

## Canonical idempotents of multiplicity-free families of algebras

Stephen DOTY, Aaron LAUVE and George H. SEELINGER

**Abstract.** Any multiplicity-free family of finite dimensional algebras has a *canonical* complete set of pairwise orthogonal primitive idempotents in each level. We give various methods to compute these idempotents. In the case of symmetric group algebras over a field of characteristic zero, the set of canonical idempotents is precisely the set of seminormal idempotents constructed by Young. As an example, we calculate the canonical idempotents for semisimple Brauer algebras.

**Mathematics Subject Classification (2010).** Primary: 16G99, 20C30; Secondary: 05E10, 16Z05.

**Keywords.** Group algebras, Brauer algebras, primitive idempotents, Jucys–Murphy elements, tower of algebras.

### Introduction

Given a finite dimensional unital associative algebra  $\mathcal{A}$  over a field  $\mathbb{k}$ , a fundamental problem is to find a *partition of unity*, i.e., a complete set of pairwise orthogonal primitive idempotents, in  $\mathcal{A}$ . (This means finding a set  $\{e_i\}_{i \in I}$  of elements satisfying  $\sum_i e_i = 1$  and  $e_i e_j = \delta_{ij} e_j$  for  $i, j \in I$ , with  $|I|$  maximal.) The corresponding problem for the center  $Z(\mathcal{A})$  is equally fundamental; in that case the partition is unique. We study these two closely related problems under the assumption that  $\mathcal{A}$  is split semisimple; i.e.,  $\mathcal{A}$  is isomorphic to a direct sum of matrix algebras over  $\mathbb{k}$ .

Our main results are for the special case where  $\mathcal{A} = \mathcal{A}_n$  fits into a multiplicity-free family  $\{\mathcal{A}_n \mid n \geq 0\}$  (see Definition 1.1), which allows for induction on  $n$ . Group algebras of symmetric groups serve as the primary motivating example. For a multiplicity-free family  $\{\mathcal{A}_n\}$ , we find that:

- (1) There is a canonical partition of unity of  $\mathcal{A}_n$  for all  $n$  (see Proposition 1.6). This fact is implicit in [OV, VO] and explicit in [GG2]; we feel it deserves to be more widely known.
- (2) The two problems (calculating the canonical partitions of unity in  $\mathcal{A}_n$  and in  $Z(\mathcal{A}_n)$  for all  $n$ ) are equivalent.
- (3) Both problems can be solved recursively by “Lagrange interpolation” methods, in terms of the eigenvalues of a Jucys–Murphy sequence on a Gelfand–Tsetlin basis of the irreducible representations.
- (4) Both problems reduce to the computation of certain polynomials in the  $n$ th Jucys–Murphy element, for all  $n$ . The polynomials depend only on a pair  $(\lambda, \mu)$  of isomorphism classes of irreducible representations, one for  $\mathcal{A}_n$  and the other for  $\mathcal{A}_{n-1}$ .

Many of the results of the paper are straightforward extensions of known results scattered through the literature. Our approach is based on the insights of Vershik and Okounkov [OV, VO] for symmetric group algebras; see also [DJ, GdlHJ, RW, HR, LR, Ram4, Ram3, DG, OP, Gar, Mat, CSST, GG2] for related work. Probably [GG2] overlaps the most with this paper.

The general theory of Lagrange interpolation methods for multiplicity-free families is presented in Sections 1–3; this theory extends known results from symmetric group algebras in characteristic zero to arbitrary multiplicity-free families. Examples of multiplicity-free families abound in the literature (e.g., partition algebras, Temperley–Lieb algebras, various families of Weyl groups and their associated Hecke algebras, Birman–Murakami–Wenzl algebras) so these results should have wide applicability. For many of these families, suitable candidates for Jucys–Murphy sequences (in our sense) have been found, which should bring all of items (1)–(4) above to bear on their study. Due to space constraints, we treat only two illustrative examples here: in Sections 4 and 5 we apply our methods to study the symmetric group algebras and Brauer algebras, respectively. Although we have chosen to avoid the language of cellular algebras, in order to keep the exposition as elementary as possible, readers interested in applying these results to other diagram algebras would be well-advised to utilize the axiomatic framework of [GG2] and the related results of [GG1].

Appendix A outlines an alternative method of computing the partition of unity of  $Z(\mathcal{A})$  in characteristic zero, based on trace characters instead of interpolation. This is valid without any assumption that the split semisimple algebra  $\mathcal{A}$  fits into a multiplicity-free family; however, it requires inverting a possibly large matrix.

## 1. Multiplicity-free families of algebras

Let  $\mathbb{k}$  be a field and  $\mathcal{A}$  an algebra over  $\mathbb{k}$ . All the algebras considered in this paper are assumed to be finite dimensional, semisimple, associative, unital, and split over  $\mathbb{k}$ . Write  $\text{Irr}(\mathcal{A})$  for the set of isomorphism classes of irreducible left<sup>1</sup>  $\mathcal{A}$ -modules and  $V^\lambda$  for a representative of the class  $\lambda \in \text{Irr}(\mathcal{A})$ . That is,  $[V^\lambda] = \lambda$ .

The general Wedderburn–Artin theorem expresses  $\mathcal{A}$  as a finite direct sum of matrix algebras over division rings; our assumption that  $\mathcal{A}$  is split over  $\mathbb{k}$  means that each of the division rings is  $\mathbb{k}$  (this is automatic if  $\mathbb{k}$  is algebraically closed), so

$$(1.1) \quad \mathcal{A} = \bigoplus_{\lambda \in \text{Irr}(\mathcal{A})} \varepsilon(\lambda)\mathcal{A} \cong \bigoplus_{\lambda \in \text{Irr}(\mathcal{A})} \text{End}_{\mathbb{k}}(V^\lambda).$$

In the isomorphism (1.1), the central idempotent  $\varepsilon(\lambda) \in \mathcal{A}$  acts as the identity in  $\text{End}_{\mathbb{k}}(V^\lambda)$  and zero in the other components, so  $\{\varepsilon(\lambda) \mid \lambda \in \text{Irr}(\mathcal{A})\}$  is the (unique) partition of unity of the center  $Z(\mathcal{A})$ .

The main objective of this paper is to study the situation where  $\mathcal{A} = \mathcal{A}_n$  fits into an infinite family of algebras satisfying the following properties.

**Definition 1.1.** A family  $\{\mathcal{A}_n \mid n \geq 0\}$  of finite dimensional split semisimple algebras over a field  $\mathbb{k}$  is a *multiplicity-free family of algebras* if the following axioms hold:

- (a) (Triviality)  $\mathcal{A}_0 \cong \mathbb{k}$ .
- (b) (Embedding) For each  $n$ , there is a unity preserving algebra embedding  $\mathcal{A}_n \hookrightarrow \mathcal{A}_{n+1}$ .
- (c) (Branching) The restriction to  $\mathcal{A}_{n-1}$  of an irreducible  $\mathcal{A}_n$ -module  $V$  is isomorphic to a direct sum of pairwise non-isomorphic irreducible  $\mathcal{A}_{n-1}$ -modules.

Whenever (c) above holds, we say that restriction from  $\mathcal{A}_n$  to  $\mathcal{A}_{n-1}$  is multiplicity-free. The following general criterion characterizes this property.

**Proposition 1.2** ([VO, Prop. 1.4]). *Restriction from  $\mathcal{A}_n$  to  $\mathcal{A}_{n-1}$  is multiplicity-free if and only if the centralizer algebra*

$$Z(\mathcal{A}_{n-1}, \mathcal{A}_n) = \{x \in \mathcal{A}_n \mid xy = yx, \text{ for all } y \in \mathcal{A}_{n-1}\}$$

*is commutative.*

---

<sup>1</sup> We could just as well work with right modules, and will do so in Sections 4, 5.

To ease notation, whenever we have a multiplicity-free family we write  $\text{Irr}(n)$  short for  $\text{Irr}(\mathcal{A}_n)$ . Extending [OV, VO], we define the *branching graph*  $\mathbf{B}$  (or *Bratteli diagram*) of the given family to be the directed graph with vertices and edges as follows:

- the vertices are the isomorphism classes  $\bigsqcup_{n \geq 0} \text{Irr}(n)$ ;
- there is an edge  $\mu \rightarrow \lambda$  from the vertex  $\mu$  to the vertex  $\lambda$  if and only if  $V^\mu$  is isomorphic to a direct summand of the restriction of  $V^\lambda$ .

Given  $\lambda \in \text{Irr}(n)$ , let  $\text{Tab}(\lambda)$  denote the set of *paths* in the branching graph starting from the unique element  $\emptyset \in \text{Irr}(0)$  and terminating at  $\lambda$ .<sup>2</sup> Concretely, an element of  $\text{Tab}(\lambda)$  has the form

$$\mathsf{T} = (\lambda_0 \rightarrow \lambda_1 \rightarrow \lambda_2 \rightarrow \cdots \rightarrow \lambda_{n-1} \rightarrow \lambda_n),$$

where  $\lambda_0 = \emptyset$  and  $\lambda_n = \lambda$ . Set  $\text{Tab}(n) = \bigsqcup_{\lambda \in \text{Irr}(n)} \text{Tab}(\lambda)$ . We say that  $\mathsf{T} \in \text{Tab}(n)$  is a path of *length*  $n$  (a path on  $n + 1$  vertices). We sometimes write  $\mathsf{T} \mapsto \lambda$  to indicate that  $\mathsf{T} \in \text{Tab}(\lambda)$ . We also write  $\bar{\mathsf{T}}$  for the path in  $\text{Tab}(n-1)$  obtained from  $\mathsf{T}$  by deleting its last edge,  $\lambda_{n-1} \rightarrow \lambda_n$ .

We now describe how to use branching to produce bases of irreducible modules. Let  $V$  be a given irreducible  $\mathcal{A}_n$ -module. By the branching rule 1.1(c) and Schur's Lemma, the decomposition

$$(1.2) \quad \text{res}_{\mathcal{A}_{n-1}} V = \bigoplus_{[W] \rightarrow [V]} W$$

is canonical. Decomposing each  $W$  on the right hand side upon restriction to  $\mathcal{A}_{n-2}$  and continuing inductively all the way down to  $\mathcal{A}_0 \cong \mathbb{k}$ , we obtain a canonical decomposition

$$(1.3) \quad \text{res}_{\mathcal{A}_0} V = \bigoplus_{\mathsf{T}} V_{\mathsf{T}}$$

into irreducible  $\mathcal{A}_0$ -modules, which are the 1-dimensional subspaces  $V_{\mathsf{T}}$ , where the index  $\mathsf{T}$  runs over the set of  $\mathsf{T} \in \text{Tab}(n)$  terminating in  $[V]$ . Note that the  $\mathcal{A}_k$ -submodule of  $V$  generated by  $V_{\mathsf{T}}$  is isomorphic to  $V^{\lambda_k} = \varepsilon(\lambda_k) \cdots \varepsilon(\lambda_n) V$ , where  $\lambda_k$  is the  $k$ th vertex in the path  $\mathsf{T}$ , for each  $k = 0, 1, \dots, n-1, n$ . Choosing a nonzero vector  $v_{\mathsf{T}} \in V_{\mathsf{T}}$  for each  $\mathsf{T}$  in  $\text{Tab}(n)$ , we get a basis

$$\{v_{\mathsf{T}} \mid \mathsf{T} \mapsto [V]\}$$

of each  $V$ , called the *Gelfand–Tsetlin basis*; this idea goes back to [GT2, GT1]. We note that the choice of  $v_{\mathsf{T}}$  is uniquely determined only up to a scalar multiple.

<sup>2</sup> The set  $\text{Tab}(\lambda)$  is analogous to the set of standard tableaux of shape  $\lambda$  in the representation theory of symmetric groups.

In what follows, an important role is played by the *Gelfand–Tsetlin* subalgebra  $\mathcal{X}_n$  ( $n \geq 1$ ). Following [VO], this is the subalgebra of  $\mathcal{A}_n$  generated by the centers

$$Z(\mathcal{A}_1), Z(\mathcal{A}_2), \dots, Z(\mathcal{A}_n).$$

It is easy to see that  $\mathcal{X}_n$  is a commutative subalgebra of  $\mathcal{A}_n$ , for all  $n$ . Clearly  $\mathcal{X}_n \subseteq \mathcal{X}_{n+1}$ , for all  $n$ .

**Definition 1.3.** To each path  $\top : \emptyset = \lambda_0 \rightarrow \lambda_1 \rightarrow \dots \rightarrow \lambda_n$  of length  $n$  in the branching graph, we associate a unique element  $\varepsilon_\top := \varepsilon(\lambda_1)\varepsilon(\lambda_2)\cdots\varepsilon(\lambda_n)$  of the Gelfand–Tsetlin subalgebra  $\mathcal{X}_n$ .

**Remark 1.4.** Equivalently,  $\varepsilon_\top$  can be defined recursively by:

$$\varepsilon_\top = \begin{cases} \varepsilon_{\bar{\top}} \cdot \varepsilon(\lambda_n) & \text{if } n > 0, \\ 1 & \text{if } n = 0 \end{cases}$$

in terms of the notation  $\bar{\top}$  introduced above.

Given an irreducible module  $V \cong V^\lambda$  for  $\mathcal{A}_n$  and any  $\top \mapsto \lambda$ , the element  $\varepsilon_\top \in \mathcal{A}_n$  is the projection mapping  $V$  onto  $V_\top$ . In [VO, Prop. 1.1], Vershik and Okounkov use these canonical projections to prove the following result.

**Proposition 1.5.** *The Gelfand–Tsetlin algebra  $\mathcal{X}_n$  is the algebra of all elements of  $\mathcal{A}_n$  that act diagonally on the Gelfand–Tsetlin basis  $\{v_\top\}$  for each irreducible  $\mathcal{A}_n$ -module  $V$ . In particular, the algebra  $\mathcal{X}_n$  is a maximal commutative subalgebra of  $\mathcal{A}_n$ .*

*Proof.* Suppose that  $\top \mapsto \lambda \in \text{Irr}(n)$ . Since  $\varepsilon_\top$  projects  $V \cong V^\lambda$  onto its one-dimensional subspace  $V_\top$ , it follows that  $\varepsilon_\top$  sends  $v_\top$  to itself. Also,  $\varepsilon_\top$  acts as zero on all  $v_s$  such that  $s \neq \top$ . So with respect to the Gelfand–Tsetlin basis  $\{v_\top\}$  for  $V$ , the operators  $\varepsilon_\top$  are diagonal matrices. In view of (1.1), the algebra generated by  $\{\varepsilon_\top \mid \top \in \text{Tab}(n)\}$  is a maximal commutative subalgebra of  $\mathcal{A}_n$ . Since  $\mathcal{X}_n$  is commutative and contains this subalgebra, we have equality, which completes the proof.  $\square$

The following result did not explicitly appear in [VO], although it is implicit in their setup. It provides an explicit and canonical partition of unity in  $\mathcal{A}_n$  for each  $n$ , in terms of the primitive central idempotents.

**Proposition 1.6.** *The set  $\{\varepsilon_\top \mid \top \in \text{Tab}(n)\}$  is a family of pairwise orthogonal primitive idempotents in  $\mathcal{A}_n$  that sums to 1 (the unit in  $\mathcal{A}_n$ ). It is also a  $\mathbb{k}$ -basis for the Gelfand–Tsetlin subalgebra  $\mathcal{X}_n$ .*

*Proof.* It is clear from Definition 1.3 that  $\varepsilon_T = \varepsilon(\lambda_1) \cdots \varepsilon(\lambda_n)$  is idempotent for any  $T$ , since its factors commute. The commutativity of the factors is also used to check that  $\varepsilon_T \varepsilon_{T'} = 0$  if either

$$T \mapsto \lambda \text{ and } T' \mapsto \lambda' \text{ with } \lambda \neq \lambda'$$

or

$$T \mapsto \lambda \text{ and } T' \mapsto \lambda \text{ with } T \neq T'.$$

So the idempotents are pairwise orthogonal.

For any  $T \mapsto \lambda$ ,  $\varepsilon_T$  acts as one on  $V_T^\lambda$  and zero on all  $V_S^\lambda$ , for  $S \neq T$ . Since  $V^\lambda = \bigoplus_{T \mapsto \lambda} V_T^\lambda$ , it follows that  $\sum_{T \mapsto \lambda} \varepsilon_T$  and  $\varepsilon(\lambda)$  both act as one on  $V^\lambda$ . Furthermore, both act as zero on  $V^\mu$ , for each  $\lambda \neq \mu \in \text{Irr}(n)$ . This shows that  $\sum_{T \mapsto \lambda} \varepsilon_T = \varepsilon(\lambda)$ . It follows that  $\sum_{T \in \text{Tab}(n)} \varepsilon_T = \sum_{\lambda \in \text{Irr}(n)} \varepsilon(\lambda) = 1$ .

Finally, the various  $\varepsilon_T$  are primitive since we have precisely the right number, namely  $\sum_{\lambda \in \text{Irr}(n)} \dim_{\mathbb{k}} V^\lambda = |\text{Tab}(n)|$ .

The last claim in the proposition follows from the proof of Proposition 1.5, since the  $\varepsilon_T$  are linearly independent and  $\dim_{\mathbb{k}} \mathcal{X}_n = \sum_{\lambda \in \text{Irr}(n)} \dim_{\mathbb{k}} V^\lambda$ .  $\square$

**Corollary 1.7.** *The canonical idempotents  $\{\varepsilon_T \mid T \in \text{Tab}(n)\}$  satisfy the following properties:*

- (1)  $\sum_{T \mapsto \lambda} \varepsilon_T = \varepsilon(\lambda)$ , for all  $\lambda \in \text{Irr}(n)$ .
- (2)  $\varepsilon_{\bar{T}} \varepsilon_T = \varepsilon_T$ , for all  $T \mapsto \lambda$ ,  $\lambda \in \text{Irr}(n)$ .

Furthermore,  $\{\varepsilon_T \mid T \in \text{Tab}(n), n \geq 0\}$  is the unique set of pairwise orthogonal idempotents satisfying these two properties.

