

## On the covering type of a space

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**Abstract.** We introduce the notion of the “covering type” of a space, which is more subtle than the notion of the Lusternik–Schnirelman category. It measures the complexity of a space which arises from coverings by contractible subspaces whose non-empty intersections are also contractible.

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From the point of view of Algebraic Topology, the simplest spaces are the contractible ones. A crude measure of the complexity of a space  $X$  is the size of a finite open covering of  $X$  by contractible subspaces; this idea goes back to the work of Lusternik and Schnirelman in 1934. A more subtle invariant is the size of *good covers*, i.e., covers by contractible subspaces such that each of their non-empty intersections is also contractible. The idea of a good cover appears in a 1952 paper by André Weil [Wei], but is preceded by Leray’s notion of a *convexoid cover* [Ler, p. 139] which uses closed covers with acyclic intersections.

We define the *covering type* of  $X$  to be the minimum size of a good cover of a space homotopy equivalent to  $X$ . See Definition 1.2 below for a precise description using open covers; for finite CW complexes, an equivalent version using closed covers by subcomplexes is given in Theorem 2.5.

We will see that this is an interesting measure of the complexity of a space. For connected graphs, the covering type is approximately  $\sqrt{2h}$ , where  $h$  is the number of circuits. Thus it is very different from the Lusternik–Schnirelman category (which is 2 for non-tree graphs), and Farber’s topological complexity [Far] (which is at most 3 for graphs).

The covering type of a surface is related to its chromatic number. The *chromatic number* of a surface is the smallest number  $n$  such that every map on the surface is  $n$ -colorable (see Definition 6.1); it was first described in 1890 by Haewood

[Hae]. Finding the chromatic number of a surface was long known as the *map coloring problem*. The chromatic number of the 2-sphere is 4; this is the “four color theorem,” settled in 1976. The other cases of the map coloring problem were settled in 1968; see [Rin2]. For the 2-sphere, torus and projective plane, the covering type equals the chromatic number: 4, 7 and 6, respectively.

Special solutions to the map coloring problem give an upper bound for the covering type of a surface: it is at most one more than the chromatic number. Combinatorial and topological considerations of any space, such as its Betti numbers, yield general lower bounds for its covering type. This approach shows that the covering type of a surface is always more than half the chromatic number. Although our upper and lower bounds for the covering type of a surface differ as functions of its genus  $g$ , both are linear in  $\sqrt{g}$  (see Section 5).

**Open problem.** Except for the sphere, torus and projective plane, determining the covering type of a surface is an open problem. For example, we do not know if the covering type of the Klein bottle is 7 or 8. For the 2-holed torus (oriented surface of genus 2), we only know that the covering type is between 6 and 10.

**Motivation.** Leray’s motivation for introducing his convexoid covers was to easily compute homology; see [Ler, pp. 153–159].

Weil’s motivation for introducing special open coverings was to prove de Rham’s theorem for a manifold [Wei]. Here is a formalization of his idea in the language of cohomology theories. Let  $h^*$  and  $k^*$  be two cohomology theories and let  $T : h^* \rightarrow k^*$  be a natural transformation such that the kernel and cokernel of the homomorphism  $T_X : h^*(X) \rightarrow k^*(X)$  are of order at most  $m$  when  $X$  is a point. Using Mayer-Vietoris sequences and induction on  $n$ , we see that for a space  $X$  of covering type  $n$ , the kernel and cokernel of  $T_X$  have order at most  $m^{2^n}$ . This general principle was applied by Weil in the case where  $h^*$  is singular cohomology,  $k^*$  is de Rham cohomology and  $m = 0$  (Poincaré’s lemma). We used the same idea with  $m = 2^t$  in a preliminary version of our paper [KSW], comparing the algebraic Witt group of a real algebraic variety with a purely topological invariant; this was in fact our initial motivation for investigating the notion of covering type.

Here are our definitions of a good cover and the covering type:

**Definition 1.1.** *Let  $X$  be a topological space. A family of contractible open subspaces  $U_i$  forms a **good (open) cover** if every nonempty intersection  $U_{i_1} \cap \dots \cap U_{i_n}$  is contractible.*

**Definition 1.2.** *The strict covering type of  $X$  is the minimum number of elements in a good cover of  $X$ . The covering type of  $X$ ,  $ct(X)$ , is the minimum of the strict covering types of spaces  $X'$  homotopy equivalent to  $X$ .*

Thus  $ct(X) = 1$  exactly when  $X$  is contractible, and  $ct(X) = 2$  exactly when  $X$  is the disjoint union of two contractible spaces. It is easy to see that a circle has covering type 3, and only slightly harder to see that the 2-sphere and the figure eight have covering type 4 (small neighborhoods of the faces of a tetrahedron give a good cover of the 2-sphere).

The difference between the strict covering type and the covering type is illustrated by bouquets of circles.

**Example 1.3.** *The strict covering type of a bouquet of  $h$  circles is  $h + 2$ , since 3 subcomplexes are needed for each circle. For  $h = 2$  and 3,  $S^1 \vee S^1$  and  $S^1 \vee S^1 \vee S^1$  both have covering type at most 4, since Figure 1 indicates good covers of homotopy equivalent spaces. (The  $U_i$  are small neighborhoods of the three outer edges,  $X_1, X_2, X_3$  and the inside curve  $X_4$ ). We will see in Proposition 4.1 that the covering type is exactly 4 in both cases.*

If  $X$  is a CW complex, we may replace ‘open subspace’ by ‘closed subcomplex’ in Definition 1.1 to obtain the analogous notion of a *good closed cover* of  $X$ , and the concomitant notion of a closed covering type. We will show in Theorem 2.5 that the closed and open covering types of a finite CW complex agree. This simplifies our illustrations.

