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Holding convex polyhedra by circular rings

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Abstract. In 1995, T. Zamfirescu proved that most convex bodies can be held by circles, that is, for most convex bodies $\mathcal{B}$ it is possible to attach a hinged circular ring of appropriate size to $\mathcal{B}$ so that it cannot slip out of $\mathcal{B}$. Since then, many results have been obtained concerning the existence of such circles for various convex polyhedra, and the sizes of such circles when they exist. It seems, however, that these results were obtained individually by ad hoc methods. In this paper we develop a unified concept and methods enabling a systematic presentation of these results, and we also obtain a few new results. A complete survey on the topic is also presented.

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1. Introduction and a survey of related results

A *convex body* is a compact convex set with interior points in $\mathbb{R}^3$. How is it possible to hold a convex body by a hinged circular ring (see Figure 1.1) of suitable size? This paper is an attempt at a rather systematic treatise concerning this problem and variants thereof, especially for convex polyhedra.

In applied disciplines like robotics (and subfields thereof, such as *motion planning*) one is confronted with many geometric problems, and also their solutions need a lot of geometric intuition. This implies that typical questions from computational, discrete, and convex geometry can also yield basic knowledge for very applied situations. The general type of question investigated here can be described as follows: given some geometric object A (e.g., a compact point set, called "body") and some system B of barriers (described as a geometric configuration, like a finite point set, a family of compact sets, or the complement of it), A should pass B with respect to the group of motions (or its subgroup

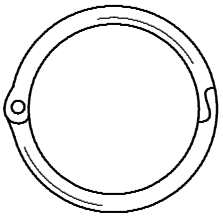

FIGURE 1.1
A hinged circular ring

of translations) remaining completely in the complement of B, with or without friction. Contrarily, one can ask for a system B sufficient to block A in some optimal sense (e.g., for B a finite set having, for instance, smallest cardinality to do so). Choosing A as convex body, B as family of translates of A or as finite set, and using the translation group, we enter combinatorial geometry, i.e., we refer then to notions like *blocking numbers*, *fixing systems* and *hindering systems* (see [Zon], [BMS, § 4], and [BM]). Extending this to the group of motions, we are in the small field of *immobilizing* (*convex*) *shapes* which is investigated mainly in computational geometry (cf., e.g., [BFMM] and [CSU]) and more related to our investigations here. The *piano mover's problem* is even more general: one has to find a continuous motion that will take a given body or a family of bodies, presented by A, from a given initial position to a desired final position, but with strong geometric constraints which forbid the bodies to come in contact with the fixed barrier system B and with each other (see, e.g., [SS] and, for an even more general concept, [Daw]). This problem is also nicely presented in the problem book [CFG], see G5 there.

The problem that we will discuss here is closely related to these concepts: given a convex body A, find a non-extensible string forming a net B around A (which can, in particular, consist only of a circle) such that A cannot slip out. And describe, somehow contrarily, a related situation where A unexpectedly can slip out. Looking at the existing references, this small field might be called "circles (and cages) holding convex bodies against continuous motions", and is mainly developed in 3-space. It is our aim to survey first the recent state of knowledge, to develop then a unified concept which allows a convenient approach to and a new presentation of existing results, and to derive also various new results. In a few cases, results described in the following sections in a detailed way are, for the sake of completeness, already shortly mentioned in the survey starting now.

In 1920, Zindler [Zin] studied problems on circular cylinders C of smallest possible radius r_1 which cover a convex body A. He observed that if A is for

example an affine cube, then one can move a circle, whose radius r_0 is smaller than r_1, "over A". Conversely, one can interpret that circle as fixed object B, and then A as the body that can be moved "through B". Zindler posed the interesting question for the smallest possible ratio $\frac{r_0}{r_1}$ still guaranteeing that this process is possible. Zindler's contribution can be seen as the starting point for the small field that we discuss here. More precisely, we ask for the optimal total length of circles that can hold (certain types of) convex bodies against motions.

Zamfirescu [Zam1] defined that a convex body A in $\mathbb{R}^3$ is said to *be held by a circle* B if the intersection of B and the interior of A is empty and it is not possible to continuously and rigidly move B away from A without intersecting the interior of A (we say then that B *holds* A). He proved that the family of convex bodies which cannot be held by some circle form a nowhere dense subset of the space of all three-dimensional convex bodies with respect to the Hausdorff metric (see [Schn, §1.8]). A single, but suggestive result was derived in [Tan1] (see also [Tan2]): the regular triangular prism with all edges of length 1 can be held by a circle. In [Fru] it was shown that if a circle of diameter d holds a convex body of minimum width w, then $\frac{d}{w} > \frac{2}{3}$, which is sharp. The author also claims that this can be generalized to $\mathbb{R}^n, n > 2$, for holding spheres of dimension $n-2$, and he derives respective inequalities in terms of d and w. In [Zam3] it is proved that, in the sense of *Baire category* (cf. [Gru] and [Schn, §2.6]), for *most* convex bodies in $\mathbb{R}^3$ Zindler's observation is true: they can be pushed through a circle whose radius is smaller than that of the smallest circumscribed circular cylinder. (From now on we use the word "most" in this sense.) If we imagine this circle as a circular hole in a wall, the natural question occurs which influence then the thickness of this wall has. This is studied in [Zam2], and it turns out that in most cases it has influence. This type of results is clearly related to embeddings of convex bodies A into infinitely long cylinders perpendicular to the holes in walls. See [Mae1] for regular tetrahedra in circular cylinders, and [MT2] for regular tetrahedra in regular triangular cylinders. In the first case all tetrahedra have equivalent positions (i.e., they can be superposed by a rigid motion within the respective prism), in the second case not, and the non-equivalent positions are described in [MT2]. The analogous question for square prisms seems not to be settled. Coming back to holding circles, Maehara [Mae4] proved that, for A being the regular icosahedron, the range of the space of all circles (defined via diameter) holding A has two components. This was generalized by Bárány and Zamfirescu. They showed in [BZ2] and [BZ1] that for most convex bodies the space of their holding circles has infinitely many components, and that various "counterintuitive" relations between extremal radii of holding circles exist. Another result from [BZ1] refers to the replacement of holding circles by planar closed convex curves, called *holding frames*. It says

that if the holding frame is neither a triangle (no triangle holds any convex body) nor a circle (any circle fixes some convex bodies, e.g., tetrahedra), then some tetrahedron in $\mathbb{R}^3$ is fixed by this frame without motion. The latter means that even the rotation is excluded which is trivially possible for holding circles. Continuing the study of holding frames, it is proved in [BMT] that a convex body can pass through a triangular hole iff it can do so by a translation along a line perpendicular to the hole. As an application, the minimum size of an equilateral triangular hole through which a regular tetrahedron with unit edge-length can pass is determined. Again the fact that no triangular frame can hold a convex body is used, and it is shown that every non-triangular frame can fix some tetrahedron. The authors of [ITZ] determine the smallest circular and the smallest square hole in a plane of $\mathbb{R}^3$ through which a regular tetrahedron of fixed size can pass. Extensions of these problems to higher dimensions are given in [IZ] and [MTl]. In the first paper diameters and minimal widths of convex hyperplanar holes in dimensions 3, 4, and 5 are determined, through which respective regular simplices can pass. And [MTl] refers to n-dimensional simplices which can be pushed through hyperplanar holes whose shapes are given by $(n-1)$-dimensional regular simplices, cubes, and balls.

It is clearly impossible to hold a ball in $\mathbb{R}^3$ by a circular ring. So we continue by recalling some classical results on holding a unit ball by other constraints. We now do this in greater detail, since this part of the field is no longer discussed in the following sections.

Theorem 1.1 (Besicovitch [Besl]). *The length of an inextensible string to construct a net around a unit ball so that the ball cannot slip out of it is greater than* 3π, *and it is possible to bring it as near to* 3π *as we like.*

Figure 1.2 shows that for any $\varepsilon > 0$ there is a net of total length smaller than $3\pi + \varepsilon$ that holds a unit ball. Indeed, since the length of the string used in any "3-cycle" of the net in Figure 1.2 is less than 2π, the ball cannot slip out of the

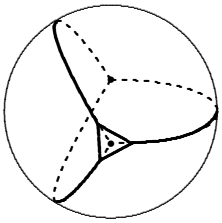

FIGURE 1.2
A net that holds a unit ball

net. Croft [Cro] proved the same result with a different method, see also [Ste]. By allowing that some of the six great circular arcs (see again Figure 1.2) can break, in [Cro] also a more general question is studied.

The smallest cube that contains a unit ball must have dimensions $2 \times 2 \times 2$. Hence the sum of its edge-lengths is 24. L. Fejes Tóth [8, p. 143] conjectured that the total length of the edges of a convex polyhedron that contains a unit ball is greater than or equal to 24, with equality only when the polyhedron is a cube. This conjecture was proved by Besicovitch and Eggleston.

Theorem 1.2 (Besicovitch and Eggleston [BE]). *The total length of the edges of a convex polyhedron that contains a unit ball is at least* 24*, and* 24 *is attained only by a cube.*

By a *cage* we mean the one-skeleton of a convex polyhedron; this notion creates several interesting problems in combinatorial geometry (see, e.g., [Schr]). Coxeter asked for the minimum of the total edge-length of a cage that can hold a unit ball. For a right triangular prism, all whose edges are of length $\sqrt{3}$, the distance from the center of the prism to the midpoint of each edge is equal to 1. Hence the 1-skeleton of this triangular prism is a cage that can hold a unit ball, and the total length of edges of this cage is $9\sqrt{3} \approx 15.5884$. Coxeter conjectured in his review of the paper [BE] (see MR0095448 and also [Cox]) that this is the smallest value of the total length of edges of a cage that can hold a unit ball. His conjecture was refuted by Besicovitch.

Theorem 1.3 (Besicovitch [Bes2], Aberth [Abe]). *The total length of the edges of a cage that can hold a unit ball is greater than* $\gamma = \frac{8}{3}\pi + 2\sqrt{3} \approx 11.84$*, and* γ *is the greatest lower bound.*

Besicovitch constructed a cage of total length $\gamma + \varepsilon$ that holds a unit ball, and Aberth proved that γ is the greatest lower bound of the total length of edges of such a cage. Figure 1.3 shows Besicovitch's cage. In the review of [Bes2] (see MR0155236) Coxeter repeats his conjecture restricted to polyhedra with the property that all their edges have to touch the enclosed sphere.

In [Zam4] Zamfirescu extends the representations of usual segments with two endpoints to "segments" between two convex bodies in their space with respect to the Hausdorff metric. A path in this space consisting of k consecutive segments is then called a *k-move*. He shows that if a convex body A is held by a cage B, it can migrate through a 2-move to a translate A' of A outside B, keeping its diameter constant on the way. Also further results of this type are verified in [Zam4], and two interesting research problems on cages are formulated in [MZ].