*Proof.* Property (a) was proved already in the proof of the previous proposition. Property (b) follows immediately from the definition of  $\varepsilon_T$  and the definition of  $\bar{T}$ .

Suppose that  $\{g_T \mid T \in \text{Tab}(n), n \geq 0\}$  is another set such that for each fixed  $n$ , the set  $\{g_T \mid T \in \text{Tab}(n)\}$  is a set of pairwise orthogonal idempotents in  $\mathcal{A}_n$  satisfying properties (a) and (b). For the unique path  $\emptyset$  of length 0, we have  $g_\emptyset = \varepsilon_\emptyset = 1$ . Proceeding by induction on  $n$ , suppose that  $n > 0$  is fixed and assume that  $g_S = \varepsilon_S$  for all paths  $S$  of length strictly less than  $n$ . Then for  $T \in \text{Tab}(n)$  with  $T \mapsto \lambda$ , we have

$$\begin{aligned} \varepsilon_T &= \varepsilon_{\bar{T}} \varepsilon(\lambda) = g_{\bar{T}} \sum_{S \mapsto \lambda} g_S = g_{\bar{T}} \sum_{S \mapsto \lambda} g_{\bar{S}} g_S \\ &= \sum_{S \mapsto \lambda} g_{\bar{T}} g_{\bar{S}} g_S = \sum_{S \mapsto \lambda} \delta_{\bar{T}, \bar{S}} g_{\bar{S}} g_S = g_{\bar{T}} g_T = g_T. \end{aligned}$$

Note that the penultimate equality above is valid because  $T$  is the only path of shape  $\lambda$  whose restriction of length  $n-1$  is  $\bar{T}$ .  $\square$

It is illuminating to introduce a *global Gelfand–Tsetlin basis* for  $\mathcal{A}_n$  at this point.

Fix a Gelfand–Tsetlin basis  $\{v_\top \mid \top \mapsto \lambda\}$  for each irreducible  $V^\lambda$ ,  $\lambda \in \text{Irr}(n)$ . We may identify the algebra  $\text{End}_{\mathbb{k}}(V^\lambda)$  with the matrix algebra  $\text{Mat}_{\dim V^\lambda}(\mathbb{k})$  by means of the basis. Let  $\varphi_{S,\top}^\lambda$  be the  $\mathbb{k}$ -linear endomorphism of  $V^\lambda$  mapping  $v_\top$  to  $v_S$  and all other  $v_{\top'}$  to 0. The set

$$\{\varphi_{S,\top}^\lambda \mid S, \top \mapsto \lambda\}$$

is a basis of  $\text{End}_{\mathbb{k}}(V^\lambda)$ ; under the identification  $\text{End}_{\mathbb{k}}(V^\lambda) \cong \text{Mat}_{\dim V^\lambda}(\mathbb{k})$ , it corresponds to the basis of matrix units. The desired global Gelfand–Tsetlin basis of  $\mathcal{A}_n$  under the isomorphism (1.1) is the disjoint union

$$(1.4) \quad \bigsqcup_{\lambda \in \text{Irr}(n)} \{\varphi_{S,\top}^\lambda \mid S, \top \mapsto \lambda\}.$$

This basis is uniquely determined by the choice of Gelfand–Tsetlin basis  $\{v_\top\}$  for each  $V^\lambda \in \text{Irr}(n)$ , but it depends on those choices. Note that  $\varphi_{S,\top}^\lambda \cdot \varphi_{S',\top'}^\mu = 0$  for  $\lambda \neq \mu$ ; this follows from the equality  $\text{Hom}_{\mathcal{A}_n}(V^\lambda, V^\mu) = 0$ , which is true by Schur’s Lemma. Hence the basis (1.4) satisfies

$$(1.5) \quad \varphi_{S,\top}^\lambda \cdot \varphi_{S',\top'}^\mu = \delta_{\lambda,\mu} \delta_{\top,S'} \varphi_{S,\top}^\lambda$$

where  $\delta$  is the usual Kronecker delta. In particular, each  $\varphi_{\top,\top}^\lambda$  is an idempotent. We note that (1.5) implies that the basis (1.4) is a cellular basis in the sense of [GL].

The above allows us to model the algebra  $\mathcal{A}_n$  isomorphically as the matrix algebra consisting of all  $N \times N$  block diagonal matrices, where  $N = \sum_{\lambda \in \text{Irr}(n)} \dim_{\mathbb{k}} V^\lambda$ , such that the block indexed by each  $\lambda$  is a full matrix algebra of  $d \times d$  matrices over  $\mathbb{k}$ , where  $d = \dim V^\lambda$ . Of course, since the Gelfand–Tsetlin bases of the irreducible representations are unique only up to choice of scalars, this model depends on those choices. However, being products of the unique central idempotents, the  $\varepsilon_\top$  themselves are independent of the choices.

**Corollary 1.8.** *Under the identification  $\mathcal{A}_n \cong \bigoplus_{\lambda \in \text{Irr}(n)} \text{End}_{\mathbb{k}}(V^\lambda)$  of (1.1), the primitive central idempotent  $\varepsilon(\lambda)$  corresponding to any  $\lambda \in \text{Irr}(n)$  satisfies the identity*

$$\varepsilon(\lambda) = \sum_{\top \mapsto \lambda} \varphi_{\top,\top}^\lambda.$$

*Likewise, for any path  $\top \mapsto \lambda$  in  $\mathbf{B}$  we have the identity*

$$\varepsilon_\top = \varphi_{\top,\top}^\lambda.$$

*Proof.* To prove that  $\varepsilon(\lambda) = \sum_{\mathbb{T} \rightarrow \lambda} \varphi_{\mathbb{T}, \mathbb{T}}^\lambda$  observe that both sides act as one on  $V^\lambda$  and as zero on all other irreducibles  $W \not\cong V^\lambda$ . Similarly, the equality  $\varepsilon_{\mathbb{T}} = \varphi_{\mathbb{T}, \mathbb{T}}^\lambda$  follows from the fact that both sides act the same on all  $V^\mu$  ( $\mu \in \text{Irr}(n)$ ).  $\square$

- Remarks 1.9.** (a) Let  $\mathcal{X}_n$  be the maximal commutative subalgebra of  $\mathcal{A}_n$  defined above. By Proposition 1.6, it is spanned by the idempotents  $\varepsilon_{\mathbb{T}}$ . Then it is clear from (1.5) and Corollary 1.8 that the global Gelfand–Tsetlin basis  $\{\varphi_{\mathbb{S}, \mathbb{T}}^\lambda\}$  is a basis consisting of simultaneous (left or right) eigenvectors for the action of  $\mathcal{X}_n$  by left or right multiplication. To be explicit: an arbitrary element  $\sum_{\mathbb{U}} c_{\mathbb{U}} \varepsilon_{\mathbb{U}}$  of  $\mathcal{X}_n$  acts on  $\varphi_{\mathbb{S}, \mathbb{T}}^\lambda$  by left multiplication as the scalar  $c_{\mathbb{S}}$  and by right multiplication as the scalar  $c_{\mathbb{T}}$ .
- (b) Similarly, as already noted in Proposition 1.5, the basis  $\{v_{\mathbb{T}} : \mathbb{T} \mapsto \lambda\}$  of  $V^\lambda$  is a basis of simultaneous eigenvectors for the action of  $\mathcal{X}_n$ . To be explicit, the element  $\sum_{\mathbb{T}} c_{\mathbb{T}} \varepsilon_{\mathbb{T}}$  as above acts as  $c_{\mathbb{T}}$  on the basis element  $v_{\mathbb{T}}$ , for each  $\mathbb{T}$ .
- (c) The decomposition  $\mathcal{A}_n = \bigoplus_{\mathbb{T} \in \text{Irr}(n)} \mathcal{A}_n \varepsilon_{\mathbb{T}}$ , which is a decomposition of  $\mathcal{A}_n$  into a direct sum of irreducible left ideals, is actually a “weight space” decomposition for the action of  $\mathcal{X}_n$  by right multiplication, in the sense that each element of  $\mathcal{A}_n \varepsilon_{\mathbb{T}}$  is an eigenvector for the right action of an arbitrary element  $\sum_{\mathbb{U} \in \text{Irr}(n)} c_{\mathbb{U}} \varepsilon_{\mathbb{U}}$  of  $\mathcal{X}_n$ , of eigenvalue  $c_{\mathbb{T}}$ . A similar remark, with left and right interchanged, holds for the decomposition  $\mathcal{A}_n = \bigoplus_{\mathbb{T} \in \text{Irr}(n)} \varepsilon_{\mathbb{T}} \mathcal{A}_n$ .

Thus, we see that in some sense the role of the Gelfand–Tsetlin algebra  $\mathcal{X}_n$  in the theory of multiplicity-free families is analogous to that of a Cartan subalgebra in the theory of Lie algebras.

## 2. Central idempotents via interpolation

The primitive central idempotents can be computed by a type of Lagrange interpolation, provided that a generator of the center is available. This applies to an arbitrary split semisimple finite dimensional algebra  $\mathcal{A}$ , so we temporarily drop the assumption that the algebra fits into a multiplicity-free family.

Note that  $\{\varepsilon(\lambda) \mid \lambda \in \text{Irr}(\mathcal{A})\}$  is a basis for the center  $Z(\mathcal{A})$ . So any element  $z \in Z(\mathcal{A})$  is uniquely expressible in the form

$$z = \sum_{\lambda \in \text{Irr}(\mathcal{A})} a_\lambda \varepsilon(\lambda).$$

It follows that  $z \cdot \varepsilon(\lambda) = a_\lambda \varepsilon(\lambda)$  for all  $\lambda \in \text{Irr}(\mathcal{A})$ . Call the tuple  $(a_\lambda)_{\lambda \in \text{Irr}(\mathcal{A})}$  the (eigen)spectrum of  $z$ . A spectrum is *simple* if it has no repeated entries.



- Lemma 2.1.** (a) *An element  $z \in Z(\mathcal{A})$  generates  $Z(\mathcal{A})$  if and only if its spectrum is simple.*
- (b) *If  $\mathbb{k}$  has at least as many elements as  $|\text{Irr}(\mathcal{A})|$ , the center  $Z(\mathcal{A})$  is generated (as an algebra) by a single element.*

*Proof.* (a) Regarded as a linear operator on  $Z(\mathcal{A})$  by multiplication, the element  $z$  is diagonal with respect to the basis  $\{\varepsilon(\lambda) \mid \lambda \in \text{Irr}(\mathcal{A})\}$ . Let  $S = \{a_\lambda \mid \lambda \in \text{Irr}(\mathcal{A})\}$  be the set of *distinct* eigenvalues of  $z$ . The minimal polynomial of  $z$  is  $\prod_{a \in S} (z - a)$ . Let  $m = |\text{Irr}(\mathcal{A})| = \dim_{\mathbb{k}} Z(\mathcal{A})$ . Clearly, the element  $z$  generates  $Z(\mathcal{A})$  if and only if the set  $\{1, z, \dots, z^{m-1}\}$  is linearly independent. This is true if and only if the minimal polynomial of  $z$  has degree  $m$ . So  $z$  generates  $Z(\mathcal{A})$  if and only if it has simple spectrum.

(b) Choose  $m$  distinct elements of  $\mathbb{k}$ , say  $a_1, \dots, a_m$ . Choose any enumeration  $\lambda_1, \dots, \lambda_m$  of the elements of  $\text{Irr}(\mathcal{A})$ . Then  $z = a_1 \varepsilon(\lambda_1) + \dots + a_m \varepsilon(\lambda_m)$  has simple spectrum, hence generates  $Z(\mathcal{A})$ . □

- Remarks 2.2.** (a) If  $z = \sum_{\lambda} a_{\lambda} \varepsilon(\lambda)$  is a generator of  $Z(\mathcal{A})$  then the change of basis matrix expressing the powers  $1, z, \dots, z^{m-1}$  in terms of the idempotents  $\varepsilon(\lambda)$  is a Vandermonde matrix in the  $a_{\lambda}$ 's.
- (b) If the field  $\mathbb{k}$  is large compared to  $m = |\text{Irr}(\mathcal{A})|$ , there are many generators of  $Z(\mathcal{A})$ . In fact, if  $\mathbb{k}$  is a finite field of  $q$  elements, then the probability  $P(q)$  that a randomly chosen element of  $Z(\mathcal{A})$  actually generates the center is

$$P(q) = \frac{q(q-1) \cdots (q-m+1)}{q^m} = \left(1 - \frac{1}{q}\right) \cdots \left(1 - \frac{m-1}{q}\right),$$

Evidently,  $\lim_{q \rightarrow \infty} P(q) = 1$ .

The lemma leads immediately to an interpolation formula for the  $\varepsilon(\lambda)$ , provided that one can find a generator and compute its spectrum.

**Proposition 2.3.** *Suppose that  $z$  is a generator of  $Z(\mathcal{A})$ , with spectrum  $(a_{\lambda} \mid \lambda \in \text{Irr}(\mathcal{A}))$ . Then the polynomial*

$$Q_{\lambda}(z) = \prod_{\mu \in \text{Irr}(\mathcal{A}): \mu \neq \lambda} \frac{z - a_{\mu}}{a_{\lambda} - a_{\mu}}$$

*is equal to  $\varepsilon(\lambda)$ , for each  $\lambda \in \text{Irr}(\mathcal{A})$ .*

*Proof.* This is immediate from the fact that  $\prod_{\mu \in \text{Irr}(\mathcal{A})} (z - a_{\mu}) = 0$ , which implies that

$$Q_\lambda(z) \cdot \varepsilon(\mu) = \delta_{\lambda,\mu} \varepsilon(\lambda).$$

Hence  $Q_\lambda(z) = Q_\lambda(z) \cdot 1 = \sum_{\mu \in \text{Irr}(\mathcal{A})} Q_\lambda(z) \cdot \varepsilon(\mu) = \varepsilon(\lambda)$ .  $\square$

The formula in Proposition 2.3 is useful only if we have a way of retrieving  $z$ 's spectrum without already knowing the central idempotents. At least in characteristic zero, this can be done whenever the irreducible trace characters are known.

**Proposition 2.4** ([Ram4, Lemma 1.9]). *Let  $\chi^\lambda$  be the trace character of  $V^\lambda$  for any  $\lambda \in \text{Irr}(\mathcal{A})$ . Suppose that  $\mathbb{k}$  has characteristic zero. Writing  $z = \sum_\lambda a_\lambda \varepsilon(\lambda)$ , we have  $a_\lambda = \frac{\chi^\lambda(z)}{\chi^\lambda(1)}$ .*

*Proof.* For any  $v \in V^\lambda$ ,  $z$  acts as  $a_\lambda$ ; i.e.,  $z \cdot v = a(\lambda) v$ , so  $\text{trace}(z)$  on  $V^\lambda$  is equal to  $(\dim_{\mathbb{k}} V^\lambda) a_\lambda$ . In other words,  $\chi^\lambda(z) = \chi^\lambda(1) a_\lambda$ .  $\square$

The above analysis leads to a probabilistic algorithm for computing the primitive central idempotents.

**Algorithm 2.5.** Suppose that  $\mathbb{k}$  has characteristic zero. Then to compute all the central idempotents  $\varepsilon(\lambda)$ ,

- (a) Pick a random  $z \in Z(\mathcal{A})$  and compute its spectrum (using Proposition 2.4 or otherwise). If the spectrum is not simple, try again.
- (b) Once a generator  $z$  with simple spectrum is found, use Proposition 2.3 to compute the  $\varepsilon(\lambda)$  for all  $\lambda \in \text{Irr}(\mathcal{A})$ .

If this can be carried out, the formulas thus obtained will express the  $\varepsilon(\lambda)$  in terms of polynomial expressions in some random central element. One would usually prefer to have expressions for the  $\varepsilon(\lambda)$  in terms of elements that are understood in some explicit way. At the least, one would prefer to understand how the chosen central generator  $z$  interacts with some set of standard generators for the algebra  $\mathcal{A}$ .

**Example 2.6.** Let  $\mathcal{A} = \mathcal{H}_q(n)$  be the Iwahori–Hecke algebra corresponding to the symmetric group  $\mathfrak{S}_n$ , over a field  $\mathbb{k}$  such that  $0 \neq q \in \mathbb{k}$ , and assume that  $\mathcal{H}_q(n)$  is split semisimple. In [DJ], certain  $q$ -analogues of the original Jucys–Murphy elements in  $\mathbb{k}\mathfrak{S}_n$  were constructed in  $\mathcal{H}_q(n)$ . As pointed out in [OP, §8.1], their sum  $Z_n$  has simple spectrum, hence is a generator of the center  $Z(\mathcal{H}_q(n))$ . Thus the formula in Proposition 2.3 computes the primitive central idempotents  $\varepsilon(\lambda) \in \mathcal{H}_q(n)$  for each  $\lambda \vdash n$ .

**Example 2.7.** Let  $\mathcal{A} = \mathbb{k}\mathfrak{S}_n$  be the group algebra of a symmetric group over a field  $\mathbb{k}$  of characteristic zero. Let  $z_n$  be the formal sum of all the transpositions in  $\mathfrak{S}_n$ , regarded as an element of  $\mathbb{k}\mathfrak{S}_n$ . This is precisely the element obtained from the element  $Z_n$  in the previous example, if  $q$  is specialized to 1. The central element  $z_n$  generates  $Z(\mathbb{k}\mathfrak{S}_n)$  for  $n = 2, 3, 4$ , and 5, but for  $n = 6$  it fails to do so. The eigenvalues of  $z_6$  on the irreducible modules indexed by partitions  $(4, 1^2)$  and  $(3, 3)$  coincide. Likewise for  $(3, 1^3)$  and  $(2^3)$ . At this writing, we do not know of any satisfactory uniform choice of elements  $z_n \in \mathbb{k}\mathfrak{S}_n$  generating the respective centers. This seems to be an interesting open problem.

To conclude this section, we mention an alternative approach to computing the primitive central idempotents for  $\mathcal{A}$ . Recall that if  $\mathcal{A} = \mathbb{C}G$  for a finite group  $G$ , Frobenius gave a formula for  $\varepsilon(\lambda)$  in terms of the simple character  $\chi_G^\lambda$ . This result was extended to split semisimple finite dimensional algebras in characteristic zero by Kilmoyer; however, it involves inverting a  $(\dim \mathcal{A}) \times (\dim \mathcal{A})$  matrix. See Appendix A for a brief exposition.

