1^{\mathcal{Q}}(K, E^n, Q)$. ☐

Notation 10.5. For all $\beta \in \Phi_0$ and all $t \in E$, let $x_\beta(t)$ be the elements of $\mathrm{O}(q_0)$ defined by applying 8.2 and 8.4 with the interval $[1, n]$ replaced by the interval $[0, n]$. Thus, in particular, the restriction of $x_\beta(t)$ to V is as it was in the previous section for all $\beta \in \Phi$ and all $t \in E$, $x_{\alpha_0}(t)$ is the unique element of $\mathrm{O}(q_0)$ that fixes the elements e_k and f_m of $\mathcal{B}_0$ for all $k \neq 1$ and all $m \neq 0$ and maps e_1 to $e_1 + te_0$ and f_0 to $f_0 - tf_1$ for all $t \in E$ and $x_{\tilde{\alpha}}(t)$ is the unique element of $\mathrm{O}(q_0)$ that fixes the elements e_k and f_m of $\mathcal{B}_0$ for all $k \in [0, n]$ and all $m \in [2, n]$ and maps f_0 to $f_0 - te_1$ and f_1 to $f_1 + te_0$ for all $t \in E$, where $\tilde{\alpha}$ is the highest root of Φ_0 with respect to the basis $\alpha_0, \ldots, \alpha_n$.

Proposition 10.6. *Let Ω_0 be as in 10.2 and let $\tilde{\alpha}$ be the highest root of the root system $\Phi = \langle \alpha_1, \ldots, \alpha_n \rangle \cap \Phi_0$ of type D_n. Then*

$$x_{\alpha_0}(t)^{\Omega_0} = x_{\tilde{\alpha}}(\eta_1 \bar{t})$$

and

$$x_{\alpha_i}(t)^{\Omega_0} = x_{-\alpha_i}(-\eta_i^{-1}\eta_{i+1}\bar{t})$$

for all $t \in E$ and all $i \in [1, n-1]$ as well as

$$x_{\alpha_n}(t)^{\Omega_0} = x_{-\alpha_n}(-\eta_{n-1}^{-1}\eta_n^{-1}\bar{t})$$

for all $t \in E$.

Proof. The first identity holds by (10.2), 10.5 and some computation, and the remaining identities hold by 9.7. □

11. The quadrangles of type E_8

Our goal in this section is to prove 11.21. Let Φ be a root system of type E_7 and let $\alpha_1, \ldots, \alpha_7$ and $\tilde{\alpha}$ be as in [Bou, Plate VI]. Let W be the Weyl group of Φ, let S be the set of reflections s_{α_i} for $i \in [1, 7]$, let Φ_1 be the root system $\langle \alpha_2, \ldots, \alpha_7 \rangle \cap \Phi$ of type D_6, let $S_1 = S \backslash \{s_{\alpha_1}\}$ and let $W_1 = \langle S_1 \rangle$.

The pair (W_1, S_1) is a Coxeter system of type D_6. Let w_1 denote the longest element in W_1 with respect to the set of generators S_1. Since $\tilde{\alpha}$ is orthogonal to α_i for all $i \in [2, 7]$, we have

$$w_1(\tilde{\alpha}) = \tilde{\alpha}. \tag{11.1}$$

By 2.10, $w_1(\alpha_i) = -\alpha_i$ for all $i \in [2, 7]$. Applying w_1 to the equation

$$\tilde{\alpha} = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7, \tag{11.2}$$

we conclude that

$$\tilde{\alpha} = w_1(\alpha_1) + \alpha_1. \tag{11.3}$$

Thus

$$w_1(\alpha_1) = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7. \tag{11.4}$$

Notation 11.5. We denote by Δ the building $\mathsf{E}_7(E)$. Let Σ be an apartment of Δ, let c be a chamber of Σ and let Δ_1 be the unique residue of Δ of type D_6 containing c. Thus $\Delta_1 \cong \mathsf{D}_6(E)$ and $\Sigma_1 := \Delta_1 \cap \Sigma$ is an apartment of Δ_1. We identify the root system Φ with the set of roots of Σ and $\mathrm{Aut}(\Phi)$ with a subgroup of $\mathrm{Aut}(\Sigma)$ as in 4.1. This gives an identification of Φ_1 with the roots of Σ_1.

Notation 11.6. Let $\tilde{\Delta}$, $\tilde{\Sigma}$, $\tilde{c}$, $\tilde{\alpha}_1, \dots, \tilde{\alpha}_6$ and $\{\tilde{x}_\beta\}_{\beta \in \Phi_1}$ be the building, the apartment, the chamber, the set of roots and the coordinate system called Δ, Σ, c, $\alpha_1, \dots, \alpha_6$ and $\{x_\beta\}_{\beta \in \Phi}$ in 8.3 and 8.4 with $n = 6$. There exists an isomorphism ν from $\tilde{\Delta}$ to Δ_1 mapping $\tilde{\Sigma}$ to Σ_1, $\tilde{c}$ to c and the root $\tilde{\alpha}_i$ to $\alpha_{\pi(i)}$ for all $i \in [1,6]$, where π is the map sending the sequence $1, 2, \dots, 6$ to the sequence $7, 6, 5, 4, 2, 3$. Let $x_\beta = \nu^{-1} \cdot \tilde{x}_\beta \cdot \nu$ for all $\beta \in \Phi_1$. Then $\{x_\beta\}_{\beta \in \Phi_1}$ is a coordinate system for Δ_1. By 4.11, we can extend this coordinate system to a coordinate system $\{x_\beta\}_{\beta \in \Phi}$ for Δ.

The root $\tilde{\alpha}$ is orthogonal to the root α_i for all $i \in [2,7]$. Thus $[U_{\pm\alpha_i}, U_{\tilde{\alpha}}] = 1$ for all $i \in [2,7]$ by 4.2(ii). By 3.2 and 9.11, there exists an element $\hat{\Omega}_1$ in

$$\langle U_\beta \mid \beta \in \Phi_1 \rangle \subset \mathrm{Aut}(\Delta)$$

stabilizing Δ_1 and Σ_1 and centralizing $U_{\tilde{\alpha}}$ such that

$$x_{\alpha_i}^{\hat{\Omega}_1} = x_{-\alpha_i}(-t) \tag{11.7}$$

for all $i \in [2,7]$.

Let R be the unique residue such that $R \cap \Sigma$ and Σ_1 are opposite residues of Σ. For each root β in Φ_1, there exist chambers of Σ_1 not in β. Thus each root of Φ_1 contains chambers of R (by [Weil, 5.2]) and hence the corresponding root group stabilizes R. Therefore the element $\hat{\Omega}_1$ stabilizes R. Since it also stabilizes Σ_1, it stabilizes $\mathrm{proj}_R(\Sigma_1)$. By [Weil, 5.14(i)], $\mathrm{proj}_R(\Sigma_1) = R \cap \Sigma$. Hence $\hat{\Omega}_1$ stabilizes the convex closure of Σ_1 and $R \cap \Sigma$. By [Weil, 8.9 and 9.2], this convex closure is Σ. We conclude that $\hat{\Omega}_1$ stabilizes Σ. Since w_1 and $\hat{\Omega}_1$ have the same restriction to Σ_1, the restriction of $\hat{\Omega}_1$ to Σ is w_1. By 4.14, therefore, there exist $\kappa_1, \dots, \kappa_7 \in E^*$ such that

$$\hat{\Omega}_1 = g_{w_1, \kappa_1, \dots, \kappa_7, \mathrm{id}}.$$

Thus, in particular, we have

$$x_{\alpha_1}(t)^{\hat{\Omega}_1} = x_{w_1(\alpha_1)}(\kappa t) \tag{11.8}$$

for $\kappa = \kappa_1$ and for all $t \in E$. By (11.7), $\kappa_i = -1$ for all $i \in [2,7]$. By 4.7(ii), there exists $\rho \in E^*$ such that

$$x_{w_1(\alpha_1)}(t)^{\hat{\Omega}_1} = x_{\alpha_1}(\rho t) \tag{11.9}$$

for all $t \in E$. By 4.2(i) and (11.3), there exists $\delta \in \{1,-1\}$ such that

$$\left[x_{\alpha_1}(s), x_{w_1(\alpha_1)}(t)\right] = x_{\tilde{\alpha}}(\delta st) \tag{11.10}$$

for all $s,t \in E$. Applying $\hat{\Omega}_1$ to this identity, we find that

$$\left[x_{w_1(\alpha_1)}(\kappa s), x_{\alpha_1}(\rho t)\right] = x_{\tilde{\alpha}}(\delta st)$$

for all $s,t \in E$. Thus

$$\left[x_{\alpha_1}(\rho t), x_{w_1(\alpha_1)}(\kappa s)\right] = x_{\tilde{\alpha}}(-\delta st)$$

for all $s,t \in E$. Applying (11.10) to the left-hand side of this identity, we conclude that

$$\kappa\rho = -1. \tag{11.11}$$

Notation 11.12. Let σ, $x \mapsto \overline{x}$ and K be as in 8.15, let $\lambda_1, \eta_1, \dots \eta_6 \in K^*$ and let Q be as in 9.4 with $n = 6$. We set

$$\hat{\Omega} = g_{\lambda_1,\lambda_2\dots,\lambda_7,\sigma} \cdot \hat{\Omega}_1,$$

where $\lambda_2 = \eta_5^{-1}\eta_6$, $\lambda_3 = \eta_5^{-1}\eta_6^{-1}$, $\lambda_4 = \eta_4^{-1}\eta_5$, $\lambda_5 = \eta_3^{-1}\eta_4$, $\lambda_6 = \eta_2^{-1}\eta_3$, $\lambda_7 = \eta_1^{-1}\eta_2$ and $g_{\lambda_1,\dots,\lambda_7,\sigma}$ is as in 4.7(i). Thus

$$\lambda_2^2\lambda_3^3\lambda_4^4\lambda_5^3\lambda_6^2\lambda_7 = \eta_1^{-1}\cdots\eta_6^{-1}. \tag{11.13}$$

Notation 11.14. Let $\nu\colon \tilde{\Delta} \to \Delta_1$ be as in 11.6 and let $\tilde{\Omega}$ be the automorphism of $\tilde{\Delta}$ in (9.6) with $n = 6$ and $\eta_1, \dots, \eta_6$ be as in 11.12. We denote by Ω the automorphism $\nu^{-1} \cdot \tilde{\Omega} \cdot \nu$ of Δ_1. The automorphism Ω satisfies the identities in 9.7 with $n = 6$ and with the roots $\alpha_1, \dots, \alpha_6$ replaced by the roots $\alpha_7, \alpha_6, \alpha_5, \alpha_4, \alpha_2, \alpha_3$ of Φ_1 (in that order).

Proposition 11.15. *The automorphism $\hat{\Omega}$ stabilizes Δ_1, the restriction of $\hat{\Omega}$ to Δ_1 is the automorphism Ω defined in 11.14 and Ω is an involution.*

Proof. Since $\hat{\Omega}_1$ and $g_{\lambda_1,\dots,\lambda_7,\sigma}$ both stabilize Δ_1, so does $\hat{\Omega}$. The second claim holds by (9.9) and the third claim by 9.5(i). $\square$

Proposition 11.16. *The automorphism* $\hat{\Omega}$ *is an involution if and only if*

$$N(\lambda_1) = -\eta_1 \cdots \eta_6, \tag{11.17}$$

where N *is as in* 9.1.