We have structured this article as follows. In Section 2, we show that we may also define the covering type of a finite CW complex using covers by contractible subcomplexes. In Section 3, we establish useful lower bounds on covering type using homology, and use these bounds in Section 4 to determine the covering type of any graph in Proposition 4.1.

In Sections 5 and 6 we determine the covering type of some classical surfaces, and use graphs on an arbitrary surface to give upper bounds for its covering type. For oriented surfaces of genus  $g > 2$ , the covering type lies between  $2\sqrt{g}$  and  $3.5\sqrt{g}$ ; see Proposition 6.2. Similar bounds hold for non-oriented surfaces; see Proposition 6.4.

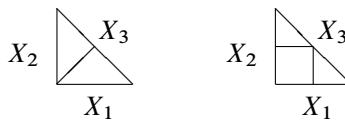


FIGURE 1  
 $ct(X) = 4$  for bouquets of 2 or 3 circles

We conclude in Section 7 with some upper bounds for the covering type in higher dimensions, showing for instance that  $ct(\mathbb{R}\mathbb{P}^m)$  is between  $m + 2$  and  $2m + 3$ .

## 2. Open vs. closed covers

Any cover of a manifold by geodesically convex open subsets is a good cover; by convexity, every nonempty intersection of them is contractible. Thus compact manifolds have finite strict covering type. (An elementary proof is given in [KL, VI.3].)

Finite simplicial complexes also have finite strict covering type, because the open stars of the vertices form a good cover. Finite polyhedra also have finite strict covering type, as they can be triangulated. Any finite CW complex is homotopy equivalent to a finite polyhedron by [Whi, Thm. 13], so its covering type is also finite.

**Proposition 2.1.** *If a paracompact space has finite covering type  $n$ , it is homotopy equivalent to a finite CW complex whose strict covering type is  $n$ .*

*Proof.* Suppose that  $X$  is a paracompact space with a finite open cover  $\mathcal{U} = \{U_i\}$  and let  $N$  be its geometric nerve;  $N$  is a simplicial complex whose vertices are the indices  $i$ , and a set  $J$  of vertices spans a simplex of  $N$  if  $\bigcap_{i \in J} U_i \neq \emptyset$ . The Alexandroff map  $X \rightarrow N$  is a continuous function, given by the following standard construction; see [AH, IX.3.4] or [Dug, VIII.5.4]. Choose a partition of unity  $\{v_i\}$  associated to the open cover. Any point  $x \in X$  determines a set  $J_x$  of indices (the  $i$  such that  $x \in U_i$ ), and the Alexandroff map sends a point  $x$  to the point with barycentric coordinates  $v_i(x)$  in the simplex spanned by  $J_x$ .

If  $\{U_i\}$  is a good cover, the Alexandroff map  $X \rightarrow N$  is a homotopy equivalence, by the “Nerve Lemma” (see [Hat, 4G.3]). If  $\mathcal{U}$  has  $n$  elements then  $N$  has  $n$  vertices, and the open stars of these vertices form a good cover of  $N$ . If  $ct(X) = n$  then  $N$  has no smaller good cover, and hence  $N$  has strict covering type  $n$ .  $\square$

**Example 2.2.** *If  $ct(X) = 3$ ,  $X$  is either the disjoint union of 3 contractible sets or is homotopy equivalent to the circle, by the Alexandroff map. To see this, suppose that  $\{U_1, U_2, U_3\}$  is a good cover of a connected  $X$ . By a case by case inspection, we see that  $H^1(X) = 0$  and  $ct(X) < 3$  unless  $U_i \cap U_j \neq \emptyset$  for all  $i, j$  and  $U_1 \cap U_2 \cap U_3 = \emptyset$ . In this case, the Alexandroff map  $X \rightarrow S^1$  is a homotopy equivalence and  $\dim H^1(X) = 1$ .*

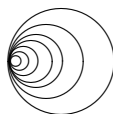


FIGURE 2  
Hawaiian earring

Similarly, a case by case inspection shows that (up to homotopy) the only connected spaces with covering type 4 are the 2-sphere and the bouquets of 2 or 3 circles illustrated in Figure 1.

**Remark 2.3.** The “Hawaiian earring” [Hat, 1.25] is the union of the circles  $(x - 1/n)^2 + y^2 = 1/n^2$  in the plane; see Figure 2. It is a compact space which has no good open cover; its strict covering type is undefined. It follows from Proposition 2.1 and compactness that its covering type is also undefined.

**Definition 2.4.** If  $X$  is a finite CW complex, a **good closed cover** is a family of contractible subcomplexes  $\{X_i\}$  such that every intersection of the  $X_i$  is either empty or contractible.

For example, the maximal simplices of any simplicial complex form a good closed cover. We will see more examples in Sections 4 and 5.

For our next result, we need the classifying space  $BP$  of a finite poset  $P$ . It is a simplicial complex whose vertices are the elements of  $P$ , and whose simplices are the totally ordered subsets of  $P$ .

**Theorem 2.5.** If  $X$  is a finite CW complex, the covering type of  $X$  is the minimum number of elements in a good closed cover of some complex homotopy equivalent to  $X$ .