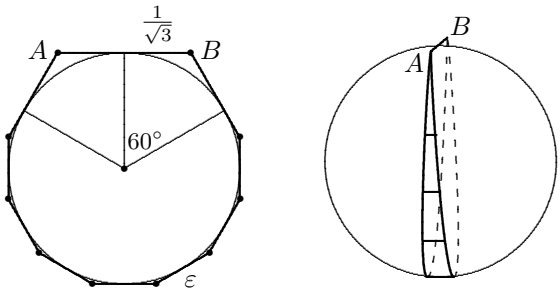

FIGURE 1.3
Construction of Besicovitch's cage

Let us return here to convex bodies and circles. To make our arguments clear, we define some notions as follows. For a given closed domain $\mathcal{D} \subset \mathbb{R}^3$, two circles in $\mathbb{R}^3 \setminus \mathrm{int}(\mathcal{D})$ are said to be *isotopic to each other over* $\mathcal{D}$ if one of these circles can be continuously and congruently moved in $\mathbb{R}^3 \setminus \mathrm{int}(\mathcal{D})$ so that it coincides with the other one, where $\mathrm{int}(*)$ denotes the *interior* of $*$. Thus, circles isotopic in that sense are congruent. A circle Γ is said to be *attached to* $\mathcal{D} \subset \mathbb{R}^3$ if $\Gamma \cap \mathrm{int}(\mathcal{D}) = \varnothing$ and $\mathrm{conv}(\Gamma) \cap \mathcal{D} \neq \varnothing$, where $\mathrm{conv}(*)$ denotes the *convex hull* of $*$. If a circle Γ attached to a convex body $\mathcal{B}$ in $\mathbb{R}^3$ is isotopic over $\mathcal{B}$ to a circle Γ' satisfying $\mathrm{conv}(\Gamma') \cap \mathcal{B} = \varnothing$, then we say that Γ can *slip out of* $\mathcal{B}$. If Γ cannot slip out of $\mathcal{B}$, then we say that Γ *holds* $\mathcal{B}$. A convex body $\mathcal{B}$ is called *circle-free* if no circle can hold $\mathcal{B}$.

Balls and ellipsoids in $\mathbb{R}^3$ are clearly circle-free. It is also not difficult to see that every right circular cylinder is circle-free. Every right circular cone is also circle-free. For two nonempty subsets $U, V \subset \mathbb{R}^3$, the *Minkowski sum* $U + V$ is defined as

$$U + V = \{u + v : u \in U,\ v \in V\}.$$

It is known (Maehara [Mae3]) that *for every compact convex set* X *contained in a plane in* $\mathbb{R}^3$, *the Minkowski sum* $X + \mathbf{B}$ *is circle free, where* $\mathbf{B}$ *is a ball of arbitrary radius centered at the origin.* Thus, a *sausage* (i.e., the Minkowski sum of a line-segment and a ball) is also circle-free.

Is there a convex body that can be held by a circle? Surprisingly, most convex bodies (in the sense of Baire categories, see again [Gru] and [Schn, §1.8]) can be held by circles, as was proved by Zamfirescu.

Theorem 1.4 (Zamfirescu [Zam1]). *The set of circle-free convex bodies forms a subset in the space of all convex bodies in* $\mathbb{R}^3$ *which is nowhere dense with respect to Hausdorff metric.*

In the sequel, we mainly concentrate on holding circles of convex polyhedra. In §2, we introduce “trunks of convex polyhedra" and “transversal disks of trunks" as basic notions, give several examples, and present a so-called “Symmetrization Lemma" and an “Isotopy Lemma" as key lemmas. In §3, various results on circles holding convex polyhedra are shown by using these notions and lemmas, and in §4 the key lemmas are proved.

2. Holding a convex polyhedron by a circle

2.1. Trunks of a convex polyhedron. In the sequel, a set of points $A, B, C, \ldots$ in $\mathbb{R}^3$ and its convex hull are both denoted by the juxtaposition $ABC\ldots$.

A *trunk* $\mathcal{E}$ of a convex polyhedron Π in $\mathbb{R}^3$ is a nonempty set of those edges of Π that are cut by a single plane passing through no vertex of Π. Since such a plane divides the endpoints of the edges into two nonempty sets, a trunk can be represented as $\mathcal{E} = (U, V)$, where U is the set of endpoints on one side of the plane, and V is the set of endpoints on the other side of the plane. The convex hull of $\mathcal{E}$ (i.e., the convex hull of $U \cup V$) is denoted by $\langle \mathcal{E} \rangle$. (Note that $\langle \mathcal{E} \rangle$ is a convex polyhedron, and $\mathcal{E}$ can be regarded as a trunk of $\langle \mathcal{E} \rangle$.) For example, in a tetrahedron $ABCD$ in $\mathbb{R}^3$, the pair (AB, CD) represents a trunk of the tetrahedron. A circle is said to be *attached to a trunk* of a convex polyhedron if the disk bounded by the circle intersects all edges of the trunk.

Let us recall here two types of quadratic surfaces that we use in the following. Let g, l be a pair of lines in $\mathbb{R}^3$, and suppose that $g \nparallel l$ (*non-parallel*) and that g does not lie in a plane perpendicular to l. By rotating g around l, we obtain a surface. If g and l intersect, then we have a (double) *circular cone* with *axis* l; otherwise, we have *one-sheet hyperboloids of revolution* with *axis* l, see Figure 2.1. These surfaces are *ruled surfaces*, represented by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{b^2} = c^2 .$$

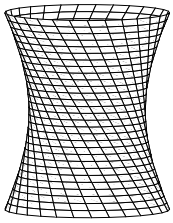

FIGURE 2.1
A one-sheet hyperboloid of revolution

(If $c = 0$, then this equation represents the surface of a (double) circular cone, and otherwise a one-sheet hyperboloid of revolution is represented.) Note that the latter surface is also "constricted" at $z = 0$. A one-sheet hyperboloid of revolution divides $\mathbb{R}^3$ into two parts, and the one that contains the axis of the surface is called the *inside of the surface*.

The next lemma is obvious, but useful.

Lemma 2.1. *Let $\mathcal{H}$ be a circular cone or a one-sheet hyperboloid of revolution.*

(1) *A section of $\mathcal{H}$ by a plane is a circle if and only if the plane is perpendicular to the axis of $\mathcal{H}$.*

(2) *If a section of $\mathcal{H}$ is an ellipse, then its minor axis lies on a plane perpendicular to the axis of $\mathcal{H}$.* □

The length of a line segment XY in $\mathbb{R}^3$ is denoted by $|XY|$. For a point X and a line g in $\mathbb{R}^3$, the distance $d(X, g)$ from X to g is defined by $d(X, g) = \min\{|XY| : Y \in g\}$. The distance $d(l, g)$ between two lines l, g is defined by $d(l, g) = \min\{|XY| : X \in l, Y \in g\}$. For a family of lines $g_1, g_2, \ldots, g_n$ $(n > 2)$, a line l that satisfies

$$X \in l \Rightarrow d(X, g_1) = \cdots = d(X, g_n)$$

is called an *equidistant line* of the family $\{g_1, \ldots, g_n\}$. For example, for a family of lines $g_1, \ldots, g_n$ lying on a one-sheet hyperboloid of revolution $\mathcal{H}$, it can be proved by using Lemma 2.1 (1) that the axis l of $\mathcal{H}$ is an equidistant line of $\{g_1, \ldots, g_n\}$.

Theorem 2.1. *If a family of lines $\{g_1, \ldots, g_n\}$ has an equidistant line l such that l does not lie on a plane perpendicular to g_1 and $l \nparallel g_1$, $d(l, g_1) > 0$, then $g_1, \ldots, g_n$ lie on a one-sheet hyperboloid of revolution with axis l.*

Proof. We use the following fact, without proof.

> ⋆ For two disjoint lines l, g, let $P, X \in l$, $Q, Y \in g$ be points that satisfy $|PQ| = d(g, l)$ and $XY \perp l$. Then, (i) $l \perp PQ \perp g$ and (ii) $|XY|$ is uniquely determined by $|PQ|$, $|PX|$ and $d(X, g)$.

Let $\mathcal{H}$ be the hyperboloid obtained by rotating g_1 around l. Since l is the equidistant line of $g_1, \ldots, g_n$, there exist $P \in l$ and $Q_i \in g_i$ such that $|PQ_1| = \cdots = |PQ_n| = d(l, g_1)$. Then $PQ_i \perp l$, and hence each Q_i lies on $\mathcal{H}$. For a point $X \in l$, let $Y_i \in g_i$ satisfy that $XY_i \perp l$. By (ii) of the above fact ⋆, we have $|XY_1| = \cdots = |XY_n|$. Hence each Y_i lies on $\mathcal{H}$. Therefore each g_i lies on $\mathcal{H}$. □

A trunk $\mathcal{E} = (U, V)$ is called *hyperboloidal* (resp. *conic*) if all edges of the trunk lie on a one-sheet hyperboloid of revolution (resp. on a circular cone). In a right pyramid with apex P whose base is a regular polygon, the set of edges emanating from P is clearly a conic trunk.

Example 2.1. In the regular icosahedron $\mathcal{I}$ shown in Figure 2.2 left, the trunk $\mathcal{E} = (ABCDE, A^*B^*C^*D^*E^*)$ is hyperboloidal. (Indeed, the line FF^* is an equidistant line of the lines determined by the edges in $\mathcal{E}$.) Thus, by rotating $\mathcal{I}$ around the line FF^*, we have a non-convex figure as shown in Figure 2.2 right. Note that $\mathcal{E}$ contains pairs mutually symmetric to the center of the icosahedron, say (AD^*, A^*D), (BE^*, B^*E), etc. Let Γ be the minimal circle attached to $\mathcal{E}$ at its most "constricted" part, and let $\Gamma'(\neq \Gamma)$ be any other circle attached to $\mathcal{E}$. If the plane determined by Γ' is perpendicular to FF^*, then clearly Γ' has larger diameter than Γ. If the plane of Γ' is not perpendicular to FF^*, then the plane cuts a pair of edges of $\mathcal{E}$ that are symmetric to each other with respect to the center of the icosahedron, at a pair of points with distance greater than the diameter of Γ. Hence the diameter of Γ' is larger than that of Γ. Therefore $\mathcal{I}$ can be held by a circle.

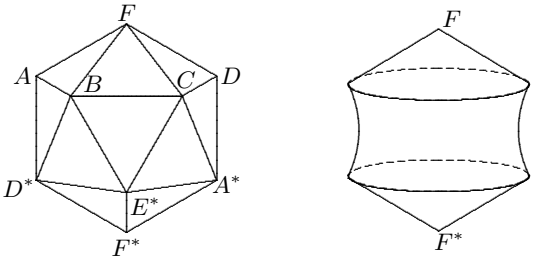

FIGURE 2.2
A hyperboloidal trunk of a regular icosahedron

Example 2.2. Similarly, a regular tetrahedron $\mathcal{T}$, a cube $\mathcal{C}$, and a regular octahedron $\mathcal{O}$ have hyperboloidal trunks, and they can be held by circles as shown in Figure 2.3. A regular dodecahedron has two different types of hyperboloidal trunks as indicated by attached circles in Figure 2.4, and it is also not circle-free.

Remark 2.1. Even if a convex polyhedron has a hyperboloidal trunk, the smallest circle attached to the hyperboloidal trunk does not necessarily hold the convex polyhedron. For example, in a right triangular pyramid $P\text{-}ABC$ with equilateral triangular base ABC, its trunk (PA, BC) is hyperboloidal by Lemma 3.2 in §3,

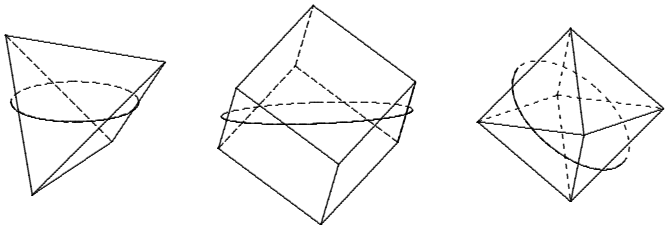

FIGURE 2.3
Holding circles

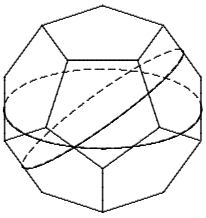

FIGURE 2.4
Circles attached to hyperboloidal trunks of a dodecahedron

but if the height of the pyramid is very small, then the pyramid is circle-free, as proved in Theorem 3.3 in §3.