Proof. The automorphism $\hat{\Omega}$ is an extension of Ω and $\Omega^2 = 1$. Thus $\hat{\Omega}^2$ centralizes U_{α_i} for all $i \in [2,7]$. By the uniqueness assertion in 4.7(i), therefore, $\hat{\Omega}$ is an involution if and only if $\hat{\Omega}^2$ centralizes U_{α_1}. We have

$$\begin{aligned} x_{\alpha_1}(t)^{\hat{\Omega}^2} &= x_{w_1(\alpha_1)}(\kappa\lambda_1\bar{t})^{\hat{\Omega}} && \text{by (11.8)} \\ &= x_{w_1(\alpha_1)}(\lambda_1 \cdot \eta_1^{-1} \cdots \eta_6^{-1} \cdot \kappa\overline{\lambda_1 t})^{\hat{\Omega}_1} && \text{by 4.7(ii), (11.4) and (11.13)} \\ &= x_{\alpha_1}(\rho\kappa \cdot N(\lambda_1)\eta_1^{-1} \cdots \eta_6^{-1} t) && \text{by (11.9)} \\ &= x_{\alpha_1}(-N(\lambda_1)\eta_1^{-1} \cdots \eta_6^{-1} t) && \text{by (11.11).} \end{aligned}$$

Thus $\hat{\Omega}$ is an involution if and only if (11.17) holds. □

Corollary 11.18. *Suppose the quadratic form* Q *in* 11.12 *is anisotropic and that* (11.17) *holds. Then* $\hat{\Omega}$ *is a Galois involution and* Δ_1 *is a* $\langle\hat{\Omega}\rangle$*-chamber.*

Proof. The first claim holds by 11.16 and the second claim holds by 9.5(iii) and 11.15. □

Proposition 11.19. *Suppose the quadratic form* Q *in* 11.12 *is anisotropic and that* (11.17) *holds. Then* $\Delta^{\langle\hat{\Omega}\rangle}$ *is a Moufang set with non-abelian root groups.*

Proof. By 11.18, $\hat{\Omega}$ is an involution and by 4.7(ii), (11.2) and (11.13), we have

$$x_{\tilde{\alpha}}(t)^{\hat{\Omega}} = x_{\tilde{\alpha}}(-\lambda_1\overline{\lambda_1}^{-1}\bar{t})^{\hat{\Omega}_1} = x_{\tilde{\alpha}}(-\lambda_1\overline{\lambda_1}^{-1}\bar{t}) \tag{11.20}$$

for all $t \in E$. Let T be the trace of the extension E/K and let

$$X = \{(t,u) \in E^2 \mid T(\overline{\lambda_1}u) + \kappa\delta N(\lambda_1 t) = 0\}.$$

It follows from (11.8), (11.10) and (11.20) that for all $(t,u) \in X$, the element

$$g_{t,u} := x_{\alpha_1}(t)x_{w_1(\alpha_1)}(\kappa\lambda_1\bar{t})x_{\tilde{\alpha}}(u)$$

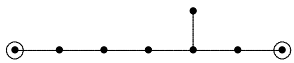
is centralized by $\hat{\Omega}$.

The roots of Σ cutting Δ_1 (as defined in 3.5) are the roots in Φ_1. All the other positive roots of Φ contain $\Delta_1 \cap \Sigma$. In particular, α_1, $w_1(\alpha_1)$ and $\tilde{\alpha}$ all contain $\Delta_1 \cap \Sigma$. By 6.12(v), the root group of $\Delta^{\langle\hat{\Omega}\rangle}$ fixing the $\langle\hat{\Omega}\rangle$-chamber Δ_1 is isomorphic to the centralizer of $\hat{\Omega}$ in the group generated by all the roots of Φ containing $\Delta_1 \cap \Sigma$. Thus $\langle g_{u,t} \mid (u,t) \in X \rangle$ is contained in this root group. For each $t \in E$, we can choose $u_t \in E$ such that $(t, u_t) \in X$. Applying (11.10) and the identities [TW, 2.2], we find that

$$[g_{s,u_s}, g_{t,u_t}] = x_{\tilde{\alpha}}(\delta\kappa\lambda_1(s\bar{t} - \bar{s}t))$$

for all $s, t \in E$. Thus not all of the elements g_{t,u_t} commute with each other. □

Theorem 11.21. *Let $\Lambda = (K, V, Q)$ be a quadratic space of type E_8. Then there exists a separable quadratic extension E/K such that Q_E is hyperbolic and for each such extension E/K, there exists a Galois involution Ω of the building $\Delta = \mathsf{E}_8(E)$ such that the Tits index of the group $\Gamma := \langle \Omega \rangle$ is*



and the fixed point building Δ^Γ is isomorphic to $\mathsf{B}_2^{\mathcal{E}}(\Lambda)$.

Proof. By 5.5, we can choose a separable quadratic extension E/K such that Q_E is hyperbolic and we can assume that $V = E^6$ and there exists $\eta_1, \ldots, \eta_6 \in K$ such that

$$Q(u_1, \ldots, u_6) = \eta_1 N(u_1) + \cdots + \eta_6 N(u_6)$$

for all $(u_1, \ldots, u_6) \in V$, where N is the norm of the extension E/K, and

$$-\eta_1\eta_2\cdots\eta_6 \in N(E). \tag{11.22}$$

Let $\Delta = \mathsf{E}_8(E)$, let Σ be an apartment of Δ and let c be a chamber of Σ. Let Φ be the root system of type E_8 and let $\alpha_1, \ldots, \alpha_8$ be as in [Bou, Plate VII]. We identify Φ with the set of roots of Σ and $\mathrm{Aut}(\Phi)$ with a subgroup of $\mathrm{Aut}(\Sigma)$ as in 4.1 and choose a coordinate system $\{x_\beta\}_{\beta\in\Phi}$ for Δ. Let A be the unique subset of S spanning a subdiagram of Π of type D_6, let w_1 denote the longest element in the Coxeter group W_A with respect to the set of generators A, let R denote the unique A-residue of Δ containing c, let R_1 be the unique residue of type D_7 containing R and let R_2 be the unique residue of type E_7 containing R.

By (11.22), we can choose λ_1 so that (11.17) holds. Let κ be as in (11.8) and let $\lambda_2, \ldots, \lambda_7$ be as in 11.12. We then set $\kappa_1 = \kappa\lambda_1$, $\kappa_i = -\lambda_i$ for all $i \in [2, 7]$, $\kappa_8 = \eta_1$ and

$$\Omega = g_{w_1,\kappa_1,\ldots,\kappa_8,\sigma},$$

where σ is the non-trivial element in $\mathrm{Gal}(E/K)$ and $g_{w_1,\kappa_1,\ldots,\kappa_8,\sigma}$ is as in 4.13. Let $\Gamma = \langle \Omega \rangle$. Since w_1 stabilizes $R \cap \Sigma$, it also stabilizes $R_1 \cap \Sigma$ and $R_2 \cap \Sigma$. Hence R, R_1 and R_2 are Γ-residues.

By 4.11 with R_1 in place of R and 11.6, we can assume that the coordinate system $\{x_\beta\}_{\beta\in\Phi}$ was chosen so that there are two isomorphisms, one from R_1 to the building Δ_0 in 10.3 with $n = 6$ carrying the automorphism Ω_0 defined in 10.2 to the restriction of Ω to R_1 and the other from R_2 to the building Δ in 11.5 carrying the automorphism $\hat{\Omega}$ defined in (11.12) to the restriction of Ω to R_2. By 10.6 applied to the restriction of Ω to R_1, Ω^2 centralizes U_{α_i} for all $i \in [2, 8]$ and R is a Γ-chamber. By 11.18 applied to the restriction of Ω to R_2, Ω^2 also centralizes U_{α_1}. Thus Ω is a Galois involution. By 6.5, therefore, Γ is a descent group of Δ. By 6.11 and 6.12(iii), Δ^Γ is a building of type B_2, and thus by 6.12(iv), Δ^Γ is a Moufang quadrangle. Let $\mathbb{M}_1$ and $\mathbb{M}_2$ be as in 5.16 applied to Δ^Γ. By 6.15, 10.4 and 11.19, one of these two Moufang sets is isomorphic to $\mathsf{B}_1^{\mathcal{Q}}(K, E^6, Q)$ and the other has non-abelian root groups. By 5.16(a), it follows that $\Delta^\Gamma \cong \mathsf{B}_2^{\mathcal{E}}(\Lambda)$. □

12. The exceptional buildings of type A_2

Our goal in this section is to prove 12.11.

Notation 12.1. Let $\Delta = \mathsf{D}_5(E)$ and let Σ, c, Φ, $\alpha_1, \dots, \alpha_5$, $\tilde{\alpha}$, (W, S), the identification of Φ with the set of roots of Σ and the identification of $\mathrm{Aut}(\Phi)$ with a subgroup of $\mathrm{Aut}(\Sigma)$ be as in 4.1. Let $S_1 = S\backslash\{s_{\alpha_1}\}$, let $W_1 = \langle S_1\rangle$, let Φ_1 be the root system $\langle\alpha_2, \dots, \alpha_5\rangle \cap \Phi$ of type D_4 and let Δ_1 be the unique residue of type D_4 containing c.

Notation 12.2. Let $\tilde{\Delta}$, $\tilde{\Sigma}$, $\tilde{c}$, $\tilde{\alpha}_1, \dots, \tilde{\alpha}_4$ and $\{\tilde{x}_\beta\}_{\beta\in\Phi_1}$ be the building, the apartment, the chamber, the set of roots and the coordinate system called Δ, Σ, c, $\alpha_1, \dots, \alpha_4$ and $\{x_\beta\}_{\beta\in\Phi_1}$ in 8.3 and 8.4 with $n = 4$. There exists an isomorphism ν from $\tilde{\Delta}$ to Δ_1 mapping $\tilde{\Sigma}$ to Σ_1, $\tilde{c}$ to c and the root $\tilde{\alpha}_i$ to $\alpha_{\pi(i)}$ for all $i \in [1, 4]$, where π is the map sending the sequence $1, 2, 3, 4$ to the sequence $5, 3, 4, 2$. Let $x_\beta = \nu^{-1} \cdot \tilde{x}_\beta \cdot \nu$ for all $\beta \in \Phi_1$. Then $\{x_\beta\}_{\beta\in\Phi_1}$ is a coordinate system for Δ_1. By 4.11, we can extend this coordinate system to a coordinate system $\{x_\beta\}_{\beta\in\Phi}$ for Δ.

The pair (W, S) is a Coxeter system of type D_5 and the pair (W_1, S_1) is a Coxeter system of type D_4. Let w_1 denote the longest element in W_1 with respect to the set of generators S_1 and let Φ_0 be the root system of type D_6 obtained by applying 10.1 to Φ. By 8.6 applied to Φ_0, we have

$$w_1(\alpha_1) = \tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5. \tag{12.3}$$

We also know that

$$w_1(\alpha_i) = -\alpha_i \tag{12.4}$$

for all $i \in [2,5]$. By 4.16 and 8.14, there exists $\delta \in \{1,-1\}$ such that

$$\hat{\Omega}_1 := g_{w_1,\delta,-1,-1,-1,-1,\mathrm{id}} \tag{12.5}$$

is an involution, where $g_{w_1,\delta,-1,-1,-1,-1,\mathrm{id}}$ is as in 4.13.