### 3. Generalized Jucys–Murphy sequences

Now we return to the study of multiplicity-free families  $\{\mathcal{A}_n\}_{n \geq 0}$  and the problem of computing the canonical idempotents  $\{\varepsilon_T \mid T \in \text{Tab}(n)\}$ , which form a basis of the (commutative) Gelfand–Tsetlin subalgebra  $\mathcal{X}_n$  ( $n \geq 0$ ). Extracting key elements from the work of Jucys and Murphy, we show how a carefully selected sequence of elements (one from each  $\mathcal{X}_n$ ) can be used to effectively solve this problem.

Before we begin, we apply the results of the previous section to this end. Given a sequence  $(z_n \in \mathcal{X}_n \mid n \geq 1)$  of center-generating elements, i.e., elements satisfying  $\langle z_n \rangle = Z(\mathcal{A}_n)$ , we reach the  $\varepsilon_T$  in two steps:

- (i) use Propositions 2.3 and 2.4 to compute the  $\varepsilon(\lambda)$ ;
- (ii) use Definition 1.3 to compute the  $\varepsilon_T$ .

So the problem is solved, provided one can find a sequence of center-generating elements. However, as noted in Example 2.7, even for the family of group algebras of symmetric groups, such a sequence is not known.

Murphy [Mur] found a non center-generating sequence of elements – known independently to Young [You] and Jucys [Juc] – and applied them to give a new construction of Young’s seminormal form of symmetric group algebras. In recent years, analogues of such elements have been found in a number of other multiplicity-free families. The two key properties of the Young–Jucys–Murphys elements are abstracted in the next definition. But first, some notation.

From the definition of  $\mathcal{X}_n$  we have a sequence of inclusions

$$(3.1) \quad \mathcal{X}_1 \subset \cdots \subset \mathcal{X}_{n-1} \subset \mathcal{X}_n.$$

Now given  $J_1 \in \mathcal{X}_1$ ,  $J_2 \in \mathcal{X}_2, \dots, J_n \in \mathcal{X}_n$ , the inclusions (3.1) imply that  $J_1, J_2, \dots, J_n \in \mathcal{X}_n$ . Since  $\{\varepsilon_\top\}$  is a basis of  $\mathcal{X}_n$ , we have scalars  $c_\top(k) \in \mathbb{k}$ , for each  $k = 1, \dots, n$ , such that

$$(3.2) \quad J_k = \sum_{\top \in \text{Tab}(n)} c_\top(k) \varepsilon_\top.$$

In this way we associate an  $n$ -tuple  $c_\top$  to each  $\top \in \text{Tab}(n)$ ,

$$(3.3) \quad c_\top = (c_\top(1), \dots, c_\top(n)) \in \mathbb{k}^n.$$

We call this  $n$ -tuple the  $\top$ -content for the sequence  $J_1, \dots, J_n$ .

**Definition 3.1.** Let  $(J_n \mid n \in \mathbb{N})$  be a sequence of elements such that  $J_n \in \mathcal{X}_n$  for each  $n$ . We say that the sequence is:

- (a) *additively central*<sup>3</sup> if the  $n$ th partial sum  $J_1 + \cdots + J_{n-1} + J_n$  belongs to  $Z(\mathcal{A}_n)$ , for all  $n \in \mathbb{N}$ ; and
- (b) *separating* if  $\mathcal{X}_n = \langle J_1, J_2, \dots, J_n \rangle$ , for all  $n \in \mathbb{N}$ .

The sequence is a *Jucys–Murphy sequence* (JM-sequence for short) if it is both additively central and separating.

To explain our terminology, we mention that additively central sequences allow for ease of computation of the content vectors (Proposition 3.3), while separating sequences allow content vectors to distinguish different paths  $S, \top \in \text{Tab}(n)$  (Proposition 3.5).

**Proposition 3.2.** *JM-sequences in multiplicity free families always exist, provided that the underlying field is infinite.*

*Proof.* Let  $(z_n)_{n \in \mathbb{N}}$  be a center-generating sequence, which exists by Lemma 2.1(b). Then putting  $J_n := z_n - z_{n-1}$  (and stipulating that  $z_0 = 0$ ), it is easy to check that  $(J_n)_{n \in \mathbb{N}}$  is a sequence that is additively central. Assuming inductively that  $\mathcal{X}_{n-1} = \langle J_1, \dots, J_{n-1} \rangle$ , we have

$$\mathcal{X}_n = \langle \mathcal{X}_{n-1}, Z(\mathcal{A}_n) \rangle = \langle \mathcal{X}_{n-1}, z_n \rangle = \langle \mathcal{X}_{n-1}, J_n \rangle = \langle J_1, \dots, J_n \rangle.$$

This shows that  $(J_n)_{n \in \mathbb{N}}$  is also separating. □

<sup>3</sup> In some multiplicity-free families, one can find *multiplicatively central* sequences. These sequences, which were considered in [GG2], have the property that the partial product  $J_1 J_2 \cdots J_n$  belongs to  $Z(\mathcal{A}_n)$ , for all  $n \in \mathbb{N}$ , and furthermore that it acts as a nonzero scalar on each  $V^\lambda$ ,  $\lambda \in \text{Irr}(n)$ . The results in this section are equally valid in the multiplicative case, modulo a few adjustments that we leave to the reader.

We next investigate the independent notions of additively central and separating sequences before returning to JM-sequences for our main result (Theorem 3.11).

Let  $(J_k)_{k \in \mathbb{N}}$  be an additively central sequence. As  $z_n = \sum_{k=1}^n J_k \in Z(\mathcal{A}_n)$ , it acts as some scalar  $a_\lambda$  on any irreducible representation  $V^\lambda$ , for  $\lambda \in \text{Irr}(n)$ . Similarly,  $z_{n-1} = \sum_{k=1}^{n-1} J_k \in \mathcal{A}_{n-1}$ , so  $z_{n-1}$  acts as a scalar  $a_\mu$  on any  $V^\mu$ , for  $\mu \in \text{Irr}(n-1)$ . The next proposition shows how these scalars determine the  $n$ th eigenvalue  $c_\tau(n)$ . (Recall from Section 1 the construction of  $\bar{\tau} \in \text{Tab}(n-1)$  for any  $\tau \in \text{Tab}(n)$ .)

**Proposition 3.3.** *Let  $(J_k \mid k \in \mathbb{N})$  be an additively central sequence of elements in a given multiplicity-free family  $\{\mathcal{A}_n \mid n \geq 0\}$ . For any  $n$ , let  $\lambda \in \text{Irr}(n)$ . For any  $\tau \mapsto \lambda$  we have  $c_\tau(n) = a_\lambda - a_\mu$ , where  $\bar{\tau} \mapsto \mu$ .*

*Proof.* By hypothesis, we have  $J_n = z_n - z_{n-1}$  for all  $n \in \mathbb{N}$  (where we set  $z_0 = 0$ ). Then  $z_n$  acts by right multiplication as  $a_\lambda$  on  $v_\tau$ . By hypothesis we have  $v_\tau = v_{\bar{\tau}}$  since  $\tau$  has the form

$$\tau = (\lambda_0 \rightarrow \lambda_1 \rightarrow \cdots \rightarrow \lambda_{n-1} \rightarrow \lambda_n),$$

where  $\lambda_n = \lambda$  and  $\lambda_{n-1} = \mu$ . So the element  $z_{n-1}$  acts as the scalar  $a_\mu$  on  $v_\tau$ , and thus by linearity  $J_n = z_n - z_{n-1}$  acts on  $v_\tau$  as the scalar  $a_\lambda - a_\mu$ . The result is proved.  $\square$

**Remark 3.4.** Note that Proposition 3.3 says that the eigenvalue  $c_\tau(n)$  depends only on the last edge  $\lambda_{n-1} \rightarrow \lambda_n$  of the path  $\tau$  in the branching graph. So, whenever we have an additively central sequence in our multiplicity-free family, it makes sense to label each edge in level  $n$  of the branching graph by its corresponding eigenvalue  $c_\tau(n)$ . Figure 1 gives an example of such a labeled graph.

Let  $(J_n)_{n \in \mathbb{N}}$  be any sequence with  $J_n \in \mathcal{X}_n$  for each  $n$ . In the proof of the following result, which characterizes separating sequences, we will focus on the  $\tau$ -contents one coordinate at a time. Put  $\text{Wt}(k) := \{c_\tau(k) \mid \tau \in \text{Tab}(n)\}$ . This is the set of eigenvalues of the operator  $J_k$  acting on the various  $\varepsilon_\tau$ .

**Proposition 3.5.** *Given a sequence  $J_1 \in \mathcal{X}_1, J_2 \in \mathcal{X}_2, \dots, J_n \in \mathcal{X}_n$ , and corresponding content vectors  $c_\tau$ , the following are equivalent.*

(a) For all  $S, \tau \in \text{Tab}(n)$ ,  $S = \tau \iff c_S = c_\tau$ .

(b)  $\langle J_1, \dots, J_n \rangle = \mathcal{X}_n$ .

*Proof.* (a)  $\Rightarrow$  (b). We aim to show that  $\varepsilon_T \in \langle J_1, \dots, J_n \rangle$  for each  $T \in \text{Tab}(n)$ , which would complete this half of the proof. To this end, note that the polynomial  $E_T(J) = E_T(J_1, \dots, J_n)$  defined by

$$E_T(J) = \prod_{k=1}^n \prod_{\substack{c \in \text{Wt}(k) \\ c \neq c_T(k)}} \frac{J_k - c}{c_T(k) - c}$$

is well-defined as an operator on  $\mathcal{X}_n$  (acting by multiplication). Given  $S \neq T$ ,  $E_T(J)$  acts on the basis element  $\varepsilon_S$  as

$$E_T(J) \varepsilon_S = \prod_{k=1}^n \prod_{\substack{c \in \text{Wt}(k) \\ c \neq c_T(k)}} \frac{(c_S(k) - c) \varepsilon_S}{c_T(k) - c} = 0,$$

since  $c_S(k)$  is among the  $c \in \text{Wt}(k)$  and  $c_S(k) \neq c_T(k)$  for at least one  $k$ , by (a). A similar calculation shows that  $E_T(J) \varepsilon_T = \varepsilon_T$ . Hence

$$E_T(J) = E_T(J) \cdot \sum_{S \in \text{Tab}(n)} \varepsilon_S = \sum_{S \in \text{Tab}(n)} E_T(J) \varepsilon_S = \varepsilon_T,$$

and hence  $\varepsilon_T \in \langle J_1, \dots, J_n \rangle$ , as required.

(b)  $\Rightarrow$  (a). Assume that  $S, T$  are paths such that  $c_S = c_T$ . We show that  $S = T$ . Note that for any polynomial  $F(J_1, \dots, J_n) \in \langle J_1, \dots, J_n \rangle$ ,  $F$  acts on  $v_T$  and  $v_S$  by the same scalar, namely  $F(c_T(1), \dots, c_T(n)) = F(c_S(1), \dots, c_S(n))$ . Under the hypothesis (b), i.e.,  $\langle J_1, \dots, J_n \rangle = \mathcal{X}_n$ , we know that  $\varepsilon_T$  is such a polynomial. Since

$$\varepsilon_T \cdot v_T = v_T \quad \text{and} \quad \varepsilon_T \cdot v_S = \delta_{T,S} v_S,$$

we must have  $\delta_{T,S} = 1$ . So  $S = T$ . The converse implication, that  $S = T$  implies  $c_S = c_T$ , is trivial.  $\square$

Note that if  $\mathcal{X}_n = \langle J_1, \dots, J_n \rangle$  then the above proof gives the explicit formula

$$(3.4) \quad \varepsilon_T = E_T(J) = \prod_{k=1}^n \prod_{\substack{c \in \text{Wt}(k) \\ c \neq c_T(k)}} \frac{J_k - c}{c_T(k) - c},$$

expressing the canonical idempotents  $\varepsilon_T$  in terms of the separating sequence. (A similar interpolation formula appears in [Mur] in the context of symmetric group algebras.) We find another interpolating polynomial for the  $\varepsilon_T$ , having significantly lower degree than this one, in Theorem 3.8.

**Proposition 3.6.** *Let  $(J_k \mid k \in \mathbb{N})$  be a separating sequence in the multiplicity-free family  $\{\mathcal{A}_k \mid k \geq 0\}$ . Suppose that  $S, T \in \text{Tab}(n)$ . Then*

- (a)  $c_{\top}(k) = c_{\bar{\top}}(k)$  for all  $k < n$ .  
 (b) If  $\bar{S} = \bar{\top}$  but  $S \neq \top$  then  $c_S(n) \neq c_{\top}(n)$ .

*Proof.* (a) This follows from (3.2) and its analog for  $\text{Tab}(n-1)$ , and the recursive description  $\varepsilon_{\top} = \varepsilon_{\bar{\top}} \varepsilon(\lambda)$  in Remark 1.4. Specifically, we have

$$c_{\top}(k) \varepsilon_{\top} = J_k \cdot \varepsilon_{\top} = J_k \cdot \varepsilon_{\bar{\top}} \varepsilon(\lambda) = c_{\bar{\top}}(k) \varepsilon_{\bar{\top}} \varepsilon(\lambda) = c_{\bar{\top}}(k) \varepsilon_{\top}$$

for any  $k < n$ .

(b) Since  $\bar{S} = \bar{\top}$ , it follows from part (a) that  $c_S(k) = c_{\top}(k)$  for all  $k < n$ . If  $S \neq \top$  and  $c_S(n) = c_{\top}(n)$ , then  $c_S = c_{\top}$  and we reach a contradiction with Proposition 3.5.  $\square$

Proposition 3.6 implies that the following polynomial is well-defined.

**Definition 3.7.** Let  $(J_n \mid n \in \mathbb{N})$  be a separating sequence in a multiplicity-free family. For any  $\top \in \text{Tab}(n)$ , put

$$P_{\top}(J_n) := \prod_{\substack{S \in \text{Tab}(n) \\ S \neq \top, \bar{S} = \bar{\top}}} \frac{J_n - c_S(n)}{c_{\top}(n) - c_S(n)}.$$

The next theorem shows that these polynomials can be used to recursively compute the idempotents  $\varepsilon_{\top}$ . It extends [Gar, Theorems 3.4, 3.5] from symmetric group algebras to multiplicity-free families. First, we record some basic properties of the  $P_{\top}(J_n)$ . Given  $S, \top \in \text{Tab}(n)$ , we have

$$(3.5) \quad P_{\top}(J_n) \cdot \varepsilon_{\top} = \varepsilon_{\top}$$

$$(3.6) \quad P_{\top}(J_n) \cdot \varepsilon_S = 0 \text{ if } S \neq \top \text{ but } \bar{S} = \bar{\top}.$$

The proof is an easy calculation from the definition.

Let  $\top = (\lambda_0 \rightarrow \lambda_1 \rightarrow \cdots \rightarrow \lambda_n) \in \text{Tab}(n)$ . In the next result,  $\top[k]$  denotes the subpath up to vertex  $\lambda_k$  of the path  $\top$ , so, e.g.,  $\top[n-1] = \bar{\top}$ .

**Theorem 3.8.** Assume that  $(J_n \mid n \in \mathbb{N})$  is a separating sequence. Then, for any  $\top \in \text{Tab}(n)$ ,  $\varepsilon_{\top} = \varepsilon_{\bar{\top}} P_{\top}(J_n)$ . Hence,  $\varepsilon_{\top} = \prod_{k=1}^n P_{\top[k]}(J_k)$ .

*Proof.* We prove the first equality, as the second equality follows immediately from the first by induction on  $n$ . We claim that  $\varepsilon_{\bar{\top}} P_{\top}(J_n)$  acts the same as  $\varepsilon_{\top}$  on all basis elements  $\{\varepsilon_S \mid S \in \text{Tab}(n)\}$  of  $\mathcal{X}_n$ . That is,

$$\varepsilon_{\bar{\top}} P_{\top}(J_n) \cdot \varepsilon_S = \delta_{\top, S} \varepsilon_S$$

for any  $S \in \text{Tab}(n)$ . There are three cases to the claim. First, if  $S = T$ , then by (3.5) we have

$$\varepsilon_{\bar{T}} P_T(J_n) \cdot \varepsilon_T = \varepsilon_{\bar{T}} \varepsilon_T = \varepsilon_{\bar{T}} \varepsilon_{\bar{T}} \varepsilon(\lambda) = \varepsilon_T,$$

which proves the claim in case  $S = T$ . Next, if  $S \neq T$  but  $\bar{S} = \bar{T}$  then the claim is immediate from (3.6). So only the case  $S \neq T$  and  $\bar{S} \neq \bar{T}$  remains. In this case we note that  $\mathcal{X}_{n-1} \subset \mathcal{X}_n$ , and so  $\varepsilon_{\bar{T}}$  and  $J_n$  commute. Then

$$\varepsilon_{\bar{T}} P_T(J_n) \cdot \varepsilon_S = P_T(J_n) \varepsilon_{\bar{T}} \varepsilon_{\bar{S}} \varepsilon(\mu) = 0,$$

where  $S \mapsto \mu$ . This completes the proof of the claim. The recursion formula now follows, since the claim implies that

$$\varepsilon_{\bar{T}} P_T(J_n) = \varepsilon_{\bar{T}} P_T(J_n) \cdot 1 = \sum_{S \in \text{Tab}(n)} \varepsilon_{\bar{T}} P_T(J_n) \cdot \varepsilon_S = \varepsilon_T,$$

as required.  $\square$

We now return to JM-sequences. Our main result (Theorem 3.11) is a recursive formula for the primitive central idempotents analogous to Theorem 3.8. The following lemma holds the key ingredients. If  $T \in \text{Tab}(\lambda)$  we say that  $T$  has type  $\lambda$  and write  $\text{type}(T) = \lambda$ .

