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The Cartan–Hadamard Theorem for metric spaces with local geodesic bicombings

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Abstract. We prove the Cartan–Hadamard Theorem for spaces which are not necessarily uniquely geodesic but locally possess a suitable selection of geodesics, a so-called convex geodesic bicombing.

Furthermore, we deduce a local-to-global theorem for injective (or hyperconvex) metric spaces, saying that under certain conditions a complete, simply-connected, locally injective metric space is injective. A related result for absolute 1-Lipschitz retracts follows.

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1. Introduction

Local-to-global principles are spread all-around in mathematics. The classical Cartan–Hadamard Theorem from Riemannian geometry was generalized by W. Ballmann [Bal] for metric spaces with non-positive curvature, and by S. Alexander and R. Bishop [AB] for locally convex metric spaces, i.e., for spaces that locally satisfy the Busemann property, meaning that $d \circ (\gamma_1, \gamma_2)$ is convex for all constant speed geodesics γ_1, γ_2. As a normed vector space satisfies the Busemann property if and only if its norm is strictly convex, this property is not preserved under limit processes. This motivates to look at an even weaker notion of non-positive curvature, where we only request convexity for a certain choice of geodesics, compare [Kle, Section 10].

We use the convention of [ABr, Lan] to call such a collection of paths a bicombing, a term originally coined by W. Thurston in the context of geometric group theory. The following definition is for instance satisfied by the linear segments $(1-t)x + ty$ in a normed vector space. In a metric space (X, d), a *geodesic bicombing* is a selection of a geodesic between each pair of points. This is a map $\sigma\colon X \times X \times [0,1] \to X$ such that, for all $x, y \in X$, the path $\sigma_{xy} := \sigma(x, y, \cdot)$ is a geodesic from x to y, i.e. $\sigma_{xy}(0) = x$, $\sigma_{xy}(1) = y$ and $d(\sigma_{xy}(s), \sigma_{xy}(t)) = |s - t| d(x, y)$ for all $s, t \in [0,1]$. Moreover, we assume that this choice is consistent in the sense that $\sigma_{pq}([0,1]) \subset \sigma_{xy}([0,1])$ for all $p, q \in \sigma_{xy}([0,1])$ with $d(x, p) \leq d(x, q)$. A geodesic bicombing σ is called *convex* if the function $t \mapsto d(\sigma_{xy}(t), \sigma_{\bar{x}\bar{y}}(t))$ is convex for all $x, y, \bar{x}, \bar{y} \in X$. Furthermore, we say that σ is *reversible* if $\sigma_{yx}([0,1]) = \sigma_{xy}([0,1])$ for all $x, y \in X$. A metric space admits a *local geodesic bicombing*, if such a selection exists in a neighborhood $U(x, r_x)$ of each point $x \in X$, see Section 2 for the exact definition.

Metric spaces with a geodesic bicombing resemble hyperbolic spaces after U. Kohlenbach [Koh, KL], which specify W-convex metric spaces considered by W. Takahashi [Tak] and S. Itoh [Ito]. Geodesic bicombings were recently studied by D. Descombes and U. Lang in [Des, DL1, DL2] and also by G. Basso in [Bas], where they show that several results for CAT(0) and Busemann spaces carry over to spaces with convex geodesic bicombings. Here we will contribute to these studies by proving the following Cartan–Hadamard Theorem.

Theorem 1.1. *Let X be a complete, simply-connected metric space with a convex local geodesic bicombing σ. Then the induced length metric on X admits a unique convex geodesic bicombing $\tilde{\sigma}$ which is consistent with σ. As a consequence, X is contractible. Moreover, if the local geodesic bicombing σ is reversible, then $\tilde{\sigma}$ is reversible as well.*

As we show in a subsequent paper joint with G. Basso [BM], Theorem 1.1 leads to a uniqueness result for convex geodesic bicombings on convex subsets of certain Banach spaces.

Important examples of spaces with convex geodesic bicombings are given by injective metric spaces, which include the real line, $\mathbb{R}$-trees and $l_\infty(I)$ for any index set I. Recall that every metric space X possesses an injective hull, i.e., a smallest injective metric space into which X embeds [Isb]. Injective metric spaces play a crucial role in the theory of mapping extensions [AP] and fixed point theory [Sin, Soa], see also [EK] and the references therein.

A metric space X is *injective* if for all metric spaces A, B with $A \subset B$ and every 1-Lipschitz map $f\colon A \to X$, there is a 1-Lipschitz extension $\bar{f}\colon B \to X$, i.e. $\bar{f}|_A = f$. In fact, D. Descombes and U. Lang show in their work that every proper, injective metric space of finite combinatorial dimension admits a (unique) convex geodesic bicombing [DL1, Theorem 1.2]. Such spaces occur, for instance, as injective hulls of hyperbolic groups [Lan, Theorem 1.4] and therefore, every hyperbolic group acts properly and cocompactly by isometries on a space with a convex geodesic bicombing [DL1, Theorem 1.3].

