Invariants of the special orthogonal group and an enhanced Brauer category

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Abstract. We first give a short intrinsic, diagrammatic proof of the First Fundamental Theorem (FFT) of invariant theory for the special orthogonal group $SO_m(\mathbb{C})$, given the FFT for $O_m(\mathbb{C})$. We then define, by means of a presentation with generators and relations, an enhanced Brauer category $\widetilde{\mathcal{B}}(m)$ by adding a single generator to the usual Brauer category $\mathcal{B}(m)$, together with four relations. We prove that our category $\widetilde{\mathcal{B}}(m)$ is actually (and remarkably) *equivalent* to the category of representations of SO_m generated by the natural representation. The FFT for SO_m amounts to the surjectivity of a certain functor $\mathcal{F}$ on Hom spaces, while the Second Fundamental Theorem for SO_m says simply that $\mathcal{F}$ is injective on Hom spaces. This theorem provides a diagrammatic means of computing the dimensions of spaces of homomorphisms between tensor modules for SO_m (for any m).

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1. Introduction

1.1. History. If a group H acts on a vector space W, the element $w \in W$ is said to be *invariant* if $hw = w$ for all $h \in H$. The set of invariants W^H is a subspace of W, and in general the problem of finding a spanning set for W^H is known as the first fundamental problem of invariant theory, and that of describing all relations among the elements of the spanning set is the second fundamental problem. When W has extra structure, e.g., if it is an associative algebra, or a commutative algebra, the first and second problems are recast as asking for generators and relations respectively in the appropriate category. In the two examples given, one would ask for algebra generators of the invariants, together with generators of the ideal of relations among them.

If $V = \mathbb{C}^m$, $H = \mathrm{GL}(V)$ and $W = \mathrm{End}_{\mathbb{C}}(V^{\otimes r})$ for some positive integer $r \geq 1$, Schur [Sch1] completely solved the fundamental problems, and proved what are now known as the first and second fundamental theorems (FFT and SFT) of invariant theory. Note that in this case, W is an associative, non-commutative (if $r > 1$) algebra. It was subsequently shown that his theorem is equivalent to solving the corresponding problem for the commutative algebra $\mathbb{C}[W]$, where $W = V^{\oplus r} \oplus V^{*\oplus s}$, which is just the symmetric algebra on W^*. These results are well known, and we shall not rehearse them here. Suffice it to say that they have led to applications in the representation theory of Lie groups (cf. [Wey]) and to an enormous literature in what has become known as Schur–Weyl duality (see, e.g., [DDS, KM, Aiz]), in many and varied settings.

When V above is endowed with a non-degenerate symmetric (resp. skew) bilinear form $(\, , \,)$ and $H = \mathrm{O}(V)$ (resp. $\mathrm{Sp}(V)$) is the full isometry group of $(V, (\, , \,))$, Weyl [Wey] has proved results which could be interpreted as versions of the FFT and SFT for the H action on the commutative algebra $\mathbb{C}[V^{\oplus r}]$. His FFT asserts that the space of invariant functions is generated as a commutative algebra by contractions using the given form, while his SFT asserted that all relations among these generators are generated by the obvious determinantal condition.

This work has an important reinterpretation in terms of diagrams, which was given in 1937 by Brauer [Bra]. With V and H as in the previous paragraph, he defined certain algebras $\mathcal{B}_r$ in terms of diagrams, as well as homomorphisms $\eta_r : \mathcal{B}_r \longrightarrow \mathrm{End}_H(V^{\otimes r})$ $(r = 1, 2, 3, \dots)$ whose surjectivity is equivalent to the FFT in this setting. Brauer left open the SFT in this formulation, which amounts to describing the kernel of η_r, an ideal of $\mathcal{B}_r$. This question has only recently been resolved [LZ1, LZ2] by the present authors, who produced an explicit ideal generator in $\mathcal{B}_r$ for the kernel of η_r, and in this way, provided a presentation of $\mathrm{End}_H(V^{\otimes r})$.

The above circle of ideas are most efficiently expressed in the language of the Brauer category [LZ2].

There are now many vast generalisations of this work in which the base ring $\mathbb{C}$ is replaced by other domains, and in which 'group' may be replaced by Lie algebra, quantum group, or another algebraic object which has an action or co-action (see, e.g., below). These generalisations are often of central importance in addressing difficult questions concerning multiplicities and character formulae (see [Jan] for a good survey) and are related to geometric questions such as intersection cohomology of the flag varieties (cf. [Spr, Lus, DL]) . They are part of the vast subject of 'invariant theory', which includes geometric invariant theory, a subject distilled by Mumford [MFK] from many persistent mathematical themes through many ages. If H is an algebraic group over $\mathbb{C}$ acting on an

affine variety X, then H acts on the coordinate ring $\mathbb{C}[X]$, and the properties of the 'orbit variety' $X/\!/H$ are reflected in the properties of the ring $\mathbb{C}[X]^H$ of H-invariants.

In this work we shall be concerned with a particular issue in the orthogonal case. Consider the situation described above, when the form (,) is symmetric. Taking $W = \operatorname{End}_{\mathbb{C}}(V^{\otimes r})$ as above, but with $H = \mathrm{SO}(V)$ rather than $\mathrm{O}(V)$, evidently there are more invariants, but the structure of the ring of invariants has always been somewhat imperfectly understood. It is this case and its generalisations upon which this work is focussed.

1.2. This work. Many results in invariant theory have a categorical formulation in terms of the Brauer category, see, e.g., [LZl, LZ2, LZ3, Del].

In this note we give a short diagrammatic argument to deduce the FFT for $\mathrm{SO}(V)$ from that for $\mathrm{O}(V)$. This is direct, and involves no induction on dimension, which appears in traditional proofs such as those in [FH, Ri]. In order to facilitate the study of homomorphisms between tensor modules for $\mathrm{O}(V)$ and $\mathrm{SO}(V)$, we recall the definition of the Brauer category in §3.2. The FFT for $\mathrm{O}(V)$ amounts to the statement that the functor from the Brauer category to the category of tensor representations of $\mathrm{O}(V)$ is full.

However this functor is not full when regarded as a functor to representations of $\mathrm{SO}(V)$, because extra morphisms are needed. We therefore define an enhancement of the Brauer category in §5.1, which is again described in terms of a presentation using diagrams, with the generators and relations which define the Brauer category (cf. [LZ2]) plus one extra generator and several extra relations which involve it. We prove a FFT for SO_m ($= \mathrm{SO}(V)$ for $V = \mathbb{C}^m$) for our new category, i.e., we show that the functor from our enhanced category to the category of tensor representations of $\mathrm{SO}(V)$ is full, but the surprising fact to emerge is that one can prove that the new relations at the categorical level include the generator of the kernel of the functor from the Brauer category to the category of tensor representations. This implies that the maps induced on Hom spaces by the functor to representations, are all automatically isomorphisms, i.e. that our new category is equivalent to the category of tensor representations of SO_m. In particular, the dimensions of the endomorphism algebras may be computed formally in our category. That is, we may determine the Hom spaces (for example their dimensions) between $\mathrm{SO}(V)$-tensor modules by a formal computation using just the generators and relations in our enhanced category.