Notation 12.6. Let $\eta_1,\dots,\eta_4$ and Q be as in 9.4 with $n=4$, let σ, K, etc., be as in 8.15, let ν and $\tilde{\Delta}$ be as in 12.2 and let $\tilde{\Omega}$ be the automorphism of $\tilde{\Delta}$ in (9.6) with $n=4$. We denote by Ω the automorphism $\nu^{-1}\cdot\tilde{\Omega}\cdot\nu$ of Δ_1. The automorphism Ω satisfies the identities in 9.7 with $n=4$ and with the roots $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ replaced by the roots $\alpha_5,\alpha_3,\alpha_4,\alpha_2$ of Φ_1 (in that order).

Notation 12.7. Suppose that there exists $\lambda_1 \in E$ such that $N(\lambda_1) = \eta_1\eta_2\eta_3\eta_4$ and let

$$\hat{\Omega} = g_{\lambda_1,\dots,\lambda_5,\sigma}\cdot\hat{\Omega}_1 = g_{w_1,\delta\lambda_1,-\lambda_2,-\lambda_3,-\lambda_4,-\lambda_5,\sigma},$$

where $\lambda_2 = \eta_3^{-1}\eta_4^{-1}$, $\lambda_3 = \eta_2^{-1}\eta_3$, $\lambda_4 = \eta_3^{-1}\eta_4$, $\lambda_5 = \eta_1^{-1}\eta_2$, $\hat{\Omega}_1$ and δ are as in (12.5) and $g_{\lambda_1,\dots,\lambda_5,\sigma}$ is as in 4.7(i). We have

$$\lambda_1\lambda_2^2\lambda_3^2\lambda_4\lambda_5 = \overline{\lambda_1}^{-1}. \tag{12.8}$$

Theorem 12.9. *Suppose that $\eta_1\eta_2\eta_3\eta_4 \in N(E)$ and that the quadratic form Q in 12.6 is anisotropic. Let $\hat{\Omega}$ be as in 12.7 and let Δ_1 be the unique residue of type D_4 containing the chamber c. Then $\hat{\Omega}$ is a Galois involution of Δ stabilizing Δ_1 but not any proper residue of Δ_1.*

Proof. By (12.3) and (12.4), we have

$$x_{\alpha_1}^{\hat{\Omega}}(t) = x_{\tilde{\alpha}}(\delta\lambda_1\bar{t})$$

for all $t \in E$ and

$$x_{\alpha_i}^{\hat{\Omega}}(t) = x_{-\alpha_i}(-\lambda_i\bar{t})$$

for all $t \in E$ and all $i \in [2,5]$. Since $\hat{\Omega}_1$ is an involution, we have

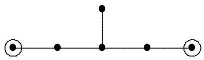
$$x_{\tilde{\alpha}}(t)^{\hat{\Omega}_1} = x_{\alpha_1}(\delta t) \tag{12.10}$$

for all $t \in E$. Therefore

$$\begin{aligned} x_{w_1(\alpha_1)}(t)^{\hat{\Omega}} &= x_{\tilde{\alpha}}(\lambda_1\lambda_2^2\lambda_3^2\lambda_4\lambda_5\bar{t})^{\hat{\Omega}_1} && \text{by 4.7(ii) and (12.3)}\\ &= x_{\tilde{\alpha}}(\overline{\lambda_1}^{-1}\bar{t})^{\hat{\Omega}_1} && \text{by (12.8)}\\ &= x_{\alpha_1}(\delta\overline{\lambda_1}^{-1}\bar{t}) && \text{by (12.10)} \end{aligned}$$

for all $t \in E$. Hence $\hat{\Omega}^2$ centralizes U_{α_1}. Thus $\hat{\Omega}$ is an involution (and hence a Galois involution). Since w_1 stabilizes $\Sigma \cap \Delta_1$, $\hat{\Omega}$ stabilizes Δ_1. The restriction of $\hat{\Omega}$ to Δ_1 coincides with the automorphism Ω defined in 12.6. By 9.5(iii), it follows that $\hat{\Omega}$ stabilizes no proper residue of Δ_1. $\square$

Theorem 12.11. *Let D be an octonion division algebra over a field K and let E/K be a separable quadratic extension such that D_E is split. Then there exists a Galois involution Ω of the building $\Delta = \mathsf{E}_6(E)$ such that the Tits index of the group $\Gamma := \langle \Omega \rangle$ is*



and the fixed point building Δ^Γ is isomorphic to $\mathsf{A}_2(D)$.

Proof. Let $\Delta = \mathsf{E}_6(E)$, let Σ be an apartment of Δ and let c be a chamber of Σ. Let Φ and $\alpha_1, \dots, \alpha_6$ be as in [Bou, Plate V]. We identify Φ with the set of roots of Σ as in 4.1 and choose a coordinate system $\{x_\beta\}_{\beta\in\Phi}$ for Δ. Let A be the unique set of vertices of the Coxeter diagram Π spanning a subdiagram of type D_4, let w_1 denote the longest element in the Coxeter group W_A with respect to the generating set A, let R denote the unique A-residue of Δ containing c and let R_1 and R_2 be the two maximal residues containing R.

There exist $\eta_1, \dots, \eta_4 \in K$ such that $\eta_1 \cdots \eta_4 \in N(E)$ and the quadratic form Q defined in 9.4 is similar to the norm of D. Let $\lambda_1, \cdots, \lambda_5$ and δ be as in 12.7. We set $\kappa_1 = \delta\lambda_1$, $\kappa_2 = -\lambda_4$, $\kappa_3 = -\lambda_2$, $\kappa_4 = -\lambda_3$, $\kappa_5 = -\lambda_5$ and $\kappa_6 = \eta_1$. Next, we set

$$\Omega_0 = g_{w_1,\kappa_1,\dots,\kappa_6,\sigma},$$

where σ is the non-trivial element in $\mathrm{Gal}(E/K)$ and $g_{w_1,\kappa_1,\dots,\kappa_6,\sigma}$ is as in 4.13. Finally, we set $\Gamma = \langle \Omega_0 \rangle$.

By 4.11 with R_2 in place of R and 12.6, we can assume that the coordinate system $\{x_\beta\}_{\beta\in\Phi}$ was chosen so that there are two isomorphisms, one from R_1 to the building Δ in 12.1 carrying the automorphism $\hat{\Omega}$ in 12.7 to the restriction of Ω to R_1 and the other from R_2 to the building Δ_0 in 10.3 with $n = 5$ carrying the automorphism Ω_0 defined in (10.2) to the restriction of Ω to R_2. Since w_1 stabilizes $R \cap \Sigma$, Γ stabilizes R. Hence Γ stabilizes the residues of Δ that contain R. By 12.9, therefore, Ω_0^2 centralizes U_{α_i} for all $i \in [1, 5]$ and R is a Γ-chamber, and by 10.6, Ω_0^2 centralizes U_{α_6}. It follows that Ω_0 is a Galois involution. By 6.5, therefore, $\Gamma := \langle \Omega_0 \rangle$ is a descent group of Δ. By 6.11 and 6.12(iii), Δ^Γ is a building of type A_2, and thus by 6.12(iv), Δ^Γ is a Moufang triangle. By [TW, 17.2–17.3], there exists a field, a skew-field or an octonion division algebra D_1 such that $\Delta^\Gamma \cong \mathsf{A}_2(D_1)$. Thus the Moufang set induced by the stabilizer of a panel of Δ^Γ in the automorphism group of Δ^Γ is isomorphic to $\mathsf{A}_1(D_1)$. By 6.15 and 10.4, it follows that

$$\mathsf{A}_1(D_1) \cong \mathsf{B}_1^{\mathcal{Q}}(K, E^4, Q).$$

Hence by [Wei3, 31.21], D_1 is an octonion division algebra whose norm is similar to Q. Therefore $D_1 \cong D$ (by [TW, 20.28], for example). □

13. The quadrangles of type E_7

Our goal in this section is to prove 13.12.

Notation 13.1. Let $\Delta = \mathsf{D}_6(E)$, Σ, c, Φ, $\alpha_1, \dots, \alpha_6$, (W, S), the identification of the set of roots of Σ with Φ, the identification of $\mathrm{Aut}(\Phi)$ with a subgroup of $\mathrm{Aut}(\Sigma)$, etc., be as in 4.1. Let $S_1 = S \backslash \{s_{\alpha_1}, s_{\alpha_2}\}$, let $W_1 = \langle S_1 \rangle$ and let Φ_1 be the root system $\langle \alpha_3, \dots, \alpha_6 \rangle \cap \Phi$ of type D_4. Let Δ_1 be the unique residue of type D_4 containing c.

Notation 13.2. Let $\tilde{\Delta}$, $\tilde{\Sigma}$, $\tilde{c}$, $\tilde{\alpha}_1, \dots, \tilde{\alpha}_4$ and $\{\tilde{x}_\beta\}_{\beta \in \Phi_1}$ be the building, the apartment, the chamber, the set of roots and the coordinate system called Δ, Σ, c, $\alpha_1, \dots, \alpha_4$ and $\{x_\beta\}_{\beta \in \Phi_1}$ in 8.3 and 8.4 with $n = 4$. There exists an isomorphism ν from $\tilde{\Delta}$ to Δ_1 mapping $\tilde{\Sigma}$ to Σ_1, $\tilde{c}$ to c and the root $\tilde{\alpha}_i$ to $\alpha_{\pi(i)}$ for all $i \in [1, 6]$, where π is the map sending the sequence $1, 2, 3, 4$ to the sequence $6, 4, 5, 3$. Let $x_\beta = \nu^{-1} \cdot \tilde{x}_\beta \cdot \nu$. Then $\{x_\beta\}_{\beta \in \Phi_1}$ is a coordinate system for Δ_1. By 4.11, we can extend this coordinate system to a coordinate system $\{x_\beta\}_{\beta \in \Phi}$ for Δ.

The pair (W, S) is a Coxeter system of type D_6 and the pair (W_1, S_1) is a Coxeter system of type D_4. Let w_1 denote the longest element in W_1 with respect to the set of generators S_1 and let $w_0 = s_{\alpha_1} w_1$. By (8.7), (8.9) and (8.10), we have

$$w_0(\alpha_2) = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 \tag{13.3}$$

and

$$w_0(\alpha_i) = -\alpha_i$$

for all $i \in [1, 6]$ other than 2. By 4.16 and 8.11 with $n = 6$, there exists $\omega \in \{1, -1\}$ such that

$$\hat{\Omega}_1 := g_{w_0,1,\omega,-1,-1,-1,-1,\mathrm{id}} \tag{13.4}$$

is an involution, where $g_{w_0,1,\omega,-1,-1,-1,-1,\mathrm{id}}$ is as in 4.13.

Notation 13.5. Let $\eta_1,\dots,\eta_4$ and Q be as in 9.4 with $n=4$, let σ, K, etc., be as in 8.15, let ν and $\tilde{\Delta}$ be as in 13.2 and let $\tilde{\Omega}$ be the automorphism of $\tilde{\Delta}$ in (9.6) with $n=4$. We denote by Ω the automorphism $\nu^{-1}\cdot\tilde{\Omega}\cdot\nu$ of Δ_1. The automorphism Ω satisfies the identities in 9.7 with $n=4$ and with the roots $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ replaced by the roots $\alpha_6,\alpha_4,\alpha_5,\alpha_3$ of Φ_1 (in that order).