Recall that injective metric spaces are complete, geodesic and contractible. Now, knowing that under the above conditions injective metric spaces possess a convex geodesic bicombing, we deduce the following local-to-global theorem for injective metric spaces.

Theorem 1.2. *Let X be a complete, locally compact, simply-connected, locally injective length space with locally finite combinatorial dimension. Then X is an injective metric space.*

It is well known that injective metric spaces are the same as absolute 1-Lipschitz retracts. For Lipschitz retracts, the weaker notion of absolute Lipschitz uniform neighborhood retracts is common, see Section 4 for more details. Absolute 1-Lipschitz uniform neighborhood retracts are locally injective but the converse is not true as we will see in Example 4.2. In fact, it turns out that the following holds.

Theorem 1.3. *Let X be a locally compact absolute 1-Lipschitz uniform neighborhood retract with locally finite combinatorial dimension. Then X is an absolute 1-Lipschitz retract.*

This paper is organized as follows. We start Section 2 by studying spaces with local geodesic bicombings, establish an appropriate exponential map and finally prove Theorem 1.1. In Section 3, we first show that every uniformly locally injective metric space with a reversible, convex geodesic bicombing is injective. Afterwards, we describe how to construct a reversible, convex local geodesic bicombing on locally injective metric spaces, which extends to a convex geodesic bicombing by Theorem 1.1. Thereby we establish Theorem 1.2. Finally in Section 4, we then investigate absolute 1-Lipschitz neighborhood retracts and prove Theorem 1.3.

2. Local geodesic bicombings

Let us first fix some notation. In a metric space X, we denote by

$$U(x,r) := \{y \in X : d(x,y) < r\}$$

the open ball of radius r around $x \in X$ and by

$$B(x,r) := \{y \in X : d(x,y) \leq r\}$$

the closed one.

Let X be a metric space and $\gamma\colon [0,1] \to X$ a continuous curve. The *length* of γ is given by

$$L(\gamma) := \sup\Big\{ \sum_{k=1}^{n} d(\gamma(t_{k-1}), \gamma(t_k)) : 0 = t_0 < \ldots < t_n = 1 \Big\}.$$

Then

$$\bar{d}(x,y) := \inf\big\{ L(\gamma) : \gamma\colon [0,1] \to X, \gamma(0) = x, \gamma(1) = y \big\}$$

defines a metric on X, called the *induced length metric*. If we have $d = \bar{d}$, we say that (X,d) is a *length space*.

For a metric space X, let $\mathcal{G}(X) := \{c\colon [0,1] \to X\}$ be the set of all geodesics in X, i.e. continuous maps $c\colon [0,1] \to X$ with $d(c(s), c(t)) = |s-t| \cdot d(c(0), c(1))$ for all $s,t \in [0,1]$. Note that geodesics need not be parametrised by arc-length. We equip $\mathcal{G}(X)$ with the metric

$$D(c,c') := \sup_{t \in [0,1]} d\big(c(t), c'(t)\big).$$

Let $c \in \mathcal{G}(X)$ and $0 \leq a \leq b \leq 1$, then we denote by $c_{[a,b]}$ the reparametrized geodesic given by $c_{[a,b]}\colon [0,1] \to X$ with $c_{[a,b]}(t) := c((1-t)a + tb)$.

Definition 2.1. A *local geodesic bicombing* on a metric space X is a local selection of geodesics, i.e., a map $\sigma\colon U \times [0,1] \to X$, $(y,z,t) \mapsto \sigma_{yz}(t)$, with $U \subset X \times X$ open and the following properties:

(i) For all $x \in X$, there is some $r_x > 0$ such that, for all $y,z \in U(x,r_x)$, there is a geodesic $\sigma_{yz}\colon [0,1] \to U(x,r_x)$ from y to z, and

$$U = \big\{(y,z) \in X \times X : y,z \in U(x,r_x) \text{ for some } x\big\}.$$

(ii) The selection is consistent with taking subsegments of geodesics, i.e.,

$$\sigma_{\sigma_{yz}(s_1)\sigma_{yz}(s_2)}(t) = \sigma_{yz}\big((1-t)s_1 + ts_2\big)$$

for $(y,z) \in U$, $0 \leq s_1 \leq s_2 \leq 1$ and $t \in [0,1]$.

We call a local geodesic bicombing σ *convex* if it is locally convex, i.e. for $y,z,y',z' \in U(x,r_x)$, it holds that

$$t \mapsto d\big(\sigma_{yz}(t), \sigma_{y'z'}(t)\big)$$

is a convex function. Furthermore, σ is *reversible* if

$$\sigma_{zy}(t) = \sigma_{yz}(1-t)$$

for all $(y,z) \in U$ and $t \in [0,1]$.

Remark. Observe that, by property (ii), a local geodesic bicombing is convex if and only if

$$d\big(\sigma_{yz}(t), \sigma_{y'z'}(t)\big) \leq (1-t)d(y,y') + t d(z,z')$$

for all $y, z, y', z' \in U(x, r_x)$ and $t \in [0,1]$.