The calculation which yields that the ideal of relations we give here includes the kernel of the maps from the Hom spaces in our abstract category to those in the category of tensor representations of SO_m appears for $m = 3$ in [BE], which was part of our inspiration for this work.

Although our results here are framed for spaces over $\mathbb{C}$, they actually apply without extra effort to any field of characteristic $p > m = \dim V$, and with a little more effort to any field of odd characteristic. That is, our main theorems hold if $\mathbb{C}$ is replaced by an algebraically closed field of characteristic $p > m$, as an analogue of the argument in [LZl, Thm. 9.4] shows. If $\mathbb{C}$ is replaced by an algebraically closed field of smaller odd characteristic, the methods of [HX] may be applied.

In a future work we intend to apply similar methods to the case of the orthosymplectic Lie super algebra $\mathfrak{osp}(m|2n)$. In this case, the space of super Pfaffians (cf. [LZ4]) will come into play as the harmonic generators of those invariants of $\mathfrak{osp}(m|2n)$ which are semi invariant for the super group scheme $\mathrm{OSp}(m|2n)$.

2. Preliminaries

Let $V = \mathbb{C}^m$ be equipped with a non-degenerate symmetric bilinear form $(\ ,\ )$. Let $G := \mathrm{O}(V)$ be its isometry group, and $G' = \mathrm{SO}(V)$, the identity component of G. We refer to V as the natural module for G or G', and we shall be concerned with the invariants $(V^{\otimes r})^{G'}$.

Assuming the first fundamental theorem (FFT) of invariant theory for G, we give a short proof of the FFT of the invariant theory for the special orthogonal group G', in the context of tensor powers of the natural module. That is, we shall give a spanning set for $(V^{\otimes r})^{G'}$. The result is of course classical, but our proof is particularly short and depends only on diagrammatic ideas.

If M is any finite dimensional G-module, then $M^{G'} = M^G \oplus M^{G,\det}$, where

$$M^{G,\det} = \{m \in M \mid g(m) = \det(g)m, \ \forall g \in G\}.$$

Our aim is to explicitly describe $M^{G'}$ when $M = V^{\otimes r}$. We assume that $(V^{\otimes r})^G$ is known, and given by the FFT for G as stated in Theorem 3.2 below; hence we need only consider $(V^{\otimes r})^{G,\det}$. To state our theorem, we need the element $\check{C} \in V \otimes V$ and its dual $\hat{C}$ introduced in §3.2 below. We note also that we shall be dealing with the Hom spaces of the category whose objects are the vector spaces $V^{\otimes r}$, $r = 0, 1, 2, \ldots$. Since $V^{\otimes 0} = \mathbb{C}$, an element D of $V^{\otimes r}$ may be thought of as an element of $\mathrm{Hom}(V^{\otimes 0}, V^{\otimes r})$, and an element $\sigma \in \mathrm{Sym}_r$ (the symmetric group) as an element of $\mathrm{Hom}(V^{\otimes r}, V^{\otimes r})$.

With this understanding, an element $D \in V^{\otimes r}$ is *harmonic* if

$$\big(\hat{C} \otimes (\mathrm{id}_V)^{\otimes r-2}\big) \circ \sigma \circ D = 0 \text{ for all } \sigma \in \mathrm{Sym}_r.$$

Theorem 2.1. *Maintain the above notation. Assume that for some integer $r_c > 0$, there exists a harmonic element Λ in $(V^{\otimes r_c})^{G,\det}$ satisfying the condition $(\Lambda, \Lambda) \neq 0$. Then $(V^{\otimes r})^{G,\det} \neq 0$ if and only if $r - r_c \geq 0$ is even, and in this case,*

$$(V^{\otimes r})^{G,\det} = \mathbb{C}\mathrm{Sym}_r(\Lambda \otimes \check{C}^{\otimes \frac{r-r_c}{2}}).$$

Here $\mathbb{C}\mathrm{Sym}_r$ is the group ring of the symmetric group Sym_r over the complex numbers. Note that $\Lambda \otimes \check{C}^{\otimes \frac{r-r_c}{2}} \in V^{\otimes r}$, so that $\mathbb{C}\mathrm{Sym}_r(\Lambda \otimes \check{C}^{\otimes \frac{r-r_c}{2}})$ denotes the space of all linear combinations of $\Lambda \otimes \check{C}^{\otimes \frac{r-r_c}{2}}$, composed with a permutation.

Remark 2.2. (1) Since any nonzero element $\Lambda \in \wedge^{\dim V} V$ evidently satisfies the conditions of Theorem 2.1, we recover the FFT of invariant theory of the special orthogonal group (see [FH, Appendix F] and [Pro, §10.2]).

(2) One of our motivations in formulating the result in this way is to stress the role of harmonicity, because that is what generalises to other cases, such as the orthosymplectic Lie super algebra.

We prepare for the proof by explaining a diagrammatic way of viewing the problem.

3. The category $\mathcal{T}$

Let $\mathcal{T}$ be the full subcategory of the category of finite dimensional modules for $G' = \mathrm{SO}(V)$ with objects $V^{\otimes r}$, where $r \in \mathbb{N}$. For any $i, j \in \mathbb{N}$, let

$$\mathcal{T}_i^j := \mathrm{Hom}_{G'}(V^{\otimes i}, V^{\otimes j}).$$

Remark 3.1. Note that $\mathcal{T}$ is naturally a tensor category: if $A \in \mathcal{T}_i^j$ and $B \in \mathcal{T}_r^s$, then $A \otimes B \in \mathcal{T}_{i+r}^{j+s}$ is defined in the obvious way. All categories in this work have a similar tensor structure, which is realised by juxtaposition of either tensors or of diagrams, depending on the context. The symbol $\otimes$ will always denote this tensor product, or juxtaposition, and should be distinguished from composition of morphisms.
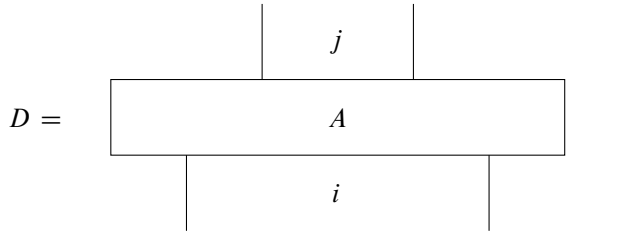
We also have a natural duality on $\mathcal{T}$ arising from the fact that the form $(\ ,\ )$ extends naturally to a non-degenerate symmetric bilinear form, also denoted $(\ ,\ )$ on $V^{\otimes r}$ for any r, so that the dual space $(V^{\otimes r})^*$ is naturally identified with $V^{\otimes r}$. If $\underline{v} \in V^{\otimes r}$, denote by $\underline{v}^*$ the element of $(V^{\otimes r})^*$ defined by $\underline{v}^*(\underline{w}) = (\underline{v}, \underline{w})$ for all $\underline{w} \in V^{\otimes r}$. Note that if $g \in \mathrm{O}(V)$, then $g(\underline{v}^*) = (g\underline{v})^*$.