**Lemma 3.9.** *Assume that  $(J_n \mid n \in \mathbb{N})$  is a JM-sequence in a multiplicity-free family  $\{\mathcal{A}_n \mid n \geq 0\}$ . Given any  $T \in \text{Tab}(n)$ , the content  $c_T(n)$  depends only on  $\text{type}(T)$  and  $\text{type}(\bar{T})$ . In particular, the polynomial  $P_T(J_n)$  in Definition 3.7 depends only on  $\text{type}(T)$ ,  $\text{type}(\bar{T})$ .*

*Proof.* The first statement follows immediately from Proposition 3.3. The second is immediate from the first and the definition of the  $P_T$ .  $\square$

Hence, the following notation is well-defined.

**Definition 3.10.** Suppose  $(J_n \mid n \in \mathbb{N})$  is a JM-sequence and  $T \in \text{Tab}(n)$ . If  $\lambda = \text{type}(T)$  and  $\mu = \text{type}(\bar{T})$  then we write  $P_\mu^\lambda(J_n) = P_T(J_n)$ .

We now arrive at the promised recursive description of the central idempotents  $\varepsilon(\lambda)$ .

**Theorem 3.11.** *Assume that  $(J_n \mid n \in \mathbb{N})$  is a JM-sequence in a multiplicity-free family. For any  $\lambda \in \text{Irr}(n)$ , we have*

$$\varepsilon(\lambda) = \sum_{\mu} P_\mu^\lambda(J_n) \cdot \varepsilon(\mu),$$

where  $\mu$  varies over the set of immediate predecessors of  $\lambda$  in the branching graph **B**.



*Proof.* We have  $\varepsilon(\lambda) = \sum_{\text{type}(\tau)=\lambda} \varepsilon_\tau$ . By Theorem 3.8 and the above lemma, we have

$$\varepsilon(\lambda) = \sum_{\tau: \text{type}(\tau)=\lambda} \varepsilon_{\bar{\tau}} P_\tau(J_n) = \sum_{\mu} \left( \sum_{\substack{\tau: \text{type}(\tau)=\lambda, \\ \text{type}(\bar{\tau})=\mu}} \varepsilon_{\bar{\tau}} \right) P_\mu^\lambda(J_n).$$

To complete the proof, it suffices to show that

$$\sum_{\substack{\tau: \text{type}(\tau)=\lambda, \\ \text{type}(\bar{\tau})=\mu}} \varepsilon_{\bar{\tau}} = \varepsilon(\mu).$$

This conclusion is justified since any path  $\bar{\tau} \in \text{Tab}(n-1)$  of type  $\mu$  extends uniquely to a path  $\tau \in \text{Tab}(n)$  of type  $\lambda$  by the branching rule 1.1(c). Thus, the sum on the left hand side above is a complete sum over all paths in  $\text{Tab}(n-1)$  of type  $\mu$ . The result follows.  $\square$

#### 4. Application: Symmetric group algebras

Let  $\mathfrak{S}_n$  be the symmetric group on  $n$  letters and  $\mathbb{k}$  a field of characteristic zero. It is well known that the family  $\{\mathbb{k}\mathfrak{S}_n \mid n \geq 0\}$  is multiplicity-free (we take  $\mathbb{k}\mathfrak{S}_0 = \mathbb{k}$ ); see [VO, Theorem 2.1] for a proof of this fact from first principles. Indeed, this multiplicity-free family is the motivating example for our paper.

Vershik and Okounkov [VO] give a complete and compelling account of the representation theory of symmetric groups from the multiplicity-free inductive viewpoint. In particular, they

- Compute the spectrum of the Young–Jucys–Murphy generators.
- Show that the set of standard tableaux with  $n$  boxes is in bijection with the set of all paths in the branching graph of length  $n$ ; this also proves the branching rule.
- Construct Young’s seminormal and (when  $\mathbb{k} = \mathbb{C}$ ) orthogonal forms for the irreducible representations.
- Compute the irreducible characters (Murnaghan–Nakayama rule).

We cannot improve upon their story. But our story is about idempotents in multiplicity-free families, so we are content to explain just enough representation theory to be able to compute the canonical idempotents

$$\{\varepsilon_\tau \mid \tau \text{ a standard tableau with } n \text{ boxes}\}$$

constructed in Definition 1.3. We also show that these idempotents coincide with the classical seminormal idempotents constructed by Young.

We work with right modules in this section, in deference to Schur–Weyl duality, discussed in the first paragraph of Section 5. Writing  $i^\sigma$  for the image of  $i$  under a permutation  $\sigma$ , we define the product  $\sigma\tau$  of two permutations by  $i^{\sigma\tau} = (i^\sigma)^\tau$ , in order that products of permutations agree with products of their Brauer diagrams. We take for granted that the partitions of  $n$  index the isomorphism classes of irreducible representations and the set of standard tableaux with  $n$  boxes is in bijection with the set of all paths in the branching graph of length  $n$ . Under this bijection, the path  $\bar{T}$  as defined in Section 1 corresponds to the standard tableau  $\bar{T}$  obtained from the standard tableau  $T$  by discarding the box containing the number  $n$ . The labeled branching graph for this family is depicted in Figure 1.

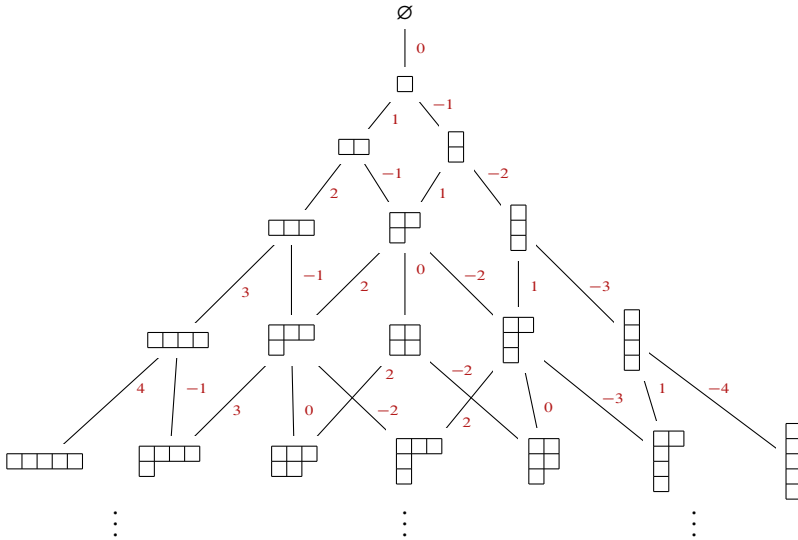


FIGURE 1  
 Branching graph for the multiplicity-free family  $\{\mathbb{k}\mathfrak{S}_n\}$ .  
 The edge labels are computed in Proposition 4.3.

Young’s construction of the irreducible representations is in terms of the so-called Young symmetrizers. Let us recall the definitions; see, e.g., [Ful].

**Definition 4.1.** Given a tableau  $T$  of  $n$  boxes, let  $R(T)$  be the subgroup of  $\mathfrak{S}_n$  consisting of all  $w$  which stabilize the rows of  $T$ ; similarly, let  $C(T)$  be the subgroup of  $\mathfrak{S}_n$  consisting of all  $w$  which stabilize the columns of  $T$ . Put

$$a_T = \sum_{w \in R(T)} w, \quad b_T = \sum_{w \in C(T)} \text{sgn}(w) w.$$

The sums  $a_\tau, b_\tau$  and their products  $a_\tau b_\tau, b_\tau a_\tau$  taken in either order are called Young symmetrizers. We put

$$y_\tau = \frac{1}{h(\lambda)} a_\tau b_\tau = \frac{1}{h(\lambda)} \sum_{v \in R(\tau)} v \sum_{w \in C(\tau)} \text{sgn}(w) w,$$

where  $h(\lambda) = \frac{n!}{n(\lambda)}$  and  $n(\lambda)$  is the number of standard tableaux of shape  $\lambda$ . (Then  $h(\lambda)$  is equal to the product of all the hook lengths in  $\tau$ ; this depends only on the shape  $\lambda$  of  $\tau$ .)

If  $w \in \mathfrak{S}_n$  and  $\tau$  is a tableau with  $n$  boxes then  $w \cdot \tau$  is the tableau obtained by replacing each number  $i \in \tau$  by  $w(i)$ , for  $i = 1, \dots, n$ . We have  $R(w \cdot \tau) = wR(\tau)w^{-1}$  and  $C(w \cdot \tau) = wC(\tau)w^{-1}$ . Thus

$$(4.1) \quad a_{w \cdot \tau} = w a_\tau w^{-1}, \quad b_{w \cdot \tau} = w b_\tau w^{-1}$$

for any  $w \in \mathfrak{S}_n$ . The  $y_\tau$  are idempotents in  $\mathbb{k}\mathfrak{S}_n$ ; these idempotents are sometimes called *Young's idempotents*. The right ideal

$$(4.2) \quad S^\lambda = y_\tau \mathbb{k}\mathfrak{S}_n$$

is an irreducible  $\mathbb{k}\mathfrak{S}_n$ -module, where  $\tau$  is any standard tableau of shape  $\lambda$ . It is known that  $y_\tau \mathbb{k}\mathfrak{S}_n \cong y_{\tau'} \mathbb{k}\mathfrak{S}_n$  if and only if  $\tau, \tau'$  have the same shape, so the isomorphism type of the right ideal  $y_\tau \mathbb{k}\mathfrak{S}_n$  depends only on the shape of  $\tau$ . It is also well known (see, e.g., [CR2, §28]) that

$$(4.3) \quad \mathbb{k}\mathfrak{S}_n = \bigoplus_{\tau} y_\tau \mathbb{k}\mathfrak{S}_n,$$

where the sum is taken over the set of all standard tableaux  $\tau$  of  $n$  boxes. This is a decomposition as a direct sum of simple right ideals, but unfortunately the family  $\{y_\tau\}$  of primitive idempotents is *not* pairwise orthogonal,<sup>4</sup> as already noted by Young; see [Ste] for an explicit counterexample.

We put  $z_n$  equal to the formal sum of all transpositions in  $\mathfrak{S}_n$ , regarded as an element of  $\mathbb{k}\mathfrak{S}_n$ . This conjugacy class sum is an element of the center of  $\mathbb{k}\mathfrak{S}_n$  for each  $n$ . We wish to show that the elements

$$(4.4) \quad J_n = z_n - z_{n-1} = (1, n) + (2, n) + \dots + (n-1, n)$$

(written in the cycle notation for permutations) define a JM-sequence in the sense of Definition 3.1.

---

<sup>4</sup> Not quite, but almost! It can be shown (see, e.g., [Ste, Prop. 1]) that for each pair  $\tau, \tau'$  of standard tableaux of  $n$  boxes, at least one of the products  $y_\tau \cdot y_{\tau'}, y_{\tau'} \cdot y_\tau$  must be zero.

**Proposition 4.2.** *Let  $\lambda$  be a partition of  $n$  and  $\tau$  a standard tableau of shape  $\lambda$ . Then the central element  $z_n$  acts by right multiplication on  $y_\tau \mathbb{k} \mathfrak{S}_n$  as the scalar  $a_\lambda = \xi(\lambda) - \xi(\lambda')$ , where  $\lambda'$  is the transpose of  $\lambda$  and  $\xi(\lambda) = \sum \binom{\lambda_m}{2}$ , summed over the parts  $\lambda = (\lambda_1, \lambda_2, \dots)$  of  $\lambda$ .*

*Proof.* Since  $z_n$  is in the center of  $\mathbb{k} \mathfrak{S}_n$ , we know there is a scalar  $a_\lambda \in \mathbb{k}$  with  $v \cdot z_n = a_\lambda v$ , for all  $v \in y_\tau \mathbb{k} \mathfrak{S}_n$ . In particular,  $y_\tau \cdot z_n = a_\lambda y_\tau$ . By definition of  $y_\tau$ , this equality becomes

$$\sum_{\alpha \in R(\tau)} \sum_{\beta \in C(\tau)} \operatorname{sgn}(\beta) \alpha \beta \cdot \sum_{1 \leq i < j \leq n} (i, j) = a_\lambda \sum_{\alpha \in R(\tau)} \sum_{\beta \in C(\tau)} \operatorname{sgn}(\beta) \alpha \beta,$$

where  $(i, j)$  denotes the transposition interchanging  $i$  and  $j$ . To compute  $a_\lambda$  we compare coefficients of the identity permutation on both sides of the equation, which gives

$$\sum_{\alpha \in R(\tau)} \sum_{\beta \in C(\tau)} \sum_{1 \leq i < j \leq n} \operatorname{sgn}(\beta) \delta_{\alpha\beta, (i, j)} = a_\lambda,$$

where  $\delta_{\sigma, \tau} = 1$  if  $\sigma = \tau$  and 0 otherwise (for  $\sigma, \tau \in \mathfrak{S}_n$ ). It is easy to see that  $\delta_{\alpha\beta, (i, j)} = 0$  unless  $i$  and  $j$  lie in the same row or column of  $\tau$ , since otherwise the product  $\alpha\beta$  must change more than just  $i$  and  $j$ . So we are reduced to counting solutions of the equation  $\alpha\beta = (i, j)$  of the form  $\alpha = (i, j)$  where  $i, j$  lie in the same row of  $\tau$  or  $\beta = (i, j)$  where  $i, j$  lie in the same column of  $\tau$ . In other words, we need to count the number of pairs  $(i, j)$  with  $i < j$  in a row of  $\tau$ , and, with opposite sign, the number of pairs  $(i, j)$  with  $i < j$  in a column of  $\tau$ . This gives the desired result  $a_\lambda = \xi(\lambda) - \xi(\lambda')$ .  $\square$

Here is a combinatorial procedure for computing the statistic  $\xi(\lambda)$  for a given shape  $\lambda$ . Insert the numbers  $0, 1, 2, \dots, \lambda_m - 1$  in order into the  $m$ th row of the diagram of shape  $\lambda$ , for each  $m$ . Then clearly  $\xi(\lambda)$  is equal to the sum of the numbers in the boxes. Note that the insertion process just described is equivalent to inserting a  $j - 1$  in each box of the  $j$ th column of the diagram. So  $\xi(\lambda)$  is the sum of all the numbers in this numbering.

On the other hand, if we insert  $i - 1$  in each box of the  $i$ th row of the diagram of shape  $\lambda$ , then  $\xi(\lambda')$  is the sum of all the numbers in this numbering, where  $\lambda'$  is the transpose of  $\lambda$ .

This implies that if we attach the statistic  $(j - 1) - (i - 1) = j - i$  to the box in row  $i$  and column  $j$  in  $\tau$  (this statistic is called the *content* of the box) then the sum of all the statistics is  $a_\lambda = \xi(\lambda) - \xi(\lambda')$ . Thus, we see that

$$(4.5) \quad a_\lambda = \sum_{(i, j)} j - i,$$

where the sum is taken over the positions  $(i, j)$  indexing all the boxes in the diagram of shape  $\lambda$ . This interpretation of  $a_\lambda$  will be used to prove the following result.

**Proposition 4.3.** *Suppose that  $\mathsf{T} \in \text{Tab}(n)$  has shape  $\lambda$ . For any  $1 \leq k \leq n$ , the eigenvalue  $c_{\mathsf{T}}(k)$  of the action of  $J_k$  on the Gelfand–Tsetlin basis element  $v_{\mathsf{T}}$  indexed by  $\mathsf{T}$  is the content  $j - i$ , where the box containing  $k$  is located in row  $i$  and column  $j$  in the tableau  $\mathsf{T}$ .*

*Proof.* We proceed by induction on  $n$ . For  $n = 1$  the result is clear:  $c_{\mathsf{T}}(1) = 0$  as  $J_1 = 0$ . Let  $n > 1$  and let  $\mathsf{T} \in \text{Tab}(n)$  be a standard tableau of shape  $\lambda$ , some partition of  $n$ . By the inductive hypothesis,  $c_{\overline{\mathsf{T}}}(k)$  has the desired value for any  $k \leq n - 1$ . By Proposition 3.6(a),  $c_{\mathsf{T}}(k) = c_{\overline{\mathsf{T}}}(k)$  for all  $k < n$ , so  $c_{\mathsf{T}}(k)$  has the desired value for all  $k < n$ . Thus, it suffices to compute the value  $c_{\mathsf{T}}(n)$ .

By Proposition 3.3 we have  $c_{\mathsf{T}}(n) = a_{\lambda} - a_{\mu}$ , where  $\overline{\mathsf{T}} \mapsto \mu$ . By equation (4.5), it follows that  $c_{\mathsf{T}}(n) = a_{\lambda} - a_{\mu} = j - i$ , where the box in  $\mathsf{T}$  containing  $n$  occurs in position  $(i, j)$ . The result is proved.  $\square$

**Remark 4.4.** If we record the statistic  $j - i$  in each box  $(i, j)$  of the Young diagram of shape  $\lambda$ , then the resulting tableau is constant along diagonals. Recall that a box in a Young diagram of shape  $\lambda$  is *removable* if excising it results in another Young diagram. Similarly, a box not in the shape  $\lambda$  is *addable* if including it results in a Young diagram. Since removable boxes are always the last box in their row or column, it is clear that no two removable boxes in  $\lambda$  can lie on the same diagonal. The same conclusion applies to addable boxes. Hence, no two removable (or addable) boxes for a shape  $\lambda$  can have the same content. This is needed in the proof of Corollaries 4.5 and 5.10.

**Corollary 4.5.** *The sequence  $(J_k \mid k \in \mathbb{N})$  is a JM-sequence in the sense of Definition 3.1.*

*Proof.* Since  $J_k = z_k - z_{k-1}$  and  $z_k \in Z(\mathbb{k}\mathfrak{S}_k)$  for all  $1 \leq k \leq n$ , it follows that each  $J_k \in \mathcal{X}_n$  and that  $(J_k)_{k \in \mathbb{N}}$  is additively central. We use Proposition 3.5 to verify that it is also a separating sequence. Proposition 4.3 computes the content vectors  $c_{\mathsf{T}} = (c_{\mathsf{T}}(1), \dots, c_{\mathsf{T}}(n))$  for each  $\mathsf{T} \in \text{Tab}(n)$ .