*Proof.* Suppose that  $ct(X) = n$ . By Proposition 2.1, we may suppose that  $X$  is a simplicial complex with  $n$  vertices. We need to show that  $X$  has a good closed cover with  $n$  elements. Let  $X_i$  denote the closed subcomplex of  $X$  consisting of points whose  $i^{th}$  barycentric coordinate is  $\geq 1/n$ . Then  $\{X_i\}$  is a closed cover of  $X$ , because at least one barycentric coordinate must be  $\geq 1/n$ . To see that it is a good closed cover, choose a subset  $J$  of  $\{1, \dots, n\}$  for which  $X_J = \bigcap_{j \in J} X_j \neq \emptyset$ . Then  $X_J$  is contractible because, if  $p$  is the point whose  $i^{th}$  barycentric coordinate is  $1/|J|$  for  $i \in J$  and 0 otherwise, the formula  $h_t(x) = tx + (1-t)p$  defines a deformation retraction from  $X_J$  to the point  $p$ . (To see that  $h_t(x)$  is in  $X_J$  for all  $t$  with  $0 \leq t \leq 1$ , note that for each  $j \in J$ ,

the  $j^{th}$  coordinate of  $h_j(x)$  is at least  $1/n$ .) Thus  $\{X_i\}_{i=1}^n$  is a good closed cover of  $X$ , as required.

Conversely, given a good closed cover  $\{X_i\}_{i=1}^n$ , the nerve  $N$  of this cover has  $n$  vertices, and has a good open cover of size  $n$ . We will construct a poset  $P$  and homotopy equivalences  $X \leftarrow BP \rightarrow N$ , showing that  $ct(X) \leq n$ . For this, we may assume  $X$  is connected.

Let  $S$  be the family of subsets  $J$  of  $\{1, \dots, n\}$  with  $X_J = \bigcap_{i \in S} X_i \neq \emptyset$ ; it is a poset ( $I \leq J$  if  $J \subseteq I$ ) and its realization  $BS$  is the geometric nerve of the cover,  $N$ . There is a functor from  $S$  to contractible spaces, sending  $J$  to  $X_J$ : if  $I \subseteq J$  then  $X_J \subseteq X_I$ . Following Segal [Seg, 4.1], the disjoint union of all nonempty intersections  $X_J$  is a topological poset  $P$ , and the obvious functor  $P \rightarrow S$  yields a continuous function from the geometric realization  $BP$  to  $N = BS$ . By the Lemma on p. 98 of [Qui] (the key ingredient in the proof of Quillen's Theorems A and B),  $BP \rightarrow BS$  is a homotopy equivalence. There is a canonical proper function  $BP \xrightarrow{p} X$  obtained from the functor from  $P$  to the trivial topological category with  $X$  its space of objects. It is an isomorphism on homology, because the inverse image of a point  $x \in X$  is a simplex, whose vertices correspond to the  $i$  with  $x \in X_i$ .

Consider the universal covering space  $\tilde{X} \rightarrow X$ ; the inverse image of each  $X_i$  is a disjoint union of spaces  $X_{i,\alpha}$ , each homeomorphic to  $X_i$ , and the  $\{X_{i,\alpha}\}$  form a good cover of  $\tilde{X}$ . (For each  $i$ , there is a non-canonical bijection between  $\{(i,\alpha)\}$  and  $\pi_1 X$ .) If  $\tilde{P}$  denotes the topological poset of intersections of the  $\{X_{i,\alpha}\}$  then, as above,  $\tilde{p} : B\tilde{P} \rightarrow \tilde{X}$  is an isomorphism on homology. Since both spaces are simply connected, the Whitehead Theorem implies that  $\tilde{p}$  is a homotopy equivalence. By inspection,  $B\tilde{P}$  is the universal covering space of  $BP$ , so  $\pi_1(BP) \cong \pi_1(\tilde{X})$ . This implies that  $p$  is a homotopy equivalence, since for  $n > 1$  we have  $\pi_n BP \cong \pi_n B\tilde{P} \cong \pi_n \tilde{X} \cong \pi_n X$ . □

### 3. Lower bounds for the covering type

In general, the covering type of  $X$  is not so easy to compute. A simple lower bound is provided by the proposition below, derived via a Mayer-Vietoris argument for the homology of  $X$ . We omit the proof, since it will follow from the more general Theorem 3.3 below.

**Proposition 3.1.** *Fix a field  $k$  and let  $hd(X)$  denote the homological dimension of  $X$ , i.e., the maximum number such that  $H_m(X, k)$  is nonzero. Then, unless  $X$  is empty or contractible,*

$$ct(X) \geq hd(X) + 2.$$

**Example 3.2.** *The sphere  $S^m$  has  $ct(S^m) = m + 2$ . This is clear for  $S^0$  and  $S^1$ , and the general case follows from the upper bound in Proposition 3.1, combined with the observation that  $S^m$  is homeomorphic to the boundary of the  $(m + 1)$ -simplex, which has  $m + 2$  maximal faces.*

*Alternatively, we could get the upper bound from the suspension formula  $ct(SX) \leq 1 + ct(X)$ . This formula is a simple exercise which will be generalized later on (Theorem 7.1).*

A stronger lower bound for  $ct(X)$  uses the Poincaré polynomial

$$P_X(t) = h_0 + h_1t + \dots + h_mt^m$$

where  $h_i$  is the rank of the homology  $H_i(X)$  or cohomology  $H^i(X)$  (with coefficients in a field). We partially order the set of polynomials in  $\mathbb{Z}[t]$  by declaring that  $P \leq Q$  if and only if all the coefficients of  $P$  are smaller or equal to the respective coefficients of  $Q$ . We now have the following theorem:

**Theorem 3.3.** *Let  $P_X(t)$  be the Poincaré polynomial of  $X$  and let  $n$  be its covering type. If  $X$  is not empty then:*

$$P_X(t) \leq \frac{(1+t)^{n-1} - 1}{t} + 1 = n + \binom{n-1}{2}t + \binom{n-1}{3}t^2 + \dots + t^{n-2}.$$

*That is,  $h_0 \leq n$ ,  $h_1 \leq \binom{n-1}{2}, \dots, h_{n-2} \leq 1$  and  $h_i = 0$  for  $i \geq n - 1$ .*

*Proof.* We proceed by induction on  $n = ct(X)$ . If  $n$  is 1 or 2 then  $P_X(t)$  is 1 or 2, respectively, and the inequality is trivial.