For $A \in \mathcal{T}_i^j$, the dual $A^* \in \mathcal{T}_j^i$ is defined by
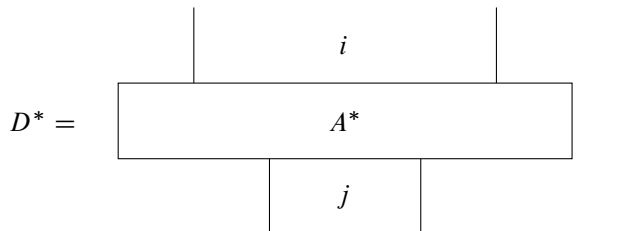
$$(A\underline{v}, \underline{w}) = (\underline{v}, A^*\underline{w})$$

for all $\underline{v} \in V^{\otimes i}$ and $\underline{w} \in V^{\otimes j}$.

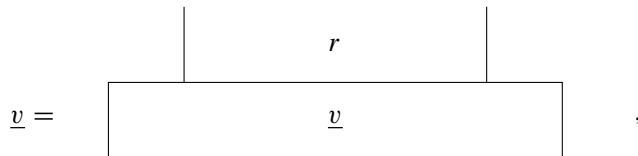**3.1. Diagrammatics.** If $A \in \mathcal{T}_i^j$, we denote A by a diagram D:

$$D = \quad \boxed{A} \quad .$$

(with j labelling strands above the box A and i labelling strands below.)

Then A^* is represented by the reflection D^* in a horizontal of the above diagram:

$$D^* = \quad \boxed{A^*} \quad .$$

(with i labelling strands above the box A^* and j labelling strands below.)

Composition of morphisms is thought of as concatenation of diagrams.

An element $\underline{v} \in (V^{\otimes r})^{G'}$ may be thought of as the map in $\mathcal{T}_0^r$ which sends $1 \in \mathbb{C}$ to $\underline{v}$. Then $\underline{v}^*$ is the map in $\mathcal{T}_r^0$ which takes $\underline{w}$ to $(\underline{v}, \underline{w})$. Diagramatically:

$$\underline{v} = \quad \boxed{\underline{v}} \quad ,$$

(with r labelling the strands above the box $\underline{v}$.)

and

$$\underline{v}^* = \quad \boxed{\underline{v}^*} \quad .$$

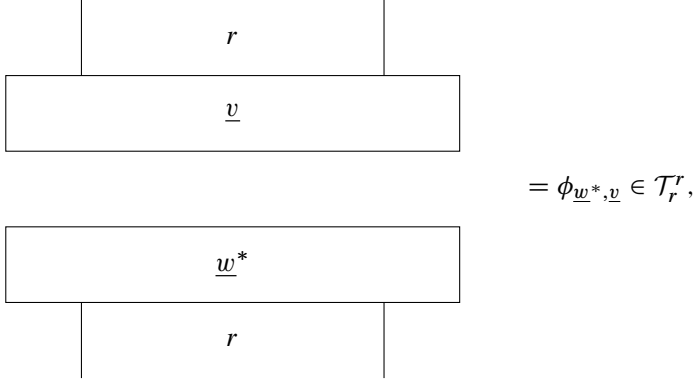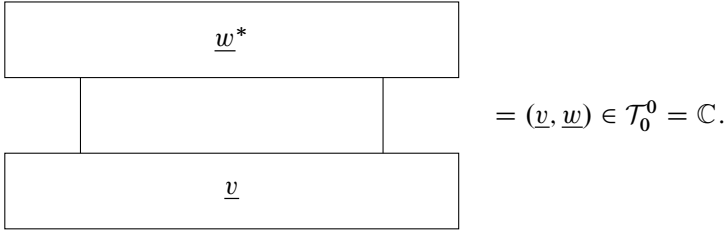(with r labelling the strands below the box $\underline{v}^*$.)
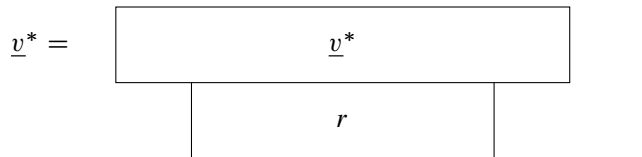
If $\underline{v}, \underline{w} \in (V^{\otimes r})^{G'}$, we may form the compositions $\underline{v} \circ \underline{w}^* \in \mathcal{T}_r^r$ and $\underline{w}^* \circ \underline{v} \in \mathcal{T}_0^0$, and it is easily verified that

$$\text{(3.1)} \qquad \begin{aligned} &\underline{v} \circ \underline{w}^* : \underline{x} \mapsto (\underline{w}, \underline{x})\underline{v} \text{ and} \\ &\underline{w}^* \circ \underline{v} : 1 \mapsto (\underline{w}, \underline{v}). \end{aligned}$$

In terms of diagrams, (3.1) may be written

$$= \phi_{\underline{w}^*,\underline{v}} \in \mathcal{T}_r^r,$$

where $\phi_{\underline{w}^*,\underline{v}}(\underline{x}) = (\underline{w}, \underline{x})\underline{v}$ for all $\underline{x} \in V^{\otimes r}$, and

$$= (\underline{v}, \underline{w}) \in \mathcal{T}_0^0 = \mathbb{C}.$$

3.2. The Brauer subcategory $B \subset \mathcal{T}$. Let $e_1, \dots, e_m$ be an orthonormal basis of V. The element $\check{C} := \sum_i e_i \otimes e_i \in V \otimes V$ is independent of the basis, and $\mathrm{O}(V)$-invariant. Regarded as an element of $\mathcal{T}_0^2 := \mathrm{Hom}_{\mathcal{T}}(0, 2)$, its dual is $\check{C}^* = \hat{C} \in \mathcal{T}_2^0$, where for $v, w \in V$, $\hat{C}(v \otimes w) = (v, w)$ (since $(\check{C}, v \otimes w) = (v, w)$). Note that the relations (3.1) show that $\hat{C} \circ \check{C} = m$ and $\check{C} \circ \hat{C} : v \otimes w \mapsto (v, w)\check{C}$.