A (local) geodesic $c\colon [0,1] \to X$ is *consistent* with the local geodesic bicombing σ if

$$c_{[a,b]}(t) = \sigma_{c(a)c(b)}(t)$$

for all $0 \leq a \leq b \leq 1$ with $(c(a), c(b)) \in U$.

To prove Theorem 1.1, we roughly follow the structure of Chapter II.4 in [BH]. Adapting the methods of S. Alexander and R. Bishop [AB], we can prove the following key lemma.

Lemma 2.2. *Let X be a complete metric space with a convex local geodesic bicombing σ and let c be a local geodesic from x to y which is consistent with σ. Then, there is some $\epsilon > 0$ such that, for all $\bar{x}, \bar{y} \in X$ with $d(x,\bar{x}), d(y,\bar{y}) < \epsilon$, there is a unique local geodesic $\bar{c}$ from $\bar{x}$ to $\bar{y}$ with $D(c,\bar{c}) < \epsilon$ which is consistent with σ. Moreover, we have*

$$L(\bar{c}) \leq L(c) + d(x,\bar{x}) + d(y,\bar{y})$$

and if $\tilde{c}$ is another consistent geodesic from $\tilde{x}$ to $\tilde{y}$ with $D(c,\tilde{c}) < \epsilon$, then

$$t \mapsto d\big(\tilde{c}(t), \bar{c}(t)\big)$$

is convex.

Proof. Let $\epsilon > 0$ be such that $\sigma\big|_{U(c(t),2\epsilon)\times U(c(t),2\epsilon)\times[0,1]}$ is a convex geodesic bicombing for all $t \in [0,1]$. Now, let $\mathsf{P}(A)$ be the following statement:

$\mathsf{P}(A)$: For all $a, b \in [0,1]$ with $0 \leq b - a \leq A$ and for all $p, q \in X$ with $d(c(a), p), d(c(b), q) < \epsilon$, there is a unique local geodesic $\bar{c}_{pq}\colon [0,1] \to X$ from p to q with $D(c_{[a,b]}, \bar{c}_{pq}) < \epsilon$ which is consistent with σ. Moreover, for all such local geodesics the map $t \mapsto d(\bar{c}_{pq}(t), \bar{c}_{p'q'}(t))$ is convex.

By our assumption, $\mathsf{P}(\frac{\epsilon}{l(c)})$ holds. Therefore, let us show $\mathsf{P}(A) \Rightarrow \mathsf{P}(\frac{3A}{2})$.

Given $a, b \in [0,1]$ with $0 \leq b - a \leq \frac{3A}{2}$, define $p_0 := c(\frac{2}{3}a + \frac{1}{3}b)$ and $q_0 := c(\frac{1}{3}a + \frac{2}{3}b)$. Then, by $\mathsf{P}(A)$, there are consistent local geodesics c_1 from p to q_0 and c_1' from p_0 to q. Inductively, we set $p_n := c_n(\frac{1}{2})$ and $q_n := c_n'(\frac{1}{2})$, where c_n is a consistent local geodesic from p to q_{n-1} and c_n' from p_{n-1} to q. Observe that, by convexity of the c_n, c_n', we have $d(p_{n-1}, p_n), d(q_{n-1}, q_n) < \frac{\epsilon}{2^n}$ and hence the sequences $(p_n)_n$ and $(q_n)_n$ converge to some p_∞ and q_∞, respectively, and we have $d(p_\infty, p_0), d(q_\infty, q_0) < \epsilon$. Furthermore, by convexity, the c_n, c'_n converge to the consistent local geodesics c_∞ from p to q_∞ and c'_∞ from p_∞ to q, which coincide between $p_\infty = c_\infty(\frac{1}{2})$ and $q_\infty = c'_\infty(\frac{1}{2})$. Hence, they define a new local geodesic c_{pq} from p to q which is consistent with σ and $p_\infty = c_{pq}(\frac{1}{3})$, $q_\infty = c_{pq}(\frac{2}{3})$.

Now, given two local geodesics c_{pq} and $c_{p'q'}$ with $D(c_{[a,b]}, c_{pq}) < \epsilon$ and $D(c_{[a,b]}, c_{p'q'}) < \epsilon$, set $s := d(p, p')$, $t := d(q, q')$, $s' := d(c_{pq}(\frac{1}{3}), c_{p'q'}(\frac{1}{3}))$ and $t' := d(c_{pq}(\frac{2}{3}), c_{p'q'}(\frac{2}{3}))$. Then we have $s' \leq \frac{s+t'}{2}$, $t' \leq \frac{s'+t}{2}$ and therefore $s' \leq \frac{s}{2} + \frac{s'}{4} + \frac{t}{4}$, i.e. $s' \leq \frac{2s+t}{3}$ and similarly $t' \leq \frac{s+2t}{3}$ follows. Hence, we get convexity of $t \mapsto d(c_{pq}(t), c_{p'q'}(t))$ and therefore also uniqueness follows.

