

The derived Maurer–Cartan locus

Ezra GETZLER

Abstract. The derived Maurer–Cartan locus is a functor MC^\bullet from differential graded Lie algebras to cosimplicial schemes. If L is a differential graded Lie algebra, let L_+ be the truncation of L in positive degrees $i > 0$. We prove that the differential graded algebra of functions on the cosimplicial scheme $\mathrm{MC}^\bullet(L)$ is quasi-isomorphic to the Chevalley–Eilenberg complex of L_+ .

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1. Introduction

Derived algebraic geometry is a non-linear analogue of homological algebra. Just as homological algebra studies modules M through projective resolutions

$$\dots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0,$$

derived algebraic geometry studies algebraic schemes through resolutions by derived schemes. In this paper, we will only concern ourselves with affine derived schemes.

Different frameworks may be employed in the study of derived schemes in characteristic zero: they may be represented as differential graded schemes or as cosimplicial schemes. The former approach is due to Tate [Tat] in the affine case; its globalization is due to Ciocan–Fontanine and Kapranov [CK]. The second approach is largely due to Quillen [Quil].

If A^* is a differential graded algebra, we denote by A^\sharp its underlying graded algebra. In this paper, we only consider differential graded commutative algebras A^* such that $A^i = 0$ for $i > 0$. If E^* is a vector space concentrated in negative degrees, let $\mathrm{Sym} E$ be the free graded commutative algebra generated by E^* : this is a polynomial algebra in generators in negative even degrees tensored with an exterior algebra in generators in negative odd degrees.

An affine differential graded scheme \mathcal{X} over a field K of characteristic zero is characterized by its ring of functions $\mathcal{O}(\mathcal{X})$, which is a differential graded commutative algebra over K , with differential $d: \mathcal{O}^*(\mathcal{X}) \rightarrow \mathcal{O}^{*+1}(\mathcal{X})$, satisfying the following conditions:

- (a) $\mathcal{O}^i(\mathcal{X}) = 0$ in positive degree $i > 0$, and there is a regular affine variety X such that $\mathcal{O}^0(\mathcal{X}) \cong \mathcal{O}(X)$;
- (b) there is a graded vector bundle

$$E^* = E^{-1} \oplus E^{-2} \oplus \dots$$

over X , and an isomorphism of graded commutative algebras

$$\mathcal{O}^*(\mathcal{X})^\# \cong \Gamma(X, \text{Sym } E)$$

over $\mathcal{O}(X)$.

The condition that the underlying variety X be *regular* is not usually taken to be part of the definition: Ciocan–Fontanine and Kapranov call differential graded schemes satisfying this additional condition *differential graded manifolds*. But this condition will be satisfied in all cases of interest. (In the language of homotopical algebra, it is a condition of fibrancy: it is analogous to restricting attention to projective resolutions in homological algebra.)

The *classical locus* $\pi^0(\mathcal{X}) \subset X$ of a derived scheme is the vanishing locus of the sheaf of ideals

$$\text{im}(d: \mathcal{O}^{-1}(\mathcal{X}) \rightarrow \mathcal{O}^0(\mathcal{X})),$$

or equivalently, the spectrum of the quotient ring

$$H^0(\mathcal{O}^*(\mathcal{X}), d) = \text{coker}(d: \mathcal{O}^{-1}(\mathcal{X}) \rightarrow \mathcal{O}^0(\mathcal{X})).$$

In the special case that the cohomology of $\mathcal{O}(\mathcal{X})$ is concentrated in degree 0, the differential graded scheme \mathcal{X} should be thought of as a resolution of the vanishing locus $\pi^0(\mathcal{X})$, in the same way as a projective resolution resolves a module.

Any regular affine scheme is a differential graded affine scheme, but there are many more examples. Tate proved in [Tat] that given any finitely generated Noetherian commutative algebra R over a field K of characteristic 0, there is a differential graded commutative ring \mathcal{R} of the above type such that $\mathcal{R}^\#$ is a finitely generated free graded commutative algebra and

$$H^i(\mathcal{R}) \cong \begin{cases} R, & i = 0, \\ 0, & i < 0. \end{cases}$$

In effect, \mathcal{R} is the ring of functions on an affine differential graded scheme \mathcal{X} with $\pi^0(\mathcal{X}) \cong \text{Spec}(R)$. Tate also proved that this affine differential graded scheme is essentially unique, in the sense that given any two differential graded algebras \mathcal{R}_0 and \mathcal{R}_1 with the above properties, there is a morphism of differential graded algebras from \mathcal{R}_0 to \mathcal{R}_1 such that the following diagram commutes:

$$\begin{array}{ccc}
 \mathcal{R}_0 & & \\
 \downarrow & \searrow & \\
 & & R \\
 \downarrow & \nearrow & \\
 \mathcal{R}_1 & &
 \end{array}$$

Let L^* be a differential graded Lie algebra. This means that L^* is a cochain complex, with differential $\delta: L^* \rightarrow L^{*+1}$, with a bilinear bracket

$$[-, -]: L^i \times L^j \rightarrow L^{i+j},$$

which is graded symmetric,

$$[x, y] = -(-1)^{ij}[y, x], \quad x \in L^i, \quad y \in L^j,$$

satisfies the graded Jacobi identity,

$$[x, [y, z]] = [[x, y], z] + (-1)^{ij}[y, [x, z]], \quad x \in L^i, \quad y \in L^j, \quad z \in L^k,$$

and the Leibniz identity,

$$\delta[x, y] = [\delta x, y] + (-1)^i[x, \delta y], \quad x \in L^i, \quad y \in L^j.$$

The differential graded Lie algebra is of *finite type* if $\dim L^i < \infty$ for all i , and vanishes for $i \ll 0$.

The function

$$F(\mu) = \delta\mu + \frac{1}{2}[\mu, \mu]$$

from L^1 to L^2 is called the *curvature*. It satisfies the *Bianchi identity*

$$(1.1) \quad \delta F(\mu) + [\mu, F(\mu)] = 0.$$

The *Maurer–Cartan locus* $\text{MC}(L) \subset L^1$ of L^* is the vanishing locus of the curvature $F(\mu) = 0$.

There is a variant of the Maurer–Cartan locus, called the *Deligne groupoid*, which takes into account the component L^0 of the differential graded Lie algebra in degree 0. The Lie algebra L^0 acts on L^1 by vector fields X_ξ , $\xi \in L^0$, given by the formula

$$X_\xi(\mu) = -\delta\xi - [\mu, \xi], \quad \mu \in L^1.$$

Let G^0 be the universal algebraic group with Lie algebra L^0 (so that all finite-dimensional representations of L^0 come from a representation of G^0), and suppose that the above action of L^0 exponentiates to an action of G^0 on L^1 : for example, this will be the case when the differential $\delta: L^0 \rightarrow L^1$ vanishes. Then this action preserved the Maurer–Cartan locus $\text{MC}(L)$: the groupoid associated to the action of G^0 on $\text{MC}(L)$ is called the Deligne groupoid of the differential graded Lie algebra L^* . Locally, most, if not all, deformation problems in algebraic geometry may be represented as Deligne groupoids.

Example 1.1. Let R be a commutative ring, and let A^* be a differential graded algebra defined over R . The Hochschild complex $\mathbf{B}^*(R, A)$ is the bigraded abelian group

$$\mathbf{B}^{j,k}(R, A) = \text{Hom}(R^{\otimes j}, A^k)$$

with differentials $d: \mathbf{B}^{j,k} \rightarrow \mathbf{B}^{j,k+1}$ and $\delta: \mathbf{B}^{j,k} \rightarrow \mathbf{B}^{j+1,k}$, given by the formulas

$$(dc)(r_1, \dots, r_{j+1}) = r_1c(r_2, \dots, r_{j+1}) + \sum_{i=1}^j (-1)^i c(r_1, \dots, r_i r_{i+1}, \dots, r_{j+1})$$

and $(\delta c)(r_1, \dots, r_j) = \delta(c(r_1, \dots, r_j))$. Furthermore, $\mathbf{B}(R, A)$ is a graded Lie algebra, with bracket

$$\begin{aligned} [c_1, c_2](r_1, \dots, r_{j_1+j_2}) &= (-1)^{j_1 k_2} c_1(r_1, \dots, r_{j_1}) c_2(r_{j_1+1}, \dots, r_{j_1+j_2}) \\ &\quad - (-1)^{(j_1+k_1)(j_2+k_2)+j_2 k_1} c_2(r_1, \dots, r_{j_2}) c_1(r_{j_2+1}, \dots, r_{j_1+j_2}), \end{aligned}$$

where $c_1 \in \mathbf{B}^{j_1, k_1}$ and $c_2 \in \mathbf{B}^{j_2, k_2}$.