We recall the definition of the Brauer category $\mathcal{B} = \mathcal{B}(m)$ (over $\mathbb{C}$) from [LZ2]. It has objects $\mathbb{N}$ and morphisms which are generated by the four morphisms $I, U, A = U^*$ and X in $\mathcal{B}_1^1, \mathcal{B}_0^2, \mathcal{B}_2^0$ and $\mathcal{B}_2^2$ respectively. Moreover we have a functor $\mathcal{F}$ from $\mathcal{B}$ to $\mathcal{T}$ which takes I to id_V, U to $\check{C}$, A to $\hat{C}$ and X to $\tau \in \mathcal{T}_2^2$ given by $\tau(v \otimes w) = w \otimes v$. We abuse language by referring to the image of this functor as the Brauer subcategory $B := \mathcal{F}(\mathcal{B}) \subset \mathcal{T}$, and to its morphisms B_i^j as Brauer morphisms. Note that $\mathbb{C}\mathrm{Sym}_r \subset B_r^r$. Diagrammatically, the morphisms $\mathrm{id}_V, \check{C}, \hat{C}$ and τ are respectively the images of the following diagrams:
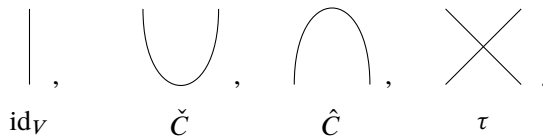
The FFT for $O(V)$ may be stated as follows.

Theorem 3.2. *In the above notation, the functor $\mathcal{F} : \mathcal{B}(m) \longrightarrow \mathcal{T}^{O(V)}$ is full, where $\mathcal{T}^{O(V)}$ is the subcategory of $\mathcal{T}$ with $O(V)$-invariant morphisms. In particular, for any non-negative integers i, j, we have*

$$(3.2) \qquad\qquad (\mathcal{T}_i^j)^{O(V)} = B_i^j.$$

Equivalently, $(V^{\otimes r})^{O(V)} = 0$ if r is odd, while if $r = 2d$ is even, then

$$(3.3) \qquad\qquad (V^{\otimes 2d})^{O(V)} = \mathbb{C}\mathrm{Sym}_{2d}(\check{C}^{\otimes d}).$$

Remark 3.3. (i) In terms of diagrams, the theorem states that $(V^{\otimes r})^{O(V)} \subseteq \mathcal{T}_0^r$ is spanned by diagrams of the following form:
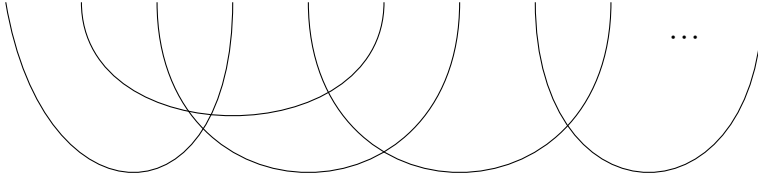
The depicted diagram has image in $\mathcal{T}$ equal to $\pi \circ (\check{C})^{\otimes d}$, where $r = 2d$ and $\pi \in \mathrm{Sym}_{2d}$ is the relevant permutation (in this case $(1)(2, 4, 6, 7, 5, 3)(8, 9)(10)$ if the $\dots$ are ignored and the nodes are numbered from 1 to 10 from left to right).

(ii) It follows from Theorem 3.2 that if $\underline{\xi} \in (V^{\otimes 2d})^G$, where $G = O(V)$ (so that $\underline{\xi}^* \in \mathcal{T}_{2d}^0$), and if $\underline{\xi}^* \circ D = 0$ for each diagram of the above form, then $\underline{\xi} = 0$.

(iii) The diagram shown represents an element of $\mathcal{T}_0^r \simeq (V^{\otimes r})^G$. Its dual in $\mathcal{T}_r^0$, which is the diagram obtained by reflecting the one shown in a horizontal line, applied to $\underline{v} = v_1 \otimes \dots \otimes v_r$ is the product of the contractions (v_i, v_j) over pairs (i, j) lying on a common arc.

4. Proof of Theorem 2.1

Suppose we have $\Lambda \in (V^{\otimes r_c})^{G,\det}$ such that $(\Lambda, \Lambda) \neq 0$ and Λ is harmonic. Let $\pi_\Lambda : V^{\otimes r_c} \longrightarrow \mathbb{C}\Lambda \subseteq V^{\otimes r_c}$ be the G-module homomorphism given by $\pi_\Lambda(\underline{w}) = \frac{(\Lambda, \underline{w})}{(\Lambda, \Lambda)}\Lambda$ for any $\underline{w} \in V^{\otimes r_c}$. The G-equivariance of π_Λ follows from the fact that $g\Lambda = \det(g)\Lambda$ for $g \in G$. For any integer $r \geq 0$, let $\pi_{\Lambda,r} = \pi_\Lambda \otimes \mathrm{id}_{V^{\otimes r}}$; then $\pi_{\Lambda,r}^2 = \pi_{\Lambda,r}$, so that $\pi_{\Lambda,r}$ is the orthogonal projection from $V^{\otimes(r_c+r)}$ to $\mathbb{C}\Lambda \otimes V^{\otimes r} = \Lambda \otimes V^{\otimes r}$.

Proof of Theorem 2.1. Since G is linearly reductive, taking G-fixed points of G-modules is an exact functor, we have an isomorphism

$$(4.1) \qquad (\Lambda \otimes V^{\otimes r})^G \simeq \pi_{\Lambda,r}\left((V^{\otimes(r_c+r)})^G\right).$$

The FFT for G enables us to determine the right hand side of (4.1) by using the harmonicity of Λ.

We show first diagrammatically that $(\Lambda \otimes V^{\otimes r})^G = 0$ if $r < r_c$. For this, observe that $(\Lambda, \Lambda)\pi_{\Lambda,r} \in \mathcal{T}_{r_c+r}^{r_c+r}$ is represented by the diagram
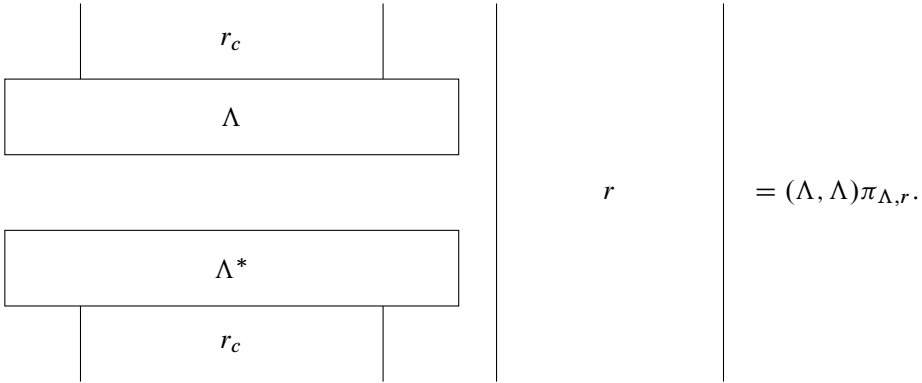
Now if $r < r_c$, it is evident that in any composition of the form $(\Lambda, \Lambda)\pi_{\Lambda,r} \circ D$, where D is the diagram in Remark 3.3, there will be an arc of the lower diagram which has both ends joined to Λ^*. By duality, the fact that Λ is harmonic implies that for any $\sigma \in \mathrm{Sym}_{r_c}$, we have $\Lambda^* \circ \sigma \circ (\check{C} \otimes (\mathrm{id})^{\otimes r_c - 2}) = 0$. It follows that by harmonicity of Λ, if $r < r_c$, any composition $(\Lambda^* \otimes \mathrm{id}_{V^{\otimes r}}) \circ \pi \circ (\check{C})^{\otimes \frac{r_c+r}{2}}$ is zero, where $\pi \in \mathrm{Sym}_{r_c+r}$, and by Remark 3.3 (ii), it follows that any element of $(\Lambda \otimes V^{\otimes r})^G$ is zero.