If  $X = \bigcup_{k=1}^n X_k$ , set  $Y = \bigcup_{k \neq 1} X_k$  and note that  $ct(Y)$ ,  $ct(Y \cap X_1)$  are at most  $n - 1$ . From the Mayer-Vietoris sequence for  $X = X_1 \cup Y$ ,

$$H_k(X_1) \oplus H_k(Y) \rightarrow H_k(X) \rightarrow H_{k-1}(X_1 \cap Y),$$

and the inductive hypothesis, we see that  $h_0(X) \leq n$  and for  $k > 0$

$$h_k(X) \leq h_k(Y) + h_{k-1}(X_1 \cap Y) \leq \binom{n-2}{k} + \binom{n-2}{k-1} = \binom{n-1}{k}. \quad \square$$

**Remark 3.4.** The lower bounds in Theorem 3.3 are not optimal for non-connected spaces, such as discrete sets, because the covering type of a non-connected space is the sum of the covering types of its components. For this reason, we shall concentrate on the covering type of connected spaces. In particular, we will see that that the bound in Theorem 3.3 is optimal when  $X$  is 1-dimensional and connected.

### 4. Graphs

Every 1-dimensional CW complex is a graph, and every connected graph is homotopy equivalent to a bouquet of  $h = 1 + E - V$  circles, where  $V$  and  $E$  are the number of vertices and edges, respectively. Since  $n = ct(X)$  is an integer, the bound in Theorem 3.3, that  $\binom{n-1}{2} \geq h$ , is equivalent to the inequality  $n \geq \lceil x \rceil$ , where  $x = \frac{3+\sqrt{1+8h}}{2}$  and the ceiling  $\lceil x \rceil$  of  $x$  denotes the smallest integer  $\geq x$ .

**Proposition 4.1.** *When  $X_h$  is a bouquet of  $h$  circles then*

$$ct(X_h) = \left\lceil \frac{3 + \sqrt{1 + 8h}}{2} \right\rceil.$$

That is,  $ct(X_h)$  is the unique integer  $n$  such that

$$\binom{n-2}{2} < h \leq \binom{n-1}{2}.$$

*Proof.* By explicitly solving the displayed quadratic inequalities, we see that the unique integer  $n$  satisfying the displayed inequalities is  $\lceil \frac{3+\sqrt{1+8h}}{2} \rceil$ . Theorem 3.3 implies that  $ct(X_h) \geq n$ . When  $h = 1$  (a circle), we saw that the lower bound  $ct = 3$  is achieved. For  $h = 2, 3$  the lower bound is 4, and we saw in Example 1.3 that it is also an upper bound, so  $ct(X_h) = 4$  in these cases.

When  $h$  is 6 or 10 we show that  $ct(X_h)$  is 5 or 6, respectively, by generalizing the pattern of Example 1.3 to introduce more L-shaped lines into the interior of a triangle, as shown in Figure 3. For a bouquet  $X_h$  of  $h = \binom{n-1}{2}$  circles, the same construction (using  $n - 3$  internal L-shaped lines) shows that  $ct(X_h) = n$ . If  $h$  and  $n$  satisfy the strict inequality of the proposition, we can construct a complex  $X'$  like Figure 3 for  $\binom{n-1}{2}$  circles and erase portions of the interior lines to obtain a complex  $X$  homotopic to a bouquet of  $h$  circles, as illustrated by Figure 4. This shows that  $ct(X_h) \leq n$ , as required. □

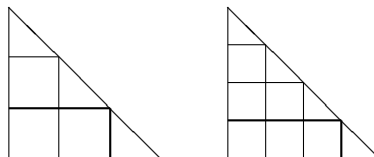


FIGURE 3  
 $ct(X_h) = 5, 6$  for bouquets of 6 and 10 circles



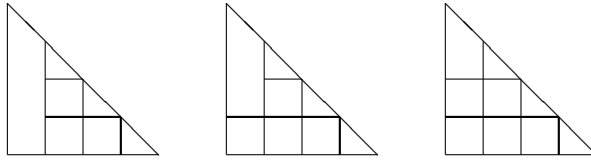


FIGURE 4  
 $ct(X_h) = 6$  for bouquets of 7–9 circles

### 5. Classical surfaces

The literature contains some pretty upper bounds for the covering type of an oriented surface of genus  $g$  (a torus with  $g$  holes). We have already seen in Example 3.2 that the 2-sphere has covering type 4; this is the case  $g = 0$ .

**Torus.** For the torus  $T$  (the case  $g = 1$ ), an upper bound  $ct(T) \leq 7$  comes from the 7-country map in Figure 5, first described by Haewood in 1890 [Hae] in connection with the map coloring problem. The map is dual to an embedding of the complete graph on 7 vertices in the torus, so each of the 7 countries is a hexagon, each pair meets in a face, and any three countries meet in a point. We shall see in Theorem 5.3 below that the covering type of the torus is indeed 7.

