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A note on o-minimal flows and the Ax–Lindemann–Weierstrass theorem for semi-abelian varieties over $\mathbb{C}$

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Abstract. In this short note we present an elementary proof of Theorem 1.2 from [UY2], and also the Ax–Lindemann–Weierstrass theorem for abelian and semi-abelian varieties. The proof uses ideas of Pila, Ullmo, Yafaev, Zannier (see, e.g., [PZ]) and is based on basic properties of sets definable in o-minimal structures. It does not use the Pila–Wilkie counting theorem.

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1. Introduction

In their article [PZ], Pila and Zannier proposed a new method to tackle problems in Arithmetic geometry, a method which makes use of model theory, and in particular the theory of o-minimal structures. They produce a new proof for the Manin-Mumford conjecture, so let us first recall the setting: An abelian variety is a projective algebraic variety, equipped with an algebraic group structure. Over the complex field it admits the structure of a compact complex Lie group. The Manin-Mumford Conjecture (proven by Raynaud, [Ray]) states that for a complex abelian variety A, if $X \subseteq A$ is an irreducible algebraic subvariety and the torsion points of the group of A are Zariski dense in X then X is a coset of an abelian subvariety of A.

The strategy of Pila and Zannier went roughly as follows: Given an n-dimensional complex abelian variety A, consider the (transcendental) uniformizing map $\pi : \mathbb{C}^n \to A$. If $V \subseteq A$ is an algebraic subvariety, with "many" torsion points, consider its pre-image $\tilde{V} = \pi^{-1}(V)$. This is an analytic subvariety $\tilde{V} \subseteq \mathbb{C}^n$, invariant under translation by the lattice $\Lambda = \ker(\pi)$. When restricted to a fundamental domain $F \subseteq \mathbb{C}^n$, the set $\tilde{V} \cap F$ is definable in the o-minimal structure $\mathbb{R}_{an}$. At the heart of the proposed method was a theorem by Pila and Wilkie, [PW], used to conclude that $\tilde{V}$ contains an algebraic variety X of positive dimension. At the last step of the proof one shows that X is contained in a coset of a $\mathbb{C}$-linear subspace L of $\mathbb{C}^n$, with $L \subseteq \tilde{V}$. Finally, the Zariski closure of $\pi(L)$ is a coset of an abelian subvariety of V (with a little more work one shows that V itself is such a coset).

Because of various analogous theorems the last ingredient of the argument became known as the "Ax–Lindemann–Weierstrass" statement for abelian varieties, which we abbreviate here ALW. Recall that the classical Lindemann-Weierstrass theorem says that if $a_1, \dots, a_n \in \mathbb{C}$ are algebraic numbers that are linearly independent over $\mathbb{Q}$ then $e^{a_1}, \dots, e^{a_n}$ are algebraically independent over $\mathbb{Q}$. In [Ax], Ax proved analogous statements for formal power series. In [PZ] ALW was proved, for abelian varieties, by a mixture of topological and o-minimal arguments.

Following the seminal paper of Pila, [Pil], on the Andre-Oort Conjecture for $\mathbb{C}^n$ it became clear that the Pila-Zannier method was very effective in attacking other problems in arithmetic geometry. Each such problem was broken-up into various parts and the ALW was isolated as a separate statement. Somewhat surprisingly, despite the fact that ALW does not seem to have a clear arithmetic content, Pila found an ingenious way to apply the Pila-Wilkie theorem again in order to prove it in the setting of the Andre-Oort conjecture for $\mathbb{C}^n$ (this is sometimes called "the hyperbolic ALW"). The method of Pila was applied extensively since then to settle several variants of ALW ([Orr], [PT], [UY1], [KUY]).

Our goal in this note is to give a simple proof of ALW for both abelian and semi-abelian varieties (recall that a semi-abelian variety is an extension of an abelian variety by $\mathbb{G}_m^n$). We believe that this simpler approach can clarify the picture substantially and eventually yield new results as well.

1.1. Geometric restatements of ALW for semi-abelian varieties. The next theorem follows from a more general theorem of Ax (see [Ax, Theorem 3]) and often is called the full Ax–Lindemann–Weierstrass Theorem (see also [Kir] for discussion). The original proof of Ax used algebraic differential methods.