Moreover, it follows by a straightforward manipulation of diagrams (see below), that if $r \geq r_c$, then again applying the FFT for G in the form (3.3), the right side of (4.1) is a linear combination of the diagrams $\Lambda \otimes \sigma \circ (\Lambda \otimes \check{C}^{\otimes \frac{r-r_c}{2}})$, where $\sigma \in \mathrm{Sym}_r$. That is, if $r \geq r_c$, then as elements of $(V^{\otimes(r+r_c)})^G \subset \mathcal{T}_0^{r_c+r}$,

$$(4.2) \qquad \pi_{\Lambda,r}\left((V^{\otimes(r+r_c)})^G\right) = \Lambda \otimes \left[\mathbb{C}\mathrm{Sym}_r \circ \left(\Lambda \otimes (V^{\otimes(r-r_c)})^G\right)\right].$$

This may be seen as follows. When a diagram D of the form in Remark 3.3 is composed with $(\Lambda, \Lambda)\pi_{\Lambda,r}$, then if the composition is non-zero, all the arcs of D have at most one end in Λ^*, and each string of Λ^* connects with an arc of D. Hence one may interpose (between D and $(\Lambda, \Lambda)\pi_{\Lambda,r}$) a permutation in $\mathrm{Sym}_{r_c} \times \mathrm{Sym}_r$ to obtain an element in $\Lambda \otimes (\Lambda \otimes (V)^{\otimes r - r_c})^G$, after sliding Λ^* and rotating through π to obtain Λ.

Next observe that evidently

$$(4.3) \qquad \Lambda \otimes (V^{\otimes r})^{G,\det} = (\Lambda \otimes V^{\otimes r})^G = \pi_{\Lambda,r}\left((V^{\otimes(r+r_c)})^G\right).$$

It follows from (4.2) and (4.3), as well as the above analysis of the case $r < r_c$, that

$$(4.4) \qquad (V^{\otimes r})^{G,\det} = \begin{cases} 0 \text{ if } r < r_c, \text{ and} \\ \mathbb{C}\mathrm{Sym}_r\left(\Lambda \otimes (V^{\otimes(r-r_c)})^G\right), \quad \forall r \geq r_c. \end{cases}$$

Applying the FFT for G (3.3) to identify $(V^{\otimes r-r_c})^G$, we obtain the theorem. $\square$

Example 4.1. Let $e_1, \ldots, e_m$ be an orthonormal basis of V, and take $\Lambda = e_1 \wedge \cdots \wedge e_m = A_m(e_1 \otimes \ldots \otimes e_m)$ where $A_m = (m!)^{-1}\sum_{\sigma \in \mathrm{Sym}_r} \varepsilon(\sigma)\sigma \in B_m^m$. Then Λ is evidently harmonic, and $(\Lambda, \Lambda) = (m!)^{-1} \neq 0$.

Corollary 4.2. *Let* $i, j \in \mathbb{N}$. *Any element* $\alpha \in (\mathcal{T}_i^j)^{\mathrm{SO}(V)}$ *is uniquely expressible in the form* $\alpha = \alpha_1 + \alpha_2$, *where* $\alpha_1 \in B_i^j$ *and* α_2 *is obtained from an element of* $(V^{\otimes r_c})^{\mathrm{O}(V),\det} \subset (\mathcal{T}_0^{r_c})^{\mathrm{SO}(V)}$ *by tensoring and composition with elements of* B, *i.e., with Brauer morphisms.*

Example 4.1 and Theorem 2.1 give a complete description of the space $(V^{\otimes r})^{\mathrm{O}(V),\det}$, so that Corollary 4.2 is the FFT for $\mathrm{SO}(V)$.

Notice that Corollary 4.2 states more than simply that all $\mathrm{SO}(V)$-morphisms between tensor powers of V are obtained from Brauer morphisms (i.e., those which commute with $\mathrm{O}(V)$) and Λ by composition and tensor product. It states that Λ need only arise at most once in any such 'word' for an invariant homomorphism.

Remark 4.3. The arguments above may be adapted to prove a corresponding result [LZ4] (where the super Pfaffian enters) for $\mathfrak{osp}_{1|2n}$.

## 5. The enhanced Brauer category $\widetilde{\mathcal{B}}(\delta)$

Let R be a commutative ring and $\delta \in R$. We fix a positive integer m. With this data, we shall define a tensor category $\widetilde{\mathcal{B}}(\delta)$, which contains a quotient of the usual Brauer category $\mathcal{B}(\delta)$ [LZ2] as a subcategory. We shall see that the relations we impose imply a relationship between δ and m, so that for each m there are only finitely many values of δ which make our relations consistent. Both categories have objects $\mathbb{N} = \{0, 1, 2, \ldots\}$, morphisms which may be described diagrammatically. There is an involution $* : \mathrm{Hom}(m, n) \longrightarrow \mathrm{Hom}(n, m)$ which is

described on diagrams by reflecting diagrams in a horizontal line, if they are drawn as we have shown in §§3,4. This map may be interpreted as a functor from the category to its opposite.

5.1. Definition of $\widetilde{\mathcal{B}}(\delta)$. We have seen that $\mathcal{B}(\delta)$ may be presented as the category with object set $\mathbb{N}$ and morphisms which are generated by the four morphisms I, U, A and X under composition, tensor product and duality, subject to certain relations, which are described in [LZ2]. In the definition below, we shall make extensive use of the total anti-symmetriser $\Sigma_r \in \mathcal{B}_r^r$, where $\mathcal{B} = \mathcal{B}(\delta)$. This is defined by

$$(5.1) \qquad \Sigma_r = \sum_{\pi \in \mathrm{Sym}_r} \varepsilon(\pi)\pi,$$

and is depicted diagramatically as
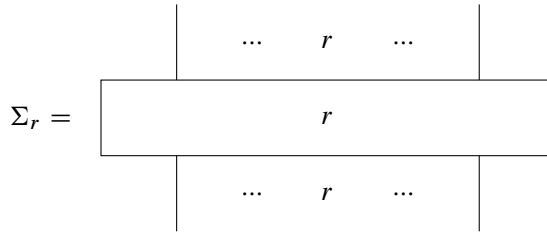
Remark 5.1 (Remark about notation). This notation is inconsistent with that of §3.1, but justified because of its simplicity, and the extensive use we make of the endomorphism Σ_r.