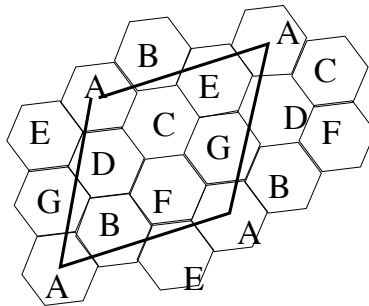


FIGURE 5  
 Torus with 7 regions

**Genus 2.** An upper bound for  $S_2$  (the oriented surface of genus 2) is 10. This comes from the example of a 10-region good covering of  $S_2$  given by Jungerman and Ringel in [JR, p. 125], and reproduced in Figure 6. To construct it, we start with a 10-region map on the 2-sphere (countries labelled 0–9), cut out the opposing shaded circles and identify their edges. We do not know the

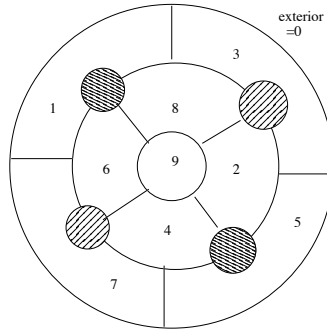


FIGURE 6  
Surface of genus 2, covered by 10 regions

precise value of  $ct(S_2)$ . We will see in Proposition 5.2 below that  $ct(S_2) \geq 6$ , so  $6 \leq ct(S_2) \leq 10$ .

Before giving more precise lower bounds for  $ct(X)$ , we record a simple result, which shows that  $ct(X) > 4$  for every closed surface  $X$  except the 2-sphere. We shall write  $H^n(X)$  for the cohomology of  $X$  with coefficients in a fixed field.

**Lemma 5.1.** *Suppose that  $X = A \cup B$  and that  $H^1(A) = H^1(B) = 0$ . Then the cohomology cup product  $H^1(X) \times H^1(X) \rightarrow H^2(X)$  is zero.*

*Proof.* Fix  $x, y \in H^1(X)$ . Since  $x$  vanishes in  $H^1(A)$ , it lifts to an element  $x'$  in  $H^1(X, A)$ . Similarly,  $y$  lifts to an element  $y'$  in  $H^1(X, B)$ . Then  $x \cup y$  is the image of  $(x', y')$  under the composition

$$H^1(X, A) \times H^1(X, B) \xrightarrow{\cup} H^1(X, A \cup B) \longrightarrow H^1(X),$$

where  $\cup$  is the relative cup product; see [Hat, p.209] Since  $X = A \cup B$ ,  $H^1(X, A \cup B) = 0$ , so  $x \cup y = 0$ . □

**Proposition 5.2.** *If the cohomology cup product is nonzero on  $H^1(X)$ , then  $ct(X) \geq 6$ .*

*Proof.* The assumption implies that  $H^2(X) \neq 0$ , so  $ct(X) \geq 4$  by Theorem 3.3. If the covering type were 4, let  $A$  be the union of the first two and  $B$  the union of the last two subspaces. Since  $A$  and  $B$  are homotopic to either one or two points, Lemma 5.1 implies that the cup product is zero in  $H^*(X)$ , contrary to fact. This shows that  $ct(X) \neq 4$ .

Now suppose that  $\{X_i\}_{i=1}^5$  is a good cover of  $X$ . If  $X_i \cap X_j \cap X_k = \emptyset$  for all  $i, j, k$  then the Čech complex associated to the cover would have zero in degree 2. Since the cohomology of this complex is  $H^*(X)$ , that would contradict the assumption that  $H^2(X) \neq 0$ . By Example 2.2, there exists  $i, j, k$  so that  $A = X_i \cup X_j \cup X_k$  has  $H^1(A) = 0$ . Let  $B$  be the union of the other two  $X_m$ ; we also have  $H^1(B) = 0$ . By Lemma 5.1, the cohomology product is zero. This contradiction shows that  $ct(X) \neq 5$ .  $\square$

**Theorem 5.3.** *The covering type of the torus  $T$  is exactly 7.*

*Proof.* By Figure 5 and Proposition 5.2, the covering type of  $T$  is either 6 or 7. Suppose the covering type were 6, i.e., that  $T$  had a good cover by 6 subcomplexes  $X_i, i = 1, \dots, 6$ . As in the proof of Proposition 5.2, the fact that  $H^2(T) \neq 0$  implies that there exist  $i, j, k$  such that  $X_i \cap X_j \cap X_k \neq \emptyset$ . By Example 2.2,  $H^1(X_i \cup X_j \cup X_k) = 0$ . Therefore, after reordering the indices of the cover, we may assume that  $H^1(B) = 0$ , with  $B = X_4 \cup X_5 \cup X_6$ . Set  $A = X_1 \cup X_2 \cup X_3$ , and note that, since  $\dim H^1(A) \leq 1$  and  $\dim H^1(X) = 2$ , there is an  $x$  in the kernel of  $H^1(X) \rightarrow H^1(A)$ . Lifting it to an element  $x'$  in  $H^1(X, A)$ , and lifting an independent element  $y$  of  $H^1(X)$  to  $H^1(X, B)$ , the proof of Lemma 5.1 shows that  $x \cup y = 0$  in  $H^2(X)$ , which is not the case.  $\square$

If  $X$  is a non-orientable surface, its *genus*  $q$  is the dimension of  $H^1(X, \mathbb{Z}/2)$ . Up to homeomorphism, there is a unique non-orientable surface  $N_q$  of genus  $q$  for each integer  $q \geq 1$ , with  $N_1$  the projective plane and  $N_2$  the Klein bottle.