Lemma 2.2. *If a hyperboloidal trunk $\mathcal{E}$ has at least five edges, then it determines a unique one-sheet hyperboloid of revolution.*

Proof. Let $\mathcal{H}_i$, $i = 1, 2$, be one-sheet hyperboloids of revolution, each containing the trunk $\mathcal{E}$, and let $l_i, i = 1, 2$, be their axes. Let $\bar{\mathcal{E}}$ denote the set of lines determined by the edges of $\mathcal{E}$. Since a quadratic surface and a line that does not lie on the surface intersect in at most two points, each $\mathcal{H}_i$ must contain $\bar{\mathcal{E}}$. Let H be a plane that is perpendicular to l_1. Then $H \cap \mathcal{H}_1$ is a circle by (1) of Lemma 2.1. Since $\bar{\mathcal{E}}$ contains at least five lines, it is possible to choose H so that $H \cap \bar{\mathcal{E}}$ contains at least five points. Then $H \cap \mathcal{H}_2$ is a quadratic curve on H that has five points in common with the circle $H \cap \mathcal{H}_1$, and hence $H \cap \mathcal{H}_2 = H \cap \mathcal{H}_1$. In this case, H is also perpendicular to l_2 by (1) of Lemma 2.1, and l_2 passes through the center of the circle $H \cap \mathcal{H}_1$. Therefore $l_1 = l_2$, and hence $\mathcal{H}_1 = \mathcal{H}_2$. □

Remark 2.2. If a hyperboloidal trunk $\mathcal{E}$ of a convex polyhedron has at most four edges, then a one-sheet hyperboloid of revolution that contains $\mathcal{E}$ is not necessarily unique. For example, consider the rectangular pyramid $B\text{-}AB^*C^*D$ inscribed in the cube $ABCD\text{-}A^*B^*C^*D^*$, see Figure 2.5. It has a hyperboloidal trunk $\mathcal{E} = (AB^*, BC^*D)$ consisting of four edges DA, AB, BB^*, B^*C^*, which

is a subset of a hyperboloidal trunk (AB^*D^*, BC^*D) of the cube $ABCD$-$A^*B^*C^*D^*$. Hence the line A^*C is an equidistant line of the family $\bar{\mathcal{E}}$ of lines determined by the edges of $\mathcal{E}$. Since the lines in $\bar{\mathcal{E}}$ are also determined by $D_1A, AB, BB^*, B^*C_1^*$, the line C_1A^* is also an equidistant line of $\bar{\mathcal{E}}$, where C_1, D_1, C_1^* are the mirror images of C, D, C^* with respect to the plane AA^*B^*B. Hence there is another one-sheet hyperboloid of revolution that contains $\mathcal{E}$ by Theorem 2.1.

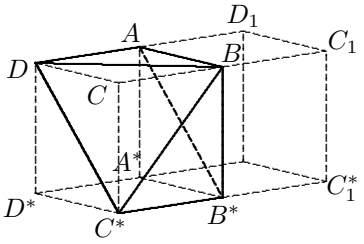

FIGURE 2.5
A rectangular pyramid inscribed in a cube

2.2. Transversal disks of a trunk. Let $\mathcal{E}$ be a trunk of a convex polyhedron in $\mathbb{R}^3$. A disk Ω is called a *transversal disk of* $\mathcal{E}$ if Ω intersects all edges in the trunk $\mathcal{E}$. (Note that Ω may intersect an edge in $\mathcal{E}$ at its endpoint.) More generally, for a set of lines $\mathcal{L}$, a plane (or a disk) is called a *transversal plane* (or a *transversal disk*) of $\mathcal{L}$ if the plane (or the disk) intersects all lines in $\mathcal{L}$. The boundary circle of a transversal disk of a trunk $\mathcal{E}$ is a circle attached to the trunk $\mathcal{E}$. If a transversal disk of $\mathcal{E}$ contains a prescribed vertex P of $\mathcal{E}$, then the disk is called a *transversal disk of the trunk* $\mathcal{E}$ *on* P. Note that a transversal disk of $\mathcal{E}$ on P is also a transversal disk of $\mathcal{E}$. Among the transversal disks of $\mathcal{E}$ (on P), one that has the minimum diameter is called a *minimal transversal disk of* $\mathcal{E}$ (on P). Since disks are compact and convex, it follows, by employing Blaschke's selection theorem (cf. [Schn, §1.8]), that for any trunk (and a prescribed vertex P), there always exists a minimal transversal disk of the trunk (on P). Note that *the boundary circle of a minimal transversal disk of a trunk* $\mathcal{E}$ *intersects* $\langle\mathcal{E}\rangle$ *in at least two points unless the disk degenerated into a point.* The diameter of a disk Ω is denoted by $d(\Omega)$.

Let us prove here the following theorem obtained by Tanoue [Tan3].

Theorem 2.2. *Every triangular right prism with equal edges is not circle-free.*

Proof. Let $ABCA^*B^*C^*$ be a triangular prism as shown in Figure 2.6 left, and let $\mathcal{E} = (ABB^*, A^*C^*C)$. Let Ω be a minimal transversal disk of $\mathcal{E}$, and Ω_A

be a minimal transversal disk of $\mathcal{E}$ on A. It is enough to show that the boundary circle of Ω cannot slip out of the triangular prism. To show this, we use the inequality

$$(*) \quad d(\Omega) < a\sqrt{7}/2 = d(\Omega_A),$$

where a is the edge-length of the prism. This is proved later. Tentatively, we assume this. Suppose that the boundary circle $\partial\Omega$ of Ω can slip out of the triangular prism. During the slipping out process, the circle $\partial\Omega$ and the disk Ω move, and Ω must meet vertices of the triangular prism. Let Z denote the first vertex that Ω meets during a slipping out process, and denote by Ω_Z the disk when Ω comes to Z. The point Z must be one of A, A^*, C, B^*. If $Z = A^*$ then, Ω_{A^*} is a transversal disk of $\mathcal{E}$ on A^*. However, the diameter of a minimal transversal disk of $\mathcal{E}$ on A^* (which is equal to $d(\Omega_A)$) is larger than $d(\Omega)$ by $(*)$, a contradiction. Suppose $Z = B^*$. Let X be the intersection of AC and Ω_{B^*}. Since $|B^*X| \geq |A^*M| = a\sqrt{7}/2$, we have $d(\Omega_{B^*}) > d(\Omega)$ by $(*)$, a contradiction. Similarly, in the cases $Z = A, C$, there arise contradictions. Hence the circle $\partial\Omega$ cannot slip out of the prism.

Now, to prove $(*)$, we show that (1) $d(\Omega_A) = a\sqrt{7}/2$ and (2) $d(\Omega) < a\sqrt{7}/2$, where a is the edge-length of the prism.

(1) Every transversal disk of $\mathcal{E}$ on A intersects the edge B^*C^* at a point Y. Then the diameter of the transversal disk is greater than or equal to $|AY|$. While Y moves on the line segment B^*C^*, the minimum value of $|AY|$ is attained when Y is at the midpoint N of B^*C^*, and $|AN| = \sqrt{a^2 + a^2(\sqrt{3}/2)^2} = a\sqrt{7}/2$. Thus $d(\Omega_A) \geq a\sqrt{7}/2$. On the other hand, denoting the midpoint of BC by M, the smallest disk that contains the rectangle AA^*NM is a transversal disk of $\mathcal{E}$ on A and has diameter $a\sqrt{7}/2$. Hence $d(\Omega_A) = a\sqrt{7}/2$.

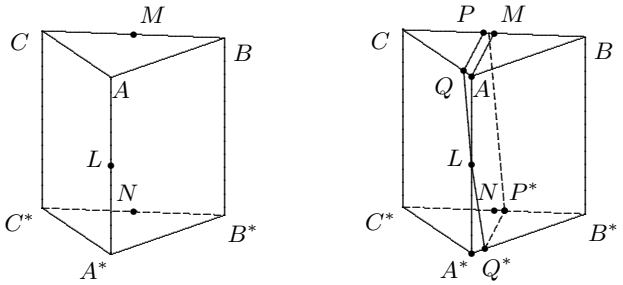

FIGURE 2.6
A triangular prism

(2) Let P and P^* be the points on the segments MC and NB^*, respectively, such that $|MP| = |NP^*| = \varepsilon/2$, where ε is a small positive number. Let

L be the midpoint of AA^*, and let Q, Q^* be the points where the plane PLP^* cuts the segments AC and A^*B^*, respectively, see Figure 2.6 right. Then $PQ \parallel P^*Q^* \parallel MA$, and PP^*Q^*Q is a rectangle. Since $|AQ| = \varepsilon$ (because $|PM| = \varepsilon/2$), we have $|QP| = (a-\varepsilon)\sqrt{3}/2$. Hence

$$|QP^*|^2 = \tfrac{3}{4}(a-\varepsilon)^2 + a^2 + \varepsilon^2 = \tfrac{7}{4}a^2 - \tfrac{3}{2}\varepsilon a + 2\varepsilon^2.$$

If ε is very small, then $-\frac{3}{2}\varepsilon a + 2\varepsilon^2 < 0$, and $|QP^*| < a\sqrt{7}/2$. Hence the diameter of the circumscribed circle of the rectangle PQQ^*P^* is smaller than $a\sqrt{7}/2$. Moreover, if ε is very small, then the midpoint L of AA^* is contained in the smallest disk that contains PP^*Q^*Q, and hence the smallest disk containing PP^*Q^*Q is a transversal disk of $\mathcal{E}$. Therefore $d(\Omega) < a\sqrt{7}/2$.

This completes the proof of the theorem. □

2.3. The Symmetrization Lemma and the Isotopy Lemma. A plane H is called a *symmetry plane of a trunk* $\mathcal{E} = (U, V)$ if both U, V are plane-symmetric to themselves and have a common symmetry plane H. For example, in the regular icosahedron in Figure 2.2 left, the plane determined by F, A, F^* is a symmetry plane of the trunk $(ABCDE, A^*B^*C^*D^*E^*)$.

The following lemma is sketchily proved by Maehara [Mae3]. We present a complete proof in §4.

Lemma 2.3 (Symmetrization Lemma). *Suppose that a trunk $\mathcal{E}$ of a convex polyhedron has a symmetry plane H, and let Ω be a transversal disk of $\mathcal{E}$.*

(1) *The boundary circle of Ω is isotopic over $\langle\mathcal{E}\rangle$ to the boundary circle of a transversal disk of $\mathcal{E}$ that is symmetric to itself with respect to the plane H.*

(2) *If Ω is not symmetric to itself with respect to H, and $\Omega \cap H \not\subset \langle\mathcal{E}\rangle$, then Ω is not a minimal transversal disk of $\mathcal{E}$.*

This lemma is also true if we replace "transversal disk of $\mathcal{E}$" by "transversal disk of $\mathcal{E}$ on P", for a vertex P of $\mathcal{E}$ lying on H.

The following conjecture was stated by Maehara [Mae4].

Conjecture 2.1. *If the diameters of two circles attached to the same trunk of a convex polyhedron are equal, then the two circles are isotopic over the convex polyhedron.*

Though we could not prove this conjecture, the following special case is useful. The proof of this special case is also given in §4. By a directed line, we mean a line, like the z-axis in $\mathbb{R}^3$, in which the $(+)$-direction is specified. Then, for

any plane that cuts the directed line, its *upper side* ((+)-side) and its *lower side* are defined naturally. For a trunk $\mathcal{E}$, the set of lines determined by the edges in $\mathcal{E}$ is denoted by $\bar{\mathcal{E}}$.

