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Some Kähler structures on products of 2-spheres

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Abstract. We consider a family of Kähler structures on products of 2-spheres, arising from complex Bott manifolds. These are obtained via iterated $\mathbb{P}^1$-bundle constructions, generalizing the classical Hirzebruch surfaces. To each such Kähler structure, we associate a Bott diagram, which is a rooted forest with an edge labelling by positive integers. We show that the Bott diagrams classify these Kähler structures up to biholomorphism.

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1. Introduction

In complex geometry, it is interesting to study the class of complex structures (or Kähler structures) supported on a fixed smooth oriented manifold M. Since the basic invariants of a complex manifold are the Chern classes, it is tempting to try and use these to distinguish complex structures on M. In complex dimension two, the Hirzebruch surfaces $\mathbb{F}_m$ are topologically either diffeomorphic to $S^2 \times S^2$ (if m is even), or to $\mathbb{P}^2 \# \overline{\mathbb{P}^2}$ (if m is odd). Focusing on the Hirzebruch surfaces diffeomorphic to $S^2 \times S^2$, a celebrated result of Hirzebruch [Hirl] shows that all the $\mathbb{F}_{2k}$ for $k \geq 0$ are distinct as complex manifolds, even though they have identical Chern classes.

In the present paper, we extend Hirzebruch's result, by considering *$\mathbb{Z}$-trivial complex Bott manifolds*. Bott manifolds first appeared in work of Bott and Samelson [BS, Ch. III, Sections 3, 4, 5], though the terminology was introduced by Grossberg and Karshon [GK, Section 2]. In complex dimension two, the $\mathbb{Z}$-trivial complex Bott manifolds are precisely the Hirzebruch surfaces $\mathbb{F}_{2k}$. In complex dimension n, these are compact Kähler manifolds diffeomorphic to $S^2 \times \cdots \times S^2 = (S^2)^n$. To each n-dimensional $\mathbb{Z}$-trivial complex Bott manifold

M, we associate a *Bott diagram*, which is a rooted forest equipped with an edge labelling by positive integers. Our main result is the following:

Main Theorem. *Every n-vertex rooted forest equipped with an edge labelling by positive integers arises as the Bott diagram of some n-dimensional $\mathbb{Z}$-trivial complex Bott manifold. A pair of n-dimensional $\mathbb{Z}$-trivial complex Bott manifolds M_1, M_2, are biholomorphic if and only if their Bott diagrams are isomorphic. Moreover, there is a diffeomorphism $\phi : M_1 \to M_2$ with the property that $\phi^*(c(M_2)) = c(M_1)$, where c denotes the total Chern classes.*

Our result provides a combinatorial classification of a certain family of Kähler structures on the products $S^2 \times \cdots \times S^2 = (S^2)^n$. When $n = 2$, the only Kähler structures on $S^2 \times S^2$ are those arising from the Hirzebruch surfaces. When $n \geq 3$, we do not know whether these products of 2-spheres support any other Kähler structures. Our result also shows that these Kähler structures are indistinguishable as far as Chern classes are concerned.

Our paper is structured as follows. We review some background material in Section 2, and prove our main theorem in Section 3. Our argument requires a reconstruction result for labelled rooted forests, a combinatorial result which is explained in the Appendix (Section A). Finally, in Section 4, we formulate a generalization of a well-known problem of Hirzebruch (recently solved by Kotschick [Kot1], [Kot2]), and explain how it led us to the results in this paper.

2. Background material

2.1. Bott manifolds. Recall that a Bott manifold M^n is a complex n-manifold that admits a *Bott tower*, namely, $M^n = B_n$ and

$$B_n \xrightarrow{\pi_n} B_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{\text{a point}\} \tag{2.1}$$

where for each $1 \leq j \leq n$, $B_j = \mathbb{P}(\mathcal{O} \oplus S_j)$ is the projectivization of the direct sum of the trivial line bundle $\mathcal{O}$ with a holomorphic line bundle S_j over B_{j-1}, with π_j the projection map.

Example. Let us consider the two dimensional Bott manifolds. Clearly $B_1 = \mathbb{P}^1$, and let us denote by $\mathcal{O}_{\mathbb{P}^1}(m)$ the line bundle over $\mathbb{P}^1$ with first Chern class $m[\mathbb{P}^1] \in H^2(\mathbb{P}^1; \mathbb{Z})$. Then B_2 is the Hirzebruch surface $\mathbb{F}_m = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}_{\mathbb{P}^1}(-m))$ over $\mathbb{P}^1$, where m is any integer. Since $\mathbb{P}(E)$ is biholomorphic to $\mathbb{P}(E \otimes L)$ for any line bundle L, tensoring with the dual bundle $(\mathcal{O}_{\mathbb{P}^1}(-m))^* = \mathcal{O}_{\mathbb{P}^1}(m)$ we see that there are canonical biholomorphisms $\mathbb{F}_m \cong \mathbb{F}_{-m}$. Thus when considering

Hirzebruch surfaces as complex manifolds, it is sufficient to consider $\mathbb{F}_m$ with $m \geq 0$. Hirzebruch showed these are all distinct as complex manifolds [Hir1]. He also showed that all $\mathbb{F}_{2k}$ are diffeomorphic to $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1 \cong S^2 \times S^2$, while all $\mathbb{F}_{2k-1}$ are diffeomorphic to $\mathbb{F}_1 = \mathbb{P}^2 \# \overline{\mathbb{P}^2}$, the one point blow up of $\mathbb{P}^2$.