It remains to prove that $L(\bar{c}) \leq L(c) + d(x, \bar{x}) + d(y, \bar{y})$. Let $\tilde{c}$ be the unique consistent local geodesic from x to $\bar{y}$ with $D(c, \tilde{c}) < \epsilon$. For t small enough we have

$$\begin{aligned} tL(\tilde{c}) &= d\big(\tilde{c}(0), \tilde{c}(t)\big) = d\big(c(0), \tilde{c}(t)\big) \\ &\leq d\big(c(0), c(t)\big) + d\big(c(t), \tilde{c}(t)\big) \leq tL(c) + td\big(c(1), \tilde{c}(1)\big), \end{aligned}$$

i.e., $L(\tilde{c}) \leq L(c) + d(y, \bar{y})$ and similarly $L(\bar{c}) \leq L(\tilde{c}) + d(x, \bar{x})$. □

Definition 2.3. Let X be a metric space with a local geodesic bicombing σ. For some fixed $x_0 \in X$, we define

$$\tilde{X}_{x_0} := \big\{c\colon [0,1] \to X \text{ local geodesic with } c(0) = x_0,\ \text{consistent with } \sigma\big\}.$$

We equip $\tilde{X}_{x_0}$ with the metric $D(c, c') = \sup_{t\in[0,1]} d(c(t), c'(t))$ and define the map

$$\exp\colon \tilde{X}_{x_0} \to X,\ \ c \mapsto c(1).$$

If X is complete, this map has the following properties.

Lemma 2.4. *Let X be a complete metric space with a convex local geodesic bicombing σ. Then the following holds:*

(i) *The map $\exp\colon \tilde{X}_{x_0} \to X$ is locally an isometry. Hence σ naturally induces a convex local geodesic bicombing $\tilde{\sigma}$ on $\tilde{X}_{x_0}$.*

(ii) *$\tilde{X}_{x_0}$ is contractible.*

(iii) *For each $\tilde{x} \in \tilde{X}_{x_0}$, there is a unique local geodesic from $\tilde{x}_0$ to $\tilde{x}$ which is consistent with $\tilde{\sigma}$, where $\tilde{x}_0$ is the constant path $\tilde{x}_0(t) = x_0$.*

(iv) *$\tilde{X}_{x_0}$ is complete.*

Proof. By Lemma 2.2, for every $c \in \tilde{X}_{x_0}$, there is some $\epsilon > 0$ such that the map $\exp\big|_{U(c,\epsilon)}\colon U(c,\epsilon) \to U(c(1),\epsilon)$ is an isometry. Hence, σ naturally induces a convex local geodesic bicombing $\tilde{\sigma}$ on $\tilde{X}_{x_0}$.

Consider the map $r\colon \tilde{X}_{x_0} \times [0,1] \to \tilde{X}_{x_0}, (c,s) \mapsto (r_s(c)\colon t \mapsto c(st))$. This defines a retraction of $\tilde{X}_{x_0}$ to $\tilde{x}_0$.

A continuous path $\tilde{c}\colon [0,1] \to \tilde{X}_{x_0}$ is a local geodesic in $\tilde{X}_{x_0}$ which is consistent with $\tilde{\sigma}$ if and only if $\exp\circ\tilde{c}$ is a local geodesic in X which is consistent with σ. Therefore, for any $c \in \tilde{X}_{x_0}$, the map $s \mapsto r_s(c)$ is the unique local geodesic from $\tilde{x}_0$ to c.

Finally, if $(c_n)_n$ is a Cauchy sequence in $\tilde{X}$, by completeness of X, for every $t \in [0,1]$, the sequences $(c_n(t))_n$ converge in X, to $c(t)$ say. Locally, i.e., inside $U(c(t), r_{c(t)})$, the subsegment $c|_{[t-\epsilon,t+\epsilon]}$ is the limit of the consistent geodesics $(c_n|_{[t-\epsilon,t+\epsilon]})_n$ and hence c is consistent with σ by the convexity of the local geodesic bicombing. □

The following criterion will ensure that exp is a covering map.

Lemma 2.5. *Let $p\colon \tilde{X} \to X$ be a map of length spaces such that*

(i) *X is connected,*

(ii) *p is a local homeomorphism,*

(iii) *for all rectifiable curves $\tilde{c}\colon [0,1] \to \tilde{X}$, we have $L(\tilde{c}) \le L(p\circ\tilde{c})$,*

(iv) *X has a convex local geodesic bicombing σ, and*

(v) *$\tilde{X}$ is complete.*

Then p is a covering map.

Proof. The proof of Proposition I.3.28 in [BH] also works in our setting. In the second step, take $U = U(x, r_x)$ and define the maps $s_{\tilde{x}}\colon U(x,r_x) \to \tilde{X}$ by $s_{\tilde{x}}(y) = \tilde{\sigma}_{xy}(1)$, where $\tilde{\sigma}_{xy}$ is the unique lift of σ_{xy} with $\tilde{\sigma}_{xy}(0) = \tilde{x}$. □

Remark. For a local isometry p, conditions (ii) and (iii) are satisfied.