Lemma 2.4 (Isotopy Lemma). *Let $\mathcal{E} = (U, V)$ be a hyperboloidal trunk of a convex polyhedron Π that lies on a one-sheet hyperboloid of revolution $\mathcal{H}$ with "directed" axis l. Suppose that* (i) *there is a transversal plane of $\mathcal{E}$ that is perpendicular to l and U lies in its upper side, and that* (ii) *$\mathcal{E}$ has a symmetric plane.*

(1) *If a circle $\Gamma = \partial\Omega_0$ attached to $\mathcal{E}$ satisfies that*
(†) *the plane of the circle cuts the axis l and U lies in its upper side,*
then the disk Ω_0 can be continuously and congruently moved, through transversal disks of $\bar{\mathcal{E}}$, to a transversal disk Ω_1 of $\bar{\mathcal{E}}$ that lies on a plane perpendicular to the axis l of $\mathcal{H}$. Hence Γ is isotopic over Π to $\partial\Omega_1$.

(2) *Two congruent circles attached to $\mathcal{E}$, both satisfying* (†), *are isotopic over Π.*

Remark 2.3. For every hyperboloidal trunk of regular polyhedra shown in Figures 2.2, 2.3, 2.4, the Isotopy Lemma can be applied and any two congruent circles attached to the trunk are isotopic over the regular polyhedron.

Example 2.3. Let $\mathcal{E} = (ABCDE, A^*B^*C^*D^*E^*)$ be a trunk of a regular icosahedron $\mathcal{I}$ as shown in Figure 2.7 left, and $\mathcal{E}' = (AA^*, CD)$ be a trunk of the tetrahedron $ACDA^*$. Then the minimal transversal disk of $\mathcal{E}$ on A coincides with the minimal transversal disk of $\mathcal{E}'$ on A.

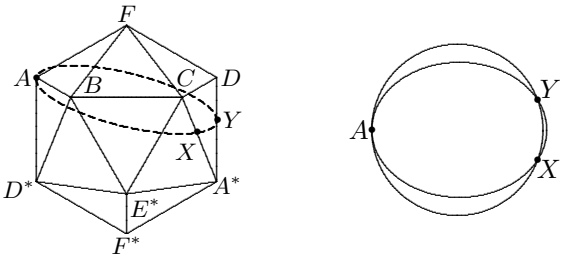

FIGURE 2.7
Example 2.3

This can be seen as follows. Let Ω be a minimal transversal disk of $\mathcal{E}$ on A. First, note that the plane H containing AFF^* is a common symmetry plane of the trunk $\mathcal{E}$ and the trunk $\mathcal{E}'$. Note also that since $H \cap \Omega \not\subset \langle\mathcal{E}\rangle$, Ω must be symmetric to itself with respect to H by (2) from the Symmetrization Lemma.

Since the trunk $\mathcal{E}$ is hyperboloidal, the intersection points of Ω and the edges in this trunk lie on an ellipse passing through A, and A is an endpoint of the major axis of this ellipse. Hence $\partial\Omega \cap \langle\mathcal{E}\rangle$ consists of A and two points X, Y on the edges A^*C, A^*D, see Figure 2.7 right. On the other hand, if X, Y are points on the edges A^*C, A^*D such that $|A^*X| = |A^*Y|$, then the smallest disk containing the triangle AXY becomes a transversal disk of $\mathcal{E}$ through A. Hence Ω coincides with the minimal transversal disk of $\mathcal{E}'$ on A.

Example 2.4. Let $P\text{-}A_1A_2\dots A_{2m}$ be a regular pyramid with apex P whose base is a regular $(2m)$-gon $A_1A_2\dots A_{2m}$. Let $\mathcal{E} = (P, A_1A_2\dots A_{2m})$, and Ω_1 be the minimal transversal disk of $\mathcal{E}$ on A_1. Then $\partial\Omega_1$ intersects $\mathcal{E}$ in only two edges PA_1, PA_{m+1}, and $d(\Omega_1) = \min\{|A_1X| : X \in PA_{m+1}\}$.

To see this, we may suppose that Ω_1 is symmetric to itself with respect to the plane PA_1A_{m+1} by (l) from the Symmetrization Lemma. Then the plane containing Ω_1 cuts the circular cone determined by $\mathcal{E}$ in an ellipse whose major axis lies in the plane PA_1A_{m+1}. Hence $\partial\Omega$ intersects only two edges PA_1, PA_{m+1} of $\mathcal{E}$. Moreover, for every X on the edge PA_{m+1}, a disk with diameter A_1X which perpendicularly intersects the plane PA_1A_{m+1} is a transversal disk of $\mathcal{E}$ on A_1. Hence $d(\Omega_1) = \min\{|A_1X| : X \in PA_{m+1}\}$.

Similarly to Example 2.4, we have the following

Example 2.5. Let $P\text{-}A_1A_2\dots A_{2m+1}$ be a regular pyramid with apex P whose base is a regular $(2m+1)$-gon $A_1A_2\dots A_{2m+1}$. Let $\mathcal{E} = (P, A_1A_2\dots A_{2m+1})$, and Ω_1 be the minimal transversal disk of $\mathcal{E}$ on A_1. Then Ω_1 coincides with the minimal transversal disk of $(PA_1, A_{m+1}A_{m+2})$ on A_1, where $(PA_1, A_{m+1}A_{m+2})$ is a trunk of the tetrahedron $PA_1A_{m+1}A_{m+2}$.

3. Various results

3.1. The range of holding circles. The *holding range* $h(\mathcal{B})$ of a convex body $\mathcal{B}$ is a subset of the reals $\mathbb{R}$ defined by

$$h(\mathcal{B}) = \{d \in \mathbb{R} : \text{there is a circle of diameter } d \text{ that holds } \mathcal{B}\}.$$

Theorem 3.1. *For a regular tetrahedron $\mathcal{T}$, a cube $\mathcal{C}$, and a regular octahedron $\mathcal{O}$, all having unit edges, we have*

$$\text{(i)} \quad h(\mathcal{T}) = [1/\sqrt{2}, 0.896\dots),$$
$$\text{(ii)} \quad h(\mathcal{C}) = [\sqrt{2}, 1.535\dots),$$
$$\text{(iii)} \quad h(\mathcal{O}) = [1, 1.1066\dots),$$

where the upper bounds of (i), (ii), (iii) *are the minimum values of the functions*

$$\frac{2(x^2-x+1)}{\sqrt{3x^2-4x+4}},\quad \frac{\sqrt{2}(x^2+2)}{\sqrt{x^2+2x+3}},\quad \frac{2(x^2+1)}{\sqrt{3x^2+2x+3}},$$

respectively.

The result (i) was obtained by Itoh et al. [ITZ], and (ii) as well as (iii) were obtained by Maehara [Mae2] and Tanoue [Tan3].

Proof. We show only the octahedron case (iii). The other cases follow similarly. Put labels A, B, C, A^*, B^*, C^* on the six vertices of $\mathcal{O}$ as in Figure 3.1. First, note that a circle attached to a trunk of $\mathcal{O}$ that is *not* hyperboloidal can always slip out of $\mathcal{O}$ by a translation. Let $\mathcal{E} = (ABC, A^*B^*C^*)$, a hyperboloidal trunk of $\mathcal{O}$. It will be clear that the minimum diameter of a holding circle of $\mathcal{O}$ is the diameter of a minimal transversal disk of $\mathcal{E}$, that is, the diameter of the circumscribed circle of the regular hexagon whose vertices are the midpoints of the edges in $\mathcal{E}$. Hence its diameter is 1.

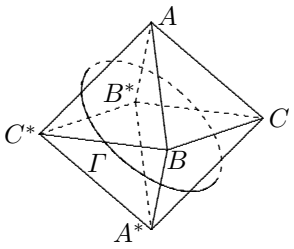

FIGURE 3.1
The attached circle Γ

Now let Γ be a circle attached to $\mathcal{E}$. If this circle can slip out of the octahedron, then during the process of slipping out, the disk $\mathrm{conv}(\Gamma)$ must meet vertices of $\mathcal{O}$. We may suppose that A is the first vertex that it meets. At the moment when it meets A, the disk $\mathrm{conv}(\Gamma)$ becomes a transversal disk of $\mathcal{E}$ on A. Hence the diameter of Γ must be at least the diameter d_0 of a minimal transversal disk of $\mathcal{E}$ on A. On the other hand, if the diameter of Γ is greater than or equal to d_0, then Γ is isotopic over $\mathcal{O}$ to the boundary circle of a transversal disk of $\mathcal{E}$ on A by the Isotopy Lemma. Then by a translation in the direction $\overrightarrow{C^*B}$, Γ can slip out of $\mathcal{O}$. Hence we have $h(\mathcal{O}) = [1, d_0)$.

Let us find the value d_0 of the diameter of a minimal transversal disk of $\mathcal{E}$ on A. Since the plane H determined by A, A^* and the midpoint of BC is a symmetry plane of $\mathcal{E}$, we may consider the diameter of a minimal transversal disk Ω of $\mathcal{E}$ on A that is symmetric to itself with respect to H by (1) from

the Symmetrization Lemma. Let K be the plane that contains the disk Ω. Let P, Q be the points where K cuts A^*B and A^*C, respectively. Since $\mathcal{E}$ is hyperboloidal, the intersections of K and the edges in $\mathcal{E}$ lie on an ellipse and A is an endpoint of the major axis of this ellipse. Hence the circle $\partial\Omega$ must pass through P, Q and A.

Let $x = |BP| = |CQ|$. Then $|PQ| = |PA^*| = 1-x$, and since $\angle ABA^* = 90°$, we have $|AP| = \sqrt{1+x^2} = |AQ|$. Now, the diameter of the circumscribed circle of the isosceles triangle APQ is computed as $\frac{2(x^2+1)}{\sqrt{3x^2+2x+3}}$, and the minimum value d_0 of this function is $d_0 = 1.106\ldots$. □

Fruchard [Fru] proved that for every convex body $\mathcal{B}$, its holding range $h(\mathcal{B})$ is a subset of the interval $(2w/3, \infty)$, where w denotes the *width* of $\mathcal{B}$, that is, the minimum distance between a pair of parallel planes bounding a strip containing $\mathcal{B}$. The lower bound $2w/3$ cannot be improved generally.

If $\mathcal{P}$ is a regular tetrahedron, or a cube, or a regular octahedron, then $h(\mathcal{P})$ is an interval as seen in Theorem 3.1. However, the holding range of a convex polyhedron is not always an interval. Indeed, it is known (Maehara [Mae4]) that *the holding range of a regular icosahedron is disconnected.* Moreover, it was shown by Bárány and Zamfirescu [BZ1] that *there are convex bodies $\mathcal{B}$ such that $h(\mathcal{B})$ has arbitrarily many connected components.*

3.2. Regular pyramids.

Lemma 3.1. *Let $P\text{-}A_1A_2\ldots A_n$ denote a regular pyramid with apex P whose base is a regular n-gon $A_1A_2\ldots A_n$, $n \geq 3$. Let $\mathcal{E}_1 = (PA_1, A_2A_3\ldots A_n)$, and denote by Ω_P, Ω_1, and Ω the minimal transversal disk of $\mathcal{E}_1$ on P, the minimal transversal disk of $\mathcal{E}_1$ on A_1, and the minimal transversal disk of $\mathcal{E}_1$, respectively. Then the following statements hold:*

(1) $d(\Omega) < d(\Omega_1)$.