Let  $\mathsf{T}[k]$  denote the standard tableau obtained from  $\mathsf{T} \in \text{Tab}(n)$  by removing all boxes containing numbers larger than  $k$ . Assume that  $\mathsf{S} \neq \mathsf{T}$ . We show  $c_{\mathsf{S}} \neq c_{\mathsf{T}}$ . Find the smallest  $k \leq n$  at which the tableaux  $\mathsf{S}, \mathsf{T}$  differ. That is,  $\mathsf{S}[k - 1] = \mathsf{T}[k - 1]$ , yet  $\mathsf{S}[k] \neq \mathsf{T}[k]$ . By Remark 4.4, the contents of the addable boxes yielding  $\mathsf{S}[k]$  and  $\mathsf{T}[k]$  differ. Appealing to Proposition 4.3, we conclude that  $c_{\mathsf{S}}(k) \neq c_{\mathsf{T}}(k)$ . This completes the proof.  $\square$

For the sake of completeness, we give another formula for the central idempotent  $\varepsilon(\lambda)$  in terms of Young symmetrizers; it was obtained by Young in his first two papers, published in 1900 and 1901. (See [Cur, Ch. II, §5] for a historical account of these developments.)

**Proposition 4.6** (Young). *For each  $\lambda \vdash n$ ,  $\varepsilon(\lambda) = \frac{1}{h(\lambda)} \sum_{\mathsf{T}} y_{\mathsf{T}}$ , where the sum is taken over all tableaux  $\mathsf{T}$  (not necessarily standard) of shape  $\lambda$ .*

The proof is an easy exercise, cf. [Sim, Cor. VI.3.7].

Recall that Young [You] found a family of primitive idempotents  $\{e_{\mathsf{T}}\}$ , also indexed by the set of standard tableaux of  $n$  boxes, which are pairwise orthogonal and sum to 1. These idempotents are part of Young's seminormal form, so we call them Young's *seminormal idempotents*. R. M. Thrall [Thr] (see also [Gar, 2.16], [Las]) found the following recursive description of the  $e_{\mathsf{T}}$ . For each standard tableau  $\mathsf{T}$  of  $n$  boxes, the element  $e_{\mathsf{T}}$  of  $\mathbb{k}\mathfrak{S}_n$  may be defined by

$$(4.6) \quad e_{\mathsf{T}} = \begin{cases} e_{\bar{\mathsf{T}}} \cdot y_{\mathsf{T}} \cdot e_{\bar{\mathsf{T}}} & \text{if } n > 1, \\ 1 & \text{if } n = 1, \end{cases}$$

where  $\bar{\mathsf{T}}$  is the standard tableau obtained from  $\mathsf{T}$  by removing the box containing  $n$ .

So we now have two families  $\{e_{\mathsf{T}}\}$ ,  $\{\varepsilon_{\mathsf{T}}\}$  of pairwise orthogonal primitive idempotents, both indexed by the set of standard tableaux of  $n$  boxes. One might ask how the two families are related. Here is the answer.

**Proposition 4.7.** *For any standard tableau  $\mathsf{T}$  of  $n$  boxes, we have  $\varepsilon_{\mathsf{T}} = e_{\mathsf{T}}$ . So the canonical idempotents of Definition 1.3 are Young's seminormal idempotents in the case of symmetric group algebras.*

*Proof.* This follows immediately from Corollary 1.7 once we observe that  $e_{\bar{\mathsf{T}}}e_{\mathsf{T}} = e_{\mathsf{T}}$  for all standard tableaux  $\mathsf{T}$ . Indeed, this relation is clear from Thrall's recursive definition of the  $e_{\mathsf{T}}$ ; see (4.6).  $\square$

**Remark 4.8.** Thrall's recursive description of the seminormal idempotents depends on the Young symmetrizers, while the simpler recursion obtained by the methods of this paper does not.

**Examples 4.9.** We compute a number of  $\varepsilon(\lambda)$  recursively using Theorem 3.11 and Proposition 4.3, referring to the branching graph in Figure 1. Of course  $\varepsilon(\square) = 1$ .

*Primitive central idempotents for  $n = 2$ :*

$$\begin{aligned} \varepsilon(\square) &= P_{\square}^{\square} \varepsilon(\square) = P_{\square}^{\square} = \frac{1}{2}(J_2 + 1) \\ \varepsilon(\boxplus) &= P_{\square}^{\boxplus} \varepsilon(\square) = P_{\square}^{\boxplus} = -\frac{1}{2}(J_2 - 1). \end{aligned}$$

*Primitive central idempotents for  $n = 3$ :*

$$\varepsilon(\square\square) = P_{\square\square}^{\square\square} \varepsilon(\square\square) = \frac{1}{3}(J_3 + 1) \varepsilon(\square\square)$$

$$\varepsilon(\boxplus) = P_{\boxminus}^{\boxplus} \varepsilon(\boxminus) + P_{\boxplus}^{\boxplus} \varepsilon(\boxplus) = -\frac{1}{3}(J_3 - 2) \varepsilon(\boxminus) + \frac{1}{3}(J_3 + 2) \varepsilon(\boxplus)$$

$$\varepsilon(\boxminus) = P_{\boxminus}^{\boxminus} \varepsilon(\boxminus) = -\frac{1}{3}(J_3 - 1) \varepsilon(\boxminus).$$

*Primitive central idempotents for  $n = 4$ :*

$$\varepsilon(\boxplus\boxplus) = P_{\boxplus\boxplus}^{\boxplus\boxplus} \varepsilon(\boxplus\boxplus) = \frac{1}{4}(J_4 + 1) \varepsilon(\boxplus\boxplus)$$

$$\varepsilon(\boxplus\boxminus) = P_{\boxplus\boxminus}^{\boxplus\boxplus} \varepsilon(\boxplus\boxplus) + P_{\boxplus\boxminus}^{\boxplus\boxminus} \varepsilon(\boxplus\boxminus) = -\frac{1}{4}(J_4 - 3) \varepsilon(\boxplus\boxplus) + \frac{1}{8}(J_4 + 2)J_4 \varepsilon(\boxplus\boxminus)$$

$$\varepsilon(\boxminus\boxplus) = P_{\boxminus\boxplus}^{\boxplus\boxplus} \varepsilon(\boxplus\boxplus) = -\frac{1}{4}(J_4 + 2)(J_4 - 2) \varepsilon(\boxplus\boxplus)$$

$$\varepsilon(\boxminus\boxminus) = P_{\boxminus\boxminus}^{\boxplus\boxplus} \varepsilon(\boxplus\boxplus) + P_{\boxminus\boxminus}^{\boxplus\boxminus} \varepsilon(\boxplus\boxminus) = \frac{1}{8}(J_4 - 2)J_4 \varepsilon(\boxplus\boxminus) + \frac{1}{4}(J_4 + 3) \varepsilon(\boxminus\boxminus)$$

$$\varepsilon(\boxminus\boxminus) = P_{\boxminus\boxminus}^{\boxminus\boxminus} \varepsilon(\boxminus\boxminus) = -\frac{1}{4}(J_4 - 1) \varepsilon(\boxminus\boxminus).$$

We note that the summands in each  $\varepsilon(\lambda)$  are the various  $\varepsilon_{\top}$  in that block, so the  $\varepsilon_{\top}$  are recoverable from the above expressions.

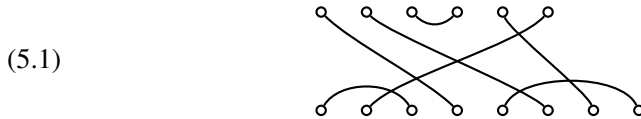
## 5. Application: Brauer algebras

In [Bra], Brauer defined a finite dimensional algebra  $\mathfrak{B}_n(m)$  over  $\mathbb{C}$  in order to quantify the invariants of orthogonal groups. If  $E$  is an  $m$ -dimensional vector space over  $\mathbb{C}$  then  $\mathrm{GL}(E) \cong \mathrm{GL}_m(\mathbb{C})$  acts naturally (on the left) on  $E$ , this action extends diagonally to one on  $E^{\otimes n}$ . The group  $\mathfrak{S}_n$  acts by place-permutation (on the right) on  $E^{\otimes n}$ . These actions commute, so by linearly extending the actions to representations, the tensor space  $E^{\otimes n}$  is a  $(\mathbb{C}\mathrm{GL}(E), \mathbb{C}\mathfrak{S}_n)$ -bimodule. Classical Schur–Weyl duality [Sch] says that the image of each representation in  $\mathrm{End}_{\mathbb{C}}(E^{\otimes n})$  is equal to the full centralizer of the other. This duality elegantly expresses the fundamental duality between the representation theories of general linear groups and symmetric groups.

Brauer extended the action of the symmetric group algebra to one of the algebra  $\mathfrak{B}_n(m)$  such that when the left action of  $\mathrm{GL}(E)$  is restricted to the orthogonal group  $\mathcal{O}(E)$ , Schur–Weyl duality also holds for the resulting  $(\mathbb{C}\mathcal{O}(E), \mathfrak{B}_n(m))$ -bimodule structure on  $E^{\otimes n}$ . This duality relates the representation theory of orthogonal groups and Brauer algebras. Brauer algebras also have connections to low-dimensional topology and knot theory; see, e.g., [Kau, BW, FG].

Let  $k, l$  be positive integers of the same parity, so that  $k + l$  is even. A Brauer  $(k, l)$ -diagram is an undirected graph with  $k + l$  vertices, such that each vertex is

an endpoint of exactly one edge. Conventionally, the vertices are arranged in two rows within a rectangle, with  $k$  vertices (the *top* vertices) along the top boundary and  $l$  vertices (the *bottom* vertices) along the bottom boundary, with the edges drawn in the interior of the rectangle in such a way that intersecting edges cross transversally. For example, the graph



is a Brauer  $(6, 8)$ -diagram. Edges connecting two vertices in the same row are called *horizontal edges*. All other edges must have one top and one bottom endpoint, such edges are *through edges*. The *rank* of a diagram is the number of through edges.

Let  $\mathbb{k}$  be a ring and  $\delta \in \mathbb{k}$  a distinguished parameter. Multiplication of Brauer diagrams is defined as follows. Given a  $(k, l)$ -diagram  $b$  and an  $(l, m)$ -diagram  $b'$ , place  $b$  above  $b'$  and identify the  $i$ th bottom vertex of  $b$  with the  $i$ th top vertex of  $b'$ . Let  $N = N(b, b')$  be the number of interior loops in the new graph and let  $b''$  be that graph with its loops and intermediate vertices omitted. Then  $b''$  is a  $(k, m)$ -diagram, and we define

$$(5.2) \quad bb' = \delta^N (b \circ b'), \quad \text{where } b \circ b' = b''.$$

The  $(k, m)$ -diagram  $b'' = b \circ b'$  is the *composite diagram* of  $b, b'$ . Note that the parameter  $\delta$  keeps track of the number of discarded interior loops. In case  $k = m$  we call the diagram  $b''$  simply an  $m$ -diagram.

The Brauer algebra over  $\mathbb{k}$  with parameter  $\delta$  is denoted by  $\mathfrak{B}_n(\delta)$ , and is defined to be the  $\mathbb{k}$ -span of the set of  $n$ -diagrams. Extended linearly, the multiplication rule  $(b, b') \mapsto bb'$  in (5.2) defines an associative multiplication on  $\mathfrak{B}_n(\delta)$ . An *identity edge* in an  $n$ -diagram is an edge connecting the  $i$ th vertices in the top and bottom rows; the  $n$ -diagram in which all edges are identity edges is the unit element of  $\mathfrak{B}_n(\delta)$ . Brauer  $n$ -diagrams in which every edge is a through edge will be identified with permutations; note that multiplication of Brauer diagrams coincides with multiplication of permutations in case both diagrams are permutations, so  $\mathbb{k}\mathfrak{S}_n$  is a subalgebra of  $\mathfrak{B}_n(\delta)$ . Clearly  $\mathfrak{B}_1(\delta) \cong \mathbb{k}$ ; we agree to interpret  $\mathfrak{B}_0(\delta) = \mathbb{k}$ .

Let  $\mathfrak{B}_{k,l}(\delta)$  be the  $\mathbb{k}$ -span of the set of  $(k, l)$ -diagrams. Multiplication of Brauer diagrams makes this into a  $(\mathfrak{B}_k(\delta), \mathfrak{B}_l(\delta))$ -bimodule with  $\mathfrak{B}_k(\delta)$  acting by left multiplication and  $\mathfrak{B}_l(\delta)$  by right multiplication. This bimodule structure will be used below to construct representations of Brauer algebras.

Semisimplicity of  $\mathfrak{B}_n(\delta)$  over  $\mathbb{C}$  was studied in [Bro] in the case when  $\delta$  is a positive integer: he showed that  $\mathfrak{B}_n(\delta)$  is semisimple if and only if  $\delta \geq n - 1$ .



Still working over  $\mathbb{C}$ , Hanlon and Wales [HW] conjectured that  $\mathfrak{B}_n(\delta)$  is always semisimple if  $\delta \in \mathbb{C}$  is not an integer; the conjecture was proved by Wenzl [Wen], who also parametrized the simple modules and established the branching diagram. Further work on semisimplicity of Brauer algebras, including semisimplicity over other fields, can be found in [DWH, Rui, CMPX].

We assume for the remainder of this section that  $\mathbb{k}$  is a field of characteristic zero and  $\delta \in \mathbb{k}$  is not an integer. This assumption ensures that  $\mathfrak{B}_n(\delta)$  is split semisimple over  $\mathbb{k}$ . Under this assumption, we show that the Brauer algebras form a multiplicity-free family, identify a JM-sequence for this family, develop eigenvalue formulas, and compute central idempotents using Theorem 3.11.

In order to simplify the notation, we suppress the parameter  $\delta$ , writing  $\mathfrak{B}_n = \mathfrak{B}_n(\delta)$  from now on. There is a natural unital embedding

$$\iota : \mathfrak{B}_n \hookrightarrow \mathfrak{B}_{n+1}$$

given by sending an  $n$ -diagram to the corresponding  $(n+1)$ -diagram obtained by appending an identity edge on the right (connecting two additional vertices). We identify  $\mathfrak{B}_n$  as a unital subalgebra of  $\mathfrak{B}_{n+1}$ , for each  $n$ , without further mention of  $\iota$ .

We write  $(i, j)$  for the  $n$ -diagram corresponding to a transposition  $(i, j) \in \mathfrak{S}_n$ ; this is the diagram with through edges connecting the  $i$ th and  $j$ th top vertices to the  $j$ th and  $i$ th bottom ones, respectively, with all other edges identity edges. Similarly,  $\overline{(i, j)}$  is the  $n$ -diagram with horizontal edges connecting the  $i$ th and  $j$ th vertices in each row, and all other edges identity edges. We set

$$(5.3) \quad s_i = (i, i+1); \quad e_i = \overline{(i, i+1)}, \quad \text{any } i < n.$$

It is easy to see that  $\mathfrak{B}_n$  is generated by the  $s_i, e_i$  for  $1 \leq i \leq n-1$ . Defining relations satisfied by these generators can be found in [Naz]. Note that  $e_i^2 = \delta e_i$ , so  $\delta^{-1}e_i$  is idempotent. Any  $e_i$  generates the two-sided ideal spanned by all diagrams with at least two horizontal edges; the quotient by this ideal is isomorphic to  $\mathbb{k}\mathfrak{S}_n$ .

Our next task is to construct the irreducible (right)  $\mathfrak{B}_n$ -modules. For this purpose it is useful to apply some general observations from [Gre, §6.2]. The applicability of these ideas to diagram algebras was demonstrated in [MS, Mar, DWH, MRH, CMPX, CDVM]; here we more or less follow the summary outline at the beginning of [CMPX]. In general, then, let  $A$  be an algebra over a field  $\mathbb{k}$  and  $e \in A$  an idempotent. The rule

$$M \mapsto Me$$

defines an exact functor  $\mathbf{F}$  (often called the ‘‘Schur functor’’) from right  $A$ -modules to right  $eAe$ -modules. The functor  $\mathbf{F}$  takes irreducible modules to irreducible modules, or zero. More precisely, we have the following result.

**Theorem 5.1** ([Gre, (6.2g)]). *Let  $\{L(\lambda) : \lambda \in \Lambda\}$  be a full set of pairwise non-isomorphic irreducible right  $A$ -modules, and let*

$$\Lambda^e = \{\lambda \in \Lambda : L(\lambda)e \neq 0\}.$$

*Then  $\{L(\lambda)e : \lambda \in \Lambda^e\}$  is a full set of pairwise non-isomorphic irreducible right  $eAe$ -modules.*

Note that right  $A$ -modules annihilated by  $e$  are equivalent to right  $A/AeA$ -modules. Thus, the irreducible right  $A$ -modules  $L(\lambda)$  with  $\lambda \in \Lambda \setminus \Lambda^e$  are a full set of irreducible  $A/AeA$ -modules. If  $A$  is finite dimensional, this reduces the problem of finding an indexing set  $\Lambda$  for the irreducible  $A$ -modules to the same problem for the smaller algebras  $eAe$ ,  $A/AeA$ .

There is another functor  $\mathbf{G}$  going from right  $eAe$ -modules to right  $A$ -modules, defined by  $\mathbf{G}(N) = N \otimes_{eAe} eA$ . This functor, which was also considered in [Gre, §6.2], is a right inverse to  $\mathbf{F}$ , i.e.,  $\mathbf{F}(\mathbf{G}(N)) \cong N$ , so  $\mathbf{G}$  is a full embedding.<sup>5</sup> Furthermore, [Gre, (6.2e)] shows that  $\mathbf{G}(N)$  always has a unique maximal proper submodule whenever  $N$  is irreducible.