Notation 13.6. Let

$$\hat{\Omega} = g_{\lambda_1,\dots,\lambda_6,\sigma}\cdot\hat{\Omega}_1 = g_{w_0,\lambda_1,\omega\lambda_2,-\lambda_3,\dots,-\lambda_6,\sigma},$$

where $\lambda_1=\eta_1\eta_2\eta_3\eta_4$, $\lambda_2=1$, $\lambda_3=\eta_3^{-1}\eta_4^{-1}$, $\lambda_4=\eta_2^{-1}\eta_3$, $\lambda_5=\eta_3^{-1}\eta_4$, $\lambda_6=\eta_1^{-1}\eta_2$ and $\hat{\Omega}_1$ and $g_{\lambda_1,\dots,\lambda_6,\sigma}$ are as in 4.7(i). Note that

$$\lambda_1\lambda_2\lambda_3^2\lambda_4^2\lambda_5\lambda_6=1. \tag{13.7}$$

Theorem 13.8. *Suppose that $\eta_1\eta_2\eta_3\eta_4\notin N(E)$ and that the quadratic form Q defined in 13.5 is anisotropic and let Δ_0 be the unique residue of type $A_1\times D_4$ containing the chamber c. Then $\hat{\Omega}$ is a Galois involution of Δ stabilizing Δ_0 but not any proper residue of Δ_0.*

Proof. We have

$$x_{\alpha_1}(t)^{\hat{\Omega}} = x_{-\alpha_1}(\lambda_1\bar{t})$$

and

$$x_{\alpha_2}(t)^{\hat{\Omega}} = x_{w_0(\alpha_2)}(\omega\bar{t})$$

for all $t\in E$ as well as

$$x_{\alpha_i}(t)^{\hat{\Omega}} = x_{-\alpha_i}(-\lambda_i\bar{t}) \tag{13.9}$$

for all $t\in E$ and all $i\in[3,6]$. We also have

$$x_{w_0(\alpha_2)}(t)^{\hat{\Omega}_1} = x_{\alpha_2}(\omega t) \tag{13.10}$$

for all $t\in E$ since $\hat{\Omega}_1$ is an involution. Therefore

$$\begin{aligned}
x_{w_0(\alpha_2)}(t)^{\hat{\Omega}} &= x_{w_0(\alpha_2)}(\lambda_1\lambda_2\lambda_3^2\lambda_4^2\lambda_5\lambda_6\bar{t})^{\hat{\Omega}_1} && \text{by 4.7(ii) and (13.3)}\\
&= x_{w_0(\alpha_2)}(\bar{t})^{\hat{\Omega}_1} && \text{by (13.7)}\\
&= x_{\alpha_2}(\omega\bar{t}) && \text{by (13.10)}
\end{aligned}$$

for all $t\in E$. Hence $\hat{\Omega}^2$ centralizes U_{α_2}. Since $\lambda_i\in K$ for all $i\in[1,6]$ and $\hat{\Omega}_1^2=1$, it follows from 4.7(ii) that

$$x_{-\alpha_1}(t)^{\hat{\Omega}} = x_{-\alpha_1}(\lambda_1^{-1}\bar{t})^{\hat{\Omega}_1} = x_{\alpha_1}(\lambda_1^{-1}\bar{t})$$

and

$$x_{-\alpha_i}(t)^{\hat{\Omega}} = x_{-\alpha_i}(\lambda_i^{-1}\bar{t})^{\hat{\Omega}_1} = x_{\alpha_i}(-\lambda_i^{-1}\bar{t})$$

for all $t \in E$ and all $i \in [3,6]$. Therefore $\hat{\Omega}^2$ centralizes U_{α_i} for all $i \in [1,6]$. Thus $\hat{\Omega}$ is a Galois involution.

The involution $\hat{\Omega}$ induces the automorphism w_0 on Σ, and w_0 stabilizes $\Delta_0 \cap \Sigma$. Therefore $\hat{\Omega}$ stabilizes Δ_0.

Let P be the 1-panel containing c, let π_P be the restriction of the projection map proj_P to Δ_0, let π denote the restriction of the projection map proj_{Δ_1} to Δ_0 and let ζ denote the restriction of $\hat{\Omega}\cdot\pi$ to Δ_1. By 3.11, 9.7 and (13.9), ζ coincides with the automorphism Ω defined in 13.5.

Suppose that R is a residue of Δ_0 stabilized by $\hat{\Omega}$. By 9.5(iii), ζ does not stabilize any proper residues of Δ_1. Therefore the image of R under the projection map π is Δ_1. By 8.13, the image of Δ_0 under π_P is a projective line over E which can be coordinatized so that $\hat{\Omega}\cdot\pi_P$ is the map $t \mapsto \lambda_1\bar{t}^{-1}$. Since $\lambda_1 = \eta_1\cdots\eta_4 \notin N(E)$, this map has no fixed points. Therefore the image of R under π_P is P. Hence $R = \Delta_0$. Thus $\hat{\Omega}$ stabilizes no proper residues of Δ_0. □

Proposition 13.11. *Suppose the quadratic form Q in 13.5 is anisotropic and that $\eta_1\eta_2\eta_3\eta_4 \notin N(E)$. Then $\Delta^{\langle\hat{\Omega}\rangle}$ is a Moufang set with non-abelian root groups.*

Proof. By (13.8), $\hat{\Omega}$ is an involution. By 4.2(i) and (13.3), there exists $\delta \in \{1,-1\}$ such that

$$\left[x_{\alpha_2}(t), x_{w_0(\alpha_2)}(s)\right] = x_{\tilde{\alpha}}(\delta st)$$

for all $s,t \in E$. Setting $s = \delta$ and conjugating by $\hat{\Omega}$, we have

$$\begin{aligned} x_{\tilde{\alpha}}(t)^{\hat{\Omega}} &= \left[x_{w_0(\alpha_2)}(\omega\bar{t}), x_{\alpha_2}(\omega\delta)\right] \\ &= x_{\tilde{\alpha}}(-\bar{t}) \end{aligned}$$

for all $t \in E$. Let T be the trace of the extension E/K and let

$$X = \left\{(t,u) \in E^2 \mid \ T(u) + \omega\delta N(t) = 0\right\}.$$

It follows from (11.10) and (11.20) that for all $(t,u) \in X$, the element

$$g_{t,u} := x_{\alpha_2}(t)x_{w_0(\alpha_2)}(\omega\bar{t})x_{\tilde{\alpha}}(u)$$

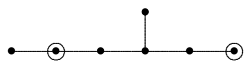
is centralized by $\hat{\Omega}$.

The roots of Σ cutting Δ_1 are the roots in $\Phi \cap \langle\alpha_1,\alpha_3,\ldots,\alpha_6\rangle$. All the other positive roots of Φ contain $\Delta_1 \cap \Sigma$. In particular, α_2, $w_0(\alpha_2)$ and $\tilde{\alpha}$ all contain $\Delta_1 \cap \Sigma$. The root group U of $\Delta^{\langle\hat{\Omega}\rangle}$ fixing the $\langle\hat{\Omega}\rangle$-chamber Δ_1 is isomorphic to the centralizer of $\hat{\Omega}$ in the group generated by all the positive roots of Φ containing $\Delta_1 \cap \Sigma$. For each $t \in E$, there exist $u_t \in E$ such that $(t, u_t) \in X$. Applying the identities [TW, 2.2], we see that

$$[g_{s,u_s}, g_{t,u_t}] = x_{\tilde{\alpha}}\big(\delta\omega(s\bar{t} - \bar{s}t)\big)$$

for all $s, t \in E$. Thus not all of the elements g_{t,u_t} commute with each other. Therefore the root group U is non-abelian. □

Theorem 13.12. *Let $\Lambda = (K, V, Q)$ be a quadratic space of type E_7. Then there exists a separable quadratic extension E/K such that Q_E is hyperbolic and for each such extension E/K, there exists a Galois involution Ω of the building $\Delta = \mathsf{E}_7(E)$ such that the Tits index of the group $\Gamma = \langle\Omega\rangle$ is*



and the fixed point building Δ^Γ is isomorphic to $\mathsf{B}_2^\xi(K, V, Q)$.

Proof. By 5.5, we can choose a separable quadratic extension E/K such that Q_E is hyperbolic and assume that $V = E^4$ and that there exists $\eta_1, \ldots, \eta_4 \in K$ such that

$$Q(u_1, \ldots, u_4) = \eta_1 N(u_1) + \cdots + \eta_4 N(u_4)$$

for all $(u_1, \ldots, u_6) \in V$, where N is the norm of the extension E/K, and

$$\eta_1\eta_2\eta_3\eta_4 \notin N(E).$$

Let σ be the non-trivial element in $\mathrm{Gal}(E/K)$, let $\Delta = \mathsf{E}_7(E)$, let Σ be an apartment of Δ and let c be a chamber of Σ. Let Φ be the root system and let $\alpha_1, \ldots, \alpha_7$ be as in [Bou, Plate VI]. We identify Φ with the set of roots of Σ as in 4.1 and choose a coordinate system $\{x_\beta\}_{\beta\in\Phi}$ for Δ. Let A be the unique subset of S spanning a subdiagram of Π of type $A_1 \times D_4$, let w_0 denote the longest element in the Coxeter group W_A with respect to the generating set A and let R denote the unique A-residue of Δ containing c. Let R_1 and R_2 be the unique residues of type D_6 and $A_1 \times D_5$ containing c, let R_3 be the unique residue of R_2 of type D_5 containing c and let ξ be the restriction of $\Omega \cdot \mathrm{proj}_{R_3}$ to R_3.

Let $\lambda_1, \ldots, \lambda_6$ be as in 13.6. We set $\kappa_1 = \eta_1$, $\kappa_2 = -\lambda_5$, $\kappa_3 = -\lambda_6$, $\kappa_4 = -\lambda_4$, $\kappa_5 = -\lambda_3$, $\kappa_6 = \delta\lambda_2$ and $\kappa_7 = \lambda_1$. We then set

$$\Omega = g_{w_0,\kappa_1,\ldots,\kappa_7,\sigma},$$

where $g_{w_0,\kappa_1,\ldots,\kappa_7,\sigma}$ is as in 4.13. Finally, we set $\Gamma = \langle\Omega\rangle$.

By 4.11 with R_3 in place of R and 13.5, we can assume that the coordinate system $\{x_\beta\}_{\beta\in\Phi}$ was chosen so that there are two isomorphisms, one from R_1 to the building Δ in 13.1 carrying the restriction of Ω to R_1 to the automorphism $\hat{\Omega}$ in 13.6 and the other from R_3 to the building Δ_0 in 10.3 with $n = 5$ carrying the map ξ to the automorphism Ω_0 defined in (10.2).

By 13.8, Ω^2 centralizes U_{α_i} for all $i \in [2,7]$ and R is a Γ-chamber. By 3.11 and 10.6, Ω^2 centralizes U_{α_1}. Thus Ω^2 centralizes U_{α_i} for all $i \in [1,7]$. Hence Ω is a Galois involution. By 6.5, therefore, Γ is a descent group of Δ. By 6.11 and 6.12(iii), Δ^Γ is a building of type B_2, and thus by 6.12(iv), Δ^Γ is a Moufang quadrangle. Let $\mathbb{M}_1$ and $\mathbb{M}_2$ be as in 5.16 applied to Δ^Γ. By 6.15, 10.4 and 13.11, one of these two Moufang sets is isomorphic to $\mathsf{B}_1^{\mathcal{Q}}(K, E^4, Q)$ and the other has non-abelian root groups. By 5.16(a), it follows that $\Delta^\Gamma \cong \mathsf{B}_2^{\mathcal{E}}(\Lambda)$. □

14. The quadrangles of type E_6

Our goal in this section is to prove 14.11.