Special cases of this construction give differential graded Lie algebras with applications to deformation theory. For example, let M be a finite-dimensional vector space and let n be a natural number, and consider the graded algebra

$$A^k = \begin{cases} \text{End}(M), & k = 0, \\ \text{Hom}(R^{\oplus n}, M), & k = 1, \\ 0, & \text{otherwise.} \end{cases}$$

The product on A^* is given by the product on $\text{End}(M)$, and the natural pairing $\text{End}(M) \otimes \text{Hom}(R^{\oplus n}, M) \rightarrow \text{Hom}(R^{\oplus n}, M)$, and otherwise it vanishes. The differential on A^* is zero. A Maurer–Cartan element of $\mathbf{B}(R, A)$ is a pair (ρ, f) , where ρ is an action of R on M , and $f: R^{\oplus n} \rightarrow M$ is a morphism of R -modules. The Deligne groupoid is given by the natural action of the semisimple algebraic group $\text{GL}(M)$ on $\text{MC}(\mathbf{B}(R, A))$, which has the effect of conjugating ρ , and composing with f .

The *Quot* scheme of projective geometry is obtained by an analogue of this construction: one takes a finite dimensional truncation R of the homogenous ring of polynomials $\mathbb{C}[x_0, \dots, x_N]$, a finite dimensional truncation M of a homogenous $\mathbb{C}[x_0, \dots, x_N]$ -module, and forms the Lie subalgebra $B_0(R, A) \subset B(R, A)$ consisting of elements of zero total homogeneity. For further details, see [CK].

Example 1.2. Our second example is the Harrison complex of a vector space R . Given natural numbers p and q , let $\text{III}(p, q)$ be the set of partitions of $\{0, \dots, p+q-1\}$ into disjoint subsets $I = (i_1 < \dots < i_p)$ and $J = (j_1 < \dots < j_q)$. Harrison cochains are multilinear maps from R to itself which vanish on shuffles: $\text{CHarr}^k(R, R)$ is the set of $c \in \text{Hom}(R^{\otimes k+1}, R)$ such that for all $0 < p < k$, we have

$$\sum_{(I, J) \in \text{III}(p, k-p)} (-1)^{i_1 + \dots + i_p} c(r_{i_1}, \dots, r_{i_p}, r_{j_1}, \dots, r_{j_{k-p}}) = 0.$$

This is a graded Lie algebra with respect to the Gerstenhaber bracket: if $c_1 \in \text{CHarr}^{k_1}(R, R)$ and $c_2 \in \text{CHarr}^{k_2}(R, R)$, the bracket equals

$$\begin{aligned} [c_1, c_2](r_0, \dots, r_{k_1+k_2}) &= \sum_{i=0}^{k_1} (-1)^{ik_2} c_1(r_0, \dots, c_2(r_i, \dots, r_{i+k_2}), \dots, r_{k_1+k_2}) \\ &\quad - \sum_{i=0}^{k_2} (-1)^{k_1k_2+ik_1} c_2(r_0, \dots, c_1(r_i, \dots, r_{i+k_1}), \dots, r_{k_1+k_2}). \end{aligned}$$

The Maurer–Cartan locus $\text{MC}(\text{CHarr}(R, R))$ consists of all bilinear maps $\mu \in \text{CHarr}^1(R, R)$ such that $[\mu, \mu] = 0$. This is the space of all commutative associative products on R .

The Lie algebra $\text{CHarr}^0(R, R)$ may be identified with $\text{End}(R)$, which is a semisimple Lie algebra with associated universal algebraic group $\text{GL}(R)$. This group acts on the graded Lie algebra $\text{CHarr}(R, R)$ by the formula

$$(g \cdot c)(r_0, \dots, r_k) = g\left(c(g^{-1}(r_0), \dots, g^{-1}(r_k))\right).$$

Thus, the Deligne groupoid of $\text{CHarr}^*(R, R)$ is the space of commutative associative products on R up to conjugation.

Example 1.3. As a final example, we sketch an application of this formalism in the study of holomorphic vector bundles. This example lies outside algebraic geometry, and requires the use of Sobolev spaces to make any sense of it.

Given a complex manifold X and a holomorphic vector bundle E on X , the Dolbeault complex $A^{0,*}(X, \text{End}(E))$ is a differential graded Lie algebra, with differential $\bar{\partial}$. The curvature of an element $\mu \in A^{0,1}(X, \text{End}(E))$ is the obstruction in $A^{0,2}(X, \text{End}(E))$ to the first-order differential operator $\bar{\partial} + \text{ad}(\mu)$ inducing a holomorphic structure on E . Thus, the Maurer–Cartan locus of $A^{0,*}(X, \text{End}(E))$ is the space of holomorphic structures (or Cauchy–Riemann operators) on E .

The graded Lie algebra $A^{0,0}(X, \text{End}(E))$ is the space of sections of the endomorphism bundle $\text{End}(E)$. The associated group is the gauge group of E , which is the space of sections of the smooth bundle of Lie groups $\text{Aut}(G)$, and the Deligne groupoid models the stack of holomorphic structures on E up to gauge equivalence.

The *differential graded Maurer–Cartan locus* $\mathcal{MC}(L)$ of a differential graded Lie algebra L^* of finite type is the affine differential graded scheme with underlying scheme the affine space L^1 , and with the graded algebra of functions

$$\mathcal{O}(\mathcal{MC}(L))^\# = \text{Sym}(L_+[1]^\vee).$$

Here, L_+^* is the truncation of L in positive degrees:

$$L_+^i = \begin{cases} L^i, & i \geq 1, \\ 0, & i < 1, \end{cases}$$

and $L_+[1]$ denotes the shift of the cochain complex L_+ down in degree by 1. This graded algebra may be identified with the graded vector space of Chevalley–Eilenberg cochains of the differential graded Lie algebra L_+^* . The differential d on $\mathcal{O}(\mathcal{MC}(L))$ is the differential of the Chevalley–Eilenberg complex: it is the sum of the adjoints of δ , which maps $(L^{i+1})^\vee$ to $(L^i)^\vee$, and of $[-, -]$, which maps $(L^k)^\vee$ to

$$\bigoplus_{i=1}^{k-1} (L^i)^\vee \otimes (L^{k-i})^\vee.$$

The image of the differential d in $\mathcal{O}^0(\mathcal{MC}(L))$ is the ideal generated by the curvature $F(x)$. This proves the following result.

Proposition 1.4. *The classical locus $\pi^0(\mathcal{MC}(L))$ of the differential graded scheme $\mathcal{MC}(L)$ is the Maurer–Cartan locus $\text{MC}(L)$.*

There is also a differential graded analogue of the Deligne groupoid. For simplicity, we consider only the case where the differential $\delta: L^0 \rightarrow L^1$ vanishes. The universal algebraic group G^0 with Lie algebra L^0 acts on the differential graded Lie algebra L_+ , and hence on the derived Maurer–Cartan locus $\text{MC}(L)$.

The groupoid in differential graded schemes associated to this action is the derived Deligne groupoid of L . For example, the derived Deligne groupoid of the differential graded Lie algebra $B_0(R, A)$ considered by Ciocan–Fontanine and Kapranov gives rise to the derived *Quot* scheme, while the derived Deligne groupoid of the graded Lie algebra $\text{CHarr}(R, R)$ leads to the derived stack of commutative associative products on R .