**Projective plane.** Here is a pretty construction of a good covering of the projective plane  $\mathbb{R}P^2$  by 6 regions. The antipode on the 2-sphere sends the regular dodecahedron to itself (preserving the 20 vertices, 30 edges and 12 faces); the quotient by this action defines a good polyhedral covering of the projective plane  $\mathbb{R}P^2$  by 6 faces (with 10 vertices and 15 edges).

This construction yields 6-colorable map on  $\mathbb{R}P^2$  whose regions are pentagons; this is illustrated in Figure 7. We remark that the boundary of this polyhedral cover is the *Petersen graph* (see [GT, Ex. 4.2.6(5)]).

**Theorem 5.4.** *The covering type of  $\mathbb{R}P^2$  is 6.*

*Proof.* The good cover associated to the embedding of the Petersen graph (giving the 6-coloring in Figure 7) shows that 6 is an upper bound for the covering type. Since the cup product on  $H^1(\mathbb{R}P^2; \mathbb{Z}/2)$  is nonzero,  $ct(\mathbb{R}P^2) \geq 6$  by Proposition 5.2.  $\square$



FIGURE 7  
The projective plane, covered by 6 regions

Exterior region 0

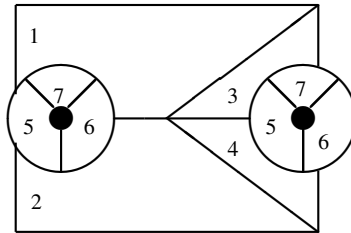


FIGURE 8  
A good covering of the Klein bottle, 8 regions

**Klein bottle.** A good covering of the Klein bottle by 8 regions is illustrated in Figure 8. To get it, we start with a cover of the 2-sphere by the five regions 0–4, add three regions in a circle along the intersection of regions 0, 1 and 2 (labelled 5–7), and another three regions in a circle at the intersection of regions 0, 3 and 4 (also labelled 5–7). Then cut out the two dark circles and identify their boundaries as indicated. Note that region 7 does not meet regions 2 and 4, and regions 3 and 6 do not meet either.

We will see in Corollary 5.6 that the covering type of the Klein bottle is either 7 or 8.

**Theorem 5.5.** *If  $q \geq 2$ , then  $ct(N_q) \geq 7$ .*

*Proof.* When  $q \geq 2$ , it is well known that the cup product is nontrivial on  $H^1(N_q, \mathbb{Z}/2)$ . For example,  $H^1(N_2, \mathbb{Z}/2)$  is 2-dimensional, and has a basis  $\{x, y\}$  such that  $y^2 = 0$ ,  $x^2 = x \cup y \neq 0$ . By Proposition 5.2, the covering type is at least 6.

Suppose that  $X = N_q$  had a good cover with 6 regions  $X_i$ . As in the proof of Proposition 5.2, the fact that  $H^2(X) \neq 0$  implies that there exist  $i, j, k$  such that  $X_i \cap X_j \cap X_k \neq \emptyset$ . Set  $B = X_i \cup X_j \cup X_k$ , and note that  $H^1(B) = 0$  by Example 2.2. Let  $A$  be the union of the other three subcomplexes in the cover. As in the proof of Theorem 5.3,  $\dim H^1(A) \leq 1$  so there is an element  $u$  in the kernel of  $H^1(X) \rightarrow H^1(A)$ . By inspection, there is an element  $v \neq u$  in  $H^1(X)$  so that  $u \cup v \neq 0$ . Lifting  $u$  to  $u' \in H^1(X, A)$  and  $v$  to  $v' \in H^1(X, B)$ , the proof of Lemma 5.1 shows that  $u \cup v = 0$ , which is a contradiction.  $\square$

**Corollary 5.6.** *The Klein bottle  $N_2$  has covering type 7 or 8.*

*Proof.* Combine Theorem 5.5 with the upper bound  $ct(N_2) \leq 8$  coming from Figure 8, which shows that.  $\square$

### 6. Bounds via genus

We may regard the covering type and the chromatic number of a surface  $S$  as functions of the genus of  $S$ . We will now show that the functions are asymptotic; for both orientable and non-orientable surfaces, their ratio lies between 1 and  $\sqrt{1/3}$  as the genus goes to  $\infty$ .

**Definition 6.1.** *The **chromatic number**  $\text{chr}(S)$  of a surface  $S$  is defined to be the smallest number  $n$  such that every map on  $S$  is colorable with  $n$  colors. By a **map** on  $S$  we mean a decomposition of  $S$  into closed polyhedral regions, called **countries**, such that the boundaries of the regions form a finite graph.*

**Orientable surfaces.** When  $S_g$  is an oriented surface of genus  $g$ , it is a famous theorem that

$$\text{chr}(S_g) = \left\lfloor \frac{7 + \sqrt{1 + 48g}}{2} \right\rfloor$$

where  $\lfloor x \rfloor$  denotes the greatest integer at most  $x$ . The case  $g = 0$  is the Four-color Theorem; see [Rin2] or [GT, Chap. 5] when  $g > 0$ .

This integer is sometimes called the *Heawood number* of  $S_g$ , after Percy Heawood who first studied  $\text{chr}(S_g)$  in the 1890 paper [Hae]. We now show that  $ct(S_g) \leq 1 + \text{chr}(S_g)$ .