Theorem 1.1 (Full ALW). *Let G be a connected semi-abelian variety of dimension d defined over $\mathbb{C}$, $\mathbf{T}_G = \mathbb{C}^d$ be the Lie algebra of G and $\exp_G : \mathbf{T}_G \to G$ the exponential map.*

Let $W \subseteq \mathbf{T}_G$ be an irreducible algebraic variety and $\bar{\xi} = (\xi_1, \dots, \xi_d) : W \to \mathbf{T}_G$ be a rational map with $\xi_i \in W(\mathbb{C})$.

Assume the image of W *under the composition* $\exp_G \circ \bar{\xi}$ *is not contained in a translate of a proper algebraic subgroup of* G*. Then the transcendence degree of* $\mathbb{C}(\exp_G(\xi_1, \ldots, \xi_d))$ *over* $\mathbb{C}(\xi_1, \ldots, \xi_d)$ *is* d*.*

Remark 1.2. The transcendence degree of $\mathbb{C}(\exp_G(\xi_1, \ldots, \xi_d))$ in the above theorem is defined to be the transcendence degree of coordinate functions of $\exp_G(\xi_1, \ldots, \xi_d)$ under some projective embedding of G. The degree is computed in the field of meromorphic functions on W.

If in the above theorem we change the conclusion to "transcendence degree of $\mathbb{C}(\exp_G(\xi_1, \ldots, \xi_d))$ over $\mathbb{C}$ is d" then we get a weaker statement that often is called Ax–Lindemann–Weierstrass theorem (ALW theorem for short).

It is not hard to see that both full ALW and ALW theorems can be interpreted geometrically (see, e.g., [Tsi] for more details).

Theorem 1.3 (ALW, Geometric Version). *Let* G *be a connected semi-abelian variety over* $\mathbb{C}$*,* $\mathbf{T}_G$ *the Lie algebra of* G *and* $\exp_G \colon \mathbf{T}_G \to G$ *the exponential map.*

Let $X \subseteq \mathbf{T}_G$ *be an irreducible algebraic variety and* $Z \subseteq G$ *the Zariski closure of* $\exp_G(X)$*. Then* Z *is a translate of an algebraic subgroup of* G*.*

We can also restate full ALW.

Theorem 1.4 (Full ALW, Geometric Version). *Let* G *be a connected semi-abelian variety over* $\mathbb{C}$*,* $\mathbf{T}_G$ *the Lie algebra of* G*,* $\exp_G \colon \mathbf{T}_G \to G$ *the exponential map, and* $\pi \colon \mathbf{T}_G \to \mathbf{T}_G \times G$ *be the map* $\pi(z) = (z, \exp_G(z))$*.*

Let $X \subseteq \mathbf{T}_G$ *be an irreducible algebraic variety and let* $Z \subseteq \mathbf{T}_G \times G$ *be the Zariski closure of* $\pi(X)$*. Then* $Z = X \times B$*, where* B *is a translate of an algebraic subgroup of* G*.*

2. Preliminaries

We work in an o-minimal expansion $\mathcal{R}$ of the real field $\mathbb{R}$, and by definable we always mean $\mathcal{R}$-definable (with parameters). For a general reference on o-minimal structures we refer the reader to [Dri]. The only property of o-minimal structures that we need is that every definable discrete subset of $\mathbb{R}^n$ is finite. We will be using the fact that the structure $\mathbb{R}_{\text{an,exp}}$ is o-minimal, see [Wie] and [DM].

If V is a finite dimensional vector space over $\mathbb{R}$ and X a subset of V then, as usual, we say that X is *definable* if it becomes definable after fixing a basis for V and identifying V with $\mathbb{R}^n$. Clearly this notion does not depend on a choice of basis.

Let $\pi\colon V \to G$ be a group homomorphisms, where V is a finite dimensional vector space over $\mathbb{R}$ and G a connected commutative algebraic group over $\mathbb{C}$. We denote the group operation of G by $\cdot$.