(2) *The inequality $d(\Omega) < d(\Omega_P)$ implies that the boundary circle $\partial\Omega$ of Ω holds the pyramid.*

Proof. (1) Denote by O the center of the base. We may suppose that Ω_1 is symmetric to itself with respect to the symmetry plane A_1PO of $\mathcal{E}_1$. Let Q be the center of Ω_1. To make our argument clear, let us consider the case $n = 5$, see Figure 3.2. (Other cases follow almost similarly.) In this case, the boundary circle $\partial\Omega_1$ intersects the edges PA_3, PA_4 at X, Y (possibly $X = P = Y$), respectively, by Example 2.5 (that is, $\partial\Omega_1$ is the circumscribed circle of the triangle A_1XY, and Q is the circumcenter of the triangle A_1XY). And the edges PA_2, PA_5 pass through the interior of the disk Ω_1. Let $\mathbf{B}$ be the ball

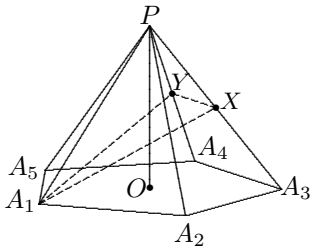

FIGURE 3.2
$\partial\Omega_1$ is circumscribed to $\triangle A_1XY$

of diameter $d(\Omega_1)$ centered at Q. Since $\angle QA_1A_2 = \angle QA_1A_5 < 90°$, both $A_1A_2 \cap \mathbf{B}$ and $A_1A_5 \cap \mathbf{B}$ are intervals. Hence we can rotate the plane containing Ω_1 around the line XY slightly so that the intersection of $\mathbf{B}$ and the rotated plane is a transversal disk of $\mathcal{E}_1$. The diameter of this disk is clearly smaller than $d(\Omega_1)$, and hence $d(\Omega) < d(\Omega_1)$.

(2) Suppose that $d(\Omega) < d(\Omega_P)$ and $\partial\Omega$ still can slip out of the pyramid. During the slipping out process, Ω meets vertices of the pyramid. Let Z be the first vertex that Ω meets, and denote by $\Omega(Z)$ the disk at the moment when Ω meets Z. Then, since $d(\Omega) < d(\Omega_1)$, Z must be P or A_2 or A_5. If $Z = P$, then $\Omega(P)$ is a transversal disk of $\mathcal{E}_1$ on P, which means $d(\Omega) = d(\Omega(P)) \geq d(\Omega_P)$, a contradiction.

Suppose that $Z = A_2$. The disk $\Omega(A_2)$ is a transversal disk of the trunk $(PA_2, A_3A_4A_5)$. By Examples 2.4 and 2.5, $d(\Omega(A_2))$ is at least the diameter of the minimal transversal disk of $(PA_2, A_3A_4A_5A_1)$ on A_2, which is equal to $d(\Omega_1)$, a contradiction. The case $Z = A_5$ is similar to the case $Z = A_2$. □

Let us define the *slope* ρ of a regular pyramid by

$$\rho = \frac{\text{height}}{\text{circumradius of the base}}.$$

Though every circular cone is circle-free, every regular pyramid of slope greater than 1 is not circle-free.

Theorem 3.2. *Every regular pyramid with slope $\rho \geq 1$ can be held by a circle.*

Remark 3.1. It is known (Maehara [Mae3]) that for every $0 < \varepsilon < 1$ and $m > 2\pi/\varepsilon^2$, a regular $(4m)$-gonal pyramid with slope $\rho = 1 - \varepsilon$ is circle-free.

Proof. To make our argument clear, let us consider again the case of a regular pyramid with pentagonal base. Let $P\text{-}A_1A_2A_3A_4A_5$ denote a regular pyramid with apex P whose base is a regular pentagon $A_1A_2A_3A_4A_5$. Define

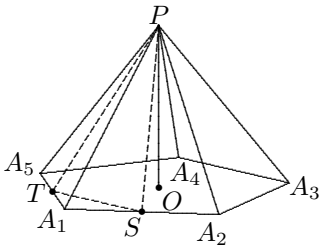

FIGURE 3.3
$\partial\Omega_P$ is circumscribed to $\triangle PST$

$\mathcal{E}_1, \Omega_P, \Omega_1, \Omega$ as in Lemma 3.1 and let $\mathcal{E} = (P, A_1A_2A_3A_4A_5)$. Note that a transversal disk of $\mathcal{E}$ on A_1 is a transversal disk of $\mathcal{E}_1$ on A_1, and vice versa.

By Lemma 3.1 (2), it is enough to show that $\rho \geq 1$ implies $d(\Omega) < d(\Omega_P)$. The minimal transversal disk Ω_P of $\mathcal{E}_1$ on P intersects the edges A_1A_2, A_1A_5 at S, T such that $|A_1S| = |A_1T| > 0$, see Figure 3.3. Now, let Q be the center of Ω_P, and $\mathbf{B}$ be the ball with center Q and diameter $d(\Omega_P)$. For every i $(2 \leq i \leq 5)$, the triangle A_1PA_i is an isosceles triangle with base A_1A_i and height greater than or equal to $|OP|$, where O is the center of the base. Thus, $\rho \geq 1$ implies that $\angle A_1PA_i \leq 90°$, and hence $\angle QPA_i < 90°$ for $i = 2, 3, 4, 5$. This implies that the edges PA_i, $i = 2, 3, 4, 5$, pass through the interior of $\mathbf{B}$. Therefore, by rotating slightly the plane PST around the line ST, we have a plane whose intersection with $\mathbf{B}$ is a transversal disk of $\mathcal{E}_1$ with diameter smaller than $d(\Omega_P)$. Hence, $d(\Omega) < d(\Omega_P)$. □

Lemma 3.2. *Let $P\text{-}ABC$ be a regular pyramid with apex P whose base is an equilateral triangle ABC, and let ρ be the slope of $P\text{-}ABC$. Let AA^* be the diameter of the circumscribed circle of ABC. Let D be a point on the edge AP such that $|AD| : |DP| = |AB| : |BP|$. Let E be a point on the line through A^* perpendicular to the plane ABC, lying in the opposite side of P with respect to the plane ABC, with $|A^*E| = |AO|/(2\rho)$, see Figure 3.4. Then*

(1) *the trunk $\mathcal{E} = (PA, BC)$ is hyperboloidal, and*

(2) *the line DE is the axis of a hyperboloid of revolution containing $\mathcal{E}$.*

Proof. By Theorem 2.1, it is enough to show that the line DE is the equidistant line of the four lines AB, AC, PB, PC. We may suppose that the circumscribed circle of $\triangle ABC$ has unit radius with center O. Then the height of the pyramid is ρ, i.e., $|PO| = \rho$. Note that $|AB| = |BC| = \sqrt{3}$, $|BO| = |BA^*| = 1$ and $|PA| = |PB| = |PA^*| = \sqrt{1 + \rho^2}$. Since $|AD| : |DP| = |AB| : |PB|$, we have

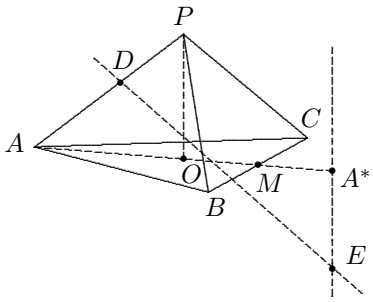

FIGURE 3.4
The line DE is an equidistant line of $\{AB, AC, PB, PC\}$

$\angle ABD = \angle PBD$. Since $\angle ABA^* = 90°$, AB is perpendicular to the plane BA^*E, and hence $AB \perp BE$. Since

$$\begin{aligned} |PE|^2 &= \big(|PO| + |A^*E|\big)^2 + |OA^*|^2 \\ &= (\rho + \tfrac{1}{2\rho})^2 + 1 = \rho^2 + 1 + 1/(2\rho)^2 + 1 \\ &= |PB|^2 + |A^*E|^2 + |BA^*|^2 \\ &= |PB|^2 + |BE|^2, \end{aligned}$$

we have $PB \perp BE$. Now, $\angle ABD = \angle PBD$, $AB \perp BE$, and $PB \perp BE$ imply together that every point on the plane DBE is equidistant to the lines AB and PB. Since $\mathcal{E}$ is symmetric to itself with respect to the plane DPE, we can deduce that the line DE is equidistant from the lines AB, AC, PB, PC. □

Theorem 3.3. *A right pyramid $P\text{-}ABC$ with apex P and equilateral triangular base ABC is circle-free if and only if*

$$\rho \le \rho_0 := \sqrt{(3\sqrt{17} - 5)/32} = 0.47988\ldots$$

where ρ is the slope of the pyramid.

Tanoue [Tan1] proved that if $\rho > \rho_0$, then $P\text{-}ABC$ can be held by a circle, and Maehara [Mae3] proved the converse.

Corollary 3.1. *The property "circle-freeness" is not affine invariant.*

Proof of Theorem 3.3. We use the same notations as in Lemma 3.2 and Figure 3.4. By Theorem 3.2, we may consider the case $\rho < 1$. Let $\mathcal{E} = (AP, BC)$, and $\mathcal{H}$ be the one-sheet hyperboloid of revolution with directed axis $l = DE$. Note that the conditions (i) and (ii) of the Isotopy Lemma hold. First, we show

that any circle Γ attached to $\mathcal{E}$ is isotopic over $\langle\mathcal{E}\rangle$ to a circle attached to $\mathcal{E}$ that satisfies the condition (†) of the Isotopy Lemma (1). We may suppose that Γ is symmetric to itself with respect to the plane $H = APO$ by the Symmetrization Lemma. The intersection $\mathrm{conv}(\Gamma) \cap \langle\mathcal{E}\rangle$ is an isosceles trapezoid $XYZW$, where X, Y, Z, W are the intersection points of $\mathrm{conv}(\Gamma)$ with the edges PB, PC, AC, AB, respectively. Let L, N be the midpoints of XY, ZW, respectively. If $\angle LNA \geq \pi/2$, then the plane of Γ clearly cuts DE, and the condition (†) of the Isotopy Lemma (1) holds. Suppose $\angle LNA < \pi/2$. Let $XYZ'W'$ be the isosceles trapezoid obtained by cutting the pyramid by the plane containing XY and being perpendicular to the line AM. Then the height of the trapezoid $XYZ'W'$ is smaller than that of $XYZW$, and $|Z'W'| < |ZW|$. Hence, by a continuous rotation of Γ around the line XY, we have an isotopy over the pyramid to a circle attached to $\mathcal{E}$ that satisfies the condition (†) of the Isotopy Lemma (1). Hence, any circles attached to $\mathcal{E}$ are isotopic over the pyramid to a circle that lies on the plane containing the minimal circle attached to $\mathcal{E}$ by the Isotopy Lemma. Therefore, to show that the pyramid is circle-free if and only if $\rho \leq \rho_0$, it is enough to show that the boundary circle of the minimal transversal disk of $\mathcal{E}$, denoted by Ω, holds the pyramid if and only if $\rho > \rho_0$. Note here that Ω is symmetric to itself with respect to the plane APM, and its boundary circle $\partial\Omega$ intersects the four edges AB, AC, PB, PC (for otherwise, by sliding Ω slightly in the direction $\overrightarrow{MA}$ or $\overrightarrow{MP}$, and squeezing its radius, we could get a transversal disk of smaller radius).