**Proposition 6.2.** *The covering type  $ct(S_g)$  of an oriented surface  $S_g$  of genus  $g \neq 2$  satisfies*

$$\left\lceil \frac{3 + \sqrt{1 + 16g}}{2} \right\rceil \leq ct(S_g) \leq \left\lceil \frac{7 + \sqrt{1 + 48g}}{2} \right\rceil.$$

If  $g = 2$ , we have  $6 \leq ct(S_2) \leq 10$ .

*Proof.* Set  $n = ct(X)$ . For  $g = 0, 1$  we have seen that  $n = 4, 7$ , respectively. For  $g = 2$ , we saw that  $n \leq 10$  (Figure 6), and we saw in Proposition 5.2 that  $n \geq 6$ . Thus we may assume that  $g > 2$ .

The lower bound  $n \geq (3 + \sqrt{1 + 16g})/2$  is just the solution of the quadratic inequality for  $h_1 = 2g$  in Proposition 3.3:

$$4g \leq 2\binom{n-1}{2} = n^2 - 3n + 2.$$

When  $g > 2$ , Jungermann and Ringel showed in [JR, Thm. 1.2] that there is a triangulation of  $S_g$  with  $n = \lceil \frac{7 + \sqrt{1 + 48g}}{2} \rceil$  vertices,  $\delta = 2n + 4(g - 1)$  triangles and  $\binom{n}{2} - t$  edges, where  $t = \binom{n-3}{2} - 6g$ . The open stars of the  $n$  vertices form a good open covering of  $S_g$ .  $\square$

**Example 6.3.** ( $g = 3$ ) Proposition 6.2 yields  $6 \leq ct(S_3) \leq 10$ . A 10-vertex triangulation of  $S_3$  is implicitly given by the orientation data on [Rin2, p. 23]; the triangulation has 42 edges and 28 triangles.

**Remark.** Any triangulation of  $S_g$  determines a graph  $\Gamma$  on  $S_g$ . The dual graph of  $\Gamma$  is formed by taking a vertex in the center of each of the  $\delta$  triangles, and connecting vertices of adjacent triangles along an arc through the edge where the triangles meet. Each country  $X_v$  in the dual map is a polygonal region, containing exactly one vertex  $v$  from the original triangulation, and the number of sides in the polygon  $X_v$  is the valence of  $v$  in the  $\Gamma$ .

The  $n$  countries in the dual map form a good closed covering of  $S_g$  because the intersection of  $X_v$  and  $X_w$  is either a face (when  $v$  and  $w$  are connected) or the empty set, and three polygons meet in a vertex exactly when the corresponding vertices form a triangle in the original triangulation.

**Non-orientable surfaces.** Similar results hold for non-orientable surfaces. For example, consider the two non-orientable surfaces of genus  $q \leq 2$ . The projective plane has chromatic number 6; see Figure 7. It is also well known that the Klein bottle has chromatic number 6; this fact was discovered by Frankin in 1934 [Fra]. However, the 6-country map on the Klein bottle is not a good covering, because some regions intersect in two disjoint edges; see [Rin2, Fig. 1.9]. Theorem 5.5 shows that there are no good covers of the Klein bottle by 6 regions. In this case,  $ct(X)$  is strictly bigger than the chromatic number of  $X$ .

For  $q \neq 1, 2$ , a famous theorem (see [Rin2, Thm. 4.10]) states that the chromatic number of a non-orientable surface of genus  $q$  is

$$\text{chr}(N_q) = \lfloor \frac{7 + \sqrt{1 + 24q}}{2} \rfloor.$$

We now show that the covering type of a non-orientable surface  $N_q$  grows (as a function of  $q$ ) at the same rate as the chromatic number of  $N_q$ . In particular,  $ct(N_q) \leq 1 + \text{chr}(N_q)$ .

**Proposition 6.4.** *The covering type  $ct(N_q)$  of a non-oriented surface  $N_q$  of genus  $q \geq 4$  satisfies*

$$5 \leq \left\lceil \frac{3 + \sqrt{1 + 8q}}{2} \right\rceil \leq ct(N_q) \leq \left\lceil \frac{7 + \sqrt{1 + 24q}}{2} \right\rceil.$$

*Proof.* The lower bound is immediate from the solution of the quadratic inequality  $\binom{n-1}{2} \geq q$  of Theorem 3.3.

The upper bounds were investigated by Ringel in 1955 [Rin1]; he considered polyhedral covers of  $N_q$  by  $\lambda$  countries, any two meeting in at most an arc and any three meeting in at most a point, and showed that  $\lambda \geq \lceil \frac{7 + \sqrt{1 + 24q}}{2} \rceil$ , with the lower bound being achieved for all  $q \neq 2, 3$ . □

**Remark 6.5.** The upper bounds for  $q < 4$  found by Ringel in [Rin1, p. 320] were:  $ct(N_2) \leq 8$  if  $q = 2$  (the Klein bottle), and  $ct(N_3) \leq 9$  if  $q = 3$ . We saw in Corollary 5.6 that  $ct(N_2)$  is 7 or 8; Ringel’s upper bound for the Klein bottle corresponds to Figure 8.

Combining Theorem 5.5 with Proposition 6.4 and Ringel’s bound for  $q = 3$ , we see that  $ct(N_3)$  and  $ct(N_4)$  are either 7, 8 or 9 and that  $7 \leq ct(N_5) \leq 10$ .