Let $\Lambda = \pi^{-1}(e)$. We say that a subset $F \subseteq V$ is *a large domain for* π if F is a connected open subset of V with $V = F + \Lambda$. If in addition the restriction of π to F is definable then we say that F is *a definable* large domain for π.

Remark 2.1. In the above setting if π is real analytic and Λ is a lattice in V then V/Λ is compact and there is a relatively compact large domain for π definable in $\mathbb{R}_{\mathrm{an}}$.

3. Key observations

In this section we fix a finite dimensional $\mathbb{C}$-vector space V, a connected commutative algebraic group G over $\mathbb{C}$ and $\pi\colon V \to G$ a complex analytic group homomorphism. We assume that $\Lambda = \pi^{-1}(e)$ is a discrete subgroup of V and that π has a definable large domain F.

Let X be a definable connected real analytic submanifold of V and let Z be the Zariski closure of $\pi(X)$ in G.

Let $\widetilde{Z} = \pi^{-1}(Z)$ and $\widetilde{Z}_F = \widetilde{Z} \cap F$. The set $\widetilde{Z}$ is a complex analytic Λ-invariant subset of V and $\widetilde{Z}_F$ is a definable subset of F.

Let

$$\Sigma_F(X) = \{v \in V : v + X \cap F \neq \varnothing \text{ and } v + X \cap F \subseteq \widetilde{Z}_F\}. \tag{3.1}$$

Clearly $\Sigma_F(X)$ is a definable subset of V.

The following is an elementary observation.

Observation 3.1. (1) If $\lambda \in \Lambda$ and $\lambda + F \cap X \neq \varnothing$ then $-\lambda \in \Sigma_F(X)$. In particular $X \subseteq F - (\Sigma_F(X) \cap \Lambda)$.

(2) If v is in $\Sigma_F(X)$ then $v + X \subseteq \widetilde{Z}$ (by analytic continuation and the connectedness of X).

As a consequence we have the following claim.

Claim 3.2. $\pi(\Sigma_F(X)) \subseteq \mathrm{Stab}_G(Z) = \{g \in G : g \cdot Z = Z\}$.

Proof. If v is in $\Sigma_F(X)$ then by Observation 3.1(2) we have $X \subseteq \widetilde{Z} - v$, and hence $\pi(X) \subseteq \pi(v)^{-1} \cdot \pi(\widetilde{Z}) = \pi(v)^{-1} \cdot Z$. Since Z is the Zariski closure of $\pi(X)$ and $\pi(v)^{-1} \cdot Z$ is a subvariety of G we have $Z \subseteq \pi(v)^{-1} \cdot Z$, hence $\pi(v)$ is in the stabilizer of Z. □

Remark 3.3. Both Observation 3.1 and Claim 3.2 hold for a complex irreducible algebraic subvariety X of V. It can be done either by a direct argument or replacing X with the set X_{reg} of smooth points on X and using the fact that X_{reg} is a connected complex submanifold of V that is dense in X.

We deduce a slight generalization of Theorem 1.2 from [UY2].

Proposition 3.4. *Let $\pi : V \to G$ be a complex analytic group homomorphism from a finite dimensional $\mathbb{C}$-vector space V to a connected commutative algebraic group G over $\mathbb{C}$. Let $\Lambda = \pi^{-1}(e)$. Assume π has a large definable domain F.*

Let $X \subseteq V$ be a definable connected real analytic submanifold (or an irreducible complex algebraic subvariety) and $Z \subseteq G$ the Zariski closure of $\pi(X)$ in G.

If X is not covered by finitely many Λ-translate of F then $\mathrm{Stab}_G(Z)$ *is infinite.*

Proof. If X is not covered by finitely many Λ-translate of F, then by Observation 3.1(1) the set $\Sigma_F(X)$ is infinite. Since it is also definable, $\pi(\Sigma_F(X))$ must be also infinite (otherwise $\Sigma_F(X)$ would be an infinite definable discrete subset contradicting o-minimality). □

The following proposition is a key in our proof of ALW.