Definition 5.2. Let R be a ring, $\delta \in R$ and $m \geq 2$ a positive integer. The category $\widetilde{\mathcal{B}}(\delta)$ has object set $\mathbb{N}$ and morphisms which are generated by I, U, A, X and one new generator $\Delta_m \in \widetilde{\mathcal{B}}_0^m$, subject to the following relations, which describe the interaction of the new generator with the Brauer morphisms.

(1) The relations [LZ2, Thm. 2.6(2)] for the generators I, U, A and X.

(2) (Harmonicity) For each positive integer r with $0 \leq r \leq m - 2$,
$(I^{\otimes r} \otimes A \otimes I^{\otimes m-r-2}) \circ \Delta_m = 0$.

(3) For each positive integer r with $0 \leq r \leq m - 2$,
$(I^{\otimes r} \otimes X \otimes I^{\otimes m-r-2}) \circ \Delta_m = -\Delta_m$.

(4) $\Delta_m \circ \Delta_m^* = \Sigma_m$.

(5) $\Delta_m \otimes I \otimes \Delta_m = (c_{m+1} \otimes I^{\otimes m}) \circ (\Delta_m \otimes \Delta_m \otimes I)$, where c_{m+1} is the $m + 1$-cycle $(m + 1, m, m - 1, \ldots, 1) \in \mathrm{Sym}_{m+1}$.

The new generator Δ_m will be depicted diagrammatically (as a morphism from 0 to m) as follows.
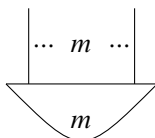
The relations above have suggestive diagrammatical interpretations, which are helpful in performing computations in the category $\widetilde{\mathcal{B}}(\delta)$. For example, the relation (4) may be depicted diagrammatically as in Fig. 1, and the relation (5) is shown in Fig. 2.
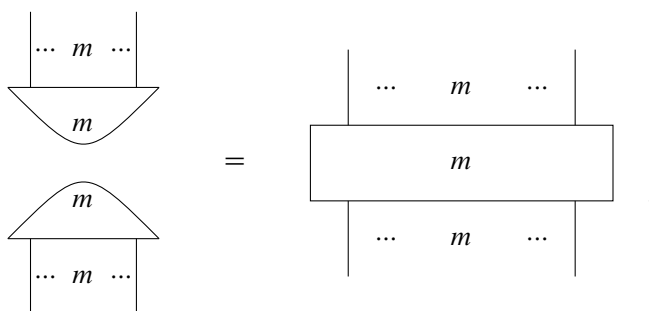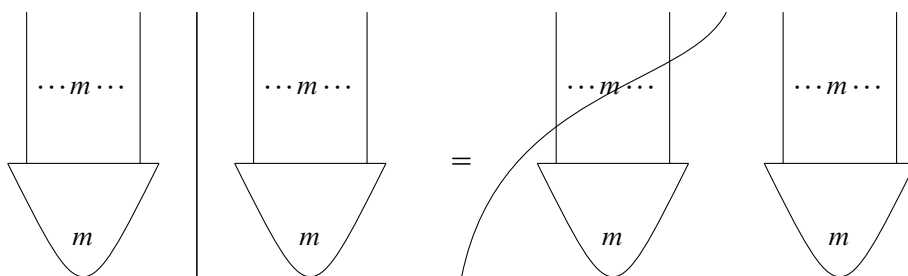

Figure 1
The relation (4)

Figure 2
The relation (5)

Remark 5.3. With our application to invariant theory in mind, we shall assume that the base ring R is an integral domain, and that $m! \neq 0$ in R.

For any integers $r > 0$ and $i = 1, 2, \ldots, r-1$, write $\sigma_i = (I^{i-1}) \otimes X \otimes (I^{\otimes r-i-1}) \in \widetilde{\mathcal{B}}_r^r$. Evidently the σ_i generate the symmetric group $\mathrm{Sym}_r \subseteq \widetilde{\mathcal{B}}_r^r$, and condition (3) of Definition 5.2 asserts that the generators σ_i of Sym_m satisfy $\sigma_i \circ \Delta_m = -\Delta_m$, whence $w \circ \Delta_m = \varepsilon(w)\Delta_m$ for $w \in \mathrm{Sym}_m$, where ε is the alternating character of Sym_m. It follows that $\Sigma_m \circ \Delta_m = m!\Delta_m$, and hence by the above assumptions, that, if $\Sigma_m = 0$, then $\Delta_m = 0$. If $\Sigma_m = 0$, the category $\widetilde{\mathcal{B}}(\delta)$ therefore is just a quotient category of $\mathcal{B}(\delta)$.

To avoid this degeneracy, we shall therefore assume that $\Sigma_m \neq 0$.

Remark 5.4. Note that although δ does not appear explicitly in the definition above, it is inherent in the definition of $\mathcal{B}(\delta)$, where it is stipulated that $U \circ A = \delta$ (note that $\mathcal{B}_0^0 = R$). The integer m enters only in the definition of $\widetilde{\mathcal{B}}$.

5.2. Some computations in the category $\widetilde{\mathcal{B}}(\delta)$. In this section, we shall perform the computation which we shall later use to show that the kernel of the (surjective) map from $\widetilde{\mathcal{B}}(m)_i^j$ to $(\mathcal{T}_i^j)^{\mathrm{SO}_m}$ is zero. We shall argue diagramatically, and recall the following result from [LZ2].

Lemma 5.5. [LZ2, Lemma 2.1(1) and (2)] *For all $r \geq 1$ we have the following relations in $\mathcal{B}(\delta)$, and hence **a fortiori** in $\widetilde{\mathcal{B}}(\delta)$.*

(6)

(7)

The next result shows that there are constraints upon the parameter δ arising from consistency questions in the category $\widetilde{\mathcal{B}}(\delta)$.

Lemma 5.6. *Assume that $m! \neq 0$ in R and that $\Sigma_m \neq 0$. Then δ satisfies the polynomial equation*

(5.2) $$\delta(\delta - 1) \ldots \big(\delta - (m-1)\big) = m!.$$

Furthermore, $\Delta_m^* \Delta_m = m! \in R$.

Proof. We shall compute $\Delta_m^* \Delta_m$ in two different ways. First, observe that by inspection of the relevant diagrams, it is evident that $(\Delta_m \Delta_m^*)^2 = (\Delta_m(\Delta_m^* \Delta_m)\Delta_m^*)$, where $\Delta_m^* \Delta_m$ is a scalar; thus applying Relation (4), $(\Delta_m \Delta_m^*)^2 = (\Delta_m^* \Delta_m)\Sigma_m$.