Corollary 2.6. *Let (X,d) be a complete, connected metric space with a convex local geodesic bicombing σ. Then $\exp\colon \tilde{X}_{x_0} \to X$ is a universal covering map.*

Proof. Consider the induced length metrics $\bar{d}$ and $\bar{D}$ on X and $\tilde{X}_{x_0}$. Since (X,d) locally is a length space, the metrics d and D locally coincide with $\bar{d}$ and $\bar{D}$, respectively. Hence p still is a local isometry with respect to the length metrics and σ is a convex local geodesic bicombing. Thus Lemma 2.5 applies. □

Proof of Theorem 1.1. First, we show that, for all $x, y \in X$, there is a unique consistent local geodesic from x to y. Since X is simply-connected, the covering map $\exp\colon \tilde{X}_x \to X$ is a homeomorphism which is a local isometry and by Lemma 2.4, there is a unique consistent local geodesic $\tilde{\sigma}_{xy}$ from x to y.

Next, we prove that $\tilde{\sigma}_{xy}$ is a geodesic. To do so, it is enough to show that, for every curve $\gamma\colon [0,1] \to X$ and every $t \in [0,1]$, we have $L(\tilde{\sigma}_{\gamma(0)\gamma(t)}) \leq L(\gamma|_{[0,t]})$. Let

$$A := \{s \in [0,1] : \forall t \in [0,s] \text{ we have } L(\tilde{\sigma}_{\gamma(0)\gamma(t)}) \leq L(\gamma|_{[0,t]})\}.$$

Clearly, A is non-empty and closed. To prove that A is open, consider $s \in A$. For $\delta > 0$ small enough, by Lemma 2.2, we have

$$\begin{aligned} L(\tilde{\sigma}_{\gamma(0)\gamma(s+\delta)}) &\leq L(\tilde{\sigma}_{\gamma(0)\gamma(s)}) + d(\gamma(s), \gamma(s+\delta)) \\ &\leq L(\gamma|_{[0,s]}) + L(\gamma|_{[s,s+\delta]}) = L(\gamma|_{[0,s+\delta]}). \end{aligned}$$

Hence, $A = [0,1]$ as desired.

Finally, we show that $t \mapsto d(\tilde{\sigma}_{xy}(t), \tilde{\sigma}_{\bar{x}\bar{y}}(t))$ is convex. By Lemma 2.2, there is a sequence $0 = t_1 < \ldots < t_n = 1$ and $\epsilon_k > 0$ such that

- the balls $U(\tilde{\sigma}_{x\bar{x}}(t_1), \epsilon_1), \ldots, U(\tilde{\sigma}_{x\bar{x}}(t_n), \epsilon_n)$ cover $\tilde{\sigma}_{x\bar{x}}$,
- the balls $U(\tilde{\sigma}_{y\bar{y}}(t_1), \epsilon_1), \ldots, U(\tilde{\sigma}_{y\bar{y}}(t_n), \epsilon_n)$ cover $\tilde{\sigma}_{y\bar{y}}$, and
- for all $p, \bar{p} \in U(\tilde{\sigma}_{x\bar{x}}(t_k), \epsilon_k)$ and $q, \bar{q} \in U(\tilde{\sigma}_{y\bar{y}}(t_k), \epsilon_k)$, the map $t \mapsto d(\tilde{\sigma}_{pq}(t), \tilde{\sigma}_{\bar{p}\bar{q}}(t))$ is convex.

Consider now a sequence $0 = s_0 < s_1 < \ldots < s_n = 1$ with

$$\begin{aligned} \tilde{\sigma}_{x\bar{x}}(s_k) &\in U(\tilde{\sigma}_{x\bar{x}}(t_k), \epsilon_k) \cap U(\tilde{\sigma}_{x\bar{x}}(t_{k+1}), \epsilon_{k+1}), \\ \tilde{\sigma}_{y\bar{y}}(s_k) &\in U(\tilde{\sigma}_{y\bar{y}}(t_k), \epsilon_k) \cap U(\tilde{\sigma}_{y\bar{y}}(t_{k+1}), \epsilon_{k+1}), \end{aligned}$$

for $k = 1, \ldots, n-1$. Then we get

$$\begin{aligned} &d\big(\tilde{\sigma}_{xy}(t), \tilde{\sigma}_{\bar{x}\bar{y}}(t)\big) \\ &\quad \leq \sum_{k=1}^{n} d\big(\tilde{\sigma}_{\tilde{\sigma}_{x\bar{x}}(s_{k-1})\tilde{\sigma}_{y\bar{y}}(s_{k-1})}(t), \tilde{\sigma}_{\tilde{\sigma}_{x\bar{x}}(s_k)\tilde{\sigma}_{y\bar{y}}(s_k)}(t)\big) \\ &\quad \leq (1-t)\Big(\sum_{k=1}^{n} d\big(\tilde{\sigma}_{x\bar{x}}(s_{k-1}), \tilde{\sigma}_{x\bar{x}}(s_k)\big)\Big) + t\Big(\sum_{k=1}^{n} d\big(\tilde{\sigma}_{y\bar{y}}(s_{k-1}), \tilde{\sigma}_{y\bar{y}}(s_k)\big)\Big) \\ &\quad = (1-t)d(x, \bar{x}) + t d(y, \bar{y}). \end{aligned}$$

Hence, $\tilde{\sigma}$ is a convex geodesic bicombing on X.