Let P_0 be the point on the line DE such that $PP_0 \perp DE$. Since $\rho < 1$, we have $|AB| > |PB|$. Since $AB \perp BE$ and $PB \perp BE$ (see the proof of Lemma 3.2), we have $|PE|^2 = |PB|^2 + |BE|^2$ and $|AE|^2 = |AB|^2 + |BE|^2$. Put $a = |AO|$. Then $|AP|^2 = a^2(1+\rho^2)$ and $|AB|^2 = 3a^2$. Hence $|AB|^2 - 2|AP|^2 = 3a^2 - 2a^2(1+\rho^2) = a^2(1-\rho^2) > 0$. Thus, $|AB|^2 > 2|AP|^2 = |AP|^2 + |PB|^2$, and hence

$$|AP|^2 + |PE|^2 = |AP|^2 + |PB|^2 + |BE|^2 < |AB|^2 + |BE|^2 = |AE|^2.$$

This implies that $\angle EPA$ is an obtuse angle, which implies that P_0 lies on the line segment DE. Therefore, the disk with center P_0 and radius $|PP_0|$ whose plane perpendicularly cuts DE at P_0, is a transversal disk of $\mathcal{E}$.

Let Q, X be the points on the lines DE, PB, respectively, such that $|QX|$ is the minimum distance between the lines DE and PB. Then the section of $\mathcal{H}$ by the plane perpendicular to l at Q is the smallest circle lying on $\mathcal{H}$ (cf. the statement $\star$ in the proof of Theorem 2.1). See Figure 3.5.

Case (a): The ray $\overrightarrow{PM}$ does not intersect the line DE.
In this case, we have $\angle P_0PM \geq \pi/2$, and hence $\angle P_0PB (= \angle P_0PC) \geq \pi/2$,

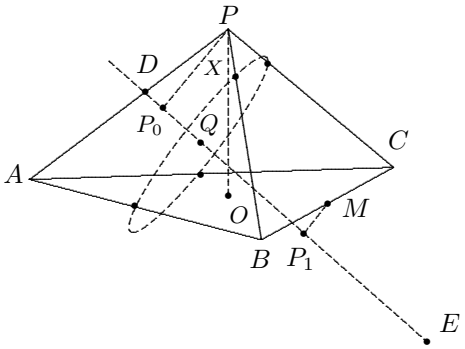

FIGURE 3.5
The minimal circle on $\mathcal{H}$

which implies that Q lies on the ray $\overrightarrow{P_0D}$, possibly with $Q = P_0$. By the Isotopy Lemma the boundary circle $\partial\Omega$ of the minimal transversal disk Ω of $\mathcal{E}$ is isotopic over the pyramid to a boundary circle Γ' of a transversal disk of $\bar{\mathcal{E}}$ that is perpendicular to l. If the plane of Γ' intersects the ray $\overrightarrow{P_0E}$, then the radius of Γ' is larger than $|P_0P|$ (since the most constricted part of $\mathcal{H}$ is the section of $\mathcal{H}$ by the plane perpendicular to l at Q). Hence $\partial\Omega$ can slip out of the pyramid, and hence the pyramid is circle-free.

Case (*b*)*:* The ray $\overrightarrow{PM}$ intersects the line DE.
In this case Q lies on the ray $\overrightarrow{P_0E}$ and $Q \neq P_0$. Let P_1 be the foot of perpendicular dropped from M to DE. Since P_1 lies between P_0 and E, we have $\angle P_1BP < \pi/2$. Hence Q lies between P_0 and P_1. Therefore the disk with center Q and radius $|QX|$ lying on the plane perpendicular to DE is the minimal transversal disk Ω of $\mathcal{E}$. The boundary circle $\partial\Omega$ intersects the edges AB, AC, PB, PC.

Let Ω_P be the minimal transversal disk of $\mathcal{E}$ on P. The boundary circle $\partial\Omega_P$ also intersects the edges AB, AC. Since $Q \neq P_0$, we have $d(\Omega) < 2|P_0P|$. If $d(\Omega_P) \geq 2|P_0P|$, then we have $d(\Omega_P) > d(\Omega)$, and the circle $\partial\Omega$ holds the pyramid by (2) of Lemma 3.1.

Suppose that $d(\Omega_P) < 2|P_0P|$. Let Z be the intersection point of l and Ω_P, and S, T be the intersection points of Ω_P with the edges AB, AC, respectively. The three points Z, S, T lie on the same side of the plane that perpendicularly intersects the line $l = DE$ at P_0. If Z lies on P_0D and $Z \neq P_0$, then $|ZS| = |ZT| > d(S, l) > |P_0P|$, and $|ZP| > |P_0P|$. Hence the radius of the circumscribed circle of the isosceles triangle PST is larger than $|P_0P|$, which implies that $d(\Omega_P) > 2|P_0P|$, contradicting the assumption $d(\Omega_P) < 2|P_0P|$. Therefore, Z lies on P_0E. Then, $\angle ZPM < \pi/2$, and hence,

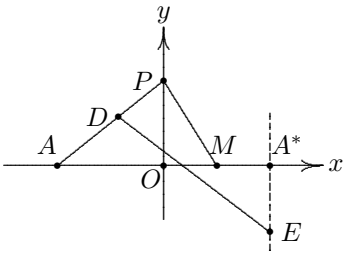

FIGURE 3.6
Projection on the plane APO

by rotating Ω_P slightly around the line ST, and squeezing its radius, we can get a transversal disk of $\mathcal{E}$. Hence $d(\Omega) < d(\Omega_P)$, and $\partial\Omega$ holds the pyramid by (2) of Lemma 3.1.

Thus, the pyramid is *not* circle-free if and only if the ray $\overrightarrow{PM}$ intersects the line DE. Now we show that the ray $\overrightarrow{PM}$ intersects the line DE if and only if $\rho > \rho_0$. To do this, let us regard the plane PAO as the xy-plane, and $A = (-1, 0), O = (0, 0)$, see Figure 3.6. Then $P = (0, \rho), M = (1/2, 0), A^* = (1, 0), E = (1, \frac{-1}{2\rho})$, and since $|AB| : |BP| = \sqrt{3} : \sqrt{1+\rho^2}$,

$$D = \left(\frac{-\sqrt{1+\rho^2}}{\sqrt{3}+\sqrt{1+\rho^2}}, \frac{\rho\sqrt{3}}{\sqrt{3}+\sqrt{1+\rho^2}} \right).$$

The ray $\overrightarrow{PM}$ intersects the line DE if and only if the slope of the line DE is greater than the slope of the line PM. The slope of PM is -2ρ, and the slope of DE is

$$\left(\frac{-1}{2\rho} - \frac{\rho\sqrt{3}}{\sqrt{3}+\sqrt{1+\rho^2}} \right) / \left(1 + \frac{\sqrt{1+\rho^2}}{\sqrt{3}+\sqrt{1+\rho^2}} \right).$$

Thus, the ray $\overrightarrow{PM}$ intersects the line DE if and only if

$$\left(\frac{-1}{2\rho} - \frac{\rho\sqrt{3}}{\sqrt{3}+\sqrt{1+\rho^2}} \right) > -2\rho \left(1 + \frac{\sqrt{1+\rho^2}}{\sqrt{3}+\sqrt{1+\rho^2}} \right),$$

and (by simplifying) if and only if

$$\sqrt{3} - 2\sqrt{3}\rho^2 < (8\rho^2 - 1)\sqrt{1+\rho^2}.$$

The left side is monotone decreasing on ρ, whereas the right side is monotone increasing on ρ. So let us find the value of ρ where both sides become equal. Putting $\xi = \rho^2$, we have $\sqrt{3} - 2\sqrt{3}\xi = (8\xi - 1)\sqrt{1+\xi}$, and after squaring both sides as well as simple calculations, we have

$$64\xi^3 + 36\xi^2 - 3\xi - 2 = 0.$$

This equation has three real solutions, namely

$$\xi = \frac{-5 \pm 3\sqrt{17}}{32}, \ \frac{-1}{4}.$$

Since $\xi > 0$, we have $\xi = \frac{-5+3\sqrt{17}}{32}$, and hence the pyramid is not circle-free if and only if $\rho > \rho_0$. □

Remark 3.2. It was proved by Maehara [Mae3] that a regular pyramid with square base can be held by a circle if and only if $\rho > \sqrt{(\sqrt{33}-3)/4} \approx 0.828$.

3.3. Holding circles with much play. Let Γ be a circle attached to a convex polyhedron Π, and PQ be an edge of Π such that $PQ \cap \mathrm{conv}(\Gamma) = \varnothing$. Suppose that there is an isotopy Γ_t, $0 \le t \le 1$, over Π with $\Gamma_0 = \Gamma$ such that (i) $Q \notin \mathrm{conv}(\Gamma_t)$ for all t, and (ii) $\mathrm{conv}(\Gamma_1)$ cuts the edge PQ into two segments. Then we say that Γ (and any circle isotopic to it over Π) *can cross over* P.

Theorem 3.4. *For every vertex P of a regular icosahedron, there is a holding circle of the regular icosahedron that can cross over P.*

Proof. We use Figure 2.7 left and the same notations as in Example 2.3. It is enough to show the case $P = A$. Let Ω be the minimal transversal disk of $\mathcal{E}$ on A. Then Ω is also a minimal transversal disk of $\mathcal{E}'$ on A by Example 2.3.

Since the trunk $\mathcal{E}$ is hyperboloidal, it is possible to rotate the circumscribed circle of the pentagon $ABCDE$ slightly around the line passing through A and being perpendicular to the plane H determined by AFF^* (see Figure 3.7), and to squeeze its radius a bit so that it is still attached to the trunk $\mathcal{E}$. Hence the diameter of Ω is smaller than the diameter of the circumscribed circle of the pentagon $ABCDE$. Let X, Y be the points where Ω cuts the edges A^*C, A^*D, respectively, and let M be the midpoint of XY. Since the boundary circle $\partial\Omega$ of Ω intersects

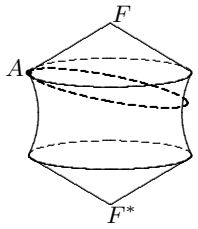

FIGURE 3.7
Move the circumscribed circle of $ABCDE$

the icosahedron only at the points A, X, Y, and $\angle MAF < \angle F^*AF = 90°$, it is possible to rotate slightly the circle $\partial\Omega$ around the line XY in either direction, without intersecting the interior of the icosahedron. Hence there is an isotopy Γ_t, $0 \le t \le 1$, over the icosahedron such that $\mathrm{conv}(\Gamma_0) \cap AF = \varnothing$, $F \notin \mathrm{conv}(\Gamma_t)$ for all t, and $\mathrm{conv}(\Gamma_1)$ cuts the edge AF dividing it into two segments. Hence Γ_0 (and Γ_1) can cross over the vertex A.