Notation 14.1. Let $\Delta = \mathsf{A}_5(E)$, let Φ be the root system of type A_5, let $\alpha_1, \dots, \alpha_5$ and $\tilde{\alpha}$ be as in [Bou, Plate I], let S be the set of reflections s_{α_i} for $i \in [1,5]$, let $W = \langle S \rangle$, let $S_1 = \{s_{\alpha_2}, s_{\alpha_3}, s_{\alpha_4}\}$, let $W_1 = \langle S_1 \rangle$, let Φ_1 denote the root system $\langle \alpha_2, \alpha_3, \alpha_4 \rangle \cap \Phi$ of type D_3 and let Δ_1 denote the unique residue of type D_3 containing c.

Notation 14.2. Let $\tilde{\Delta}$, $\tilde{\Sigma}$, $\tilde{c}$, $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3$ and $\{\tilde{x}_\beta\}_{\beta\in\Phi_1}$ be the building, the apartment, the chamber, the set of roots and the coordinate system called Δ, Σ, c, $\alpha_1, \alpha_2, \alpha_3$ and $\{x_\beta\}_{\beta\in\Phi_1}$ in 8.3 and 8.4 with $n = 3$. There exists an isomorphism ν from $\tilde{\Delta}$ to Δ_1 mapping $\tilde{\Sigma}$ to Σ_1, $\tilde{c}$ to c and the root $\tilde{\alpha}_i$ to $\alpha_{\pi(i)}$ for all $i \in [1,6]$, where π is the map sending the sequence $1, 2, 3$ to the sequence $3, 2, 4$. Let $x_\beta = \nu^{-1} \cdot \tilde{x}_\beta \cdot \nu$ for all $\beta \in \Phi_1$. Then $\{x_\beta\}_{\beta\in\Phi_1}$ is a coordinate system for Δ_1. By 4.11, we can extend this coordinate system to a coordinate system $\{x_\beta\}_{\beta\in\Phi}$ for Δ.

The pair (W, S) is a Coxeter system of type A_5 and the pair (W_1, S_1) is a Coxeter system of type D_3. Let w_1 denote the longest element of W_1 with respect to the set of generators S_1.

We have $w_1 = (s_2 s_4 s_3)^2$, from which it follows that

$$w_1(\alpha_1) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$$

and

$$w_1(\alpha_5) = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5.$$

Now let π be as in 2.9 with Φ a root system of type A_5 and let $\hat{w} = \pi \cdot w_1$. Then

$$\hat{w}(\alpha_1) = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = \tilde{\alpha} - \alpha_1 \tag{14.3}$$

and

$$\hat{w}(\alpha_5) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \tilde{\alpha} - \alpha_5 \tag{14.4}$$

as well as $\hat{w}(\alpha_i) = -\alpha_i$ for all $i \in [2, 4]$.

By 4.16 and 7.2, there exist $\delta_1, \delta_5 \in \{1, -1\}$ such that

$$\hat{\Omega}_1 := g_{\hat{w},\delta_1,-1,-1,-1,\delta_5,\mathrm{id}} \tag{14.5}$$

is an involution, where $g_{\hat{w},\delta_1,-1,-1,-1,\delta_5,\mathrm{id}}$ is as in 4.13.

Notation 14.6. Let $\lambda_1 = \eta_1$, $\lambda_2 = \eta_2^{-1}\eta_3$, $\lambda_3 = \eta_1^{-1}\eta_2$, $\lambda_4 = \eta_2^{-1}\eta_3^{-1}$ and $\lambda_5 = \eta_2$, so

$$\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 = 1, \tag{14.7}$$

and let

$$\hat{\Omega} = g_{\lambda_1,\dots,\lambda_5,\sigma} \cdot \hat{\Omega}_1 = g_{\hat{w},\delta_1\lambda_1,-\lambda_2,-\lambda_3,-\lambda_4,\delta_5\lambda_5,\sigma},$$

where $\hat{\Omega}_1$, δ_1 and δ_5 are as in (14.5), σ is as in 8.15 and $g_{\lambda_1,\dots,\lambda_5,\sigma}$ is as in 4.7(i).

Notation 14.8. Let $\tilde{\Delta}$ and ν be as in 14.2, let $\tilde{\Omega}$ be the automorphism of $\tilde{\Delta}$ defined in (9.6) with $n = 3$ and η_1, η_2, η_3 as in 14.6 and let $\Omega = \nu^{-1} \cdot \tilde{\Omega} \cdot \nu$.

Theorem 14.9. *Suppose that the quadratic form Q defined in 9.4 is anisotropic. Let $\hat{\Omega}$ be as in 14.6 and let Δ_1 be the unique S_1-residue containing the chamber c. Then $\hat{\Omega}$ is a Galois involution of Δ stabilizing Δ_1 but not any proper residue of Δ_1.*

Proof. We have

$$x_{\alpha_1}^{\hat{\Omega}}(t) = x_{\hat{w}(\alpha_1)}(\delta_1\lambda_1\bar{t})$$

for all $t \in E$. Since $\hat{\Omega}_1$ is an involution, we have

$$x_{\hat{w}(\alpha_1)}(t)^{\hat{\Omega}_1} = x_{\alpha_1}(\delta_1 t)$$

for all $t \in E$. By 4.7(ii), therefore,

$$\begin{aligned} x_{\hat{w}(\alpha_1)}(t)^{\hat{\Omega}} &= x_{\hat{w}(\alpha_1)}(\lambda_2\lambda_3\lambda_4\lambda_5\bar{t})^{\hat{\Omega}_1} \\ &= x_{\alpha_1}(\delta_1\lambda_2\lambda_3\lambda_4\lambda_5\bar{t}) \end{aligned}$$

for all $t \in E$. By (14.7), therefore, $\hat{\Omega}^2$ centralizes U_{α_1}. Similarly, $\hat{\Omega}^2$ centralizes U_{α_5}.

Since $\hat{w}_1$ stabilizes $\Sigma \cap \Delta_1$, $\hat{\Omega}$ stabilizes Δ_1. By 9.7, the restriction of $\hat{\Omega}$ to Δ_1 is the automorphism Ω defined in 14.8. Since Ω is an involution, it follows that $\hat{\Omega}^2$ centralizes U_i for all $i \in [2,4]$ (and thus for all $i \in [1,5]$ by the conclusion of the previous paragraph). We conclude that $\hat{\Omega}$ is a Galois involution and that by 9.5(iii), $\hat{\Omega}$ does not stabilize any proper residues of Δ_1. □

Proposition 14.10. *Suppose the quadratic form Q in 9.4 is anisotropic. Then $\Delta^{\langle\hat{\Omega}\rangle}$ is a Moufang set with non-abelian root groups.*

Proof. By (14.3), we have $\tilde{\alpha} = \hat{w}(\alpha_1) + \alpha_1$. Hence there exists $\omega \in \{1, -1\}$ such that

$$\big[x_{\alpha_1}(t), x_{\hat{w}(\alpha_1)}(s)\big] = x_{\tilde{\alpha}}(\omega s t)$$

for all $s, t \in E$. Setting $s = \omega$ and conjugating by $\hat{\Omega}$, we deduce that

$$\begin{aligned} x_{\tilde{\alpha}}(t)^{\hat{\Omega}} &= \big[x_{\hat{w}(\alpha_1)}(\delta_1\lambda_1\bar{t}), x_{\alpha_1}(\delta_1\lambda_2\cdots\lambda_5\omega)\big] \\ &= x_{\tilde{\alpha}}(-\bar{t}) \end{aligned}$$

for all $t \in E$. Let T be the trace of the extension E/K and let

$$X = \big\{(t,u) \in E^2 \mid \ T(u) + \omega\delta_1\lambda_1 N(t) = 0\big\}.$$

For all $(t,u) \in X$, the element

$$g_{t,u} := x_{\alpha_1}(t) x_{\hat{w}(\alpha_1)}(\delta_1\lambda_1\bar{t}) x_{\tilde{\alpha}}(u)$$

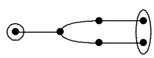
is centralized by $\hat{\Omega}$.

The roots of Σ cutting Δ_1 are the roots in $\Phi \cap \langle\alpha_2, \alpha_3, \alpha_4\rangle$. All the other positive roots of Φ contain $\Delta_1 \cap \Sigma$. In particular, α_1, $\hat{w}(\alpha_1)$ and $\tilde{\alpha}$ all contain $\Delta_1 \cap \Sigma$. The root group U of $\Delta^{\langle\hat{\Omega}\rangle}$ fixing the $\langle\Omega_0\rangle$-chamber Δ_1 is isomorphic to the centralizer of $\hat{\Omega}$ in the group generated by all the positive roots of Φ containing $\Delta_1 \cap \Sigma$. For each $t \in E$, there exists $u_t \in E$ such that $(t, u_t) \in X$. Applying the identities [TW, 2.2], we see that

$$[g_{s,u_s}, g_{t,u_t}] = x_{\tilde{\alpha}}\big(\omega\delta_1\lambda_1(s\bar{t} - \bar{s}t)\big)$$

for all $s, t \in E$. Thus not all of the elements g_{t,u_t} commute with each other. Therefore the root group U is non-abelian. □

Theorem 14.11. *Let (K, V, Q) be a quadratic space of type E_6. Then there exists a separable quadratic extension E/K such that Q_E is hyperbolic and for each such extension E/K, there exists a Galois involution Ω of the building $\Delta = \mathsf{E}_6(E)$ such that the Tits index of the group $\Gamma := \langle\Omega\rangle$ is*



and the fixed point building Δ^Γ *is isomorphic to* $\mathsf{B}_2^{\mathcal{E}}(K,V,Q)$.

Proof. By 5.5, we can choose a separable quadratic extension E/K such that Q_E is hyperbolic and assume that $V = E^3$ and that for some $\eta_1,\eta_2,\eta_3 \in K$,

$$Q(u_1,u_2,u_3) = \eta_1 N(u_1) + \eta_2 N(u_2) + \eta_3 N(u_3)$$

for all $(u_1,u_2,u_3) \in V$, where N is the norm of the extension E/K.

Let $\Delta = \mathsf{E}_6(E)$, let Σ be an apartment of Δ and let c be a chamber of Σ. Let Φ be the root system of type E_6 and let $\alpha_1,\dots,\alpha_6$ be as in [Bou, Plate V]. We identify Φ with the set of roots of Σ and $\mathrm{Aut}(\Phi)$ with a subgroup of $\mathrm{Aut}(\Sigma)$ as in 4.1 and choose a coordinate system $\{x_\beta\}_{\beta\in\Phi}$ for Δ. Let A be the unique subset of S spanning a subdiagram of Π of type D_3 that is stabilized by $\mathrm{Aut}(\Pi)$, let w_1 denote the longest element in the Coxeter group W_A with respect to the generating set A, let R denote the unique A-residue of Δ containing c, let R_1 be the unique residue of type A_5 containing R and let R_2 be the unique residue of type D_4 containing R. Let π be as in 2.9 and let $\hat{w} = \pi w_1$.

Let $\lambda_1,\dots,\lambda_5,\delta_1,\delta_5$ be as in 14.6. We set $\kappa_1 = \delta_1\lambda_1$, $\kappa_2 = \eta_1$, $\kappa_3 = -\lambda_2$, $\kappa_4 = -\lambda_3$, $\kappa_5 = -\lambda_4$ and $\kappa_6 = \delta_5\lambda_5$. We then set

$$\Omega = g_{\hat{w},\kappa_1,\dots,\kappa_6,\sigma},$$

where σ is the non-trivial element in $\mathrm{Gal}(E/K)$ and $g_{\hat{w},\kappa_1,\dots,\kappa_6,\sigma}$ is as in 4.13. Finally, we set $\Gamma := \langle\Omega\rangle$.