Definition 2.1. A Bott manifold M^n is called $\mathbb{Z}$-trivial, if its integral cohomology ring $H^*(M;\mathbb{Z})$ is (graded) ring isomorphic to $H^*((\mathbb{P}^1)^n;\mathbb{Z})$.

By the work of Masuda and Panov [MP, Theorem 5.1], a Bott manifold M^n is $\mathbb{Z}$-trivial if and only if it is diffeomorphic to $(\mathbb{P}^1)^n$. In fact, it follows from Choi and Masuda [CM, Cor. 5.1] that every graded ring isomorphism between the (integral) cohomology rings of two $\mathbb{Z}$-trivial Bott manifolds is induced by a diffeomorphism. This can also be seen directly from the corresponding statement for $(S^2)^n$.

2.2. Projectivization of vector bundles. Let us recall some general facts concerning projectivizations of vector bundles.

Let E be a holomorphic vector bundle of rank r over a compact complex manifold B, and let $\pi : M = \mathbb{P}(E) \to B$ be the projectivization of E, where π is the projection map. We adopt the algebro-geometric convention here, namely, $\pi^{-1}(x) = \mathbb{P}(E_x)$ is the set of all the *hyperplanes* (instead of *lines*) through the origin in the fiber $E_x \cong \mathbb{C}^r$. Then M is again a compact complex manifold, a holomorphic fiber bundle with fiber $\mathbb{P}^{r-1}$ over B.

Denote by L the dual of the tautological line bundle, then we have the following two short exact sequences of holomorphic vector bundles over M:

$$0 \to \mathcal{O} \to \pi^* E^* \otimes L \to T_{M|B} \to 0 \tag{2.2}$$

$$0 \to T_{M|B} \to T_M \to \pi^* T_B \to 0 \tag{2.3}$$

where $T_{M|B}$ is the relative tangent bundle, namely, the kernel of the differential of π. The first Chern class $\xi = c_1(L)$ satisfies the Grothendieck equation

$$f(\xi) := \xi^r - \xi^{r-1} \cdot \pi^* c_1(E) + \xi^{r-2} \cdot \pi^* c_2(E) - \cdots + (-1)^r \pi^* c_r(E) = 0,$$

while the cohomology ring (or the Chow ring) of M is generated by the pullback of that of B and ξ:

$$H^*(M,\mathbb{Z}) := \pi^* H^*(B,\mathbb{Z})\ [\xi]\ /\ \big(f(\xi)\big). \tag{2.4}$$

Recall that a *section* of π is a complex submanifold $Z \subseteq M$ such that $\pi|_Z : Z \to B$ is a biholomorphism. Equivalently, a section of π is given by a holomorphic map $i : B \to M$ such that $\pi \circ i = id_B$. In this case the image

$i(B)$ is the submanifold in M biholomorphic to B. Note that the sections of π correspond to quotient line bundles of E.

To see this, let Q be a holomorphic line bundle on B which is a quotient bundle of E. As we are using the hyperplane convention for projectivizations, so $\mathbb{P}(Q) \cong B$ is a submanifold of $\mathbb{P}(E) = M$, which gives a section of π. Conversely, given a section $i : B \to M$ of π, since the tautological line bundle L^* is a subbundle of $\pi^* E^*$ on M, $Q = i^* L$ would be a quotient line bundle of $i^* \pi^* E = E$ on B.

Next, let us specialize to the situation when the vector bundle on B is $E = \mathcal{O} \oplus S$, the sum of the trivial line bundle with another line bundle S. Writing $s = -\pi^* c_1(S)$, the above short exact sequences (2.2), (2.3), along with the Grothendieck equation, gives us

$$c_1(T_{M|B}) = 2\xi + s, \quad c(M) = (1 + 2\xi + s) \cdot \pi^* c(B), \quad \text{and } \xi^2 = -\xi \cdot s \tag{2.5}$$

in the cohomology (or the Chow) ring $H^*(M, \mathbb{Z})$.

2.3. Cohomology ring of Bott manifolds. Now let us apply these formula to the j-th stage $\pi_j : B_j \to B_{j-1}$, which is the projectivization of the splitting rank 2 vector bundle $\mathcal{O} \oplus S_j$ on B_{j-1}. We get the following:

$$H^*(B_j, \mathbb{Z}) = \pi_j^* H^*(B_{j-1}, \mathbb{Z})[\xi_j]/(\xi_j^2 + \xi_j s_j)$$
$$c(B_j) = (1 + 2\xi_j + s_j) \cdot \pi_j^* c(B_{j-1}),$$

where $-s_j$ and ξ_j are the first Chern class of $\pi_j^* S_j$ and $L_j = \mathcal{O}_{B_j}(1)$, the dual of the tautological line bundle on B_j.