In case  $A$  is semisimple, it follows that  $\mathbf{G}$  must take irreducible  $eAe$ -modules to irreducible  $A$ -modules (and the unique maximal proper submodule is zero). Thus, for irreducible  $A$ -modules  $M$  such that  $Me \neq 0$ , we have  $\mathbf{G}(\mathbf{F}(M)) \cong M$ . So,  $\mathbf{G}$  is also a left inverse to  $\mathbf{F}$ . Thus, in the semisimple case, the functors  $\mathbf{F}$  and  $\mathbf{G}$  implement an equivalence of categories between  $A$ -modules not killed by  $e$  and  $eAe$ -modules. Since  $A \cong AeA \oplus A/AeA$  by semisimplicity, and the  $A$ -modules killed by  $e$  are the  $A/AeA$ -modules, it follows that the  $A$ -modules not killed by  $e$  are the same as the  $AeA$ -modules. To summarize:

**Proposition 5.2.** *If  $A$  is semisimple, then:*

- (a)  $\mathbf{G}$  takes irreducible to irreducibles.
- (b) The functors  $\mathbf{F}$ ,  $\mathbf{G}$  induce an equivalence of categories between  $AeA$ -modules and  $eAe$ -modules.

Now we apply the above observations to the algebra  $\mathfrak{B}_n$ , taking  $e$  to be the idempotent  $\xi_n = \delta^{-1}e_{n-1}$ , with  $e_{n-1}$  as in (5.3). This immediately gives functors  $\mathbf{F}_n, \mathbf{G}_{n-2}$  as above, defined by the rules

$$\mathbf{F}_n(M) = M\xi_n, \quad \mathbf{G}_{n-2}(N) = N \otimes_{\xi_n \mathfrak{B}_n \xi_n} \xi_n \mathfrak{B}_n$$

for any right  $\mathfrak{B}_n$ -module  $M$ , any right  $\mathfrak{B}_{n-2}$ -module  $N$ . A crucial fact about the idempotent  $\xi_n$  is that there is an isomorphism of algebras

<sup>5</sup> In [CMPX], the functors  $\mathbf{F}, \mathbf{G}$  are called “localization” and “globalization” functors, respectively.

$$(5.4) \quad \mathfrak{B}_{n-2} \cong \xi_n \mathfrak{B}_n \xi_n$$

for each  $n \geq 2$ . The isomorphism is given by the rule  $b \mapsto \xi_n b \xi_n$ , for  $b \in \mathfrak{B}_{n-2}$ ; note that it maps the unit element of  $\mathfrak{B}_{n-2}$  to  $\xi_n$ . Furthermore,  $\xi_n$  commutes pointwise with  $\mathfrak{B}_{n-2}$ :

$$(5.5) \quad \xi_n b = b \xi_n, \quad \text{for all } b \in \mathfrak{B}_{n-2}.$$

In consequence, we have  $\mathfrak{B}_{n-2} \cong \mathfrak{B}_{n-2} \xi_n = \xi_n \mathfrak{B}_{n-2} \xi_n$ . If  $\text{Irr}(n)$  is a set indexing the irreducible  $\mathfrak{B}_n$ -modules and  $\text{Irr}^n$  a set indexing the irreducible  $\mathfrak{B}_n / \mathfrak{B}_n \xi_n \mathfrak{B}_n$ -modules, then it follows from (5.4) and the preceding remarks that

$$\text{Irr}(n) = \text{Irr}^n \sqcup \text{Irr}(n-2).$$

Since  $\mathfrak{B}_n / \mathfrak{B}_n \xi_n \mathfrak{B}_n$  is isomorphic to  $\mathbb{k} \mathfrak{S}_n$ , we can set  $\text{Irr}^n = \{\lambda \mid \lambda \vdash n\}$ . It is trivial to compute  $\Lambda_0$  and  $\Lambda_1$  (as  $\mathfrak{B}_0 \cong \mathfrak{B}_1 \cong \mathbb{k}$ ), so it immediately follows by induction on  $n$  that

$$\text{Irr}(n) = \{\lambda \mid \lambda \vdash n - 2l \quad \text{and} \quad 0 \leq 2l \leq n\}.$$

Now that we know an indexing set for the irreducible  $\mathfrak{B}_n$ -modules, we turn to the problem of constructing them. We will follow the approach of [DWH], using the  $(\mathfrak{B}_k, \mathfrak{B}_n)$ -bimodule  $\mathfrak{B}_{k,n} = \mathfrak{B}_{k,n}(\delta)$  discussed above, where  $k \leq n$  has the same parity as  $n$ . Let  $\mathfrak{B}_{k,n}^0$  be the span of the  $(k, n)$ -diagrams of rank (number of through edges) strictly smaller than  $k$ . Since multiplication of diagrams cannot increase the number of through edges,  $\mathfrak{B}_{k,n}^0$  is a sub-bimodule of  $\mathfrak{B}_{k,n}$ , and hence the quotient

$$V_k^n = \mathfrak{B}_{k,n} / \mathfrak{B}_{k,n}^0$$

is a  $(\mathfrak{B}_k, \mathfrak{B}_n)$ -bimodule. The set of  $(k, n)$ -diagrams of rank  $k$  is a complete set of representatives of the quotient. If  $k = n$ , then  $V_n^n \cong \mathbb{k} \mathfrak{S}_n$  and  $\mathbf{F}_n(V_n^n) = V_n^n \xi_n = 0$ . Furthermore, if  $k < n$ , then we have an isomorphism

$$(5.6) \quad \mathbf{F}_n(V_k^n) = V_k^n \xi_n \cong V_k^{n-2}$$

as  $(\mathfrak{B}_k, \mathfrak{B}_{n-2})$ -bimodules. The isomorphism arises from forgetting the rightmost horizontal edge in  $b \xi_n$ , for each  $(k, n)$ -diagram  $b$ . (There is a factor of  $\delta^{-1}$  which does not matter.) By restriction, since  $\mathbb{k} \mathfrak{S}_k$  is contained in  $\mathfrak{B}_k$ , the bimodule  $V_k^n$  is a  $(\mathbb{k} \mathfrak{S}_k, \mathfrak{B}_n)$ -bimodule. Therefore, if  $\lambda \vdash k$ , we define

$$M^{(\lambda, n)} = S^\lambda \otimes_{\mathbb{k} \mathfrak{S}_k} V_k^n.$$

This is a right  $\mathfrak{B}_n$ -module, where  $S^\lambda$  is the Specht module considered in the previous section. If  $\lambda \vdash n$ , then  $k = n$  and  $V_n^n \cong \mathbb{k} \mathfrak{S}_n$ , so  $M^{(\lambda, n)} \cong S^\lambda$  as right  $\mathfrak{B}_n$ -modules (with  $\mathfrak{B}_n \xi_n \mathfrak{B}_n$  acting trivially). Clearly this is an irreducible  $\mathfrak{B}_n$ -module; indeed, it is irreducible as a  $\mathbb{k} \mathfrak{S}_n$ -module.

**Proposition 5.3.** *A full set of irreducible right  $\mathfrak{B}_n$ -modules is the set of  $M^{(\lambda,n)}$  such that  $\lambda \vdash k$ ,  $0 \leq k \leq n$ , and  $n, k$  are of the same parity.*

*Proof.* Assume that  $k, n$  have the same parity. To show that the  $M^{(\lambda,n)}$  are pairwise non-isomorphic and irreducible, we proceed by induction. We consider the two cases  $k < n$  and  $k = n$ . (Modules between the two cases are non-isomorphic by Theorem 5.1.)

If  $k < n$  and  $\lambda \vdash k$ , then it follows from (5.4), (5.6), and the definition of  $M^{(\lambda,n)}$  that

$$\mathbf{F}_n(M^{(\lambda,n)}) = M^{(\lambda,n)}\xi_n = S^\lambda \otimes_{\mathbb{k}\mathfrak{S}_k} V_k^n \xi_n \cong S^\lambda \otimes_{\mathbb{k}\mathfrak{S}_k} V_k^{n-2} = M^{(\lambda,n-2)}$$

as right  $\mathfrak{B}_{n-2}$ -modules. Since  $\mathfrak{B}_n$  is semisimple, and  $M^{(\lambda,n-2)} \neq 0$  by the inductive hypothesis, it follows that

$$\mathbf{G}_{n-2}(M^{(\lambda,n-2)}) \cong M^{(\lambda,n)}$$

as right  $\mathfrak{B}_n$ -modules. Furthermore, by Proposition 5.2(a),  $M^{(\lambda,n)}$  is irreducible as a right  $\mathfrak{B}_n$ -module. Appealing to Proposition 5.2(b), we see that the distinct  $M^{(\lambda,n)}$  are pairwise non-isomorphic.

In the case  $k = n$  and  $\lambda \vdash n$ , we have  $M^{(\lambda,n)} \cong S^\lambda$ . Such modules are pairwise non-isomorphic (and irreducible) by the remarks preceding the theorem. This completes the proof.  $\square$

**Remark 5.4.** Although not needed in the sequel, to complete the picture we describe a  $\mathbb{k}$ -basis for  $M^{(\lambda,n)}$ . This requires finding a complete set of orbit representatives for the left action of  $\mathfrak{S}_k$  on the set of  $(k, n)$ -diagrams of rank  $k$ . If  $b$  is a  $(k, n)$ -diagram, we let  $\pi(b)$  in  $\mathfrak{S}_k$  be the permutation obtained from  $b$  by removing the horizontal edges and their endpoints. Recall from [FG, Xi] that a  $(k, n)$ -diagram  $b$  is a *flat  $(k, n)$ -dangle* if  $\pi(b)$  is the identity. Then the set of flat  $(k, n)$ -dangles is the desired set of representatives. Any  $(k, n)$ -diagram  $b$  is uniquely expressible as a product

$$b = \pi(b)d(b),$$

where  $d(b)$  is a flat  $(k, n)$ -dangle  $d(b)$ . It follows that the set

$$\{v \otimes d \mid v \in \widehat{S}^\lambda, d \text{ a flat } (k, n)\text{-dangle}\}$$

is a  $\mathbb{k}$ -basis for  $M^{(\lambda,n)}$ , where  $\widehat{S}^\lambda$  is any  $\mathbb{k}$ -basis of  $S^\lambda$ .

Next, we explain why the family  $\{\mathfrak{B}_n \mid n \geq 0\}$  is multiplicity-free. Recall that if  $B$  is a subalgebra of an algebra  $A$  and if  $M$  is a right  $B$ -module, then the

induced module is the right  $A$ -module  $\text{Ind}_B^A M$  defined by  $\text{Ind}_B^A M = M \otimes_B A$ . The functor  $\text{Ind}_B^A$  from  $B$ -modules to  $A$ -modules is a left adjoint to the usual restriction functor  $\text{Res}_B^A$  from  $A$ -modules to  $B$ -modules, meaning that Frobenius reciprocity holds:

$$\text{Hom}_A(\text{Ind}_B^A M, N) \cong \text{Hom}_B(M, \text{Res}_B^A N),$$

where  $M$  is any right  $B$ -module,  $N$  any right  $A$ -module.

We can apply these generalities to the inclusion  $\iota : \mathfrak{B}_{n-1} \hookrightarrow \mathfrak{B}_n$ , which identifies  $\mathfrak{B}_{n-1}$  with a subalgebra of  $\mathfrak{B}_n$ . Wenzl [Wen] observed that  $\xi_n \mathfrak{B}_n = \xi_n \mathfrak{B}_{n-1}$  and also that the map

$$(5.7) \quad \mathfrak{B}_{n-1} \rightarrow \xi_n \mathfrak{B}_n \text{ defined by } x \mapsto \xi_n x$$

gives an isomorphism  $\mathfrak{B}_{n-1} \cong \xi_n \mathfrak{B}_n$  of  $(\mathfrak{B}_{n-2}, \mathfrak{B}_{n-1})$ -bimodules. Note that  $\xi_n \mathfrak{B}_{n-1}$  is a left  $\mathfrak{B}_{n-2}$ -module since  $\xi_n$  commutes with  $\mathfrak{B}_{n-2}$ , by (5.5). Let  $\lambda \vdash k$  where  $k < n$  and  $k$  has the same parity as  $n$ . If we restrict the  $\mathfrak{B}_n$ -module isomorphism

$$M^{(\lambda, n)} \cong \mathbf{G}_{n-2}(M^{(\lambda, n-2)}) = M^{(\lambda, n-2)} \otimes_{\mathfrak{B}_{n-2}} \xi_n \mathfrak{B}_n$$

to  $\mathfrak{B}_{n-1}$ , it follows that

$$(5.8) \quad \text{Res}_{\mathfrak{B}_{n-1}}^{\mathfrak{B}_n} M^{(\lambda, n)} \cong \text{Ind}_{\mathfrak{B}_{n-2}}^{\mathfrak{B}_{n-1}} M^{(\lambda, n-2)}$$

as right  $\mathfrak{B}_{n-1}$ -modules. In light of Frobenius reciprocity this says that

$$\text{Hom}_{\mathfrak{B}_{n-1}}(\text{Res } M^{(\lambda, n)}, M^{(\mu, n-1)}) \cong \text{Hom}_{\mathfrak{B}_{n-2}}(M^{(\lambda, n-2)}, \text{Res } M^{(\mu, n-1)})$$

for any  $\mu \vdash l \leq n-1$  where  $l$  has the same parity as  $n-1$ . Here, we omitted the sub and superscripts on the restriction functors for readability. Since the algebras are semisimple, this says that

$$(5.9) \quad [\text{Res } M^{(\lambda, n)} : M^{(\mu, n-1)}]_{n-1} = [\text{Res } M^{(\mu, n-1)} : M^{(\lambda, n-2)}]_{n-2},$$

where we write  $[M : S]_n$  for the multiplicity of an irreducible  $\mathfrak{B}_n$ -module  $S$  in another  $\mathfrak{B}_n$ -module  $M$ . By induction, we may assume that the right hand side of (5.9) is always 0 or 1. This shows that restriction from  $\mathfrak{B}_n$  to  $\mathfrak{B}_{n-1}$  is multiplicity-free, at least for the case of  $k < n$ .

If  $k = n$  and  $\lambda \vdash n$ , then  $M^{(\lambda, n)} \cong S^\lambda$  with  $\mathfrak{B}_n \xi_n \mathfrak{B}_n$  acting trivially. That is, its restriction to  $\mathfrak{B}_{n-1}$  is a module with  $\mathfrak{B}_{n-1} \xi_{n-1} \mathfrak{B}_{n-1} \subset \mathfrak{B}_n \xi_n \mathfrak{B}_n$  acting trivially, so the restriction is a  $\mathbb{k}\mathfrak{S}_{n-1}$ -module. This means that the restriction rule in this case is the same as the usual restriction rule for symmetric groups (which is also multiplicity-free). This completes the proof that the family  $\{\mathfrak{B}_n \mid n \geq 0\}$  is a multiplicity-free family, in the sense of Definition 1.1.

In fact, the above analysis shows that the restriction of an irreducible  $\mathfrak{B}_n$ -module  $M^{(\lambda, n)}$  to  $\mathfrak{B}_{n-1}$  breaks up into a direct sum of irreducible  $\mathfrak{B}_{n-1}$ -modules  $M^{(\mu, n-1)}$  indexed by all partitions  $\mu$  obtained from  $\lambda$  by removing or adding one box. This justifies the branching graph for this family, which is displayed in Figure 2 below.

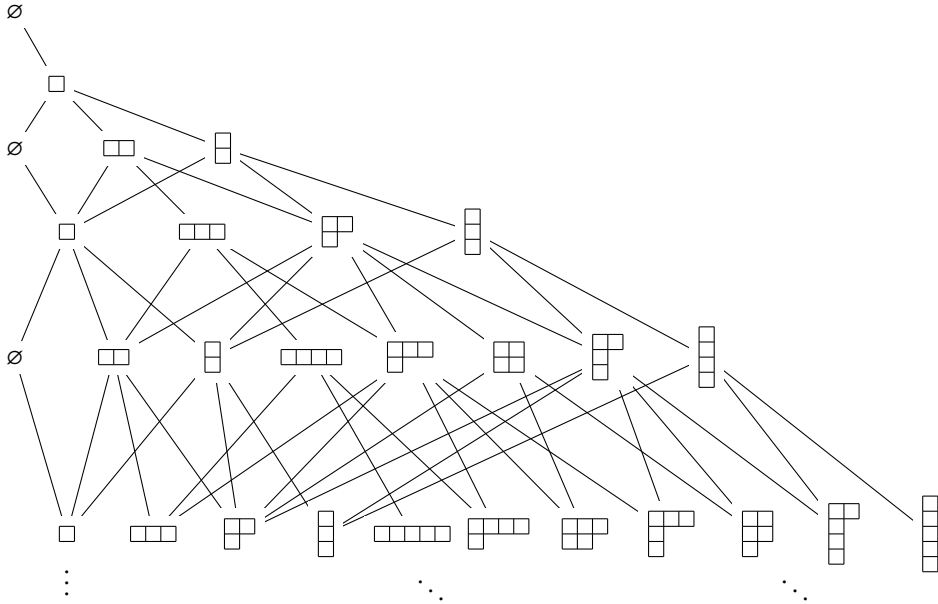


FIGURE 2  
Branching graph for the family  $\{\mathfrak{B}_n\}$

Since  $\{\mathfrak{B}_n \mid n \geq 0\}$  is a multiplicity-free family, each  $\mathfrak{B}_n$  has canonical idempotents  $\varepsilon_\tau$  given by Definition 1.3. These idempotents are indexed by paths  $\tau$  of length  $n$  in the branching graph. The set  $\text{Tab}(n)$  may be identified with the set of *up-down tableaux*, which are sequences of partitions of the form

$$(5.10) \quad \tau = (\lambda_0, \lambda_1, \dots, \lambda_{n-1}, \lambda_n)$$

such that  $\lambda_0 = \emptyset$  and, for each  $k$ , the partition  $\lambda_{k+1}$  is obtainable from the preceding partition  $\lambda_k$  by adding or removing exactly one box.

We wish to compute the idempotents  $\varepsilon_\tau$  by means of a sequence of JM-elements, according to the results of Section 3. Following Nazarov [Naz], we define elements  $J_k \in \mathfrak{B}_k$  ( $k \geq 1$ ) by

$$(5.11) \quad J_k = \sum_{i=1}^{k-1} (i, k) - \sum_{i=1}^{k-1} \overline{(i, k)}.$$

We define  $J_1$  as zero. Our definition of these elements differs slightly from Nazarov's, in that we have removed an unnecessary shift by  $(\delta - 1)/2$ . For any  $n \in \mathbb{N}$ , the elements  $J_1, \dots, J_n$  may be regarded as elements of  $\mathfrak{B}_n$  by means of the embeddings  $\mathfrak{B}_1 \subset \dots \subset \mathfrak{B}_{n-1} \subset \mathfrak{B}_n$ . The following easy results can be checked by direct computations.