By 4.11 with R_2 in place of R and 14.2, we can assume that the coordinate system $\{x_\beta\}_{\beta\in\Phi}$ was chosen so that there are two isomorphisms, one from R_1 to the building Δ in 14.1 carrying the automorphism $\hat{\Omega}$ in 14.6 to the restriction of Ω to R_1 and the other from R_2 to the building Δ_0 in 10.3 with $n = 3$ carrying the automorphism Ω_0 defined in (10.2) to the restriction of Ω to R_2. By 10.6, Ω^2 centralizes U_{α_i} for all $i \in [2,5]$ and R is a Γ-chamber. By 14.9, Ω^2 also centralizes U_{α_1} and U_{α_6}. Thus Ω is a non-type-preserving Galois involution. By 6.5, therefore, Γ is a descent group of Δ. By 6.11 and 6.12(iii), Δ^Γ is a building of type B_2, and thus by 6.12(iv), Δ^Γ is a Moufang quadrangle. Let $\mathbb{M}_1$ and $\mathbb{M}_2$ be as in 5.16 applied to Δ^Γ. By 6.15, 10.4 and 14.10, one of these two Moufang sets is isomorphic to $\mathsf{B}_1^{\mathcal{Q}}(K,E^3,Q)$ and the other has non-abelian root groups. By 5.16(a), it follows that $\Delta^\Gamma \cong \mathsf{B}_2^{\mathcal{E}}(\Lambda)$. □

15. Non-pseudo-split buildings of type F_4

In this section, we construct all buildings of type F_4 that are not pseudo-split (as defined in 15.3) and the exceptional buildings of type C_3 (see [Tit2, 9.1–9.3]) as the fixed point buildings of Galois involutions of buildings of type E_6, E_7 and E_8. Our main result is 15.4.

Theorem 15.1. *Let Δ be a simply laced and split building of type Π, let S be the vertex set of Π, let $J = S\backslash\{i\}$ for some $i \in S$, let Π_J be the subdiagram of Π spanned by J, let Δ_1 be a J-residue, let Ω_1 be a Galois involution of Δ_1 and let (Π_J, Θ_1, A) be the Tits index of $\Gamma_1 := \langle\Omega_1\rangle$. Suppose that i is adjacent in Π to a unique element of J. Then there exist an extension of Θ_1 to an automorphism Θ of Π and an extension of Ω_1 to a Galois involution Ω of Δ such that the Tits index of $\Gamma := \langle\Omega\rangle$ is (Π, Θ, A).*

Proof. By [MPW, 24.36], Ω_1 has an extension to an involution Ω of Δ and by [MPW, 29.28], Ω is a Galois involution. By 6.5, therefore, $\Gamma := \langle\Omega\rangle$ is a descent group of Δ. Let Θ denote the image of Γ in $\mathrm{Aut}(\Pi)$. The restriction of Θ to Π_J is Θ_1 and by 6.12(ii), a Γ_1-chamber is also a Γ-chamber. Thus (Π, Θ, A) is the Tits index of Γ. □

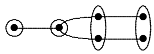
Buildings of type F_4 are all of the form $\mathsf{F}_4(D, K)$, where (D, K) is a composition algebra; see [Tit2, Thm. 10.2] and [Wei2, 30.14 and 30.15].

Notation 15.2. Let $\Lambda = (D, K)$ be a composition algebra. As in [Wei2, 30.17], we say that Λ is *of type* (i) if D/K is an inseparable extension in characteristic 2 such that $D^2 \subset K$ but D^2 equals neither K nor K^2. We say that Λ is *of type* (ii) if $D = K$ is a field. We say that Λ is *of type* (iii) if D/K is a separable quadratic extension fields; its *standard involution* in this case is the unique non-trivial element in $\mathrm{Gal}(D/K)$. We say that Λ is *of type* (iv) if D is a quaternion division algebra over K and we say that Λ is *of type* (v) if D is an octonion division algebra over K. In cases (iv) and (v), the standard involution σ is as defined in [TW, 9.6 and 9.10]. In case (v), the triple (D, K, σ) is an *honorary involutory set* as defined in [TW, 38.11] and the Moufang quadrangle $\mathsf{B}_2^{\mathcal{I}}(D, K, \sigma)$, which appears in 15.4(iii) below, is defined in [TW, 38.13].

Definition 15.3. A building $\mathsf{F}_4(D, K)$ is *split*, respectively, *pseudo-split*, if the composition algebra (D, K) is of type (ii), respectively, of type (i) or (ii), as defined in 15.2.

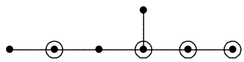
Theorem 15.4. *Let D/K be composition algebra of type (x) for $x = \mathrm{iii}$, iv or v, let σ be the standard involution of D/K and let E be a subfield of D containing K such that E/K is a separable quadratic extension. Then the following hold:*

(i) *If* $x = \text{iii}$, *then there exists a Galois involution* Ω *of the building* $\Delta = \mathsf{E}_6(E)$ *such that the Tits index of the group* $\Gamma := \langle\Omega\rangle$ *is*



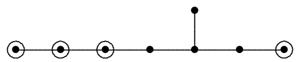
and the fixed point building Δ^Γ *is isomorphic to* $\mathsf{F}_4(D/K)$.

(ii) *If* $x = \text{iv}$, *then there exists a Galois involution* Ω *of the building* $\Delta = \mathsf{E}_7(E)$ *such that the Tits index of the group* $\Gamma := \langle\Omega\rangle$ *is*

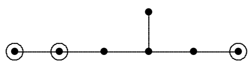


and the fixed point building Δ^Γ *is isomorphic to* $\mathsf{F}_4(D/K)$.

(iii) *If* $x = \text{v}$, *then there exists a Galois involution* Ω *of the building* $\Delta = \mathsf{E}_8(E)$ *such that the Tits index of the group* $\Gamma := \langle\Omega\rangle$ *is*



and the fixed point building Δ^Γ *is isomorphic to* $\mathsf{F}_4(D/K)$ *and there exists a residue* Δ_1 *of type* E_7 *of* Δ *stabilized by* Ω *such that the restriction* Γ_1 *of* Γ *to* Δ_1 *has Tits index*



and the fixed point building $\Delta_1^{\Gamma_1}$ *is isomorphic to* $\mathsf{C}_3^{\mathcal{I}}(\Lambda)$, *where* Λ *is the honorary involutory set* (D, K, σ).

Proof. Suppose that $x = \text{iii}$, let $\Delta = \mathsf{E}_6(E)$ and let Δ_1 be a residue of type A_5. We identify Δ_1 with the building Δ in §7 with $n = 5$ and let Ω_1 be the non-type-preserving Galois involution of Δ_1 obtained by composing the involution in 7.3 with the involution which maps $x_{\alpha_i}(t)$ to $x_{\alpha_i}(t^\sigma)$ for all $i \in [1, 5]$ and all $t \in E$. Next let Ω be a Galois involution of Δ obtained by applying 15.1 to Ω_1. By 7.4, Ω_1 fixes a chamber of Δ. It follows that the Tits index of $\Gamma = \langle\Omega\rangle$ is as in (i). By 6.11, therefore, Δ^Γ is a building of type F_4. Let J be the unique subset of S spanning a subdiagram of Π of type A_3 that is stabilized by the non-trivial automorphism of Π, let R be a J-residue stabilized by Ω and let Γ_R denote the restriction of Γ to R. By 8.17, we have

$$R^{\Gamma_R} \cong \mathsf{B}_2^{\mathcal{Q}}(K, E, N).$$

By [MPW, 22.39], R^{Γ_R} is a residue of Δ^Γ. If (E', K') is a composition algebra with norm N' such that

$$\mathsf{B}_2^{\mathcal{Q}}(K, E, N) \cong \mathsf{B}_2^{\mathcal{Q}}(K', E', N'),$$

then by [TW, 20.28 and 35.7], there is an isomorphism from E to E' mapping K to K'. Therefore

$$\Delta^\Gamma \cong \mathsf{F}_4(E, K).$$

Thus (i) holds.

Now suppose that $x = \text{iv}$, let $\Delta = \mathsf{E}_7(E)$ and let Δ_1 be a residue of type D_6. We identify Δ_1 with the building Δ in 8.3 with $n = 6$ and let Ω_1 be a Galois involution of Δ_1 obtained by applying 8.16. The Tits index of $\langle\Omega_1\rangle$ is as in (ii) with the rightmost vertex deleted. We can thus apply 15.1 to Ω_1 to obtain a Galois involution Ω of Δ such that the Tits index of $\Gamma := \langle\Omega\rangle$ is as in (ii). Therefore Δ^Γ is a building of type F_4 (by 6.11). By [MPW, 22.39] and 8.16, Δ^Γ has residues isomorphic to $\mathsf{A}_2(D)$. It follows from [TW, 35.6] that

$$\Delta^\Gamma \cong \mathsf{F}_4(D, K).$$

Thus (ii) holds.

Suppose, finally, that $x = \text{v}$. Let Ω_1 be the Galois involution of $\mathsf{E}_6(E)$ in 12.11. Applying 15.1 once and then a second time, we obtain extensions of Ω_1 to Galois involutions of $\mathsf{E}_7(E)$ and then of $\mathsf{E}_8(E)$ generating groups whose Tits indices and fixed point buildings are as in (iii). $\square$

16. Pseudo-split buildings of type F_4

The results of this section will be required in §17. They are completely parallel to the results in §4, but we formulate them separately for the sake of clarity.

Notation 16.1. Let $\Delta = \mathsf{F}_4(L, E)$, where L/E is a field extension such that $\text{char}(E) = 2$ and $L^2 \subset E$. We assume that $L \neq E$ (but we do not assume that L/E is finite dimensional). Let Φ be a root system of type F_4, let Σ be an apartment of Δ and let c be a chamber of Σ. Let $\alpha_1, \ldots, \alpha_4$ be as in [Bou, Plate VIII], let S be the set of reflections s_{α_i} for $i \in [1, 4]$ and let $W = \langle S\rangle$ be the Weyl group of Φ. We identify Φ with the set of roots of Σ and $\text{Aut}(\Phi)$ with a subgroup of $\text{Aut}(\Sigma)$ as in 4.1 so that $\alpha_1, \ldots, \alpha_4$ are the four roots of Σ containing c but not some chamber of Σ adjacent to c.

Theorem 16.2. *There exists a collection of isomorphisms $x_\beta\colon E \to U_\beta$, one for each long root β of Φ, and a collection of isomorphisms $x_\beta\colon L \to U_\beta$, one for each short root, such that for all $\alpha, \beta \in \Phi$ such that $\alpha \neq \pm\beta$ and for all $s \in E$ if α is long, all $s \in L$ if α is short, all $t \in E$ if β is long and all $t \in L$ if β is short, the following hold:*

(i) $[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(st)$ *if* α *and* β *have the same length and* $\alpha+\beta \in \Phi$.

(ii) $[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(st)x_{\alpha+2\beta}(st^2)$ *if* α *is long,* β *is short and* $\alpha+\beta \in \Phi$, *in which case also* $\alpha+2\beta \in \Phi$.

(iii) $[x_\alpha(s), x_\beta(t)] = 1$ *if* α *is orthogonal to* β.

(iv) $U_\alpha^{x_{-\alpha}(s)} = U_{-\alpha}^{x_\alpha(s^{-1})}$ *if* $s \neq 0$.