Quillen [Qui1] introduced cosimplicial schemes as an alternate foundation for the theory of derived geometry: unlike differential graded schemes, they give the correct model for derived schemes even in positive characteristic (though in this paper, we will only consider cosimplicial schemes in characteristic zero). A cosimplicial scheme X^\bullet is a functor from the category Δ of nonempty finite totally ordered sets to the category of schemes. For $n \geq 0$, denote the object

$$0 < \cdots < n$$

of Δ by $[n]$: the functor X^\bullet takes the value X^n at $[n]$.

A cosimplicial scheme X^\bullet is the spectrum of a simplicial commutative ring. Quillen proved that in characteristic zero, simplicial commutative algebras and differential graded commutative algebras have equivalent homotopy theory, in the following sense: the normalization functor N_* from simplicial vector spaces to chain complexes induces a functor from simplicial commutative algebras to differential graded commutative algebras, also denoted N_* , and this functor induces an equivalence of homotopy categories. (In fact, N_* is a right Quillen equivalence with respect to the projective closed model structures on these categories; cf. [Qui2, Section 4].) We review the construction of the functor N_* in Sections 2 and 3.

In the category Δ , we have the *coface maps*

$$d^i : [n - 1] \rightarrow [n], \quad 0 \leq i \leq n,$$

defined by

$$d^i(j) = \begin{cases} j, & j < i, \\ j + 1, & j \geq i, \end{cases}$$

and the *codegeneracy maps*

$$s^i : [n + 1] \rightarrow [n], \quad 0 \leq i \leq n,$$

defined by

$$s^i(j) = \begin{cases} j, & j < i, \\ j - 1, & j \geq i. \end{cases}$$

If X_\bullet is a cosimplicial object, we denote the induced morphisms $d^i : X^{n-1} \rightarrow X^n$ and $s^i : X^{n+1} \rightarrow X^n$ by the same symbol. If X_\bullet is a simplicial object, we denote the face and degeneracy morphisms by $\partial_i : X^n \rightarrow X^{n-1}$ and $\sigma_i : X^n \rightarrow X^{n+1}$.

The maximal augmentation of a cosimplicial scheme is the equalizer

$$\pi^0(X^\bullet) = \text{eq}\left(X^0 \begin{array}{c} \xrightarrow{d^0} \\ \xrightarrow{d^1} \end{array} X^1 \right).$$

Observe the analogy with the definition of the set of components of a simplicial set X_\bullet , which is the coequalizer

$$\pi_0(X_\bullet) = \text{coeq}\left(X_1 \begin{array}{c} \xrightarrow{\partial_0} \\ \xrightarrow{\partial_1} \end{array} X_0 \right).$$

It is the goal of this paper to make the equivalence between differential graded schemes and cosimplicial schemes in characteristic zero as explicit as possible for derived Maurer–Cartan loci. The realization of the derived Maurer–Cartan locus as a cosimplicial scheme that we propose is new. This realization may also be used in other settings, for example when affine schemes are replaced by Banach analytic spaces: it has the advantage over the differential graded Maurer–Cartan locus that it does not require making sense of the Chevalley–Eilenberg complex in the setting of differential graded Banach Lie algebras.

In order to realize the derived Maurer–Cartan locus as a cosimplicial scheme, we introduce a certain cosimplicial differential graded commutative algebra Λ^\bullet . As a graded algebra, Λ^n is the exterior algebra generated by elements $\{e_0, \dots, e_n\}$ in degree -1 : the differential on Λ^n is defined on the generators e_i by $\delta e_i = 1$. A morphism $f : [m] \rightarrow [n]$ of Δ induces a homomorphism $f : \Lambda^m \rightarrow \Lambda^n$ of differential graded commutative algebras by its action on the generators: $f(e_i) = e_{f(i)}$.

The tensor product $L^* \otimes \Lambda^n$ of a differential graded Lie algebra L^* with the differential graded commutative algebra Λ^n is again a differential graded Lie algebra, with bracket

$$[x_1 \otimes \alpha_1, x_2 \otimes \alpha_2] = (-1)^{j_2 k_1} [x_1, x_2] \otimes \alpha_1 \alpha_2,$$

where $x_1 \in L^{j_1}$, $x_2 \in L^{j_2}$, $\alpha_1 \in (\Lambda^n)^{k_1}$, and $\alpha_2 \in (\Lambda^n)^{k_2}$.

Definition 1.5. The *derived Maurer–Cartan locus* $\text{MC}^\bullet(L)$ of a differential graded Lie algebra L is the cosimplicial scheme

$$\text{MC}^n(L) = \text{MC}(L \otimes \Lambda^n).$$

We may now state our main result.

Theorem 1.6. *The normalization $N_*(\mathcal{O}(\mathrm{MC}^\bullet(L)))$ of the simplicial commutative algebra $\mathcal{O}(\mathrm{MC}^\bullet(L))$ of functions on the cosimplicial scheme $\mathrm{MC}^\bullet(L)$ is a differential graded commutative algebra concentrated in nonpositive degrees. There is a natural homomorphism of differential graded commutative algebras*

$$\Phi: \mathcal{O}(\mathcal{MC}(L)) \rightarrow N_{-*}(\mathcal{O}(\mathrm{MC}^\bullet(L))),$$

which is a quasi-isomorphism.

Thus, the two realizations of the derived Maurer–Cartan locus of L^* are equivalent.

In Section 2 of this paper, we review the Dold–Kan correspondence between simplicial abelian groups and connective chain complexes.

In Section 3, we review Eilenberg and MacLane’s formulation of the Eilenberg–Zilber theorem, in particular, the formulas for the Alexander–Whitney and shuffle maps. We show that for *abelian* differential graded Lie algebras L^* , the derived Maurer–Cartan locus $\mathrm{MC}^\bullet(L)$ may be identified with the cosimplicial vector space $K^\bullet(L_+[1])$ associated to the coconnective cochain complex $L^*_+[1]$. In this sense, $\mathrm{MC}^\bullet(L)$ is a nonlinear generalization of the functor K^\bullet realizing the equivalence of Dold and Kan between the categories of coconnective cochain complexes and cosimplicial vector spaces.

A key idea in the proof of Theorem 1.6 is the observation that the derived Maurer–Cartan locus is a *grouplike* cosimplicial scheme, in the sense of Bousfield and Kan [BK]: although $\mathrm{MC}^\bullet(L)$ is not actually a cosimplicial group scheme unless L^* is abelian, it is close to being so in a certain precise sense, as we explain in Section 4.

In particular, the underlying graded commutative algebra $N_*(\mathcal{O}(\mathrm{MC}^\bullet(L)))^\#$ of the differential graded commutative algebra $N_*(\mathcal{O}(\mathrm{MC}^\bullet(L)))$ only depends on the graded vector space $L^\#$ underlying L^* . We also prove that $N_*(\mathcal{O}(\mathrm{MC}^\bullet(L)))^\#$ is a free graded commutative algebra: the proof uses a result of Milnor and Moore [MM, Theorem 7.5] (which they ascribe to Leray), and a recent important complement to the Eilenberg–Zilber Theorem due to Ševera and Willwacher [SW].

In Section 5, we complete the proof of Theorem 1.6. In Section 6, we state the generalization of our results for nilpotent L_∞ -algebras.

2. The Dold–Kan correspondence for cosimplicial abelian groups

The *normalized chain complex* of a simplicial abelian group A_\bullet is the graded abelian group

$$N_n(A) = A_n / \sum_{i=0}^{n-1} \text{im}(\sigma_i : A_{n-1} \rightarrow A_n),$$

with differential

$$\partial = \sum_{i=0}^n (-1)^i \partial_i : N_n(A) \rightarrow N_{n-1}(A).$$

The chain complex $N_*(A)$ is *connective*: it vanishes in negative homological, or positive cohomological, degrees. (We may consider any chain complex V_* to be a cochain complex V^* , by setting $V^* = V_{-*}$.)