Proposition 3.5. *Let G be a connected commutative algebraic group over $\mathbb{C}$, $\mathbf{T}_G$ the Lie algebra of G, and $\exp_G : \mathbf{T}_G \to G$ the exponential map. Assume $\exp_G$ has a definable large domain F.*

Let $X \subseteq \mathbf{T}_G$ be a definable real analytic submanifold (or an irreducible algebraic subvariety), and $\mathbf{T}_B < \mathbf{T}_G$ the Lie algebra of the stabilizer B of the Zariski closure of $\exp_G(X)$ in G.

Then there is a finite set $S \subset \mathbf{T}_G$ such that

$$X \subseteq \mathbf{T}_B + S + F.$$

Proof. Let $\Lambda = \exp_G^{-1}(e)$. It is a discrete subgroup of $\mathbf{T}_G$.

Let $Z \subseteq G$ be the Zariski closure of $\exp_G(X)$ and B be the stabilizer of Z in G.

We define $\Sigma_F(X)$ as in (3.1).

Let B^0 be the connected component of B. It is an algebraic subgroup of G of finite index in B and satisfies: $\exp_G(\mathbf{T}_B) = B^0$, where $\mathbf{T}_B < \mathbf{T}_G$ is the Lie algebra of B.

We choose $b_1, \dots, b_n \in B$ with $B = \bigcup_{i=1} b_i \cdot B^0$, and also choose $h_1, \dots, h_n \in \mathbf{T}_G$ with $\exp_G(h_i) = b_i$. We have

$$\exp_G\Big(\bigcup_{i=1}^{n}(h_i + \mathbf{T}_B)\Big) = B,$$

hence by Claim 3.2,

$$\exp_G(\Sigma_F(X)) \subseteq \exp_G\Big(\bigcup_{i=1}^{n}(h_i + \mathbf{T}_B)\Big)$$

and

$$\Sigma_F(X) \subseteq \mathbf{T}_B + \Big(\bigcup_{i=1}^{n}(h_i + \Lambda)\Big).$$

Since Λ is a discrete subgroup of $\mathbf{T}_G$, the set $\bigcup_{i=1}^{n}(h_i + \Lambda)$ is a discrete subset of $\mathbf{T}_G$. By o-minimality, since $\Sigma_F(X)$ is definable we obtain that there is a finite set $S \subseteq \bigcup_{i=1}^{n}(h_i + \Lambda)$ with $\Sigma_F(X) \subseteq \mathbf{T}_B + S$. The proposition now follows from Observation 3.1(1). □

Remark 3.6. The above proposition immediately implies ALW Theorem for abelian varieties. Indeed let G be an abelian variety, $\exp_G \colon \mathbf{T}_G \to G$ the exponential map, $X \subseteq \mathbf{T}_G$ an irreducible algebraic subvariety, $B < G$ the stabilizer of the Zariski closure of $\exp_G(X)$ and $\mathbf{T}_B < \mathbf{T}_G$ the Lie algebra of B.

Since G is compact, there is a relatively compact fundamental domain F for $\exp_G$ definable in the o-minimal structure $\mathbb{R}_{\mathrm{an}}$.

Using Proposition 3.5, we have that $X \subseteq \mathbf{T}_B + S + F$, for some finite $S \subset \mathbf{T}_G$. Since F is relatively compact we obtain that $X \subseteq \mathbf{T}_B + K$ for some compact $K \subseteq \mathbf{T}_G$.

Let L be a $\mathbb{C}$-linear subspace of $\mathbf{T}_G$ complementary to $\mathbf{T}_B$. The projection of X to L along $\mathbf{T}_B$ is bounded. Since X is an irreducible variety, it has to be a point. It follows then that $X \subseteq \mathbf{T}_B + h$ for some $h \in \mathbf{T}_G$ and $\exp_G(X) \subseteq \exp_G(h) \cdot B$.

4. Full ALW for semi-abelian varieties

In this section we prove a general statement that implies full ALW Theorem and hence also ALW Theorem for semi-abelian varieties.