## 7. Higher dimensions

We do not know much about the covering type of higher-dimensional spaces. In this section, we give a few general theorems for upper bounds on the covering type.

**Suspensions.** The covering type of the suspension of a finite CW complex  $X$  is at most one more than the covering type of  $X$ .

To see this, recall that the cone  $CX$  is the quotient of  $X \times [0, 1]$  by the relation  $(x, 1) \sim (x', 1)$  for all  $x, x' \in X$ , while the suspension  $SX$  is the quotient of  $X \times [-1, 1]$  by the two relations  $(x, 1) \sim (x', 1)$  and  $(x, -1) \sim (x', -1)$ . If we start with a good closed cover of  $X$  by subcomplexes  $X_i$ , the cones  $CX_i$  of these form a good cover of the cone  $CX$ . Viewing the suspension  $SX$  as the union of an upper cone and a lower cone, the lower cones  $CX_i$  together with the upper cone form a good (closed) cover of  $SX$ . (This argument works for a good open cover of any topological space, provided we use open cones of the form  $X \times (-\varepsilon, 1]/(x, 1) \sim (x', 1)$ ; we leave the details as an easy exercise.)

The operation of coning off a subspace generalizes the suspension. To give a bound for the covering type in this case, we need a compatible pair of good covers.

**Theorem 7.1.** *Let  $X$  be a subcomplex of a finite CW complex  $Y$ . Suppose that  $Y$  has a good closed cover  $Y_1, \dots, Y_n$  such that  $\{X \cap Y_i\}$  is a good closed cover of  $X$ . Then the covering type of the cone  $Y \cup_X CX$  of the inclusion  $X \subset Y$  satisfies:*

$$ct(Y \cup_X CX) \leq n + 1.$$

*In particular, the suspension  $SX$  has  $ct(SX) \leq ct(X) + 1$ .*

*Proof.* The cone  $CX$  together with  $\{Y_i\}_{i=1}^n$  forms a good closed cover of  $Y \cup_X CX$ , whence the first assertion. Since the suspension  $SX$  is the cone of the inclusion of  $X$  into the upper cone  $Y$  of the suspension, the second assertion follows from the observation that, given a good cover  $\{X_i\}$  of  $X$ , the cones  $Y_i$  of the  $X_i$  satisfy the hypothesis of the theorem.  $\square$

**Covering spaces.** Another simple comparison involves the covering type of a covering space. If  $X$  is an  $n$ -sheeted covering space of  $Y$ , the covering type of  $X$  is at most  $n \cdot ct(Y)$ , because the inverse image of a contractible subcomplex of  $Y$  is the disjoint union of  $n$  contractible subcomplexes of  $X$ .

Recall that the mapping cylinder,  $\text{cyl}(f)$ , of a function  $f : X \rightarrow Y$  is the quotient of  $X \times [0, 1] \cup Y$  by the equivalence relation  $(x, 0) \sim f(x)$ ;  $\text{cyl}(f) \rightarrow Y$  is a homotopy equivalence. We may define the cone of  $f$ ,  $\text{cone } f$ , to be the cone of the inclusion of  $A = X \times \{1\}$  into  $\text{cyl}(f)$ .

**Theorem 7.2.** *If  $f : X \rightarrow Y$  is an  $n$ -sheeted covering space, the cone  $\text{cone } f$  has covering type at most  $n \cdot ct(Y) + 1$ ,*

*Proof.* If  $\{Y_k\}$  is a good cover of  $Y$ , then the inverse image of each  $Y_k$  is a disjoint union of  $n$  contractible subspaces we shall call  $X_{ik}$ , and each  $f_{ik} : X_{ik} \rightarrow Y_k$  is a homeomorphism. Then the mapping cylinders  $\text{cyl}(f_{ik})$  form a good cover of  $\text{cyl}(f)$ . Since the  $X_{ik} \times \{1\} = A \cap \text{cyl}(f_{ik})$  form a good cover of  $A = X \times \{1\}$ , the conditions of Theorem 7.1 are met. Since  $\text{cone } f$  is homotopy equivalent to  $CA \cup_A \text{cyl}(f)$ , the result follows.  $\square$

**Example 7.3.** *The bound in Theorem 7.2 is rarely sharp, although the cone of the projection from  $S^0$  to a point is  $S^1$  and indeed  $ct(S^1) = 3$ . On the other hand, if  $X$  is  $n$  discrete points and  $f$  is the projection from  $X$  to a point, the cone of  $f$  is a bouquet of  $n - 1$  circles, whose covering type is given by Theorem 4.1;  $ct(\text{cone } f)$  is smaller than  $n + 1$  when  $n \geq 4$ .*



Another example is given by  $\mathbb{R}\mathbb{P}^m$ , which is the cone of the degree 2 function  $f : S^{m-1} \rightarrow S^{m-1}$ ; Theorem 7.2 yields the upper bound  $ct(\mathbb{R}\mathbb{P}^m) \leq 2m + 3$ . This is not sharp for  $m = 2$  since we know that  $ct(\mathbb{R}\mathbb{P}^2) = 6$ , but (with Proposition 3.1) it does give the linear growth rate

$$m + 2 \leq ct(\mathbb{R}\mathbb{P}^m) \leq 2m + 3.$$

**Remark 7.4.** We do not know a good upper bound for the covering type of a product, beyond the obvious bound  $ct(X \times Y) \leq ct(X)ct(Y)$ . The torus  $T = S^1 \times S^1$  shows that this bound is not sharp, since  $ct(T) = 7$  and  $ct(S^1) = 3$ .

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