For example, the abelian group $N_k(\mathbb{Z}\Delta^n)$ is a free abelian group with generators

$$\{x_{i_0 \dots i_k} \mid 0 \leq i_0 < \dots < i_k \leq n\},$$

where $x_{i_0 \dots i_k}$ corresponds to the nondegenerate simplex $[k] \rightarrow [n]$ with vertices

$$\{i_0, \dots, i_k\} \subset \{0, \dots, n\}.$$

The differential $\partial : N_k(\mathbb{Z}\Delta^n) \rightarrow N_{k-1}(\mathbb{Z}\Delta^n)$ is given by the formula

$$\partial x_{i_0 \dots i_k} = \sum_{j=0}^k (-1)^j x_{i_0 \dots \widehat{i_j} \dots i_k}.$$

The right-adjoint of the functor N_* is the functor K_\bullet from chain complexes to simplicial abelian groups defined by Eilenberg and Mac Lane [EM2]:

$$\text{Hom}(N_*(A), Z_*) \cong \text{Hom}(A_\bullet, K_\bullet(Z)).$$

The Yoneda lemma implies that the n -simplices of the simplicial abelian group $K(Z)$ are given by the formula

$$(2.1) \quad K_n(Z) \cong \text{Hom}(N_n(\mathbb{Z}\Delta^n), Z_*).$$

Dold [Dol] and Kan [Kan] proved that the adjoint pair of functors $N \dashv K$ yields an adjoint equivalence between the categories of simplicial abelian groups and connective chain complexes. Dold and Puppe [DP] extended this equivalence to arbitrary abelian categories, with the functors N_* and K_\bullet being given by the same formulas as in the category of abelian groups.

The opposite category to the category of abelian groups is an abelian category: the corresponding categories of connective chain complexes and simplicial objects are the categories of coconnective cochain complexes (cochain complexes vanishing in negative degree) and cosimplicial abelian groups. Let us make the adjoint equivalence of Dold and Puppe more explicit in this situation.

The normalized cochain complex of a cosimplicial abelian group A^\bullet is the graded abelian group

$$N^n(A) = \bigcap_{i=0}^{n-1} \ker(s^i : A^n \rightarrow A^{n-1}),$$

with differential

$$\sum_{i=0}^{n+1} (-1)^i d^i : N^n(A) \rightarrow N^{n+1}(A).$$

The functor N^* has a left-adjoint K^\bullet , which takes cochain complexes to cosimplicial abelian groups. In fact, since it is an equivalence, it is also the right-adjoint of N^* . Let Δ_n be the cosimplicial set corepresented by the object $[n] \in \Delta$: we have

$$(\Delta_n)^k = \Delta([n], [k]).$$

Form the cosimplicial abelian group $\mathbb{Z}\Delta_n^\bullet$. If Z^\bullet is a cosimplicial abelian group, Yoneda’s Lemma implies that

$$Z^n \cong \text{Hom}(\mathbb{Z}\Delta_n^\bullet, Z^\bullet).$$

Thus, parallel to the case of simplicial abelian groups (2.1), we see that

$$(2.2) \quad \begin{aligned} K^n(Z) &\cong \text{Hom}(\mathbb{Z}\Delta_n^\bullet, K^\bullet(Z)) \\ &\cong \text{Hom}(N^*(\mathbb{Z}\Delta_n), Z^*). \end{aligned}$$

Unlike its cousin $N_*(\mathbb{Z}\Delta^n)$, the cochain complex $N^*(\mathbb{Z}\Delta_n)$ has not been discussed in the literature. Let $f_{n_0 \dots n_k} : [n] \rightarrow [k]$ be the morphism such that

$$f_{n_0 \dots n_k}^{-1}(j) = \{n_0 + \dots + n_{j-1}, \dots, n_0 + \dots + n_j - 1\}.$$

The action of the coface maps is given by

$$(2.3) \quad d^i f_{n_0 \dots n_k} = f_{n_0 \dots n_{i-1} 0 n_i \dots n_k},$$

and the action of the codegeneracy maps by

$$(2.4) \quad s^i f_{n_0 \dots n_k} = f_{n_0 \dots n_i + n_{i+1} \dots n_k}.$$

Denote by $[f_{n_0 \dots n_k}] \in N^k(\mathbb{Z}\Delta_n)$ the image of $f_{n_0 \dots n_k}$ in the normalized cochain complex.

The chain complex $N_*(A)$ may be represented as a colimit.

Lemma 2.1 (Dold [Dol], Lemma 1.11). *The quotient map from A_n to $N_n(A)$ induces a natural isomorphism of abelian groups*

$$N_n(A) \cong \bigcap_{i=1}^n \ker(\partial_i : A_n \rightarrow A_{n-1}).$$

Under this isomorphism, the differential ∂ corresponds to the operator

$$\partial_0: \bigcap_{i=1}^n \ker(\partial_i: A_n \rightarrow A_{n-1}) \rightarrow \bigcap_{i=1}^{n-1} \ker(\partial_i: A_{n-1} \rightarrow A_{n-2}).$$

In the opposite category to the category of abelian groups, this lemma yields the following corollary.

Corollary 2.2. *The abelian group $N^k(\mathbb{Z}\Delta_n)$ is a free abelian group, generated by the elements $[f_{n_0\dots n_k}]$, where $n_0 + \dots + n_k = n + 1$ and $n_i > 0$ when $i < n$. The differential is given by the formula*

$$d[f_{n_0\dots n_k}] = [f_{n_0\dots n_k 0}].$$

3. The Eilenberg–Zilber theorem

Let $A_{\bullet\bullet}$ be a bisimplicial abelian group: a contravariant functor from the category $\Delta \times \Delta$ to the category of abelian groups. Denote the maps defining the first simplicial structure by $\partial_i^{(1)}$ and $\sigma_i^{(1)}$, and those defining the second simplicial structure by $\partial_i^{(2)}$ and $\sigma_i^{(2)}$.

By the Dold–Kan theorem, the categories of bisimplicial abelian groups and first-quadrant double complexes X_{**} are equivalent. This equivalence is realized by the naturally equivalent functors $N_*^{(2)}N_*^{(1)} \cong N_*^{(1)}N_*^{(2)}$. Denote either of these functors by N_{**} .

The double complex $N_{**}(A)$ has two commuting differentials $\partial^{(1)}$ and $\partial^{(2)}$, of bidegree $(1, 0)$ and $(0, 1)$ respectively. The total chain complex of this double complex is the chain complex

$$\text{Tot}_k(N_{**}(A)) = \bigoplus_{p+q=k} N_{pq}(A),$$

with differential $\partial = \partial^{(1)} + (-1)^p \partial^{(2)}$.

The diagonal of a bisimplicial abelian group is the simplicial abelian group

$$\text{Diag}_p(A) = A_{pp}.$$

The Eilenberg–Zilber theorem [EZ] compares the chain complex $\text{Tot}_*(N_{**}(A))$ to the normalization $N_*(\text{Diag}_\bullet(A))$ of the diagonal of $A_{\bullet\bullet}$. We will use the following explicit formulation of the theorem.

Theorem 3.1 (Eilenberg and Mac Lane [EM2], Section 2). *There are natural morphisms of complexes*

$$f: N_*(\text{Diag}_\bullet(A)) \rightarrow \text{Tot}_*(N_{**}(A))$$

and

$$g: \text{Tot}_*(N_{**}(A)) \rightarrow N_*(\text{Diag}_\bullet(A))$$

and a natural homotopy

$$h: N_*(\text{Diag}_\bullet(A)) \rightarrow N_{*+1}(\text{Diag}_\bullet(A)),$$

such that fg is the identity of $\text{Tot}_*(N_{**}(A))$, $gf + \partial h + h\partial$ is the identity of $N_*(\text{Diag}_\bullet(A))$, and fh and hg vanish. In particular, the homology groups of the complexes $\text{Tot}_*(N_{**}(A))$ and $N_*(\text{Diag}_\bullet(A))$ are isomorphic.