Proposition 4.1. *Let G be a connected semi-abelian variety over $\mathbb{C}$, $\mathbf{T}_G$ the Lie algebra of G, $\exp_G\colon \mathbf{T}_G \to G$ the exponential map, V a vector group over $\mathbb{C}$ and $\pi\colon V \oplus \mathbf{T}_G \to V \times G$ the map $\pi = \mathrm{id}_V \times \exp_G$.*

Let $Y \subseteq V \oplus \mathbf{T}_G$ be an irreducible algebraic variety and $Z \subseteq \mathbf{T}_G \times G$ the Zariski closure of $\pi(Y)$. Then $Z = Z_V \times Z_G$, where Z_V is a subvariety of V and Z_G a translate of an algebraic subgroup of G.

Remark 4.2. Since Z is the Zariski closure of $\pi(Y)$, it is easy to see that if $Z = Z_V \times Z_G$ then Z_V must be the Zariski closure of $\mathrm{pr}_V(Y)$ and Z_G must be the Zariski closure of $\exp_G(\mathrm{pr}_{\mathbf{T}_G}(Y))$, where pr_V and $\mathrm{pr}_{\mathbf{T}_G}$ are the projections from $V \oplus \mathbf{T}_G$ to V and $\mathbf{T}_G$ respectively.

Before proving the proposition let's remark how it implies both versions of ALW. To get ALW we take V to be the trivial vector group 0. To get full ALW we take $V = \mathbf{T}_G$ and $Y \subseteq \mathbf{T}_G \oplus \mathbf{T}_G$ the image of X under the diagonal map, i.e., $Y = \{(u,u) \in \mathbf{T}_G \oplus \mathbf{T}_G : u \in X\}$.

We now proceed with the proof of Proposition 4.1.

Proof. Let $H = V \times G$. It is a commutative algebraic group with Lie algebra $\mathbf{T}_H = V \oplus \mathbf{T}_G$ and with exponential map $\exp_H = \pi$. Hence Z is the Zariski closure of $\exp_H(Y)$.

We denote the group operation of H by $\cdot$, and view V and G as subgroups of H. Very often for subsets $S_1 \subseteq V$ and $S_2 \subseteq G$ we write $S_1 \times S_2$ instead of $S_1 \cdot S_2$ to indicate that in this case $S_1 \cdot S_2$ can be also viewed as the Cartesian product of S_1 and S_2.

Notice that since $\exp_H$ restricted to V is the identity map we have $\exp_H^{-1}(e) = \exp_G^{-1}(e)$.

Let $\mathrm{Stab}_H(Z)$ be the stabilizer of Z in H. It is an algebraic subgroup of $V \times G$. Since V is a vector group and G is a semi-abelian variety, $\mathrm{Stab}_H(Z)$ splits as $\mathrm{Stab}_H(Z) = V_0 \times B$, where $V_0 < V$ and $B < G$ are algebraic subgroups (see [Ros, Corollary 6]).

We first show that $Z \subseteq V \times (p \cdot B)$ for some $p \in G$.

Lemma 4.3. *We have $Y - h \subseteq V + \mathbf{T}_B$ for some $h \in \mathbf{T}_H$, where $\mathbf{T}_B < \mathbf{T}_G$ is the Lie algebra of B.*

Proof of Lemma. Since G is a connected semi-abelian variety it admits a short exact sequence

$$e \to G_0 \to G \to A \to e,$$

where A is an abelian variety and G_0 is an algebraic torus, i.e., an algebraic group isomorphic to $(\mathbb{C}^*, \cdot)^k$.

We do a standard decomposition of $\mathbf{T}_G$.

Let d be the dimension of G and k the dimension of G_0. Let $\Lambda = \exp_G^{-1}(e)$. It is a discrete subgroup of $\mathbf{T}_G$ whose $\mathbb{C}$-span is $\mathbf{T}_G$. Also Λ is a free abelian group of rank $2d-k$.

Let $\mathbf{T}_0 < \mathbf{T}_G$ be the Lie algebra of G_0. It is a $\mathbb{C}$-linear subspace of $\mathbf{T}_G$ of dimension k. Let $\Lambda_0 = \Lambda \cap \mathbf{T}_0$. It is easy to see that Λ_0 is a pure subgroup of Λ (i.e., for $\lambda \in \Lambda$ and $n \in \mathbb{N}$, $n\lambda \in \Lambda_0$ implies $\lambda \in \Lambda_0$), hence it has a complementary subgroup Λ_a in Λ, i.e., a subgroup Λ_a of Λ with $\Lambda = \Lambda_0 \oplus \Lambda_a$. Let $L_a < \mathbf{T}_G$ be the $\mathbb{R}$-span of Λ_a.