Let us show that Γ_1 holds the icosahedron. The diameter of Γ_1 is equal to $d(\Omega)$. We may assume that the disk $\mathrm{conv}(\Gamma_1)$ contains no vertex, and cuts all edges of the trunk $(BCDEF, AA^*B^*C^*D^*E^*)$. Suppose Γ_1 can slip out of the icosahedron. During the slipping out process, $\mathrm{conv}(\Gamma_1)$ meets vertices of the icosahedron. Let Z be the first vertex that $\mathrm{conv}(\Gamma_1)$ meets during the slipping out process. We may suppose that $Z \neq A$. Clearly, $Z \neq F, F^*$ and $Z \neq C, D, C^*, D^*$. Is it possible that $Z = B$? If B is the first vertex that $\mathrm{conv}(\Gamma_1)$ meets during the slipping out process of Γ_1, then at the moment that $\mathrm{conv}(\Gamma_1)$ meets B, $\mathrm{conv}(\Gamma_1)$ becomes a transversal disk of the trunk (BB^*, DE) on B, and it is not symmetric to itself with respect to the plane FBF^*, which is a symmetry plane of (BB^*, DE). Hence, the diameter of $\mathrm{conv}(\Gamma_1)$ must be greater than the diameter of the minimal transversal disk of (BDE, B^*) on B, by the Symmetrization Lemma. However, the latter is equal to the diameter of the minimal transversal disk of $\mathcal{E}'$ on A, which is equal to the diameter of Ω, a contradiction. Similarly $Z \neq E$.

If $Z = A^*$, then at the moment that $\mathrm{conv}(\Gamma_1)$ meets A^*, it becomes a transversal disk of the trunk (A^*A, BF) on A^*, and analogously we have a contradiction. We can deduce $Z \neq B^*, E^*$, similarly. Hence Γ_1 cannot slip out of the icosahedron. □

Remark 3.3. The boundary circle $\Gamma = \partial\Omega$ of the minimal transversal disk of the trunk $\mathcal{E} = (ABCDE, A^*B^*C^*D^*E^*)$ on A cannot cross over F and F^*. However, since $\mathcal{E}$ is hyperboloidal with symmetry plane, Γ is isotopic over the icosahedron to the boundary circle of the minimal transversal disk of $\mathcal{E}$ on B, by the Isotopy Lemma. Hence Γ can cross over B and, similarly, can cross over C, D, E.

Now, there arises a problem. Does there exist a convex polyhedron, together with its holding circle, such that the circle can cross over every vertex of the polyhedron? The answer is yes. From the regular icosahedron shown in Figure 2.7 we get, by cutting off two pentagonal pyramids $F\text{-}ABCDE$ and $F^*\text{-}A^*B^*C^*D^*E^*$, a *regular pentagonal anti-prism* as shown in Figure 3.8.

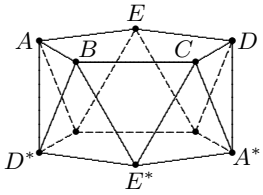

FIGURE 3.8
A regular pentagonal anti-prism

Theorem 3.5. *In case of the regular pentagonal anti-prism* (*see Figure* 3.4) *the boundary circle of the minimal transversal disk* Ω *of* $(ABCDE, A^*B^*C^*D^*)$ *on* A *can cross over every vertex of the anti-prism, but still it cannot slip out of the anti-prism.*

Proof. Let d_0 be the diameter of Ω. As seen in Remark 3.3, $\partial\Omega$ can cross over every vertex of the anti-prism.

To show that $\partial\Omega$ cannot slip out of the anti-prism, we suppose the contrary, namely that it can slip out of the anti-prism. During the slipping out process, the disk Ω crosses over vertices of the anti-prism. We may suppose that A is the first vertex that Ω crosses over, and Ω is at the position of a transversal disk of the trunk $(BCDE, AA^*B^*C^*D^*E^*)$. Let Z be the vertex that Ω meets next. We may assume $Z \neq A$. It is also clear that Z cannot be any of C, D, C^*, D^*. Suppose $Z = B$. At the moment when Ω meets B, it becomes a transversal disk of (BB^*, DE) on B. At that moment, since Ω becomes not symmetric to itself with respect to the plane determined by B, B^* and the midpoint of ED, this disk is not a minimal transversal disk of (BB^*, DE) on B (which has diameter d_0). Hence the diameter of Ω is greater than d_0, a contradiction. Thus, $Z \neq B$. Similarly, we have $Z \neq E, B^*, E^*$.

Finally, consider the case that $Z = A^*$. We use the following fact which will be proved later.

> $(\star)$ The diameter of the minimal transversal disk of (AA^*, BE) on A^* is greater than the diameter of the minimal transversal disk of (AA^*, C^*D^*) on A^*.

At the moment when Ω meets A^*, it becomes a transversal disk of $(BCDE, AA^*)$ on A^*, which is at least the diameter of the minimal transversal disk of (AA^*, DE) on A^*. Hence the diameter of this minimal transversal disk is greater than the diameter of the minimal transversal disk of (AA^*, C^*D^*) on A^* by $(\star)$, which is equal to d_0 by Example 2.3, a contradiction. Therefore, the circle $\partial\Omega$ cannot slip out of Π.

Now we show $(\star)$. Suppose the minimal transversal disk of (AA^*, BE) cuts the edges AB, AE at X, Y, respectively. We may assume that $|BX| = |EY| = t$. Let X', Y' be points on AC^*, AD^* such that $|C^*X'| = |D^*Y'| = t$. Consulting Figure 2.7 left, we can see that $|A^*B| = |A^*E| = |A^*C^*| = |A^*D^*|$ and $\angle A^*BA = \angle A^*D^*A = \angle A^*EA = \angle A^*C^*A = 90°$. Hence $|A^*X| = |A^*X'| = |A^*Y| = |A^*Y'|$. Since $|XY| > |X'Y'|$, the diameter of the circumscribed circle of the isosceles triangle A^*XY is greater than the diameter of the circumscribed circle of the isosceles triangle $A^*X'Y'$. Since the latter is greater than or equal to the diameter of the minimal transversal disk of (AA^*, C^*D^*), the proof is complete. □

4. Supplement

4.1. Proof of the Symmetrization Lemma. Let K be the plane containing Ω, and let Ω', K' be the mirror images of Ω, K with respect to H, respectively. The disk Ω' is also a transversal disk of $\mathcal{E}$.

If $K = K'$, then put $\Omega_t = \Omega + \frac{t}{2}\overrightarrow{zz'}$, where z, z' are the centers of the disks Ω and Ω', respectively. Then $\partial\Omega_t$, $t \in [0, 1]$, is an isotopy over $\langle\mathcal{E}\rangle$ and $\partial\Omega_0 = \partial\Omega$, $\partial\Omega_1$ is symmetric to itself with respect to H. Moreover, if $\Omega \neq \Omega'$, then the smallest disk containing $\Omega \cap \Omega'$ is a transversal disk of $\mathcal{E}$ that is smaller than Ω.

Now suppose that $K \neq K'$, and put $\mathcal{E} = (U, V)$. Let K_+ denote the side (half space) of K that contains U, and K_- be the other side containing V. Similarly, let K'_+ be the side of K' that contains U, and K'_- be the other side of K'. The planes K and K' together divide $\mathbb{R}^3$ into four regions

$$K_+ \cap K'_+,\ K_+ \cap K'_-,\ K_- \cap K'_+,\ K_- \cap K'_-.$$

Figure 4.1 shows the projections on the plane perpendicular to the line $H \cap K$.

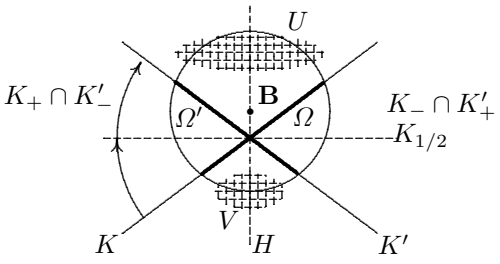

FIGURE 4.1
Projections onto the plane perpendicular to $H \cap K$

Note that $U \subset K_+ \cap K'_+$, $V \subset K_- \cap K'_-$. Since Ω' is the mirror image of Ω with respect to the plane H, the disks Ω and Ω' together determine a ball $\mathbf{B}$ such that $\mathbf{B} \cap K = \Omega$, $\mathbf{B} \cap K' = \Omega'$. Let K_t, $t \in [0,1]$, denote the uniform rotation of the plane K around the line $K \cap H$, through $(K_+ \cap K'_-) \cup (K_- \cap K'_+)$ such that $K_0 = K$ and $K_1 = K'$. Since K_t separates U from V for each $t \in [0,1]$, K_t intersects all edges in $\mathcal{E}$. Since $\mathrm{conv}(\Omega \cup \Omega') \subset \mathbf{B}$, $K_t \cap \mathbf{B}$ intersects all edges of $\mathcal{E}$, that is, $K_t \cap \mathbf{B}$ is a transversal disk of $\mathcal{E}$.

Now put $\tilde{\Omega}_t = K_t \cap \mathbf{B}$ for $t \in [0,1]$. Note that among the planes K_t, $t \in [0,1]$, the planes $K_0 = K$ and $K_1 = K'$ are those nearest to the center of $\mathbf{B}$. Hence $d(\tilde{\Omega}_t) \leq d(\Omega)$ for all $t \in [0,1]$. Therefore, replacing each $\tilde{\Omega}_t$ by the concentric disk Ω_t in K_t whose diameter equals $d(\Omega)$, we have an isotopy $\partial\Omega_t$, $t \in [0,1]$, over $\langle \mathcal{E} \rangle$. Then $\partial\Omega_{1/2}$ is symmetric to itself with respect to H.

Next, suppose that $K \neq K'$ and $\Omega \cap H \not\subset \langle \mathcal{E} \rangle$. Since $\Omega \cap H = \Omega \cap \Omega'$, it follows that $\Omega \cap \Omega' \not\subset \langle \mathcal{E} \rangle$. Thus, at least one endpoint of the line segment $\Omega \cap \Omega'$ is not contained in $\langle \mathcal{E} \rangle$, that is, $\partial\Omega \cap \partial\Omega' \not\subset \langle \mathcal{E} \rangle$. Note that $\mathrm{conv}(\Omega \cup \Omega') \cap \partial\tilde{\Omega}_{1/2} = \partial\Omega \cap \partial\Omega'$. Since

$$\langle \mathcal{E} \rangle \cap \partial\tilde{\Omega}_{1/2} = \langle \mathcal{E} \rangle \cap K_{1/2} \cap \partial\tilde{\Omega}_{1/2} \subset \mathrm{conv}(\Omega \cup \Omega') \cap \partial\tilde{\Omega}_{1/2} = \partial\Omega \cap \partial\Omega',$$

$\partial\Omega \cap \partial\Omega' \not\subset \langle \mathcal{E} \rangle$ implies that $\langle \mathcal{E} \rangle \cap \partial\tilde{\Omega}_{1/2}$ consists of at most one point. Since the boundary circle of a minimal transversal disk of $\mathcal{E}$ must intersect $\mathcal{E}$ in at least two points, $\tilde{\Omega}_{1/2}$ is not a minimal transversal disk of $\mathcal{E}$. Therefore, Ω is not a minimal transversal disk of $\mathcal{E}$, either. □

4.2. Proof of the Isotopy Lemma. (1) We may suppose that Ω_0 is symmetric to itself with respect to the symmetry plane H of $\mathcal{E} = (U, V)$. Then the center Z of Ω_0 lies on H. Let K be the plane that contains Ω_0. Let $H \cap \Gamma = \{P, Q\}$. The line segment PQ is a diameter of Ω_0 and Z is the midpoint of PQ. We may suppose that l intersects the ray $\overrightarrow{ZP}$. Let O be the intersection of l and the plane that perpendicularly bisects PQ, and let $\mathbf{B}$ be the ball with center O and radius $|OP|$, see Figure 4.2. Among the points on PQ that are obtained by the orthogonal projection of the points $\Omega_0 \cap \mathcal{E}$ on PQ, let $\check{X}$ be the one nearest to P. Let X be a point in $\Omega_0 \cap \mathcal{E}$ that is projected to $\check{X}$. Then, clearly $|OX| \leq |OP|$. Let K' be the plane containing $X\check{X}$ and perpendicular to l, and let Z' be the intersection point of K' and l. Since every line $\bar{\mathcal{E}}$ lies on $\mathcal{H}$, the circle $K' \cap \mathcal{H}$ has center Z', radius $|Z'X|$. Since $|OX| \leq |OP|$, the circle $K' \cap \mathcal{H}$ is contained in the disk $\mathbf{B} \cap K'$. Hence, every line in $\bar{\mathcal{E}}$ also passes through $\mathbf{B} \cap K'$. Let K_+ (resp. K_-) be the upper side (resp. the lower side) of K. Let K'_+ (resp. K'_-) be the upper side (resp. the lower side) of K'. Then $U \subset K_+, V \subset K_-$ and $Q \in K'_+, P \in K'_-$.