Proof. Assertions (i)–(iii) hold by [Ste, (R2) on p. 30] (or [Car, Thm. 5.2.2]) and [Tit2, 10.3.2]. Assertion (iv) holds by [Ste, (R7) on p. 30 and Lemma 59 on p. 160]. □

Remark 16.3. We call a set $\{x_\beta\}_{\beta\in\Phi}$ satisfying the four conditions in 16.2 a *coordinate system* for Δ. The assertions 4.6, 4.9 (with both τ and τ' identically equal to 1) and 4.11 all hold with the word "equivalent" replaced by "equal" in our present setting and with virtually the same proofs (but without concerns over minus signs since we are now in characteristic 2).

From now on we fix a coordinate system $\{x_\beta\}_{\beta\in\Phi}$ for Δ.

Theorem 16.4. *Let* $\gamma \in \mathrm{Aut}(\Phi)$, *let* λ_1, λ_2 *be non-zero elements of* E, *let* λ_3, λ_4 *be non-zero elements of* L *and let* σ *be an element of* $\mathrm{Aut}(L)$ *stabilizing* E. *Then the following hold:*

(i) *There exists a unique automorphism*

$$g = g_{\gamma,\lambda_1,\lambda_2,\lambda_3,\lambda_4,\sigma}$$

of Δ *that stabilizes the apartment* Σ *such that*

$$x_{\alpha_i}(t)^g = x_{\gamma(\alpha_i)}(\lambda_i t^\sigma)$$

for all $i \in [1,2]$ *and all* $t \in E$ *and*

$$x_{\alpha_i}(v)^g = x_{\gamma(\alpha_i)}(\lambda_i v^\sigma)$$

for all $i \in [3,4]$ *and all* $v \in L$.

(ii) *If*

$$\beta = \sum_{i=1}^4 c_i\alpha_i \in \Phi,$$

then

$$x_\beta(t)^g = x_{\gamma(\beta)}(\lambda_\beta t^\sigma)$$

for all $t \in E$ *if* β *is long, respectively, for all* $t \in L$ *if* β *is short, where*

$$\lambda_\beta = \prod_{i=1}^4 \lambda_i^{c_i}.$$

Proof. The existence assertion in (i) holds by [Ste, Lemma 58 on p. 158] (and the existence of field automorphisms) applied to $F_4(L)$ and restriction of scalars to E in the long root groups; uniqueness holds by [Weil, 9.7]. Assertion (ii) follows by induction from 16.2(i)–(ii) and [Hum, §10.2, Cor. to Lemma A] once it is established that it holds for $\beta = -\alpha_i$ for all $i \in [1,4]$. This can be done exactly as in the proof of 4.7(ii). □

Definition 16.5. A *Galois involution* of Δ is an element of order 2 in the coset $g_{\lambda_1,\ldots,\lambda_4,\sigma}G^\dagger$ for some $\lambda_1,\ldots,\lambda_4,\sigma$ with $\sigma \neq 1$, where $G^\dagger$ is as in 3.1. This is a special case of the notion of a Galois involution of an arbitrary Moufang building given in [MPW, 31.1].

Theorem 16.6. *If Ω is an isotropic Galois involution of Δ, then $\Gamma := \langle\Omega\rangle$ is a descent group of Δ.*

Proof. This is a special case of [MPW, 32.27]. □

17. The quadrangles of type F_4

In this section we construct the Moufang quadrangles of type F_4 as fixed point buildings of Galois involutions of pseudo-split buildings of type F_4; see 15.3 and 17.14. Our construction is essentially the same as the construction given in [MM1] except that we construct the initial anisotropic Galois involution of a pseudo-split Moufang quadrangle and verify that it is anisotropic in a simpler fashion.

Notation 17.1. Let L/E be as in 16.1, let M denote the direct sum of six copies of E and let $V = M \oplus L$, which we think of as a vector space over E. Let

$$\mathcal{B} = \{e_1, e_2, e_3, f_1, f_2, f_3\}$$

be a basis of the subspace $\{(u,0) \mid u \in M\}$ of V, let L be identified with its image under the map $v \mapsto (0,v) \in L$ and let $q\colon V \to E$ be the quadratic form given by

$$q\big(x_1e_1 + y_1f_1 + x_2e_2 + y_2f_2 + x_3e_3 + y_3f_3 + v\big) = x_1y_1 + x_2y_2 + x_3y_3 + v^2$$

for all $x_1,\ldots,y_3 \in E$ and all $v \in L$.

Notation 17.2. Let Δ_0 denote the building of type B_3 whose chambers are the maximal flags of subspaces of V that are totally isotropic with respect to q and let q_0 denote the restriction of q to $L = (0,L) \subset V$. Thus q_0 is anisotropic and totally singular and by 3.4,

$$\Delta_0 \cong \mathsf{B}_3^{\mathcal{Q}}(E, L, q_0).$$

Notation 17.3. For each ordered pair (i,j) of distinct integers i,j in the interval $[1,3]$ and each $t\in E$, let $x_{ij}(t)$ denote unique element of $\mathrm{O}(q)$ that sends e_j to e_j+te_i and f_i to f_i+tf_j, fixes all other elements of $\mathcal{B}$ and acts trivially on L. For each unordered pair $\{i,j\}$ of distinct integers i,j in $[1,3]$ and each $t\in E$, let $y_{ij}(t)$ denote the unique element of $\mathrm{O}(q)$ that sends f_j to f_j+te_i and f_i to f_i+te_j, fixes all other elements of $\mathcal{B}$ and acts trivially on L and let $z_{ij}(t)$ denote the unique element of $\mathrm{O}(q)$ that sends e_j to e_j+tf_i and e_i to e_i+tf_j, fixes all other elements of $\mathcal{B}$ and acts trivially on L. For each $i\in[1,3]$ and each $v\in L$, let $x_i(v)$ denote the unique element of $\mathrm{O}(q)$ that maps f_i to $f_i+v^2e_i+v$, fixes all other elements of $\mathcal{B}$ and acts trivially on L and let $y_i(v)$ denote the unique element of $\mathrm{O}(q)$ that maps e_i to $e_i+v^2f_i+v$, fixes all other elements of $\mathcal{B}$ and acts trivially on L.

Remark 17.4. Let Σ_0 be the apartment of Δ_0 whose chambers contain only subspaces spanned by subsets of $\mathcal{B}$. Let Φ_1 denote a root system of type B_3 and let $\alpha_1,\alpha_2,\alpha_3$ and $\varepsilon_1,\varepsilon_2,\varepsilon_3$ be as in [Bou, Plate II] with $n=3$, so that $\alpha_1=\varepsilon_1-\varepsilon_2$, $\alpha_2=\varepsilon_2-\varepsilon_3$ and $\alpha_3=\varepsilon_3$. For each $\beta\in\Phi_1$, we set $u_\beta=x_{ij}$ if $\beta=\varepsilon_i-\varepsilon_j$ for some $i,j\in[1,3]$, $u_\beta=y_{ij}$ if $\beta=\varepsilon_i+\varepsilon_j$ for some $i,j\in[1,3]$, $u_\beta=z_{ij}$ if $\beta=-\varepsilon_i-\varepsilon_j$ for some $i,j\in[1,3]$, $u_\beta=x_i$ if $\beta=\varepsilon_i$ for some $i\in[1,3]$ and $u_\beta=y_i$ if $\beta=-\varepsilon_i$ for some $i\in[1,3]$, where x_{ij}, y_{ij}, etc. are as in 17.3. Then $u_\beta(E)$ for β long and $u_\beta(L)$ for β short are root groups of Δ_0 and $\{u_\beta\}_{\beta\in\Phi_1}$ is a coordinate system for Δ_0.

Notation 17.5. Let σ be an involution in $\mathrm{Aut}(L)$ stabilizing E, let $F=\mathrm{Fix}_L(\sigma)$ and let $K=\mathrm{Fix}_E(\sigma)$. We will usually write $\overline{x}$ in place of x^σ for $x\in L$. Let N be the norm of the extension L/F. Thus F/K is a purely inseparable extension such that $F^2\subset K$ and the restriction of N to E is the norm of the extension E/K.

Notation 17.6. Let η_1,η_2 be non-zero elements of K, let $T=E\oplus E\oplus F$ considered as a vector space over K, let $Q_0\colon T\to K$ denote the quadratic form over K given by

$$Q_0(y_1,y_2,u)=\eta_1N(y_1)+\eta_2N(y_2)+u^2$$

for all $(y_1,y_2,u)\in T$ and let $Q\colon K\oplus K\oplus T\to K$ denote the quadratic form over K given by

$$Q(s,t,z)=st+Q_0(z)$$

for all $(s,t,z)\in T$.

Proposition 17.7. *Let V, $\mathcal{B}$, q, etc., be as in 17.1, let V_0 denote the subspace spanned by $\{e_2, e_3, f_2, f_3\} \cup L$, let $q_0\colon V_0 \to E$ denote the restriction of q to V_0, let $x \mapsto \overline{x}$ and F be as in 17.5, let η_1, η_2 and Q and Q_0 be as 17.6 and let $\Omega = \Omega_{\eta_1,\eta_2}$ be the σ-linear automorphism of V given by*

$$\Omega\Big(\sum_{i=1}^{3}(x_i e_i + y_i f_i) + v\Big) = \overline{x_1}e_1 + \overline{y_1}f_1 + \eta_1\overline{y_2}e_2 + \eta_1^{-1}\overline{x_2}f_2 + \eta_2\overline{y_3}e_3 + \eta_2^{-1}\overline{x_3}f_3 + \overline{v}$$

for all $x_1, x_2, x_3, x_1, y_2, y_3 \in E$ and all $v \in L$. Then the following hold:

(i) *$q(\Omega(x)) = \overline{q(x)}$ for all $x \in V$ and $\Omega^2 = 1$.*

(ii) *$q \cong Q \otimes_K E$.*

(iii) *If the quadratic form Q_0 is anisotropic, then there are no non-zero Ω-invariant subspaces of V_0 that are totally isotropic with respect to q_0.*

Proof. Assertion (i) is clear and assertion (ii) follows from 9.3. Suppose that U is a non-zero totally isotropic Ω-invariant subspace of V_0. Thus $q(v) = 0$ for all $v \in U$. Let u be a non-zero element of U. The sum $v := u + \Omega(u)$ is fixed by Ω. Replacing u by tu for some $t \in E \backslash F$ if necessary, we can assume that v is non-zero. We have

$$v = x_2e_2 + y_2f_2 + x_3e_3 + y_3f_3 + s$$

for some $x_2, x_3, y_2, y_3 \in E$ and some $s \in L$ not all zero. Since v is fixed by Ω, we have $x_i = \eta_{i-1}\overline{y_i}$ for $i \in [2,3]$ and $\overline{s} = s$. Therefore the elements y_2, y_3, s are not all zero, $s \in F$ and

$$Q_0(y_2, y_3, s) = \eta_1 y_2\overline{y_2} + \eta_2 y_3\overline{y_3} + s^2 = q(v) = 0.$$

Thus (iii) holds. □

Notation 17.8. Let Δ, Σ, c, Φ, $\alpha_1, \dots, \alpha_4$, (W, S), the identification of Φ with the set of roots of Σ and the identification of $\mathrm{Aut}(\Phi)$ with a subgroup of $\mathrm{Aut}(\Sigma)$ be as in 16.1. Let $\{x_\beta\}_{\beta\in\Phi}$ be as 16.2, let Δ_1 denote the unique $\{s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_3}\}$-residue of Δ containing c, let Σ_1 denote the apartment $\Sigma \cap \Delta_1$ of Δ_1 and let Φ_1 denote the root system $\langle\alpha_1, \alpha_2, \alpha_3\rangle \cap \Phi$ of type B_3, which we think of as the root system Φ_1 in 17.4. There exists an isomorphism ν from the building Δ_0 defined in 17.2 to Δ_1 mapping Σ_0 to Σ_1 and sending each root $\beta \in \Phi_1 \subset \Phi$ of Σ_0 to the root $\beta \cap \Sigma_1$ of Σ_1. Let $\{u_\beta\}_{\beta\in\Phi_1}$ be as in 17.4 and let $x_\beta = \nu^{-1} \cdot u_\beta \cdot \nu$ for each $\beta \in \Phi_1$. Then $\{x_\beta\}_{\beta\in\Phi_1}$ is a coordinate system for Δ_1 and by 16.3, it extends to a coordinate system $\{x_\beta\}_{\beta\in\Phi}$ for Δ. We set $\Omega_0 = \nu^{-1} \cdot \Omega \cdot \nu$, where $\Omega = \Omega_{\eta_1,\eta_2}$ is as in 17.7.