**Lemma 5.5** ([Naz, Lemma 2.1]). *For any  $k = 1, \dots, n$ , the element  $J_k$  commutes with any  $b \in \mathfrak{B}_{n-1}$ . Hence, the elements  $J_1, \dots, J_n$  pairwise commute in  $\mathfrak{B}_n$ .*

We omit the easy proof, which is given in [Naz]. The lemma immediately gives the following commutation relations between the  $J_k$  and the generators  $s_i, e_i$  defined in (5.3). Note that the relations in part (c) differ from those given by Nazarov because our definition of  $J_k$  differs slightly from his.

**Proposition 5.6** ([Naz, Prop. 2.3]). *The following relations hold in the algebra  $\mathfrak{B}_n$ :*

- (a)  $s_k J_l = J_l s_k, \quad e_k J_l = J_l e_k \quad (l \neq k, k+1)$ .
- (b)  $s_k J_k - J_{k+1} s_k = e_k - 1, \quad s_k J_{k+1} - J_k s_k = 1 - e_k$ .
- (c)  $e_k (J_k + J_{k+1}) = (1 - \delta) e_k = (J_k + J_{k+1}) e_k$ .

*Proof.* The commutation relations (a) follow from Lemma 5.5 if  $l > k+1$  and from the definitions otherwise. Furthermore, it is easy to check from the definition that the elements  $J_k$  can be defined by the recursion

$$J_1 = 0, \quad J_{k+1} = s_k J_k s_k + s_k - e_k \quad (k \geq 1).$$

This implies the relations (b). Turning to (c), we have by direct computation for any  $l = 1, \dots, k-1$  the equalities

$$e_k(k, l) = e_k(\overline{k+1, l}) \quad \text{and} \quad e_k(\overline{k, l}) = e_k(k+1, l).$$

Combining these equalities with the obvious identities  $e_k s_k = e_k, e_k^2 = \delta e_k$  and the definition of the  $J_k$  produces the leftmost equality in (c). The rightmost equality is proved similarly.  $\square$

Relations (a) and (b) of the proposition immediately imply the following.

**Corollary 5.7** ([Naz, Cor. 2.4]). *The sum  $z_n = J_1 + \dots + J_{n-1} + J_n$  is a central element of  $\mathfrak{B}_n$ .*

It remains to compute the eigenvalues of the  $J_k$  on the irreducible modules and prove that the sequence  $(J_k)_{k \in \mathbb{N}}$  is separating.

**Proposition 5.8.** *Let  $\lambda \vdash k$  where  $k = n - 2l$  and  $0 \leq 2l \leq n$ . Suppose that  $a_\lambda$  is the eigenvalue of equation (4.5). Then the central element  $z_n = J_1 + \cdots + J_n$  of  $\mathfrak{B}_n$  acts on  $M^{(\lambda, n)} = S^\lambda \otimes_{\mathbb{k}\mathfrak{S}_k} V_k^n$  as the scalar  $\beta_\lambda = a_\lambda + l(1 - \delta)$ .*

*Proof.* This argument follows the proof of [GG2, Theorems 5.3, 5.1]. We proceed by induction on  $n$ . The base cases  $n = 0, 1$  are trivial, so assume that  $n \geq 2$ . There are two cases.

If  $l = 0$ , then  $\lambda \vdash n$  and  $M^{(\lambda, n)} \cong S^\lambda$ , with the ideal  $\mathfrak{B}_n \xi_n \mathfrak{B}_n$  acting trivially. We can write

$$z_n = z_n^{\mathfrak{S}_n} - \bar{z}_n,$$

where  $z_n^{\mathfrak{S}_n} = \sum_{i < j} (i, j)$  is the sum of all the transpositions in  $\mathfrak{S}_n$  and  $\bar{z}_n = \sum_{i < j} \overline{(i, j)} \in \mathfrak{B}_n \xi_n \mathfrak{B}_n$ . It follows from Proposition 4.2 that  $z_n$  acts as the scalar  $a_\lambda$ , so the proof is complete in case  $l = 0$ .

Now suppose  $l > 0$ . In this case we use the isomorphism

$$M^{(\lambda, n)} \cong \mathbf{G}_{n-2}(M^{(\lambda, n-2)}) = S^\lambda \otimes_{\mathbb{k}\mathfrak{S}_k} V_k^{n-2} \otimes_{\xi_n \mathfrak{B}_n \xi_n} \xi_n \mathfrak{B}_n$$

from the proof of Proposition 5.3. Since the central element  $z_n$  acts by a fixed scalar on the entire module, it suffices to compute its eigenvalue on any nonzero vector in the module, so we consider its action on  $u \otimes v \otimes \xi_n$ , where  $0 \neq u \otimes v \in S^\lambda \otimes_{\mathbb{k}\mathfrak{S}_k} V_k^{n-2}$ . By induction we have

$$(u \otimes v)z_{n-2} = (a_\lambda + (l-1)(1-\delta))u \otimes v.$$

It follows that

$$\begin{aligned} (u \otimes v \otimes \xi_n)z_n &= (u \otimes v \otimes \xi_n)(z_{n-2} + J_{n-1} + J_n) \\ &= (u \otimes v \otimes \xi_n)z_{n-2} + (u \otimes v \otimes \xi_n)(J_{n-1} + J_n). \end{aligned}$$

By Proposition 5.6(a) we know that  $\xi_n = \frac{1}{8}e_{n-1}$  commutes with  $z_{n-2}$ , so the first term in the right hand side of the above is

$$(u \otimes v \otimes \xi_n)z_{n-2} = (u \otimes v)z_{n-2} \otimes \xi_n = (a_\lambda + (l-1)(1-\delta))u \otimes v \otimes \xi_n.$$

The second term in the right hand side is computed by Proposition 5.6(c) as

$$(u \otimes v \otimes \xi_n)(J_{n-1} + J_n) = (1-\delta)u \otimes v \otimes \xi_n.$$

Hence, by combining the equations in the last three displays, we obtain the equality

$$(u \otimes v \otimes \xi_n)z_n = (a_\lambda + l(1-\delta))u \otimes v \otimes \xi_n$$

and the proof is complete.  $\square$



This result will now be applied to compute the eigenvalues of the  $J_k$  on the Gelfand–Tsetlin basis of the irreducible  $\mathfrak{B}_n$ -modules.

**Proposition 5.9.** *Suppose that  $\lambda \in \text{Irr}(n)$  and  $\{v_\tau \mid \tau \mapsto \lambda\}$  is the Gelfand–Tsetlin basis of  $M^{(\lambda, n)}$ . Let  $\tau = (\lambda_0, \lambda_1, \dots, \lambda_n)$  be an up-down tableau with  $\lambda_n = \lambda$ . Suppose that  $\lambda_k$  and  $\lambda_{k-1}$  differ by a box in row  $i$  and column  $j$ . Then the eigenvalue of  $J_k$  on the eigenvector  $v_\tau$  is*

$$c_\tau(k) = \begin{cases} j - i & \text{if } \lambda_k \text{ has one more box than } \lambda_{k-1}, \\ (1 - \delta) + i - j & \text{if } \lambda_k \text{ has one fewer box than } \lambda_{k-1}. \end{cases}$$

*Proof.* Set  $z_k = \sum_{l=1}^k J_l$  and note that  $J_k = z_k - z_{k-1}$ , for any  $1 \leq k \leq n$ .

We proceed by induction on  $n$ . For  $n = 1$  the result is clear:  $c_\tau(1) = 0$  as  $J_1 = 0$ . Let  $n > 1$  and let  $\tau \in \text{Tab}(n)$ . By the inductive hypothesis,  $c_{\bar{\tau}}(k)$  has the desired value for any  $k \leq n - 1$ . By Proposition 3.6(a),  $c_\tau(k) = c_{\bar{\tau}}(k)$  for all  $k < n$ , so  $c_\tau(k)$  has the desired value for all  $k < n$ . Thus, it suffices to compute the value  $c_\tau(n)$ .

By Propositions 3.3 and 5.8 we have  $c_\tau(n) = \beta_\lambda - \beta_\mu$ , where  $\bar{\tau} \mapsto \mu$ , and  $\beta_\lambda = a_\lambda + l(1 - \delta)$ . There are two cases to consider: if  $\lambda = \lambda_n$  has one more box or one fewer box than  $\mu = \lambda_{n-1}$ . In the first case, Propositions 5.8 and 4.3 give us  $\beta_\mu = a_\mu + l(1 - \delta)$ , and

$$c_\tau(n) = \beta_\lambda - \beta_\mu = j - i.$$

In the second case,  $\beta_\mu = a_\mu + (l - 1)(1 - \delta)$ , and hence

$$c_\tau(n) = \beta_\lambda - \beta_\mu = (1 - \delta) + i - j.$$

This completes the proof. □

**Corollary 5.10.** *The sequence  $(J_k \mid k \in \mathbb{N})$  is a JM-sequence in the sense of Definition 3.1.*

*Proof.* Since  $J_k = z_k - z_{k-1}$  and  $z_k \in Z(\mathfrak{B}_k)$  for all  $1 \leq k \leq n$  (Corollary 5.7), it follows that each  $J_k \in \mathcal{X}_n$  and that  $(J_k)_{k \in \mathbb{N}}$  is additively central. To prove that it is also a separating sequence, we use Proposition 3.5. That is, we verify that  $S = T$  if and only if  $c_S = c_T$ . (One direction is automatic.)

Proposition 5.9 computes the content vectors  $c_\tau = (c_\tau(1), \dots, c_\tau(n))$  for each  $\tau \in \text{Tab}(n)$ . If  $\tau = (\lambda_0 \rightarrow \dots \rightarrow \lambda_n)$ , we write  $T[k] = (\lambda_0 \rightarrow \dots \rightarrow \lambda_k)$  for the truncated path. Assume  $S \neq T$  are distinct paths of length  $n$  and find the first level  $k \leq n$  at which the paths  $S, T$  diverge. So  $S[k - 1] = T[k - 1]$ , yet  $S[k] \neq T[k]$ . Let  $S[k] \mapsto \lambda$  and  $T[k] \mapsto \nu$  be the terminal shapes of the paths, and let  $T[k - 1] \mapsto \mu$ . There are three cases.

Case 1:  $\lambda, \nu$  are obtained by adding different boxes to  $\mu$ . Here  $c_S(k)$  and  $c_T(k)$  are both computed using the first formula in Proposition 5.9. Appealing to Remark 4.4, we see that  $c_S(k) \neq c_T(k)$ .

Case 2:  $\lambda, \nu$  are obtained by removing different boxes from  $\mu$ . We must use the second formula in Proposition 5.9. Appealing to Remark 4.4, we again have  $c_S(k) \neq c_T(k)$ .

Case 3: One of  $\lambda, \nu$  is obtained by adding a box and the other by removing one. Here  $c_S(k)$  and  $c_T(k)$  cannot possibly be equal, as Proposition 5.9 says that one value is an integer and the other is not (recall that  $\delta \in \mathbb{k} \setminus \mathbb{Z}$ ).

All cases reach the conclusion that  $c_S \neq c_T$ , so the proof is complete.  $\square$

**Examples 5.11.** To avoid ambiguity, we write  $\varepsilon^{(n)}(\lambda)$  for the primitive central idempotent  $\varepsilon(\lambda)$  in  $\mathfrak{B}_n$ . The  $\varepsilon^{(n)}(\lambda)$  can be computed recursively using Theorem 3.11 and Proposition 5.9, referring to the branching graph in Figure 2. Of course  $\varepsilon^{(1)}(\square) = 1$ .

*Primitive central idempotents for  $n = 2$ :*

$$\begin{aligned} \varepsilon^{(2)}(\emptyset) &= P_{\square}^{\emptyset} \varepsilon^{(1)}(\square) = P_{\square}^{\emptyset} = \frac{J_2-1}{(1-\delta)-1} \cdot \frac{J_2+1}{(1-\delta)+1} \\ \varepsilon^{(2)}(\square) &= P_{\square}^{\square} \varepsilon^{(1)}(\square) = P_{\square}^{\square} = \frac{J_2-(1-\delta)}{1-(1-\delta)} \cdot \frac{J_2+1}{1+1} \\ \varepsilon^{(2)}(\boxplus) &= P_{\square}^{\boxplus} \varepsilon^{(1)}(\square) = P_{\square}^{\boxplus} = \frac{J_2-(1-\delta)}{-1-(1-\delta)} \cdot \frac{J_2-1}{-1-1}. \end{aligned}$$

*Primitive central idempotents for  $n = 3$ :*

$$\begin{aligned} \varepsilon^{(3)}(\square) &= P_{\square}^{\square} \varepsilon^{(2)}(\square) + P_{\square}^{\boxplus} \varepsilon^{(2)}(\boxplus) + P_{\square}^{\emptyset} \varepsilon^{(2)}(\emptyset) \\ &= \frac{(J_3+1)(J_3-2)}{(\delta+2)(\delta-1)} \varepsilon^{(2)}(\square) + \frac{(J_3+2)(J_3-1)}{(\delta-1)(\delta-4)} \varepsilon^{(2)}(\boxplus) + 1 \cdot \varepsilon^{(2)}(\emptyset) \\ \varepsilon^{(3)}(\square\square) &= P_{\square}^{\square\square} \varepsilon^{(2)}(\square) = \frac{(J_3+\delta)(J_3+1)}{3(\delta+2)} \varepsilon^{(2)}(\square) \\ \varepsilon^{(3)}(\boxplus) &= P_{\square}^{\boxplus} \varepsilon^{(2)}(\square) + P_{\square}^{\boxplus} \varepsilon^{(2)}(\boxplus) \\ &= -\frac{(J_3+\delta)(J_3-2)}{3(\delta-1)} \varepsilon^{(2)}(\square) + \frac{(J_3+\delta-2)(J_3+2)}{3(\delta-1)} \varepsilon^{(2)}(\boxplus) \\ \varepsilon^{(3)}(\boxplus) &= P_{\square}^{\boxplus} \varepsilon^{(2)}(\boxplus) = -\frac{(J_3+\delta-2)(J_3-1)}{3(\delta-4)} \varepsilon^{(2)}(\boxplus). \end{aligned}$$

*Primitive central idempotents for  $n = 4$ :*

There are eight idempotents at this level; we compute two of them:

$$\begin{aligned} \varepsilon^{(4)}(\square) &= P_{\square}^{\square} \varepsilon^{(3)}(\square) + P_{\square\square}^{\square} \varepsilon^{(3)}(\square\square) + P_{\square}^{\square} \varepsilon^{(3)}(\square) \\ &= \frac{(J_4+\delta-1)(J_4+1)}{2\delta} \varepsilon^{(3)}(\square) + \frac{(J_4+1)(J_4-3)}{(\delta+4)\delta} \varepsilon^{(3)}(\square\square) - \frac{(J_4+\delta)(J_4^2-4)J_4}{2(\delta-2)(\delta-4)\delta} \varepsilon^{(3)}(\square) \\ \varepsilon^{(4)}(\square\square) &= P_{\square}^{\square\square} \varepsilon^{(3)}(\square) + P_{\square\square}^{\square\square} \varepsilon^{(3)}(\square\square) \\ &= \frac{(J_4+\delta)(J_4+\delta-2)(J_4+2)J_4}{8(\delta+2)\delta} \varepsilon^{(3)}(\square) - \frac{(J_4+\delta+1)(J_4-3)}{4\delta} \varepsilon^{(3)}(\square\square). \end{aligned}$$

We note that the summands in each  $\varepsilon^{(n)}(\lambda)$  are the various  $\varepsilon_{\tau}$  in that block, so the  $\varepsilon_{\tau}$  are recoverable from the above expressions.

**Remark 5.12.** The recent paper [KMP] explores a completely different technique for computing central idempotents in semisimple Brauer algebras. Their technique is specific to that context.

## A. Primitive central idempotents via trace characters

We give a brief exposition of another approach to computing the primitive central idempotents in a split semisimple finite dimensional algebra  $\mathcal{A}$ . The approach generalizes a classical formula of Frobenius for the central idempotents of group algebras  $\mathbb{C}G$  for finite groups  $G$  (see Corollary A.2 below) in terms of the irreducible characters of  $G$ . We show that the irreducible trace characters of  $\mathcal{A}$  still uniquely determine its central idempotents, provided its defining field  $\mathbb{k}$  has characteristic zero.

Here, it is not necessary that  $\mathcal{A}$  fits into a multiplicity-free family. The requirement on  $\mathbb{k}$  guarantees invertibility of the  $(\dim \mathcal{A}) \times (\dim \mathcal{A})$  matrix of the natural trace form on  $\mathcal{A}$ . A slightly more general result (due to Kilmoyer) can be found in [CRI, Proposition (9.17)]; see also [Ram1].

**Definition.** Given any (not necessarily irreducible) finite dimensional  $\mathcal{A}$ -module  $V$ , let  $\chi^V$  be the *trace character* of  $V$ , defined by

$$\chi^V(a) = \text{trace}(\varphi^V(a)),$$

where  $\varphi^V : \mathcal{A} \rightarrow \text{End}_{\mathbb{k}}(V)$  is the representation corresponding to the  $\mathcal{A}$ -module  $V$ . If  $[V] = \lambda$  for  $\lambda \in \text{Irr}(\mathcal{A})$ , we write  $\chi^\lambda$  in place of  $\chi^V$ .