The explicit formulas for the natural transformations f and g are as follows. The map f from $N_k(\text{Diag}_\bullet(A))$ to $\text{Tot}_k(N_{**}(A))$ is the Alexander–Whitney map

$$(3.1) \quad f_k = \sum_{p=0}^k \partial_{p+1}^{(1)} \dots \partial_k^{(1)} \partial_0^{(2)} \dots \partial_{p-1}^{(2)}.$$

The component g_{pq} of g mapping A_{pq} to $N_{p+q}(\text{Diag}_\bullet(A))$ is given by the formula (Eilenberg and Mac Lane [EM1], Section 5)

$$(3.2) \quad g_{pq} = \sum_{\{i_1 < \dots < i_p\} \amalg \{j_1 < \dots < j_q\} \in \text{III}(p,q)} (-1)^{\sum_{\ell=1}^p (i_\ell - \ell + 1)} \sigma_{j_q}^{(1)} \dots \sigma_{j_1}^{(1)} \sigma_{i_p}^{(2)} \dots \sigma_{i_1}^{(2)}.$$

This map is called the *shuffle map*.

A *simplicial coalgebra* is a simplicial R -module A_\bullet together with simplicial morphisms $c: A_\bullet \rightarrow A_\bullet \otimes A_\bullet$, the comultiplication, and $\varepsilon: A_\bullet \rightarrow R$, the augmentation, such that the diagram

$$\begin{array}{ccc} A_\bullet & \xrightarrow{c} & A_\bullet \otimes A_\bullet \\ c \downarrow & & \downarrow c \otimes A \\ A_\bullet \otimes A_\bullet & \xrightarrow{A \otimes c} & A_\bullet \otimes A_\bullet \otimes A_\bullet \end{array}$$

commutes (coassociativity), and both $(\varepsilon \otimes A)c$ and $(A \otimes \varepsilon)c$ equal the identity morphism of A_\bullet .

Using the Alexander–Whitney map, we may show that the normalized chain complex of a simplicial coalgebra over a commutative ring R is a differential graded coalgebra. Let $(A \boxtimes A)_{\bullet\bullet}$ be the bisimplicial R -module

$$(A \boxtimes A)_{pq} = A_p \otimes A_q.$$

In particular, $\text{Diag}_\bullet(A \boxtimes A) \cong A_\bullet \otimes A_\bullet$ and

$$\text{Tot}_*(N_{**}(A \boxtimes A)) \cong N_*(A) \otimes N_*(A).$$

The comultiplication $c: A_\bullet \rightarrow A_\bullet \otimes A_\bullet$ gives a morphism of complexes

$$N_*(c): N_*(A) \rightarrow N_*(\text{Diag}_\bullet(A \otimes A)).$$

Composing with the Alexander–Whitney map, we obtain a map

$$fN_*(c): N_*(A) \rightarrow N_*(A) \otimes N_*(A).$$

It is easily checked that this morphism of chain complexes is coassociative, and has

$$N_*(\varepsilon): N_*(A) \rightarrow N_*(R) \cong R$$

as a counit.

A *simplicial algebra* is a simplicial R -module A_\bullet together with simplicial morphisms $m: A_\bullet \otimes A_\bullet \rightarrow A_\bullet$, the multiplication, and $\eta: R \rightarrow A_\bullet$, the unit, such that the diagram

$$\begin{array}{ccc} A_\bullet \otimes A_\bullet \otimes A_\bullet & \xrightarrow{m \otimes A} & A_\bullet \otimes A_\bullet \\ \downarrow A \otimes m & & \downarrow m \\ A_\bullet \otimes A_\bullet & \xrightarrow{m} & A \end{array}$$

commutes (associativity), and both $m(\eta \otimes A)$ and $(A \otimes \eta)m$ equal the identity morphism of A_\bullet .

Using the shuffle map, we may show that the normalized chain complex of a simplicial algebra A_\bullet over a commutative ring R is a differential graded algebra. The multiplication $m: A_\bullet \otimes A_\bullet \rightarrow A_\bullet$ gives a morphism of complexes

$$N_*(m): N_*(\text{Diag}_\bullet(A \boxtimes A)) \rightarrow N_*(A).$$

Composing with the shuffle map, we obtain a morphism

$$N_*(m)g: N_*(A) \otimes N_*(A) \rightarrow N_*(A).$$

This morphism of chain complexes is associative, and has

$$N_*(\eta): N_*(R) \cong R \rightarrow N_*(A)$$

as a unit. In fact, more is true: if A_\bullet is a simplicial *commutative* algebra, then $N_*(A)$ is a differential graded commutative algebra.

Parallel constructions in the opposite category to the category of R -modules shows that the normalized cochain complex $N^*(A)$ of a cosimplicial algebra A^\bullet is a differential graded algebra, and that the normalized cochain complex $N^*(A)$ of a cosimplicial (cocommutative) coalgebra A^\bullet is a differential graded (cocommutative) coalgebra.

A *simplicial bialgebra* is a simplicial algebra A_\bullet which is at the same time a simplicial coalgebra, in such a way that the comultiplication $c: A_\bullet \rightarrow A_\bullet \otimes A_\bullet$ and augmentation $\varepsilon: A_\bullet \rightarrow R$ are morphisms of simplicial algebras. The following result is proved in Appendix A of Ševera and Willwacher [SW]: the proof is by an explicit calculation verifying the required compatibility between the Alexander–Whitney and shuffle maps.

Proposition 3.2. *The normalized chain complex $N_*(A)$ of a simplicial (commutative) bialgebra A_\bullet is a differential graded (commutative) bialgebra.*

If X_\bullet is a simplicial set, the simplicial abelian group $\mathbb{Z}X_\bullet$ is a simplicial coalgebra, and the Alexander–Whitney map makes $N_*(\mathbb{Z}X)$, the simplicial chain complex of X_\bullet , into a differential graded coalgebra. On the other hand, if X^\bullet is a cosimplicial set, the cosimplicial abelian group $\mathbb{Z}X^\bullet$ is a cosimplicial cocommutative coalgebra, and the shuffle map makes $N^*(\mathbb{Z}X)$ into a differential graded cocommutative coalgebra. In the following proposition, we analyse the differential graded coalgebra $N^*(\mathbb{Z}\Delta_n^\bullet)$.

Proposition 3.3. *The dual $N^*(\mathbb{Z}\Delta_n)^\vee$ of the differential graded coalgebra $N^*(\mathbb{Z}\Delta_n)$ is isomorphic to the differential graded commutative algebra Λ^n . This duality is induced by the following pairing between the free abelian groups $N^1(\mathbb{Z}\Delta_n)$ and $(\Lambda^n)^{-1}$:*

$$\langle [f_{i,n-i+1}], e_j \rangle = \begin{cases} 1, & i \leq j, \\ 0, & i > j. \end{cases}$$

Proof. If $\varphi: [m] \rightarrow [n]$ is a morphism of Δ , then

$$\varphi^*[f_{i,n-i+1}] = [f_{i',m-i'+1}],$$

where i' is the cardinality of the set $\varphi^{-1}(\{0, \dots, i-1\})$. It is easily seen that

$$\langle \phi^*[f_{i,n-i+1}], e_j \rangle = \langle [f_{i,n-i+1}], e_{\varphi(j)} \rangle,$$

and hence that the pairing between the simplicial abelian group $[n] \mapsto N^1(\mathbb{Z}\Delta_n)$ and the cosimplicial abelian group $[n] \mapsto (\Lambda^n)^{-1}$ is compatible with the respective actions of the category Δ . That is, the pairing descends to the colimit $N^1(\mathbb{Z}\Delta_\bullet) \otimes_\Delta (\Lambda^\bullet)^{-1}$.