Notation 17.9. Let w_1 be the longest element in the Coxeter group W_J with respect to the set of generators $J := \{s_{\alpha_2}, s_{\alpha_3}\}$. Thus $w_1 = (s_{\alpha_2}s_{\alpha_3})^2$, from which it follows that

$$w_1(\alpha_1) = \alpha_1 + 2\alpha_2 + 2\alpha_3$$

and

$$w_1(\alpha_4) = \alpha_2 + 2\alpha_3 + \alpha_4,$$

as well as $w_1(\alpha_i) = -\alpha_i$ for both $i \in \{2, 3\}$.

Proposition 17.10. *Let $\{x_\beta\}_{\phi\in\Phi}$ and Ω_0 be as in 17.8. Then*

$$x_{\alpha_i}(t)^{\Omega_0} = x_{w_1(\alpha_i)}(\lambda_i \bar{t})$$

for $i \in [1, 2]$ and all $t \in E$ and

$$x_{\alpha_3}(v)^{\Omega_0} = x_{w_1(\alpha_3)}(\lambda_i \bar{v})$$

for all $v \in L$, where $\lambda_1 = \eta_1$, $\lambda_2 = \eta_1^{-1}\eta_2$ and $\lambda_3 = \eta_2^{-1}$.

Proof. This follows from 17.4, 17.7, 17.9 and some computation. □

Notation 17.11. Let Δ_2 be the unique $\{s_{\alpha_2}, s_{\alpha_3}\}$-residue of Δ_1 containing c.

Theorem 17.12. *Suppose that $\eta_1\eta_2 = \lambda_4^2$ for some $\lambda_4 \in F$ and that the quadratic from Q_0 in 17.6 is anisotropic. Let*

$$\hat{\Omega} = g_{w_1,\lambda_1,\lambda_2,\lambda_3,\lambda_4,\sigma}$$

be as in 16.4(i) with σ as in 17.5, w_1 as in 17.9 and $\lambda_1, \lambda_2, \lambda_3$ as in 17.10, and let Δ_2 be as in 17.11. Then $\hat{\Omega}$ is a Galois involution stabilizing Δ_2 but no proper residue of Δ_2.

Proof. Since w_1 stabilizes $\Delta_2 \cap \Sigma$, $\hat{\Omega}$ stabilizes Δ_2. By 16.4(ii) and 17.9, we have

$$\begin{aligned} x_{\alpha_4}(v)^{\hat{\Omega}^2} &= x_{w_1(\alpha_4)}(\lambda_4 \bar{v})^{\hat{\Omega}} \\ &= x_{\alpha_4}(\lambda_2\lambda_3^2\lambda_4^2 v) = x_{\alpha_4}(v) \end{aligned}$$

for all $v \in L$. By 16.4(i) and 17.10, the restriction of $\hat{\Omega}$ to Δ_1 coincides with Ω_0. Since Ω_0 is an involution, we conclude that $\hat{\Omega}^2$ centralizes U_{α_i} for all $i \in [1, 4]$. Therefore $\hat{\Omega}$ is a Galois involution and by 17.7(iii), $\hat{\Omega}$ does not stabilize any proper residues of Δ_2. □

Proposition 17.13. *Suppose that the quadratic form Q_0 in 17.6 is anisotropic and that $\eta_1\eta_2 \in F^2$. Let $\hat{\Omega}$ be as in 17.12, let $\Gamma = \langle\hat{\Omega}\rangle$, let Δ_1 and Δ_2 be as in 17.8 and 17.11 and let R be the Γ-panel containing Δ_2 other than Δ_1. Then*

$$\Delta_1^\Gamma \cong \mathsf{B}_1^{\mathcal{Q}}(K, E \oplus E \oplus F, Q_0)$$

and

$$R^\Gamma \cong \mathsf{B}_1^{\mathcal{Q}}(F, M, \hat{Q})$$

for some anisotropic quadratic space $(F, M, \hat{Q})$ defined over F whose defect is non-trivial and has co-dimension 4.

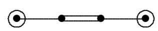
Proof. First note that by 17.12, the restrictions of $\hat{\Omega}$ to Δ_1 and to R are both Galois involutions. Let V, q and Ω be as in 17.7 and let $\hat{V} = \mathrm{Fix}_V(\Omega)$. It follows from [MPW, 2.40] (as in the proof of 10.4) that the map $W \mapsto W \cap \hat{V}$ is an inclusion- and dimension-preserving bijection from the set of all Ω-invariant subspaces of V to the set of all subspaces of $\hat{V}$, and an Ω-invariant subspace W of V is totally isotropic with respect to q if and only if $W \cap \hat{V}$ is totally isotropic with respect to Q. Since Q_0 is anisotropic, the first claim holds by 3.4. Since

$$R \cong \mathsf{B}_3^{\mathcal{Q}}(L, E^{1/2}, x \mapsto x^2),$$

the second claim holds by [MPW, 35.13]. □

In the following EF denotes the composite of the fields E and F. Thus EF/E is an extension such that $(EF)^2 \subset E$.

Theorem 17.14. *Let (K, V, φ) be a quadratic space of type F_4 and let F be as in 5.9. Then there exists a separable quadratic extension E/K such that φ_E is pseudo-split and for each such extension E/K, there exists a Galois involution Ω of the building $\Delta = \mathsf{F}_4(EF/E)$ such that the Tits index of the group $\Gamma = \langle\Omega\rangle$ is*



and the fixed point building Δ^Γ is isomorphic to $\mathsf{B}_2^{\mathcal{F}}(K, V, \varphi)$.

Proof. By 5.12, there exist separable quadratic extensions E/K such that φ_E is pseudo-split and letting E/K be any one of them, we can assume that $V = E \oplus E \oplus F$ and that for some $\eta_1, \eta_2 \in K$,

$$\varphi(y_1, y_2, u) = \eta_1 N(y_1) + \eta_2 N(y_2) + u^2$$

for all $(y_1, y_2, u) \in V$, where N is the norm of the extension E/K, and

$$\eta_1\eta_2 \in F^2.$$

Let $L = EF$, let $\Delta = \mathsf{F}_4(L, E)$, let Ω be the Galois involution called $\hat{\Omega}$ in 17.12 and let $\Gamma = \langle\Omega\rangle$. By 16.6, Γ is a descent group of Δ. By 17.12, there exist Γ-chambers of type B_2. By 6.11 and 6.12(iii), it follows that Δ^Γ is a building of type B_2, and thus by 6.12(iv), Δ^Γ is a Moufang quadrangle. Let $\mathbb{M}_1$ and $\mathbb{M}_2$ be as in 5.16 applied to Δ^Γ. By 6.15 and 17.13, one of these two Moufang sets is isomorphic to $\mathsf{B}_1^{\mathcal{Q}}(\Lambda)$ and the other is as in 5.16(b). By 5.16, therefore, we have $\Delta^\Gamma \cong \mathsf{B}_2^{\mathcal{F}}(\Lambda)$. □

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References

[Bou] N. Bourbaki, *Elements of Mathematics: Lie Groups and Lie Algebras*, Chapters 4–6. Springer 2002. Zbl 0983.17001 MR 1890629

[CdM] E. Callens and T. De Medts, Moufang sets of mixed type F_4. *Math. Nachr.* **288** (2015), 366–388. MR 3320453

[Car] R. W. Carter, *Simple Groups of Lie Type*. John Wiley & Sons 1972. Zbl 0248.20015 MR 0407163

[CP] B. Conrad and G. Prasad, *Characterization of Pseudo-Reductive Groups*. Annals of Math. Studies **191**, Princeton Univ. Press 2016.

[dMed] T. De Medts, A characterization of quadratic forms of type E_6, E_7 and E_8. *J. Algebra* **252** (2002), 394–410. Zbl 1012.11029 MR 1925144

[dMSW] T. De Medts, Y. Segev and R. M. Weiss, Tits endomorphisms and buildings of type F_4. Submitted.

[Die] J. Dieudonné, *La Géométrie des Groupes Classiques*. Springer 1955. Zbl 0067.26104 MR 0072144

[Hum] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*. Springer 1972. Zbl 0254.17004 MR 0323842

[Mue] B. Mühlherr, A geometric approach to non-embeddable polar spaces of rank 3. *Bull. Soc. Math. Belg. A* **42** (1990), 577–594. Zbl 0732.51011 MR 1316212

[MPW] B. Mühlherr, H. P. Petersson and R. M. Weiss, *Descent in Buildings*. Annals of Math. Studies **190**, Princeton Univ. Press 2015. Zbl 1338.51002 MR 3364836

[MM1] B. Mühlherr and H. Van Maldeghem, Exceptional Moufang quadrangles of type F_4. *Canad. J. Math.* **51** (1999), 347–371. Zbl 0942.51002 MR 1697148

[MM2] B. Mühlherr and H. Van Maldeghem, Moufang sets from groups of mixed type. *J. Alg.* **300** (2006), 820–833. Zbl 1101.51003 MR 2228223

[RT] M. Ronan and J. Tits, Building buildings. *Math. Ann.* **278** (1987), 291–306. Zbl 0628.51001 MR 0909229

[Ste] R. Steinberg, *Lectures on Chevalley Groups*. Yale Univ. 1968. Zbl 1196.22001 MR 0466335

[Tit1] J. Tits, Classification of algebraic semi-simple groups, in: *Algebraic Groups and Discontinuous Groups*, Boulder, 1965. *Proc. Symp. Pure Math.* **9**, Amer. Math. Soc. (1966), pp. 33–62. Zbl 0238.20052 MR 0224710

[Tit2] J. Tits, *Buildings of Spherical Type and Finite BN-pairs*. Lecture Notes in Math. **386**, Springer 1974. Zbl 0295.20047 MR 0470099

[Tit3] J. Tits, Cours 1994–1995, in: *Résumés des Cours au Collège de France, 1973–2000.* Documents Mathématiques, Soc. Math. France 2013, pp. 257–273. Zbl 1286.01001 MR 3235648

[TW] J. Tits and R. M. Weiss, *Moufang Polygons*. Springer Monographs in Math., Springer 2002. Zbl 1010.20017 MR 1938841

[Wei1] R. M. Weiss, *The Structure of Spherical Buildings*. Princeton Univ. Press 2004. Zbl 1061.51011 MR 2034361

[Wei2] R. M. Weiss, *The Structure of Affine Buildings*. Annals of Math. Studies **168**, Princeton Univ. Press 2009. Zbl 1166.51001 MR 2468338

[Wei3] S. Weiss, Integrability of Moufang foundations. Dissertation, Univ. of Giessen, 2014, 212 pp.; arXiv:1410.5211.

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