Let  $\rho = \chi^{\mathcal{A}}$  be the trace character of the left regular module; i.e., the character of  $\mathcal{A}$  regarded as a module over itself by left multiplication. Since  $\text{End}_{\mathbb{k}}(V^\lambda) \cong (V^\lambda)^* \otimes V^\lambda$ , it follows from (1.1) that

$$\mathcal{A} \cong \bigoplus_{\lambda} (\dim V^{\lambda}) V^{\lambda}$$

as left  $\mathcal{A}$ -modules. Since characters are additive on direct sums of modules and since  $\chi^{\lambda}(1) = \dim V^{\lambda}$ , it follows that

$$(A1) \quad \rho = \sum_{\lambda} (\dim V^{\lambda}) \chi^{\lambda} = \sum_{\lambda} \chi^{\lambda}(1) \chi^{\lambda}.$$

The problem of finding central idempotents  $\varepsilon(\lambda)$  is now framed as follows. Given a fixed basis  $B = B(\mathcal{A})$  of  $\mathcal{A}$ , write

$$(A2) \quad \varepsilon(\lambda) = \sum_{b \in B} c_b^{\lambda} b$$

and try to compute the coefficients  $c_b^{\lambda} \in \mathbb{k}$ . To that end, we may multiply both sides of (A2) by a basis element  $b' \in B$ , and then apply  $\rho$  to both sides to get

$$(A3) \quad \rho(\varepsilon(\lambda)b') = \sum_{b \in B} \rho(bb') c_b^{\lambda}.$$

On the other hand, we can use (A1) to express  $\rho(\varepsilon(\lambda)b')$  as

$$(A4) \quad \rho(\varepsilon(\lambda)b') = \sum_{\mu} \chi^{\mu}(1) \chi^{\mu}(\varepsilon(\lambda)b') = \chi^{\lambda}(1) \chi^{\lambda}(b').$$

(The last equality in (A4) comes by multiplying the equation  $1 = \sum_{\mu} \varepsilon(\mu)$  on the right by  $b'$ , then applying  $\chi^{\lambda}$  to both sides.) Note that  $\chi^{\lambda}(\varepsilon(\mu)b') = 0$  for  $\lambda \neq \mu$ , since  $\varepsilon(\mu)b'$  belongs to a block upon which  $\chi^{\lambda}$  acts as zero.

Finally, we combine (A3) and (A4) to obtain

$$(A5) \quad \sum_{b \in B} \rho(bb') c_b^{\lambda} = \chi^{\lambda}(1) \chi^{\lambda}(b').$$

For fixed  $\lambda$ , we may regard (A5) as a linear system (one equation for each  $b'$ ) that govern the values  $c_b^{\lambda}$ . This leads to the following result.

**Proposition A.1.** *Suppose a split semisimple finite dimensional algebra  $\mathcal{A}$  has underlying field  $\mathbb{k}$  of characteristic zero. Then the primitive central idempotents  $\varepsilon(\lambda)$  of  $\mathcal{A}$  are uniquely determined by its irreducible characters.*

*Proof.* Given a basis  $B$  of  $\mathcal{A}$ , let  $M = (\rho(bb'))_{b', b \in B}$  be the square matrix of coefficients in the linear system (A5), with rows indexed by  $b'$  and columns by  $b$ . This is just the matrix of the natural bilinear trace form, i.e.,  $(a, a') = \rho(aa') \forall a, a' \in \mathcal{A}$ , with respect to the basis  $B$ . As  $\mathcal{A}$  is split semisimple over a field of characteristic zero, a classical argument, as in [Vin, Theorem 11.54], shows that the trace form is nondegenerate. Hence  $M$  is invertible.

Let  $r^{\lambda}$  be the column vector  $(\chi^{\lambda}(1) \chi^{\lambda}(b'))_{b' \in B}$ . Then the column vector  $(c_b^{\lambda})_{b \in B}$  defining  $\varepsilon(\lambda)$  in (A2) is uniquely determined and equal to  $M^{-1}r^{\lambda}$ .  $\square$

We note that the vector  $r^\lambda$  in the proof of Proposition A.1 is just the  $\lambda$ -row of the character table of  $\mathcal{A}$ , scaled by  $\chi^\lambda(1) = \dim_{\mathbb{k}} V^\lambda$ . So we have an alternative method of producing the irreducible characters, provided this table is known. See, e.g., [Ram2] for the case of Brauer algebras.

In the case of group algebras, Proposition A.1 recovers the classical formula of Frobenius (see [Fro, III, pp. 244–274]). In that case, the matrix  $M$  of the natural trace form is easy to invert.

**Corollary A.2** (Frobenius). *Suppose that  $\mathcal{A} = \mathbb{k}G$  is a split semisimple group algebra over a field  $\mathbb{k}$  of characteristic zero, where  $G$  is a finite group. Then for any  $\lambda \in \text{Irr}(\mathbb{k}G)$ ,*

$$\varepsilon(\lambda) = \frac{1}{|G|} \sum_{g \in G} \chi^\lambda(1) \chi^\lambda(g^{-1}) g.$$

*Proof.* This follows from the observation that  $\rho(g)$  is zero for any  $g \neq 1_G$ , while  $\rho(1_G) = |G|$ , where  $1_G$  denotes the identity element of  $G$ . Indeed, let  $B(\mathcal{A}) = G$  be the basis of  $\mathcal{A}$  given by the group elements. Then the matrix  $M = (\rho(gg'))_{g', g \in G}$  in the proof of the proposition is  $|G|$  times the permutation matrix  $P = (\delta_{g^{-1}, g'})_{g', g \in G}$ , so  $M^{-1} = \frac{1}{|G|} P^T$ . Then

$$\rho(gg') = |G| \delta_{g^{-1}, g'}$$

in terms of the usual Kronecker delta. The formula for  $\varepsilon(\lambda)$  now follows by an easy calculation.  $\square$

**Acknowledgments.** This project started as an undergraduate research project by the third author, jointly mentored by the first two. The authors are grateful to the Mulcahy Scholars Program of Loyola University Chicago for support. Our work was greatly influenced by a seminar talk by Tony Giaquinto [Gia]. We would also like to thank Stuart Martin and Peter Tingley for useful conversations and advice, and the referee for suggesting substantial improvements.

## References

- [Bra] R. BRAUER, On algebras which are connected with the semisimple continuous groups. *Ann. of Math.* (2) **38** (1937), 857–872. [JFM 63.0873.02](#) [MR1503378](#)
- [Bro] W. P. BROWN, The semisimplicity of  $\omega_f^n$ . *Ann. of Math.* (2) **63** (1956), 324–335. [Zbl0070.26901](#) [MR0075931](#)
- [BW] J. S. BIRMAN and H. WENZL, Braids, link polynomials and a new algebra. *Trans. Amer. Math. Soc.* **313** (1989), 249–273. [Zbl0684.57004](#) [MR0992598](#)

- [CDVM] A. COX, M. DE VISSCHER and P. MARTIN, The blocks of the Brauer algebra in characteristic zero. *Represent. Theory* **13** (2009), 272–308. [Zbl1230.20046](#) [MR2550471](#)
- [CMPX] A. COX, P. MARTIN, A. PARKER and C. XI, Representation theory of towers of recollement: theory, notes, and examples. *J. Algebra* **302** (2006), 340–360. [Zbl1147.16016](#) [MR2236606](#)
- [CR1] C. W. CURTIS and I. REINER, *Methods of representation theory. Vol. I*. John Wiley & Sons, Inc., New York (1981). With applications to finite groups and orders; Pure and Applied Mathematics; A Wiley-Interscience Publication. [Zbl0469.20001](#) [MR1038525](#)
- [CR2] ———, *Representation theory of finite groups and associative algebras*. AMS Chelsea Publishing, Providence, RI (2006). Reprint of the 1962 original. [Zbl1093.20003](#) [MR2215618](#)
- [CSST] T. CECCHERINI-SILBERSTEIN, F. SCARABOTTI and F. TOLLI, *Representation theory of the symmetric groups*, vol. 121 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge (2010). The Okounkov–Vershik approach, character formulas, and partition algebras. [Zbl0070.26901](#) [MR0075931](#)
- [Cur] C. W. CURTIS, *Pioneers of representation theory: Frobenius, Burnside, Schur, and Brauer*, vol. 15 of *History of Mathematics*. American Mathematical Society, Providence, RI; London Mathematical Society, London (1999). [Zbl0939.01007](#) [MR1715145](#)
- [DG] P. DIACONIS and C. GREENE, *Applications of Murphy's elements* (1998). <http://statweb.stanford.edu/~cgates/PERSI/papers/EFSNSF335.pdf>. Accessed 2014. Technical Report No. 335, Department of Statistics, Stanford University.
- [DJ] R. DIPPER and G. JAMES, Blocks and idempotents of Hecke algebras of general linear groups. *Proc. London Math. Soc.* (3) **54** (1987), 57–82. [Zbl0615.20009](#) [MR0872250](#)
- [DWH] W. F. V. DORAN, D. B. WALES and P. J. HANLON, On the semisimplicity of the Brauer centralizer algebras. *J. Algebra* **211** (1999), 647–685. [Zbl0944.16002](#)
- [FG] S. FISHEL and I. GROJNOWSKI, Canonical bases for the Brauer centralizer algebra. *Math. Res. Lett.* **2** (1995), 15–26. [Zbl0842.20037](#) [MR1312974](#)
- [Fro] F. G. FROBENIUS, *Gesammelte Abhandlungen. Bände I, II, III*. Herausgegeben von J.-P. Serre. Springer-Verlag, Berlin-New York (1968). [Zbl0169.28901](#) [MR0235974](#)
- [Ful] W. FULTON, *Young tableaux*, vol. 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge (1997). [Zbl0878.14034](#) [MR1464693](#)
- [Gar] A. GARSIA, *Young's seminormal representation, Murphy elements, and content evaluations* (2003). [www.math.ucsd.edu/~garsia/somepapers/Youngseminormal.pdf](http://www.math.ucsd.edu/~garsia/somepapers/Youngseminormal.pdf). Accessed 2014. Course Lecture Notes.

- [GdlHJ] F. M. GOODMAN, P. DE LA HARPE and V. F. R. JONES, *Coxeter graphs and towers of algebras*, vol. 14 of *Mathematical Sciences Research Institute Publications*. Springer-Verlag, New York (1989). [Zbl 0698.46050](#) [MR 0999799](#)
- [GG1] F. M. GOODMAN and J. GRABER, Cellularity and the Jones basic construction. *Adv. in Appl. Math.* **46** (2011), 312–362. [Zbl 1266.20005](#) [MR 2794027](#)
- [GG2] — On cellular algebras with Jucys Murphy elements. *J. Algebra* **330** (2011), 147–176. [MR 2774622](#)
- [Gia] A. GIAQUINTO, *Orthogonal bases of irreducible symmetric group representations* (2014). Loyola University Chicago Algebra and Combinatorics Seminar, 26th February 2014.
- [GL] J. J. GRAHAM and G. I. LEHRER, Cellular algebras. *Invent. Math.* **123** (1996), 1–34. [Zbl 0853.20029](#) [MR 1376244](#)
- [Gre] J. A. GREEN, *Polynomial representations of  $GL_n$* , vol. 830 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York (1980). Second ed., 2007. [Zbl 0451.20037](#) [MR 0606556](#)
- [GT1] I. M. GELFAND and M. L. TSETLIN, Finite-dimensional representations of groups of orthogonal matrices. *Dokl. Akad. Nauk SSSR (N.S.)* **71** (1950), 1017–1020. [MR 0035774](#)
- [GT2] — Finite-dimensional representations of the group of unimodular matrices. *Dokl. Akad. Nauk SSSR (N.S.)* **71** (1950), 825–828. [MR 0034763](#)
- [HR] T. HALVERSON and A. RAM, Characters of algebras containing a Jones basic construction: The Temperley–Lieb, Okada, Brauer, and Birman–Wenzl algebras. *Adv. Math.* **116** (1995), 263–321. [Zbl 0856.16038](#) [MR 1363766](#)
- [HW] P. HANLON and D. WALES, Computing the discriminants of Brauer’s centralizer algebras. *Math. Comp.* **54** (1990), 771–796. [Zbl 0698.20006](#) [MR 1010599](#)
- [Juc] A.-A. A. JUCYS, Symmetric polynomials and the center of the symmetric group ring. *Rep. Mathematical Phys.* **5** (1974), 107–112. [Zbl 0288.20014](#) [MR 0419576](#)
- [Kau] L. H. KAUFFMAN, An invariant of regular isotopy. *Trans. Amer. Math. Soc.* **318** (1990), 417–471. [Zbl 0763.57004](#) [MR 0958895](#)
- [KMP] O. KING, P. MARTIN and A. PARKER, On central idempotents in the Brauer algebra. *J. Algebra* **512** (2018), 20–46. [Zbl 06915729](#) [MR 3841515](#)
- [Las] A. LASCoux, Young’s representations of the symmetric group (2001). pp. 94–104.
- [LR] R. LEDUC and A. RAM, A ribbon Hopf algebra approach to the irreducible representations of centralizer algebras: The Brauer, Birman–Wenzl, and type A Iwahori–Hecke algebras. *Adv. Math.* **125** (1997), 1–94. [Zbl 0936.17016](#) [MR 1427801](#)
- [Mar] P. MARTIN, The structure of the partition algebras. *J. Algebra* **183** (1996), 319–358. [Zbl 0863.20009](#) [MR 1399030](#)
- [Mat] A. MATHAS, Seminormal forms and Gram determinants for cellular algebras. *J. Reine Angew. Math.* **619** (2008), 141–173. With an appendix by Marcos Soriano. [Zbl 1152.20008](#) [MR 2414949](#)

- [MRH] P. P. MARTIN and S. RYOM-HANSEN, Virtual algebraic Lie theory: Tilting modules and Ringel duals for blob algebras. *Proc. London Math. Soc.* (3) **89** (2004), 655–675. [Zbl 1134.20301](#) [MR 2107010](#)
- [MS] P. MARTIN and H. SALEUR, The blob algebra and the periodic Temperley–Lieb algebra. *Lett. Math. Phys.* **30** (1994), 189–206. [Zbl 0799.16005](#) [MR 1267001](#)
- [Mur] G. E. MURPHY, A new construction of Young’s seminormal representation of the symmetric groups. *J. Algebra* **69** (1981), 287–297. [Zbl 0455.20007](#) [MR 0617079](#)
- [Naz] M. NAZAROV, Young’s orthogonal form for Brauer’s centralizer algebra. *J. Algebra* **182** (1996), 664–693. [Zbl 0868.20012](#) [MR 1398116](#)
- [OP] O. V. OGIEVETSKY and P. N. PYATOV, *Lecture on Hecke algebras*. Bonn (2001). <http://cds.cern.ch/record/735174>. Accessed 2016. Based on lectures presented at the International School “Symmetries and Integrable Systems”, Dubna, 8–11 June 1999.
- [OV] A. OKOUNKOV and A. VERSHIK, A new approach to representation theory of symmetric groups. *Selecta Math. (N.S.)* **2** (1996), 581–605. [Zbl 0959.20014](#) [MR 1443185](#)
- [Ram1] A. RAM, Dissertation, Chapter 1: Representation Theory (version July 4, 1990) (1990). <http://www.ms.unimelb.edu.au/~ram/Preprints/dissertationChapt1.pdf>. Accessed 2016.
- [Ram2] — Characters of Brauer’s centralizer algebras. *Pacific J. Math.* **169** (1995), 173–200. [Zbl 0839.20019](#) [MR 1346252](#)
- [Ram3] — A “second orthogonality relation” for characters of Brauer algebras. *European J. Combin.* **18** (1997), 685–706. [Zbl 0884.20009](#) [MR 1468338](#)
- [Ram4] — Seminormal representations of Weyl groups and Iwahori–Hecke algebras. *Proc. London Math. Soc.* (3) **75** (1997), 99–133. [Zbl 0876.20010](#) [MR 1444315](#)
- [Rui] H. RUI, A criterion on the semisimple Brauer algebras. *J. Combin. Theory Ser. A* **111** (2005), 78–88. [Zbl 1095.16006](#) [MR 2144855](#)
- [RW] A. RAM and H. WENZL, Matrix units for centralizer algebras. *J. Algebra* **145** (1992), 378–395. [Zbl 0761.16008](#) [MR 1144939](#)
- [Sch] I. SCHUR, Über die rationalen Darstellungen der allgemeinen linearen Gruppe. *Sitzungsber. Akad. Berlin* (1927), 58–75. [JFM 53.0108.05](#)
- [Sim] B. SIMON, *Representations of Finite and Compact Groups*, vol. 10 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI (1996). [Zbl 0840.22001](#) [MR 1363490](#)
- [Ste] J. R. STEMBRIDGE, Orthogonal sets of Young symmetrizers. *Adv. in Appl. Math.* **46** (2011), 576–582. [Zbl 1227.05268](#) [MR 2794038](#)
- [Thr] R. M. THRALL, Young’s semi-normal representation of the symmetric group. *Duke Math. J.* **8** (1941), 611–624. [Zbl 0061.04102](#) [MR 0005728](#)



- [Vin] E. B. VINBERG, *A course in algebra*, vol. 56 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI (2003). Translated from the 2001 Russian original by Alexander Retakh. [Zbl1016.00003](#) [MR1974508](#)
- [VO] A. M. VERSHIK and A. Y. OKUN'KOV, A new approach to representation theory of symmetric groups. II. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **307** (2004), 57–98, 281. [Zbl1083.20502](#) [MR2050688](#)
- [Wen] H. WENZL, On the structure of Brauer's centralizer algebras. *Ann. of Math.* (2) **128** (1988), 173–193. [Zbl0656.20040](#) [MR0951511](#)
- [Xi] C. XI, On the quasi-heredity of Birman–Wenzl algebras. *Adv. Math.* **154** (2000), 280–298. [Zbl0971.16008](#) [MR1784677](#)
- [You] A. YOUNG, On quantitative substitutional analysis. I–IX. *Proc. London Math. Soc.* (1901–1952) .

(Reçu le 30 juin 2016; version révisée reçue le 10 juillet 2018)

Stephen DOTY, Department of Mathematics and Statistics,  
Loyola University Chicago, Chicago, IL 60660 USA

*e-mail:* [doty@math.luc.edu](mailto:doty@math.luc.edu)

Aaron LAUVE, Department of Mathematics and Statistics,  
Loyola University Chicago, Chicago, IL 60660 USA

*e-mail:* [lauve@math.luc.edu](mailto:lauve@math.luc.edu)

George H. SEELINGER, Department of Mathematics, University of Virginia,  
P.O. Box 400137, Charlottesville, VA 22904 USA

*e-mail:* [ghs9ae@virginia.edu](mailto:ghs9ae@virginia.edu)