Given a coalgebra A with comultiplication $c: A \rightarrow A \otimes A$, let

$$c^{(k)} = (A^{\otimes k-2} \otimes c) \dots (A \otimes c)c: A \rightarrow A^{\otimes k}$$

be the iterated coproduct. Let p be the projection from $N^*(\mathbb{Z}\Delta_n)^{\otimes k}$ to $N^1(\mathbb{Z}\Delta_n)^{\otimes k}$. Let π be the symmetrization operator

$$\pi = \sum_{\sigma \in \mathcal{S}_n} (-1)^{\varepsilon(\sigma)} \sigma: N^1(\mathbb{Z}\Delta_n)^{\otimes k} \rightarrow N^1(\mathbb{Z}\Delta_n)^{\otimes k}.$$

The proposition is a consequence of the following formula:

$$pc^{(k)}[f_{n_0 \dots n_k}] = \pi([f_{n_0, n_1 + \dots + n_k}] \otimes [f_{n_0 + n_1, n_2 + \dots + n_k}] \otimes \dots \otimes [f_{n_0 + \dots + n_{k-1}, n_k}]) \in N^1(\mathbb{Z}\Delta_n)^{\otimes k}.$$

This formula is proved using the explicit formulas (3.2) and (2.4) for the shuffle product and for the action of the codegeneracies on $\mathbb{Z}\Delta_n^\bullet$. □

4. The derived Maurer–Cartan locus

In the last section, we introduced the simplicial differential graded cocommutative coalgebra $[n] \mapsto N_*(\Delta_n)$, and proved that it was dual to the cosimplicial differential graded commutative algebra $[n] \mapsto \Lambda^n$. As we have seen in (2.2), the inverse functor to the normalized cochains from cosimplicial abelian groups to coconnective cochain complexes may be represented in terms of Λ^\bullet :

$$K^\bullet(Z) = Z^0(Z \otimes \Lambda^\bullet).$$

Here, $Z^0(-)$ is the abelian group of 0-cocycles in the tensor product of Z^* with the cosimplicial cochain complex Λ^\bullet .

An abelian differential graded Lie algebra is the same thing as a cochain complex, and its Maurer–Cartan locus may be identified with the space of 1-cocycles of L^* . Thus, in this case, we obtain the identification

$$MC(L) \cong K^0(L_+[1]),$$

where we recall from the introduction that $L_+[1]$ is the suspended cochain complex

$$L_+[1]^i = \begin{cases} L^{i+1}, & i \geq 0, \\ 0, & i < 0. \end{cases}$$

Tensoring L^* with Λ^n , we see that the functor K^\bullet may be identified with the derived Maurer–Cartan locus of *abelian* differential graded Lie algebras:

$$MC^\bullet(L) \cong K^\bullet(L[1]).$$

This provides some motivation for our definition of the derived Maurer–Cartan locus for not necessarily abelian differential graded Lie algebras.

The graded vector space Λ^n decomposes as the direct sum of the ideal $e_0\Lambda^n$ and the image of the coface map $d^0: \Lambda^{n-1} \rightarrow \Lambda^n$. The monomials in the elements $\varepsilon_i = e_{i+1} - e_i$, $0 \leq i < n$, form a basis over \mathbb{Z} of the free abelian group Λ^n .

Lemma 4.1. *There is a natural isomorphism*

$$\mathrm{MC}^n(L) \cong \bigoplus_{k=0}^n \bigoplus_{0 \leq i_1 < \dots < i_k < n} \varepsilon_{i_1} \dots \varepsilon_{i_k} L^{k+1},$$

induced by the projection $\Lambda^n \rightarrow \Lambda^n/e_0\Lambda^n \cong \mathrm{im}(d^0)$. The element of $\mathrm{MC}^n(L)$ corresponding to

$$\xi = \sum_{k=0}^n \sum_{0 \leq i_1 < \dots < i_k < n} \varepsilon_{i_1} \dots \varepsilon_{i_k} x_{i_1 \dots i_k}, \quad x_{i_1 \dots i_k} \in L^{k+1},$$

equals $\xi - e_0F(\xi)$.

Proof. An element of $L^* \otimes \Lambda^n$ of total degree 1 has the form $\xi + e_0\eta$, where

$$\eta = \sum_{k=0}^n \sum_{0 \leq i_1 < \dots < i_k < n} \varepsilon_{i_1} \dots \varepsilon_{i_k} y_{i_1 \dots i_k}, \quad y_{i_1 \dots i_k} \in L^{k+2}.$$

Taking the curvature of this element, we obtain the expression

$$F(\xi + e_0\eta) = (F(\xi) + \eta) - e_0(\delta\eta + [\xi, \eta]).$$

Along the vanishing locus of the equation $F(\xi) + \eta = 0$, the equation $\delta\eta + [\xi, \eta] = 0$ holds automatically: it is just the Bianchi identity (1.1) for the curvature. \square

In terms of this representation for $\mathrm{MC}^\bullet(L)$, the codegeneracy morphism $s^j: \mathrm{MC}^{n+1}(L) \rightarrow \mathrm{MC}^n(L)$ is given by the formula

$$(4.1) \quad (s^j x)_{i_1 \dots i_k} = \begin{cases} x_{i_1 \dots i_\ell i_{\ell+1} \dots i_k+1} + x_{i_1 \dots i_{\ell-1} i_\ell+1 \dots i_k+1}, & i_\ell + 1 = j, \\ x_{i_1 \dots i_\ell i_{\ell+1} \dots i_k+1}, & i_\ell + 1 < j \leq i_{\ell+1}. \end{cases}$$

When $j > 0$, the coface morphism $d^j: \mathrm{MC}^{n-1}(L) \rightarrow \mathrm{MC}^n(L)$ is given by the formula

$$(4.2) \quad (d^j x)_{i_1 \dots i_k} = \begin{cases} x_{i_1 \dots i_\ell i_{\ell+1}-1 \dots i_k-1}, & i_\ell + 1 < j \leq i_{\ell+1}, \\ 0, & j \in \{i_1 + 1, \dots, i_k + 1\}. \end{cases}$$

The remaining coface map d^0 encodes the geometry of the derived Maurer–Cartan locus: it is given by the formulas

$$(4.3) \quad (d^0 x)_{i_1+1 \dots i_k+1} = x_{i_1 \dots i_k}$$

and

$$(4.4) \quad (d^0 x)_{0i_2+1 \dots i_k+1} = \sum_{\ell=1}^k \sum_{i_\ell < i < i_{\ell+1}} (-1)^\ell x_{i_2 \dots i_\ell i_{\ell+1} \dots i_k} - F(\xi)_{i_2 \dots i_k}.$$

In particular, the codegeneracy $s^0: \text{MC}^1(L) \rightarrow \text{MC}^0(L)$ is given by the formula $s^0(x, y) = x$, and the face maps

$$d^0, d^1: \text{MC}^0(L) \cong L^1 \rightarrow \text{MC}^1(L) \cong L^1 \times L^2$$

are given by the formulas $d^0x = (x, -F(x))$ and $d^1x = (x, 0)$. Thus, there is a natural identification of the classical locus $\pi^0(\text{MC}^\bullet(L))$ of the cosimplicial scheme $\text{MC}^\bullet(L)$ with the Maurer–Cartan locus $\text{MC}(L)$ of the differential graded Lie algebra L^* .

By (4.1) and (4.2), the codegeneracy maps s^i of $\text{MC}^\bullet(L)$ as well as the coface maps d^i , $i > 0$, are homomorphisms of (abelian) group schemes. Adapting the terminology of Bousfield and Kan ([BK], Chapter X, Section 4.8), we call such a cosimplicial scheme *grouplike*. Grouplike cosimplicial spaces are fibrant (op. cit. Section 4.6): we now show that an analogous property holds for grouplike simplicial schemes.

Let X^\bullet be a cosimplicial scheme. The *matching scheme* $M^n(X)$ is the equalizer

$$M^n(X) = \text{eq}\left(\prod_{0 \leq i < n} X^{n-1} \rightrightarrows \prod_{0 \leq i < j < n} X^{n-2} \right)$$

where the two maps in this diagram take $(x^i)_{0 \leq i < n}$ to $(s^i x^j)_{0 \leq i < j < n}$ and $(s^{j-1} x^i)_{0 \leq i < j < n}$.