If σ is reversible, then $\tilde{\sigma}^*_{xy}(t) := \tilde{\sigma}_{yx}(1-t)$ also defines a convex geodesic bicombing on X which is consistent with σ. Therefore, by uniqueness, $\tilde{\sigma}^*$ and $\tilde{\sigma}$ coincide, i.e. $\tilde{\sigma}$ is reversible. □

3. Locally injective metric spaces

N. Aronszajn and P. Panitchpakdi [AP] proved that injective metric spaces are exactly the *hyperconvex* metric spaces, namely metric spaces with the property that for every family of closed balls $\{B(x_i, r_i)\}_{i\in I}$ with $d(x_i, x_j) \le r_i + r_j$, for all $i, j \in I$, we have $\bigcap_{i\in I} B(x_i, r_i) \neq \varnothing$. Note that in hyperconvex metric spaces closed balls are hyperconvex.

Definition 3.1. A metric space X is *locally injective* if, for every $x \in X$, there is some $r_x > 0$ such that $B(x, r_x)$ is injective. If we can take $r_x = r$ for all x, we call X *uniformly locally injective*.

Lemma 3.2. *Let X be a metric space with the property that every closed ball $B(x, r)$ is injective, then X is itself injective.*

Proof. Let $\{B(x_i, r_i)\}_{i\in I}$ be a family of closed balls with $d(x_i, x_j) \le r_i + r_j$. Fix some $i_0 \in I$ and set $A_i := B(x_i, r_i) \cap B(x_{i_0}, r_{i_0})$. Since, for r big enough, we have $x_i, x_j \in B(x_{i_0}, r)$, we get that the A_i's are externally hyperconvex in A_{i_0} and $A_i \cap A_j \neq \varnothing$ for all $i, j \in I$. Hence, it follows

$$\bigcap_{i\in I} B(x_i, r_i) = \bigcap_{i\in I} A_i \neq \varnothing$$

by [Mie, Proposition 1.2]. □

Proposition 3.3. *Let X be a uniformly locally injective metric space with a reversible, convex geodesic bicombing σ. Then X is injective.*

Proof. Consider the following property:

$\mathsf{P}(R)$: For every family $\{B(x_i, r_i)\}_{i\in I}$ with $d(x_i, x_j) \le r_i + r_j$ and $r_i \le R$, there is some $x \in \bigcap_{i\in I} B(x_i, r_i)$.

Since X is uniformly locally injective, this clearly holds for some $R_0 > 0$. Next, we show $\mathsf{P}(R) \Rightarrow \mathsf{P}(2R)$ and therefore $\mathsf{P}(R)$ holds for any $R \ge 0$.

Let $\{B(x_i, r_i)\}_{i\in I}$ be a family of closed balls with $d(x_i, x_j) \le r_i + r_j$ and $r_i \le 2R$. For $i, j \in I$, define $y_{ij} := \sigma_{x_i x_j}(\frac{1}{2})$. By convexity of σ, we have

$$d(y_{ij}, y_{ik}) = d\big(\sigma_{x_i x_j}(\tfrac{1}{2}), \sigma_{x_i x_k}(\tfrac{1}{2})\big) \le \tfrac{1}{2} d(x_j, x_k) \le \tfrac{r_j}{2} + \tfrac{r_k}{2}.$$

Hence, for every $i \in I$, there is some $z_i \in \bigcap_{j\in I} B(y_{ij}, \frac{r_j}{2})$. Now, observe that $d(z_i, z_j) \le d(z_i, y_{ij}) + d(y_{ij}, z_j) \le \frac{r_i}{2} + \frac{r_j}{2}$ and therefore, we find

$$x \in \bigcap_{i\in I} B(z_i, \tfrac{r_i}{2}) \subset \bigcap_{i\in I} B(x_i, r_i).$$

Since all balls with center in $B(x,r)$ and radius larger than $2r$ contain $B(x,r)$, $\mathsf{P}(R)$ for $R = 2r$ implies that $B(x,r)$ is injective. Hence, by Lemma 3.2, X is injective. □

Since compact, locally injective metric spaces are always uniformly locally injective we conclude the following.