We have that $\mathbf{T}_G = \mathbf{T}_0 \oplus L_a$, and Λ_a is a lattice in L_a.

The restriction of $\exp_G$ to $\mathbf{T}_0$ is a complex Lie group homomorphism from $\mathbf{T}_0$ onto G_0 whose kernel is Λ_0. Choosing an appropriate basis for $\mathbf{T}_0$ and after identifying G_0 with $(\mathbb{C}^*,\cdot)^k$, we may assume that $\mathbf{T}_0 = \mathbb{C}^k$ and the restriction of $\exp_G$ to $\mathbf{T}_0$ has form $(z_1,\dots,z_k) \mapsto (e^{2\pi i z_1},\dots,e^{2\pi i z_k})$. In particular $\Lambda_0 = \mathbb{Z}^k$ and the restriction of $\exp_G$ to $i\mathbb{R}^k$ is definable in $\mathbb{R}_{\exp}$.

From now on we identify $\mathbf{T}_0$ with $\mathbb{C}^k$ and use decompositions

$$\mathbf{T}_G = \mathbb{C}^k \oplus L_a = \mathbb{R}^k \oplus i\mathbb{R}^k \oplus L_a \text{ and } \mathbf{T}_H = V \oplus \mathbb{R}^k \oplus i\mathbb{R}^k \oplus L_a.$$

Since both L_a/Λ_a and $\mathbb{R}^k/\mathbb{Z}^k$ are compact we can choose relatively compact large domains $F_a \subseteq L_a$ and $F_0 \subseteq \mathbb{R}^k$ for $\exp_G \restriction L_a$ and $\exp_G \restriction \mathbb{R}^k$ respectively, definable in $\mathbb{R}_{\mathrm{an}}$

It is easy to see that $F_0 + i\mathbb{R}^k + F_a$ is a large domain for $\exp_G$ and $F = V + F_0 + i\mathbb{R}^k + F_a$ is a large domain for $\exp_H$, both definable in $\mathbb{R}_{\mathrm{an,exp}}$.

Let $\mathbf{T}_B < \mathbf{T}_H$ be the Lie algebra of B. Since $\exp_H^{-1}(e) = \exp_G^{-1}(e) = \Lambda$, we apply Proposition 3.5 to Y and $\exp_H$ and get a finite $S \subset \mathbf{T}_H$ with $Y \subseteq \mathbf{T}_B + S + F$. Thus we have

$$Y \subseteq \mathbf{T}_B + S + F = V + \mathbf{T}_B + S + F_0 + i\mathbb{R}^k + F_a.$$

Since the closures of F_0 and F_a are compact, we can find a compact subset $K \subseteq \mathbf{T}_H$ with $S + F_0 + F_a \subseteq K$, and hence

$$Y \subseteq V + \mathbf{T}_B + i\mathbb{R}^k + K. \tag{4.1}$$

Let $M = V + \mathbf{T}_B + i\mathbb{R}^k$. It is an $\mathbb{R}$-linear subspace of $\mathbf{T}_H$. We first claim that $Y \subseteq M + h$ for some $h \in \mathbf{T}_H$. Indeed, using elementary linear algebra it is sufficient to show that for any $\mathbb{R}$-linear map $\xi\colon \mathbf{T}_H \to \mathbb{R}$ vanishing on M the image of Y under ξ is a point. Let $\xi\colon \mathbf{T}_H \to \mathbb{R}$ be an $\mathbb{R}$-linear map vanishing on M. From (4.1) we obtain that $\xi(Y)$ is bounded. Therefore, since Y is an irreducible algebraic variety and the map $\bar{\xi}\colon \mathbf{T}_H \to \mathbb{C}$ given by $\bar{\xi}\colon z \mapsto \xi(z) - i\xi(iz)$ is a $\mathbb{C}$-linear map, the set $\xi(Y)$ must be a point. Thus we have $Y \subseteq M + h$ for some $h \in \mathbf{T}_H$.