But again by Relation (4), $(\Delta_m \Delta_m^*)^2 = \Sigma_m^2 = m!\Sigma_m$, whence comparing coefficients of the non-zero element Σ_m, it follows that

$$(5.3) \qquad\qquad \Delta_m^* \Delta_m = m!.$$

Next, note that we have the relation (8) depicted in Fig. 3.
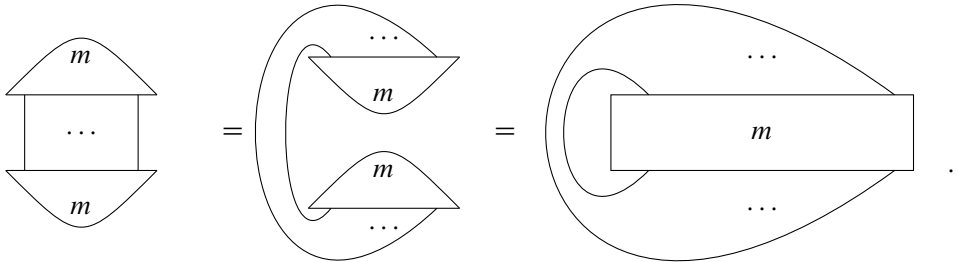

Figure 3
Relation (8)

Now the right side of relation (8) is, by m applications of Lemma 5.5, equal to $(\delta - (m-1))(\delta - (m-2))\dots(\delta - 1)\delta$, while the left side is just $\Delta_m^* \Delta_m$. The result is now clear from (5.3). $\qquad\qquad\square$

Theorem 5.7. *The assumptions of Remark 5.3 remain in force. That is, we assume $m! \neq 0$ in R, and $\Sigma_m \neq 0$.*

(1) *In the category $\widetilde{\mathcal{B}}(\delta)$ we have the equality of morphisms $\Sigma_{m+1} = f_m(\delta)\Sigma_m \otimes I$. Here m is the positive integer occurring in the definition of $\widetilde{\mathcal{B}}(\delta)$ and f_m is the polynomial in δ given by $f_m(\delta) = (\delta - (m-1))(\delta - (m-2))\dots(\delta - 1) - (m-1)!$.*

(2) *We have $\Sigma_{m+1} = 0$.*

(3) *We have $\delta = m$.*

Proof. We begin by proving part (1). In this proof we shall make liberal use, both explicit and implicit, of the mutually inverse isomorphisms $\mathbb{U}_p^q : \widetilde{\mathcal{B}}_{p+q}^r \to \widetilde{\mathcal{B}}_p^{r+q}$ and $\mathbb{A}_q^r : \widetilde{\mathcal{B}}_p^{r+q} \to \widetilde{\mathcal{B}}_{p+q}^r$ defined in [LZ2, Cor. 2.8]. Note that these isomorphisms involve only operations (tensor product and composition) with the Brauer morphisms in $\widetilde{\mathcal{B}}(\delta)$.

The relation (6) of Lemma 5.5 yields in our situation

We now replace each of the two rectangles in the second summand on the right side above by the left side of Figure 1. A little manipulation then shows that the result will follow if we prove the relation (9) in Fig. 4.

FIGURE 4
Relation (9)

Next, observe that by rotating the top half of the left side of Relation (9) in Fig. 4 anticlockwise by π and then applying the isomorphism $\mathbb{U}_0^1$ from $\mathcal{B}_1^1$ to $\mathcal{B}_0^2$, the relation (9) is equivalent to Relation (10) in Fig. 5.

Now to prove Relation (10), observe first that applying the isomorphism $\mathbb{U}_0^m$ to both sides of the relation (4) as shown in Fig. 1, we obtain the relation (11) in Fig. 6.

Then, applying $I \otimes A^{\otimes m-1} \otimes I$ to both sides of Relation (11), and applying Lemma 5.5 (7) $m-1$ times, we obtain the relation (10) of Fig. 5, and the proof of part (1) is complete.

To prove part (2), note that it follows from part (1) that

$$\Sigma_{m+1} \circ (I^{\otimes m-1} \otimes U) = f_m(\delta)(\Sigma_m \otimes I) \circ (I^{\otimes m-1} \otimes U).$$
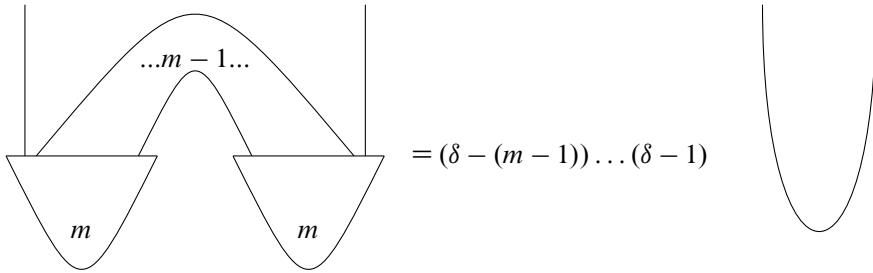
$$= (\delta - (m-1))\dots(\delta - 1)$$
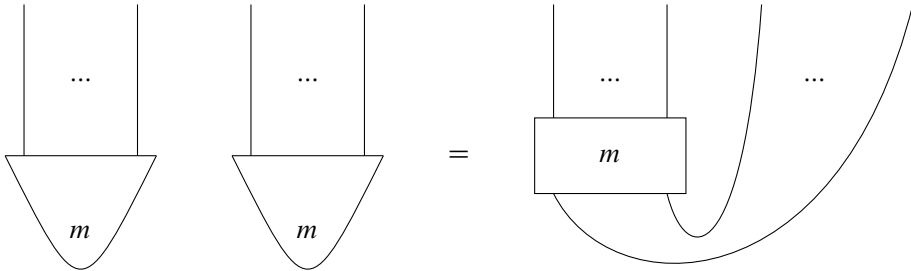
Figure 5
Relation (10)

Figure 6
Relation (11)

But the left side of this equation is evidently zero, while the right side is an invertible multiple of $f_m(\delta)\Sigma_m \otimes I$. It follows that $f_m(\delta) = 0$, and hence by (1), that $\Sigma_{m+1} = 0$.

Finally to prove part (3), observe that $f_m(\delta) = 0$ implies that $\delta(\delta - 1)\dots(\delta - (m-1)) = \delta(m-1)!$. Comparing this to the relation $\delta(\delta-1)\dots(\delta-(m-1)) = m!$ of Lemma 5.6, we see that $\delta = m$. $\qquad\qquad\square$

Remark 5.8. Although Theorem 5.7 has been proved under the assumption that $\Sigma_m \neq 0$, the conclusions of parts (1) and (2) evidently remain true if $\Sigma_m = 0$, but (3) fails. The application to invariant theory is also predicated upon this assumption.

6. A covariant functor-application to invariant theory

6.1. The main theorem. The results of the previous section indicate that the category of relevance to the invariant theory of SO_m is $\widetilde{\mathcal{B}}(m)$. Our main theorem is as follows.