Corollary 3.4. *Let X be a compact, locally injective metric space with a reversible, convex geodesic bicombing σ. Then X is injective.*

Corollary 3.5. *Let X be a proper, locally injective metric space with a reversible, convex geodesic bicombing σ. Then X is injective.*

Proof. Let $\{B(x_i,r_i)\}_{i\in I}$ be a family of balls with $d(x_i,x_j) \leq r_i + r_j$. Fix some $i_0 \in I$ and define $I_n = \{i \in I : d(x_i,x_{i_0}) \leq n\}$, for $n \in \mathbb{N}$. Since $B(x_{i_0},n)$ is compact, by the previous corollary, there is some $y_n \in \bigcap_{i\in I_n} B(x_i,r_i)$. Especially, $(y_n)_n \subset B(x_{i_0},r_{i_0})$ and hence, there is some converging subsequence $y_{n_k} \to y \in \bigcap_{i\in I} B(x_i,r_i)$. □

Remark. In [Lan], U. Lang proves that every injective metric space admits a reversible, conical geodesic bicombing (Proposition 3.8). Observe also that this is the only property of the geodesic bicombing used in the proof of Proposition 3.3. Therefore, we get the following equivalence statement (in the terminology of [Lan]): A metric space is injective if and only if it is uniformly locally injective and admits a reversible, conical geodesic bicombing.

If an injective metric space X is proper, it also admits a (possibly non-consistent) convex geodesic bicombing [DL1, Theorem 1.1] and if X has finite combinatorial dimension in the sense of A. Dress [Dre], this convex geodesic bicombing is consistent, reversible and unique [DL1, Theorem 1.2]. In our terms, this is:

Proposition 3.6. *Every proper, injective metric space with finite combinatorial dimension admits a unique reversible, convex geodesic bicombing.*

Recall that, by the Hopf–Rinow Theorem, any complete, locally compact length space is proper.

Corollary 3.7. *Let X be a locally compact, locally injective metric space with locally finite combinatorial dimension. Then X admits a reversible, convex local geodesic bicombing.*

Proof. For every $x \in X$, there is some $r_x > 0$ such that $B(x, 3r_x)$ is compact, injective and has finite combinatorial dimension. This also holds for $B(x, r_x)$ and therefore, there is a reversible, convex geodesic bicombing σ^x on $B(x, r_x)$.

We will check that for $B(x, r_x)$ and $B(y, r_y)$ with $B(x, r_x) \cap B(y, r_y) \neq \varnothing$ the two geodesic bicombings σ^x, σ^y coincide on the intersection. Assume without loss of generality that $r_x \geq r_y$ and hence $B(x, r_x), B(y, r_y) \subset B(x, 3r_x)$. Then the convex geodesic bicombing τ on $B(x, 3r_x)$ restricts to both $B(x, r_x)$ and $B(y, r_y)$ since, for $p, q \in B(z, r_z)$, we have $d(z, \tau_{pq}(t)) \leq (1-t)d(z, p) + td(z, q) \leq r_z$. Hence, by uniqueness, the geodesic bicombings σ^x, σ^y are both restrictions of τ and thus coincide on $B(x, r_x) \cap B(y, r_y)$.

Therefore σ, defined by $\sigma|_{B(x,r_x) \times B(x,r_x)} := \sigma^x|_{B(x,r_x) \times B(x,r_x)}$, is a reversible, convex local geodesic bicombing on X. □

Proof of Theorem 1.2. Let X be a complete, locally compact, simply-connected, locally injective length space with locally finite combinatorial dimension. By Corollary 3.7, X has a reversible, convex local geodesic bicombing, which induces a reversible, convex geodesic bicombing by Theorem 1.1. Hence, we can apply Corollary 3.5 and deduce that X is injective. □

4. Absolute 1-Lipschitz Neighborhood Retracts

Absolute Lipschitz uniform neighborhood retracts appear for instance in the study of approximations of Lipschitz maps, see [HJ, Section 7]. The question arises, how much absolute Lipschitz uniform neighborhood retracts differ from being an absolute Lipschitz retract. Theorem 1.3 will give a first answer in the case of absolute 1-Lipschitz retracts.

A metric space X is an *absolute* 1*-Lipschitz neighborhood retract* if, for every metric space Y with $X \subset Y$, there is some neighborhood U of X in Y and a 1-Lipschitz retraction $\rho\colon U \to X$. Furthermore, if we can take $U = U(X, \epsilon)$ for some $\epsilon > 0$, we call X an *absolute* 1*-Lipschitz uniform neighborhood retract*. In this case, ϵ can be chosen independent of Y; see [HJ, Proposition 7.78]. Finally, if there is always a 1-Lipschitz retraction $r\colon Y \to X$, then X is an *absolute* 1*-Lipschitz retract*. This is equivalent to X being an injective metric space [AP, Theorem 8].

Lemma 4.1. *Let X be an absolute* 1*-Lipschitz (uniform) neighborhood retract. Then X is (uniformly) locally injective.*

Proof. Consider $X \subset l_\infty(X)$. Since X is an absolute 1-Lipschitz neighborhood retract, there is some neighborhood U of X and a 1-Lipschitz retraction $\rho\colon U \to X$. For $x \in X$, there is some $r_x > 0$ such that $B(x, r_x) \subset U$. Let now $\{B(x_i, r_i)\}_{i\in I}$ be a family of closed balls with $x_i \in B(x, r_x) \cap X$ and $d(x_i, x_j) \leq r_i + r_j$. Then, since $l_\infty(X)$ is injective, there is some $y \in B(x, r_x) \cap \bigcap_{i\in I} B(x_i, r_i) \subset U$. Hence, we have $\rho(y) \in B(x, r_x) \cap \bigcap_{i\in I} B(x_i, r_i) \cap X$ and therefore $B(x, r_x) \cap X$ is injective.