We will use the following fact that it is not difficult to prove.

Fact 4.4. Let $Y' \subseteq \mathbf{T}_H$ be an irreducible complex analytic subset containing the origin. If $W \subseteq \mathbf{T}_H$ is the $\mathbb{R}$-span of Y' (i.e. the smallest $\mathbb{R}$-linear subspace containing Y') then W is a $\mathbb{C}$-linear subspace of $\mathbf{T}_H$.

In particular if $Y' \subseteq U$ for some $\mathbb{R}$-linear subspace U of $\mathbf{T}_H$ then $Y' \subseteq iU$.

Applying the above fact to $Y' = Y - h$ we obtain

$$Y - h \subseteq M \cap iM = (V + \mathbf{T}_B + i\mathbb{R}^k) \cap (V + \mathbf{T}_B + \mathbb{R}^k). \tag{4.2}$$

Thus to finish the proof of Lemma, it remains to show that

$$(V + \mathbf{T}_B + i\mathbb{R}^k) \cap (V + \mathbf{T}_B + \mathbb{R}^k) = V + \mathbf{T}_B. \tag{4.3}$$

Since B is a semi-abelian subvariety of G, the intersection $B_1 = B \cap G_0$ is an algebraic torus with the Lie algebra $\mathbf{T}_{B_1} = \mathbf{T}_B \cap \mathbb{C}^k$. Since B_1 is an algebraic subtorus of G_0, $\mathbf{T}_{B_1}$ has a $\mathbb{C}$-basis in $\Lambda \cap \mathbb{C}^k = \mathbb{Z}^k \subset \mathbb{R}^k$.

It follows then that $\mathbf{T}_{B_1}$ has the form $E \oplus iE$ for some $\mathbb{R}$-linear subspace $E \subseteq \mathbb{R}^k$, and hence

$$\mathbf{T}_B \cap (\mathbb{R}^k + i\mathbb{R}^k) = E \oplus iE.$$

We are now ready to show (4.3). Let $\alpha \in (V + \mathbf{T}_B + i\mathbb{R}^k) \cap (V + \mathbf{T}_B + \mathbb{R}^k)$. Then

$$\alpha = v_1 + u_1 + w_1 = v_2 + u_2 + iw_2$$

for some $v_1, v_2 \in V, u_1, u_2 \in \mathbf{T}_B, w_1, w_2 \in \mathbb{R}^k$. Since $\mathbf{T}_H = V \oplus \mathbf{T}_G$, we get $v_1 = v_2$, and $(u_1 - u_2) = -w_1 + iw_2$.

Thus $-w_1 + iw_2 \in \mathbf{T}_B \cap (\mathbb{R}^k + i\mathbb{R}^k) = E \oplus iE$, so $w_1, w_2 \in E$, and hence $w_1, w_2 \in \mathbf{T}_B$. It implies that $\alpha \in V + \mathbf{T}_B$, that shows (4.3). It finishes the proof of Lemma. □

We choose $p \in G$ with $V \cdot \exp_H(h) = V \cdot p$ and obtain

$$\exp_H(Y) \subseteq V \times (p \cdot B) \text{ for some } p \in G$$

hence

$$Z \subseteq V \times (p \cdot B). \tag{4.4}$$

Let $Z_V = \{v \in V : v \times p \in Z\}$. It is an algebraic subvariety of V and we claim that $Z = Z_V \times (p \cdot B)$.

If $v \in Z_V$ then $v \cdot p \in Z$, and since B lies in the stabilizer of Z we have $v \times (p \cdot B) \subseteq Z$. Hence $Z_V \times (p \cdot B) \subseteq Z$.

Let $v \in V, g \in G$ with $v \cdot g \in Z$. Since B lies in the stabilizer of Z we have $v \times (g \cdot B) \subseteq Z$. By (4.4), $v \times (g \cdot B) \subseteq V \times (p \cdot B)$, hence $g \cdot B = p \cdot B$, $v \cdot p \in Z$, $v \in Z_V$ and $v \cdot g \in Z_v \times (p \cdot B)$. It shows that $Z \subseteq Z_V \times (p \cdot B)$. □

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