A cosimplicial scheme X^\bullet is *fibrant* if for each $n \geq 0$, the morphism $X^n \rightarrow M^n(X)$ given by the formula $x^i = s^i x$ is smooth. The proof of the following proposition is modeled on Moore’s proof that simplicial groups are fibrant.

Proposition 4.2 (cf. [BK], Proposition 4.9). *A grouplike cosimplicial scheme is fibrant.*

Proof. In characteristic zero, a morphism of group schemes is smooth if it has a section. We define morphisms $y^i: M^n(X) \rightarrow X^n$, $0 \leq i \leq n + 1$, by induction on i : $y^0 = 1$ and

$$y^{i+1} = y^i d^i ((s^i y^i)^{-1} x^i).$$

It is easily proved, by induction on i , that $s^j y^i = x^j$ for $j < i$. The desired section is $y^{n+1}: M^n(X) \rightarrow X^n$. □

The graded commutative algebra $N_*(\mathcal{O}(\text{MC}^\bullet(L)))^\#$ is actually a free commutative algebra: there is a graded vector space V_* and an isomorphism of graded commutative algebras

$$N_*(\mathcal{O}(\text{MC}^\bullet(L)))^\# \cong \text{Sym } V.$$

This is because the multiplicative structure of $N_*(\mathcal{O}(\text{MC}^\bullet(L)))$ does not depend on the coface maps of $\text{MC}^\bullet(L)$, but only on its codegeneracies, and thus there is an isomorphism

$$(4.5) \quad N_*\left(\mathcal{O}(\text{MC}^\bullet(L))\right)^\# \cong N_*\left(\mathcal{O}(\text{MC}^\bullet(L^\natural))\right)^\#,$$

where L^\natural is the underlying cochain complex of the differential graded Lie algebra L^* . We now apply the following result.

Proposition 4.3. *Let W_* be a connective chain complex. Then there is a connective graded vector space V_* and an isomorphism of graded commutative algebras*

$$N_*(\text{Sym } K(W))^\# \cong \text{Sym } V.$$

Proof. The proof makes use of the fact that $\text{Sym } K(W)$ is a simplicial commutative bialgebra. Proposition 3.2 implies that $N_*(\text{Sym } K(W))^\#$ is a graded commutative bialgebra.

Let W_*^+ be the chain complex

$$W_i^+ = \begin{cases} W_i, & i > 0, \\ 0, & i = 0. \end{cases}$$

There is a natural isomorphism of graded commutative bialgebras

$$N_*(\text{Sym } K(W))^\# \cong \text{Sym } W_0 \otimes N_*(\text{Sym } K(W^+))^\#.$$

Let $\bar{N}_*(\text{Sym } K(W^+))$ be the augmentation ideal of $N_*(\text{Sym } K(W^+))$, that is, the chain complex of elements of positive degree, and let

$$\begin{aligned} Q(N_*(\text{Sym } K(W^+))) \\ \cong \bar{N}_*(\text{Sym } K(W^+)) / (\bar{N}_*(\text{Sym } K(W^+)) \cdot \bar{N}_*(\text{Sym } K(W^+))) \end{aligned}$$

be the chain complex of indecomposables.

Theorem 7.5 of Milnor and Moore [MM], which holds over any field of characteristic zero, states that $N_*(\text{Sym } K(W^+))^\#$ is a free graded commutative algebra, generated by any section of the quotient morphism

$$\bar{N}_*(\text{Sym } K(W^+))^\# \rightarrow Q(N_*(\text{Sym } K(W^+)))^\#. \quad \square$$

5. Proof of Theorem 1.6

This section is the heart of this paper: we prove that the differential graded Maurer–Cartan locus $\mathcal{MC}(L)$ is equivalent to the derived Maurer–Cartan locus

$\text{MC}^\bullet(L)$. Since both functors only depend on L_+^* , we will assume in this section that $L^* = L_+^*$, in other words, that L^i vanishes unless $i \geq 1$.

The normalization $N_*(\mathcal{O}(\text{MC}^\bullet(L)))$ of the simplicial commutative algebra $\mathcal{O}(\text{MC}^\bullet(L))$ is the differential graded commutative algebra of functions on an affine differential graded scheme. If $n > 0$, there is a natural linear map $\alpha \mapsto \Phi(\alpha)$ from the vector space $(L^{n+1})^\vee$ to $\mathcal{O}(\text{MC}^n(L))$, which takes $\alpha \in (L^{n+1})^\vee$ to the linear form $\alpha(x_0 \dots x_{n-1})$ on $\text{MC}^n(L)$. (Here, we use the coordinate system of Lemma 4.1.) The explicit formula (4.2) for the coface maps $d^i: \text{MC}^{n-1}(L) \rightarrow \text{MC}^n(L)$, $1 \leq i \leq n$, shows that the function $\Phi(\alpha)$ lies in

$$\bigcap_{i=1}^n \ker \left(\partial_i: \mathcal{O}(\text{MC}^n(L)) \rightarrow \mathcal{O}(\text{MC}^{n-1}(L)) \right),$$

and thus determines an element of $N_n(\mathcal{O}(\text{MC}^\bullet(L)))$. The resulting linear map from $\bigoplus_{n=0}^\infty (L^{n+1})^\vee$ to $\bigoplus_{n=0}^\infty N_{-n}(\mathcal{O}(\text{MC}^\bullet(L)))$ induces a morphism of graded commutative algebras

$$(5.1) \quad \Phi: \mathcal{O}(\mathcal{MC}(L)) \rightarrow N_{-*}(\mathcal{O}(\text{MC}^\bullet(L))).$$

Lemma 5.1. *The morphism Φ is compatible with the differentials on the differential graded algebras $\mathcal{O}(\mathcal{MC}(L))$ and $N_{-*}(\mathcal{O}(\text{MC}^\bullet(L)))$.*

Proof. The differential $d: \mathcal{O}^{n-1}(\mathcal{MC}(L)) \rightarrow \mathcal{O}^n(\mathcal{MC}(L))$ is the sum of differentials d_1 and d_2 , given by the formulas $(d_1\alpha)(x) = \alpha(\delta x)$ and $(d_2\alpha)(x, y) = (-1)^{|x|} \alpha([x, y])$, where $x, y \in L^*$.

Using the explicit formula for the codegeneracy map (4.1), we may show that the product of the linear forms $\Phi(\beta)$ and $\Phi(\gamma)$ associated to the one-cochains $\beta \in (L^{p+1})^\vee$ and $\gamma \in (L^{q+1})^\vee$ on L^* is represented by the following quadratic polynomial on $\text{MC}^{p+q}(L)$:

$$(\Phi(\beta)\Phi(\gamma))(x) = \sum_{\substack{I=\{i_1 < \dots < i_p\} \\ J=\{j_1 < \dots < j_q\} \\ I \sqcup J = \{0, \dots, p+q-1\}}} (-1)^{\sum_{\ell=1}^p (i_\ell - \ell + 1)} \beta(x_{i_1 \dots i_p}) \gamma(x_{j_1 \dots j_q}).$$

The differential

$$d: N_{-n}(\mathcal{O}(\text{MC}^\bullet(L))) \rightarrow N_{-n+1}(\mathcal{O}(\text{MC}^\bullet(L)))$$

equals the pullback by the morphism $d^0: \text{MC}^{n-1}(L) \rightarrow \text{MC}^n(L)$. Applied to $\Phi(\alpha)$, where $\alpha \in (L^{n+1})^\vee$, (4.4) gives

$$d\Phi(\alpha)(x) = -\alpha(\delta x_0 \dots x_{n-2}) - \frac{1}{2} \sum_{I \sqcup J = \{0, \dots, n-2\}} (-1)^{\sum_{\ell=1}^{|I|} (i_\ell - \ell + 1)} \alpha([x_I, x_J]).$$

The first and second terms inside the parentheses correspond to d_1 and d_2 respectively: in the case of d_2 , we use the explicit formula for $\Phi(\beta)\Phi(\gamma)$ to make this identification. \square