Claim: No line in $\bar{\mathcal{E}}$ passes through $\mathrm{int}(K_+ \cap K'_-)$.

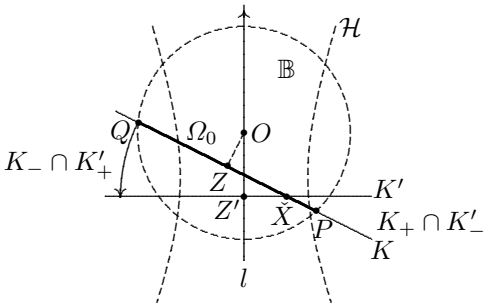

FIGURE 4.2
Projection on the plane H

To see this, suppose, on the contrary, a line g in $\bar{\mathcal{E}}$ passes through $\mathrm{int}(K_+ \cap K'_-)$. Then the projection $\check{g}$ of g on the plane H never intersects the segment $\check{X}P$. For otherwise, we have a contradiction to the definition of $\check{X}$. Hence $\check{g}$ must intersect the line segment $\check{X}Q$. Let AB $(A \in U, B \in V)$ be the edge of $\mathcal{E}$ that determines the line g. Then, since the line g never passes through $\mathrm{int}(K_- \cap K'_-)$ and $B \in K_- \cap K'_+$, it follows that there is no transversal plane of $\mathcal{E}$ that is perpendicular to l, contradicting the assumption (i) of the theorem.

Let Ω_t $(0 \le t \le 1)$ denote the continuous rotation of Ω_0 around the line $X\check{X}$ (if $X = \check{X}$, then around the line through $\check{X}$ and perpendicular to H) as shown by the curved arrow in Figure 4.2 such that Ω_1 lies on the plane K'. Since each line in $\bar{\mathcal{E}}$ passes through both Ω and $K' \cap \mathbf{B}$, and since no line of $\bar{\mathcal{E}}$ passes through $\mathrm{int}(K_+ \cap K'_-)$, Ω_t $(0 \le t \le 1)$ are all transversal disks of $\bar{\mathcal{E}}$. Since Π is enclosed by the planes determined by the "lateral faces" of $\langle \mathcal{E} \rangle$, we have $(\mathbf{B} \cap K_+ \cap K'_-) \cap \mathrm{int}(\Pi) = \varnothing$. Hence Γ is isotopic over Π to $\partial\Omega_1$.

(2) Let Γ_1, Γ_2 be congruent circles attached to $\mathcal{E}$, each lying on the plane perpendicular to l. Consider the tube obtained as the trajectory of the translation of Γ_1 to Γ_2. Since each line in $\bar{\mathcal{E}}$ passes through $\mathrm{conv}(\Gamma_i)$, $i = 1, 2$, $\mathrm{int}(\Pi)$ does not intersect this tube. Hence Γ_1 and Γ_2 are isotopic over Π. Now, (2) follows from (1).

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References

[Abe] O. Aberth, An isoperimetric inequality for polyhedra and its application to an extremal problem. *Proc. London Math. Soc.* (*3*) **13** (1963), 322–336. Zbl 0113.16204 MR 0149371

[BMT] I. Bárány, H. Maehara, and N. Tokushige, Tetrahedra passing through a triangular hole, and tetrahedra fixed by a planar frame. *Comput. Geom.* **45** (2012), 14–20. Zbl 1242.52003 MR 2842618

[BZ1] I. Bárány and T. Zamfirescu, Holding circles and fixing frames. *Discrete Comput. Geom.* **50** (2013), 1101–1111. Zbl 1283.52002 MR 3138148

[BZ2] —— Circles holding typical convex bodies. *Lib. Math.* (*N.S.*) **33** (2013), 21–25. Zbl 1308.52002 MR 3177479

[Bes1] A. S. Besicovitch, A net to hold a sphere. *Math. Gaz.* **41** (1957), 106–107. Zbl 0079.14503 MR 0085556

[Bes2] —— A cage to hold a unit-sphere. *Proc. Sympos. Pure Math.*, Vol. VII, pp. 19–20. Amer. Math. Soc., Providence, R.I., 1963. Zbl 0141.19903 MR 0155236

[BE] A. S. Besicovitch and H. G. Eggleston, The total length of the edges of a polyhedron. *Quart. J. Math. Oxford Ser.* (*2*) **8** (1957), 172–190. Zbl 0083.17302 MR 0095448

[BM] V. Boltyanski and H. Martini, Fixing and hindering systems in combinatorial geometry, *Beitr. Algebra Geom.* **40** (1999), 551–563. Zbl 0962.52001 MR 1720123

[BMS] V. Boltyanski, H. Martini, and P. S. Soltan, *Excursions into Combinatorial Geometry*, Universitext. Springer-Verlag, Berlin, 1997. Zbl 0877.52001 MR 1439963

[BFMM] J. Bracho, H. Fetter, D. Mayer and L. Montejano, Immobilization of solids and Mondriga quadratic forms. *J. London Math. Soc.* (2) **51** (1995), 189–200. Zbl 0819.52005 MR 1310731

[CFG] H. T. Croft, K. J. Falconer, and R. K. Guy, *Unsolved Problems in Geometry*. Corrected reprint of the 1991 original version. Problem Books in Mathematics. Unsolved Problems in Intuitive Mathematics, II. Springer-Verlag, New York, 1994. Zbl 0748.52001 MR 1316393

[Cox] H. S. M. Coxeter, Unsolved problem. In: *Proc. Sympos. Pure Math.*, Vol. VII, p. 495. Amer. Math. Soc., Providence, R.I., 1963.

[Cro] H. T. Croft, A net to hold a sphere. *J. London Math. Soc.* **39** (1964), 1–4. Zbl 0122.41104 MR 0161226

[CSU] J. Czyzowicz, I. Stojmenovic, and J. Urrutia, Immobilizing a shape. *Internat. J. Comput. Geom. Appl.* **9** (1999), 181–206. Zbl 1035.68526 MR 1681242

[Daw] R. J. MacG. Dawson, On the mobility of bodies in R^n. *Math. Proc. Cambridge Philos. Soc.* **98** (1985), 403–412. Zbl 0587.53011 MR 0803600

[FeT] L. Fejes Tóth, *Lagerungen in der Ebene, auf der Kugel und im Raum*. Springer-Verlag, Heidelberg 1972. Zbl 0229.52009

[Fru] A. Fruchard, Small holding circles. *Geom. Dedicata* **157** (2012), 397–405. Zbl 1275.52004 MR 2893495

[Gru] P. M. Gruber, Baire categories in convexity. *Handbook of Convex Geometry*, Vol. B, pp. 1327–1346. North-Holland, Amsterdam, 1993. Zbl 0791.52002 MR 1243011

[ITZ] J. Itoh, Y. Tanoue, and T. Zamfirescu, Tetrahedra passing through a circular or square hole. *Rend. Circ. Mat. Palermo* (*2*) *Suppl.* **77** (2006), 349–354. Zbl 1104.52010 MR 2245680

[IZ] J. Itoh and T. Zamfirescu, Simplices passing through a hole. *J. Geom.* **83** (2005), 65–70. Zbl 1112.52003 MR 2193228

[Mae1] H. Maehara, On congruent embeddings of a tetrahedron into a circular cylinder. *Yokohama Math. J.* **55** (2010), 171–177. Zbl 1203.52009 MR 2666143

[Mae2] —— On the diameter of a circle to hold a cube. *Lecture Notes Comput. Sci. 7033*, pp. 147–153, 2011. Zbl 1349.52002 MR 2927319

[Mae3] —— Holding a regular pyramid by a circle. *J. Geom.* **102** (2011), 133–148. Zbl 1246.52004 MR 2904623

[Mae4] —— Necks of regular polyhedra. *J. Geom.* **5** (2013), 329–340. Zbl 1278.52005 MR 3089785

[MT1] H. Maehara and N. Tokushige, Regular simplices passing through holes. *Geom. Dedicata* **145** (2010), 19–32. Zbl 1193.52004 MR 2600942

[MT2] —— Classification of the congruent embeddings of a tetrahedron into a triangular prism. *Graphs Combin.* **27** (2011), 451–463. Zbl 1255.52012 MR 2787430

[MZ] L. Montejano and T. Zamfirescu, Two problems on cages for discs. In *Convexity and Discrete Geometry Including Graph Theory*, pp. 263–264, Springer Proc. Math. Stat., 148, Springer, Cham, 2016. Zbl 1373.52004 MR 3516718

[Schn] R. Schneider, *Convex Bodies: The Brunn–Minkowski Theory*. Cambridge University Press, Cambridge, 1993. Zbl 0798.52001 MR 1216521

[Schr] O. Schramm, How to cage an egg? *Invent. Math.* **107** (1992), 543–560. Zbl 0726.52003 MR 1150601

[SS] J. T. Schwartz and M. Sharir, On the piano movers' problem. V. The case of a rod moving in three-dimensional space amidst polyhedral obstacles. *Comm. Pure Appl. Math.* **37** (1984), 815–848. Zbl 0554.51009 MR 0762874

[Ste] S. K. Stein, On a theorem of Besicovitch concerning curves on the sphere. *Math. Gaz.* **47** (1963), 44. Zbl 0107.39904

[Tan1] Y. Tanoue, Regular triangular pyramids held by a circle. *J. Geom.* **94** (2009), 151–157. Zbl 1222.52002 MR 2534415

[Tan2] —— Circles holding a regular triangular prism. *An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat.* **19** (2011), 275–283. Zbl 1224.52006 MR 2785703

[Tan3] —— Circles holding convex polyhedra. Thesis, Kumamoto University, 2011.

[Zam1] T. Zamfirescu, How to hold a convex body? *Geom. Dedicata* **54** (1995), 313–316. Zbl 0834.52003 MR 1326735

[Zam2] —— Polytopes passing through circles. *Period. Math. Hungar.* **57** (2008), 227–230. Zbl 1199.52024 MR 2469608

[Zam3] —— Pushing convex and other bodies through rings and holes. *An. Univ. Vest. Timis., Ser. Mat.-Inf.* **48** (2010), 299–306. Zbl 1240.52005 MR 2849343

[Zam4] —— Escaping from a cage. *Lib. Math.* (*N.S.*) **35** (2015), 43–49. Zbl 1365.52003 MR 3538141

[Zin] K. Zindler, Über konvexe Gebilde. *Monatsh. Math. Phys.* **30** (1920), 87–102. JFM 47.0681.06 MR 1549047

[Zon] C. Zong, The kissing number, blocking number and covering number of a convex body. *Surveys on Discrete and Computational Geometry*, pp. 529–548, *Contemp. Math.*, **453**, *Amer. Math. Soc.*, Providence, RI, 2008. Zbl 1144.52020 MR 2405694

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