Theorem 6.1. *Let V be $\mathbb{C}^m$, equipped with a nondegenerate symmetric bilinear form $(,)$, and let $G = SO(V)$. Let $\mathcal{T}$ be the full tensor subcategory of finite dimensional representations of G generated by V under tensor product.*

*There is an **equivalence** of categories $\mathcal{F} : \widetilde{\mathcal{B}}(m) \longrightarrow \mathcal{T}$ defined by $\mathcal{F}(r) = V^{\otimes r}$ for $r \in \mathbb{N}$, $\mathcal{F}(I) = \mathrm{id} : V \to V$, $\mathcal{F}(X) : V \otimes V \to V \otimes V$ is interchange of tensor factors, $\mathcal{F}(A) : V \otimes V \to \mathbb{C}$ is the map $v \otimes w \mapsto (v, w)$, $\mathcal{F}(U) : 1 \mapsto \check{C}$ and $\mathcal{F}(\Delta_m) = \Lambda$, where Λ is the harmonic homomorphism in Example 4.1.*

Before giving the proof, we shall make some elementary observations concerning the structure of $\widetilde{\mathcal{B}}(m)$.

Lemma 6.2. *Let $\widetilde{\mathcal{B}}_0$ be the subcategory of $\widetilde{\mathcal{B}}(m)$ generated by all Brauer diagrams (i.e., by the morphisms I, X, A and U).*

(1) *Each diagram of $\widetilde{\mathcal{B}}(m)$ is either in $\widetilde{\mathcal{B}}_0$ or is obtained from Δ_m by tensoring and composing with elements of $\widetilde{\mathcal{B}}_0$.*

(2) *Let $s, t \in \mathbb{N}$. Then*

$$\widetilde{\mathcal{B}}_s^t = \widetilde{\mathcal{B}}_{s,0}^t \oplus \widetilde{\mathcal{B}}_{s,1}^t,$$

where $\widetilde{\mathcal{B}}_{s,0}^t$ is the span of the Brauer diagrams in $\widetilde{\mathcal{B}}_s^t$, and $\widetilde{\mathcal{B}}_{s,1}^t$ is the span of diagrams of the second type described in (1).

Proof. If the diagram $D \in \widetilde{\mathcal{B}}(m)$ is not in $\widetilde{\mathcal{B}}_0$, then it may be expressed as a 'word' in the generators I, X, A, U and Δ_m, with connectives $\otimes$ (tensor product) and $\circ$ (composition), since $\Delta_m^* = \mathbb{A}_m^0(\Delta_m)$. But the relation (5) in Fig. 2 above shows that any diagram with two occurrences of Δ_m, is equal in $\widetilde{\mathcal{B}}$ to an element where the occurrences are adjacent. Hence by the relation (5) in Fig. 5, this diagram is equal to one in $\widetilde{\mathcal{B}}_0$. Thus we may assume that there is precisely one occurrence of Δ_m in the word expression for D. This proves (1).

The statement (2) is an immediate consequence of (1), since each Hom space is spanned by diagrams, and the two types of diagrams in (1) are complementary. $\square$

Proof of Theorem 6.1. We have seen [LZ2] that the relations satisfied by I, X, A and U are satisfied by their images under the functor $\mathcal{F}$. It is easily checked that the relations (1) to (5) of definition 5.2 are satisfied with Δ_m replaced by Λ, so that we do have a functor.

It remains only to see that $\mathcal{F}$ defines isomorphisms on Hom spaces. But Theorem 2.1 states precisely that $\mathcal{F}$ is surjective on Hom spaces (the FFT). We are therefore reduced to proving the injectivity of $\mathcal{F}$ on Hom spaces, which is the SFT for SO_m.

By Lemma 6.2(2), each element $\beta \in \ker(\mathcal{F}_s^t) : \mathrm{Hom}_{\widetilde{\mathcal{B}}(m)}(s,t) \longrightarrow \mathrm{Hom}_G(V^{\otimes s}, V^{\otimes t})$ is uniquely of the form $\beta = \beta_0 + \beta_1$, where $\beta_i \in \widetilde{\mathcal{B}}_{s,i}^t$ $(i = 0, 1)$. Moreover $\mathcal{F}_s^t$ maps $\widetilde{\mathcal{B}}_{s,0}^t$ to $\mathrm{Hom}_{O(V)}(V^{\otimes s}, V^{\otimes t})$, and $\widetilde{\mathcal{B}}_{s,1}^t$ to the space of skew invariants for $O(V)$. It follows that $\beta \in \ker(\mathcal{F}_s^t)$ if and only if $\beta_i \in \ker(\mathcal{F}_s^t)$ for $i = 0, 1$.

Now [LZ2, Thm. 4.8] states that the image of an element γ of $\widetilde{\mathcal{B}}(m)_0$ under $\mathcal{F}$ is zero if and only if γ is in the ideal $\langle \Sigma_{m+1} \rangle$ of morphisms generated under the operations of a tensor category by Σ_{m+1}.

This proves that β_0 is in $\langle \Sigma_{m+1} \rangle$.

As for β_1, note that because of its form, we have $\beta_1 \circ (\Delta_m^* \otimes I^{\otimes r}) \in \widetilde{\mathcal{B}}_0$ and hence lies in $\langle \Sigma_{m+1} \rangle$ for some r, and so $\beta_1 = (m!)^{-1} \left(\beta_1 \circ (\Delta_m^* \otimes I^{\otimes r}) \right) \circ (\Delta_m \otimes I^{\otimes r})$ is also in the ideal $\langle \Sigma_{m+1} \rangle$. Hence $\beta \in \langle \Sigma_{m+1} \rangle$. By Theorem 5.7, $\langle \Sigma_{m+1} \rangle$ is zero in $\widetilde{\mathcal{B}}(m)$, and the proof is complete. □

6.2. Dimensions of Hom spaces. The dimension of the space $\mathrm{Hom}_G(V^{\otimes s}, V^{\otimes t})$, where G is either $O(V)$ or $SO(V)$, depends only on $r := s + t$. Let $d(r) = \dim(\mathrm{Hom}_{O(V)}(\mathbb{C}, V^{\otimes r}))$ $(= \dim(\mathrm{Hom}_{O(V)}(V^{\otimes s}, V^{\otimes t}))$ when $s + t = r$). The following statement is easily deduced from Lemma 6.2 and Theorems 6.1 and 2.1.

Corollary 6.3. *If $s + t = r$, we have*

$$\dim\left(\mathrm{Hom}_{SO(V)}(V^{\otimes s}, V^{\otimes t})\right) = \dim \widetilde{\mathcal{B}}_s^t \le \binom{r}{m} d(r - m) + d(r).$$

Note that since $d(r) = 0$ when r is odd, depending on the parity of m, either or both terms on the right side of the above formula may vanish.

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