Let $F^1\mathcal{O}(\mathcal{MC}(L))$ be the augmentation ideal of the algebra of Chevalley–Eilenberg cochains of L^* (that is, cochains of negative degree), and let $F^1\mathcal{O}(\mathcal{MC}^\bullet(L))$ be the augmentation ideal of the simplicial commutative algebra $\mathcal{O}(\mathcal{MC}^\bullet(L))$ (that is, polynomials with vanishing constant term). For $k > 1$, let $F^k\mathcal{O}(\mathcal{MC}(L))$ and $F^k\mathcal{O}(\mathcal{MC}^\bullet(L))$ be the k th powers of $F^1\mathcal{O}(\mathcal{MC}(L))$, and $F^1\mathcal{O}(\mathcal{MC}^\bullet(L))$, and let $F^k N_*(\mathcal{O}(\mathcal{MC}^\bullet(L))) = N_*(F^k\mathcal{O}(\mathcal{MC}^\bullet(L)))$. The morphism Φ of (5.1) is compatible with the filtrations on $\mathcal{O}(\mathcal{MC}(L))$ and $N_*(\mathcal{O}(\mathcal{MC}^\bullet(L)))$, and the induced morphism

$$\mathrm{gr}_F \Phi: \mathrm{gr}_F \mathcal{O}(\mathcal{MC}(L)) \rightarrow \mathrm{gr}_F N_{-*}(\mathcal{O}(\mathcal{MC}^\bullet(L)))$$

may be identified with the morphism

$$(5.2) \quad \Phi: \mathcal{O}(\mathcal{MC}(L^{\natural})) \rightarrow N_{-*}(\mathcal{O}(\mathcal{MC}^\bullet(L^{\natural})))$$

of differential graded commutative algebras. Theorem 1.6 is thus a consequence of the following lemma.

Lemma 5.2. *The morphism (5.2) is a quasi-isomorphism.*

Proof. Let Z_* be the connective chain complex $L^{\natural}[1]^\vee$: as a vector space, we have $Z_n \cong (L^{n+1})^\vee$. The iterated shuffle product $g^{\otimes n}$ induces a morphism of simplicial abelian groups

$$K_\bullet g^{(n)}: K_\bullet(Z^{\otimes n}) \rightarrow K_\bullet(Z)^{\otimes n},$$

which is a quasi-isomorphism by the Eilenberg–Zilber theorem. This morphism is equivariant with respect to the action of the symmetric group S_n . Taking invariants, summing over n , and taking normalized chains, we obtain a quasi-isomorphism of differential graded commutative algebras

$$\mathrm{Sym} Z \rightarrow N_*(\mathrm{Sym} K_\bullet(Z)).$$

This may be identified with the morphism Φ of (5.2). \square

6. Generalization to nilpotent L_∞ -algebras

The definition of the differential graded scheme $\mathcal{MC}(L)$ extends to L_∞ -algebras: these are a generalization of differential graded Lie algebras in which the Jacobi rule is only satisfied up to a hierarchy of higher homotopies.

An operation $[x_1, \dots, x_k]$ on a graded vector space L^* is *graded antisymmetric* if

$$[x_1, \dots, x_i, x_{i+1}, \dots, x_k] + (-1)^{|x_i||x_{i+1}|} [x_1, \dots, x_{i+1}, x_i, \dots, x_k] = 0$$

for all $1 \leq i \leq k - 1$.

An L_∞ -algebra is a graded vector space L^* with graded antisymmetric operations a sequence $[x_1, \dots, x_n]$ of degree $2 - n$, $n > 0$, such that for each n ,

$$\sum_{k=1}^n \sum_{\substack{I=\{i_1 < \dots < i_k\} \\ J=\{j_1 < \dots < j_{n-k}\} \\ I \cup J = \{1, \dots, n\}}} (-1)^{\varepsilon + \sum_{i=1}^{n-k} (j_i - i)} [[x_{i_1}, \dots, x_{i_k}], x_{j_1}, \dots, x_{j_{n-k}}] = 0.$$

Here, the sign $(-1)^\varepsilon$ is the sign associated by the Koszul sign convention to the action of π on the elements x_1, \dots, x_n of L^* . The 1-bracket $x \mapsto [x]$ is a differential on L^* , so an L_∞ -algebra is in particular a cochain complex.

An L_∞ -algebra L^* is *nilpotent* if it has a decreasing filtration $F_k L^*$ such that for each $i \in \mathbb{Z}$, $F_k L^i = 0$ if $k \gg 0$, and for each $n > 0$,

$$[F_{k_1} L^{i_1}, \dots, F_{k_n} L^{i_n}] \subset F_{k_1 + \dots + k_n + 1} L^{i_1 + \dots + i_n - n + 2}.$$

In particular, every differential graded Lie algebra concentrated in degrees ≥ 1 is nilpotent.

Lemma 6.1. *Let L^* be an L_∞ -algebra. Its truncation L^*_+ is an L_∞ -algebra, which is nilpotent if and only if the curvature*

$$F(x) = \sum_{n=1}^{\infty} \frac{1}{n!} [x^{\otimes n}]$$

is a polynomial map from L^1 to L^2 .

The Maurer–Cartan locus $\text{MC}(L) \subset L^1$ of an L_∞ -algebra L^* is the vanishing locus of the Maurer–Cartan equation $F(x) = 0$. The differential graded Maurer–Cartan locus $\mathcal{MC}(L)$ of L is the affine differential graded scheme with underlying scheme the affine space L^1 , and with the differential graded algebra of functions

$$\mathcal{O}(\mathcal{MC}(L)) = \text{Sym}(L_+[1]^\vee).$$

The differential d on $\mathcal{O}(\mathcal{MC}(L))$ is the differential of the generalization of the Chevalley–Eilenberg complex to L_∞ -algebras: it is the sum of the adjoints of the n -fold brackets $[-, \dots, -]$, which maps $(L^k)^\vee$ to

$$\bigoplus_{\substack{i_1 + \dots + i_n = k - n + 2 \\ i_1, \dots, i_n \geq 1}} (L^{i_1})^\vee \otimes \dots \otimes (L^{i_n})^\vee.$$

Example 6.2. If $F: V \rightarrow W$ is an arbitrary polynomial which vanishes at $0 \in V$, we may form an L_∞ -algebra L^* with $L^1 = V$, $L^2 = W$, and all other vector spaces L^i vanishing. The brackets of L^* are the polarizations of the homogeneous components of the polynomial $F: L^1 \rightarrow L^2$. The Maurer–Cartan locus of L^* is the vanishing locus of the polynomial F , and $\mathcal{O}(\mathcal{MC}(L))$ is the Koszul complex of F .

Thus, the differential graded Maurer–Cartan locus for nilpotent L_∞ -algebras generalizes at the same time the differential graded Maurer–Cartan locus for differential graded Lie algebras and the Koszul complex for a polynomial map between finite dimensional vector spaces.

The tensor product $L^* \otimes \Lambda^n$ of a nilpotent L_∞ -algebra L^* with Λ^n is again a nilpotent L_∞ -algebra, with brackets

$$[x_1 \otimes \alpha_1, \dots, x_n \otimes \alpha_n] = (-1)^{\sum_{i>j} k_i \ell_j} [x_1, \dots, x_n] \otimes \alpha_1 \dots \alpha_n$$

for $x_i \in L^{k_i}$ and $\alpha_i \in (\Lambda^n)^{\ell_i}$.

Definition 6.3. The derived Maurer–Cartan locus $\mathcal{MC}^\bullet(L)$ of a nilpotent L_∞ -algebra is the cosimplicial scheme

$$\mathcal{MC}^n(L) = \mathcal{MC}(L \otimes \Lambda^n).$$

The statement and proof of Theorem 1.6 extend without difficulty to nilpotent L_∞ -algebras. The only twist in the proof is the verification of Lemma 5.1 in this more general setting, that Φ is a morphism of complexes. We leave this task to the motivated reader.

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Ezra GETZLER, Department of Mathematics, Northwestern University, Evanston, IL 60208-2730, USA

e-mail: getzler@northwestern.edu