If X is an absolute 1-Lipschitz uniform neighborhood retract, we have $U = U(X, \epsilon)$ for some $\epsilon > 0$ and therefore, we can choose $r_x = \frac{\epsilon}{2}$ for all $x \in X$. □

The converse is not true, as the following example shows.

Example 4.2. Consider the unit sphere S^1 endowed with the inner metric. Since, for every $x \in S^1$ and $\epsilon \in (0, \frac{\pi}{2}]$, the ball $B(x, \epsilon)$ is isometric to the interval $[-\epsilon, \epsilon]$, the unit sphere S^1 is uniformly locally injective.

But S^1 is not an absolute 1-Lipschitz neighborhood retract. Fix some inclusion $S^1 \subset l_\infty(S^1)$. We choose three points $x, y, z \in S^1$ with $r := d(x, y) = d(x, z) = d(y, z) = \frac{2\pi}{3}$. Let U be a neighborhood of S^1 in $l_\infty(S^1)$. As U is open, there is some $\epsilon \in (0, \frac{r}{2})$ such that $B(x, \epsilon) \subset U$. By hyperconvexity of $l_\infty(S^1)$, there is some

$$p \in B(x, \epsilon) \cap B(y, r - \epsilon) \cap B(z, r - \epsilon) \subset U.$$

But since

$$B(x, \epsilon) \cap B(y, r - \epsilon) \cap B(z, r - \epsilon) \cap S^1 = \varnothing,$$

there is no 1-Lipschitz retraction $\rho\colon S^1 \cup \{p\} \to S^1$.

In fact, the notion of an absolute 1-Lipschitz uniform neighborhood retract is quite restrictive.

Lemma 4.3. *Let X be an absolute 1-Lipschitz uniform neighborhood retract. Then X is*

(i) *complete,*

(ii) *geodesic, and*

(iii) *simply-connected.*

Proof. Fix some inclusion $X \subset l_\infty(X)$ and $r := \frac{\epsilon}{2} > 0$ such that there is a 1-Lipschitz retraction $\rho\colon U(X, \epsilon) \to X$.

First, if $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X, it converges to some $x \in U(X, \epsilon)$. It follows that $x = \rho(x) \in X$.

Next, assume that there is a geodesic in X between points at distance less than d. By Lemma 4.1, this is clearly true for $d = r$. Consider two points $x, y \in X$ with $d(x, y) \leq d + r$. Now, since $l_\infty(X)$ is geodesic, there is some $z \in l_\infty(X)$ with $d(x, y) = d(x, z) + d(z, y)$, $d(x, z) \leq r$ and $d(z, y) \leq d$. But then, we have $\rho(z) \in X$ with $d(x, y) = d(x, \rho(z)) + d(\rho(z), y)$ and, by our hypothesis, there are geodesics from x to $\rho(z)$ and from $\rho(z)$ to y which combine to a geodesic from x to y.

Finally, since X is locally simply-connected, every curve is homotopic to a curve of finite length and hence it is enough to consider loops of finite length. We show that every such loop in X is contractible.

Let γ be a loop in X of length $L(\gamma) = 2\pi R$ with $R > r$. Denote by $S^2_R := \{x \in \mathbb{R}^3 : |x| = R\}$ the sphere of radius R endowed with the inner metric and let $A := \{x \in S^2_R : 0 \leq x_3 \leq R \sin(\frac{r}{R})\}$ be the region bounded by the two circles $c := \{x \in S^2_R : x_3 = 0\}$ and $c' := \{x \in S^2_R : x_3 = R \sin(\frac{r}{R})\}$. Let $f : c \to X$ be a parametrization of γ by arclength and let $\bar{f} : A \to l_\infty(X)$ be a 1-Lipschitz extension. Then $\gamma' := \rho \circ \bar{f}(c')$ is a loop of length $L(\gamma') \leq L(c') = 2\pi R'$ with $R' := R \cos(\frac{r}{R})$, which is homotopic to γ. Since $\cos(\frac{r}{R'}) \leq \cos(\frac{r}{R})$, we find inductively a loop γ_n with $L(\gamma_n) \leq 2\pi R \cos(\frac{r}{R})^n$, which is homotopic to γ.

If $L(\gamma) = 2\pi R$ with $R \leq r$, we can use the same argument with A replaced by the upper halfsphere of radius R to show that γ is contractible. □

We conclude that an absolute 1-Lipschitz uniform neighborhood retract is a complete, simply-connected, locally injective length space and therefore Theorem 1.3 follows directly from Theorem 1.2.

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