Given a Bott manifold M^n with Bott tower (2.1), let us write

$$x_j = (\pi_{j+1} \circ \cdots \circ \pi_n)^* \xi_j$$
$$h_j = (\pi_{j+1} \circ \cdots \circ \pi_n)^* s_j$$

for each $1 \leq j \leq n$. Note that x_1 is the first Chern class of the pull back to M of $\mathcal{O}_{\mathbb{P}^1}(1)$ on B_1, and $h_1 = 0$. By an inductive argument, we obtain the following:

$$H^*(M, \mathbb{Z}) = \mathbb{Z}[x_1, \ldots, x_n]/(x_1^2, x_2^2 + x_2 h_2, \ldots, x_n^2 + x_n h_n) \tag{2.6}$$
$$c(M) = (1 + 2x_1)(1 + 2x_2 + h_2) \cdots (1 + 2x_n + h_n) \tag{2.7}$$

where $x_1, \ldots, x_n$ is a set of generators for $H^2(M, \mathbb{Z}) \cong \mathbb{Z}^n$, and each h_j satisfies

$$h_j = a_{j1} x_1 + a_{j2} x_2 + \cdots + a_{j,j-1} x_{j-1} \tag{2.8}$$

where all a_{jk} are integers.

Example. In the special case where all the line bundles S_j are trivial, we get the product $P = (\mathbb{P}^1)^n$ of n-copies of the complex projective line $\mathbb{P}^1$. In this case, all $h_j = 0$ and we will denote the corresponding x_j by y_j. The above computations give us:

$$H^*(P, \mathbb{Z}) = \mathbb{Z}[y_1, \ldots, y_n]/(y_1^2, \ldots, y_n^2) \tag{2.9}$$

$$c(P) = (1 + 2y_1) \cdots (1 + 2y_n). \tag{2.10}$$

3. Proof of the Main Theorem

This entire section is devoted to the proof of the Main Theorem. Throughout this section, all Bott manifolds will be $\mathbb{Z}$-trivial.

3.1. The structure of $\mathbb{Z}$-trivial Bott manifolds. We start by analyzing how the $\mathbb{Z}$-triviality condition affects the cohomology elements h_j.

For a given a Bott tower on M^n, assume that $2 \mid h_j$ and $h_j^2 = 0$ for all j. Write $z_j = x_j + \frac{1}{2}h_j$. Since $h_1 = 0$, and for each $2 \leq j \leq n$ the corresponding h_j is generated by $x_1, \ldots, x_{j-1}$, it follows that $\{z_1, \ldots, z_n\}$ generates $H^*(M, \mathbb{Z})$. Also, each $z_j^2 = 0$ by the Grothendieck equation. So defining $\phi(y_j) = z_j$ gives a graded ring isomorphism $\phi : H^*(P; \mathbb{Z}) \to H^*(M^n; \mathbb{Z})$, and M^n is $\mathbb{Z}$-trivial. By the result of Choi and Masuda [CM], there is a diffeomorphism $\Phi : M^n \to P \cong (S^2)^n$, which induces $\Phi^* = \phi$. Moreover, by the Chern class formula, we see that $\phi(c(P)) = c(M)$.

Conversely, if there exists an isomorphism $\phi : H^*(P, \mathbb{Z}) \to H^*(M, \mathbb{Z})$, then we claim that $2 \mid h_j$ and $h_j^2 = 0$ for all j. To see this, let us write $\phi(y_j) = z_j$. We have

$$H^*(M, \mathbb{Z}) = \mathbb{Z}[z_1, \ldots, z_n]/(z_1^2, \ldots, z_n^2).$$

For each $1 \leq k \leq n$, the group $H^{2k}(M, \mathbb{Z})$ is a free abelian group generated by products $z_I = z_{i_1} \cdots z_{i_k}$ for all multi-indices $I = (i_1 \cdots i_k)$ of length k, where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Note that for any integer linear combination $z = a_1 z_1 + \cdots + a_n z_n$, if $z^2 = 0$, then $a_i a_j = 0$ for all $i \neq j$. Thus at most one of these a_i could be non-zero.

Now we proceed to show that $2 \mid h_j$ and $h_j^2 = 0$, by induction on j, where $j \in A := \{1, 2, \ldots, n\}$. First we have $h_1 = 0$. For $j = 2$, since $x_1^2 = 0$, we know that there must be a unique $i_1 \in A$ such that $x_1 = \varepsilon_1 z_{i_1}$, where $\varepsilon_1 = \pm 1$ since x_1 is a primitive element in $H^2(M, \mathbb{Z})$. Write $x_2 = a z_{i_1} + z$, where z is a linear combination of z_j for $j \in A \setminus \{i_1\}$. We have $h_2 = b z_{i_1}$ since h_2 is a multiple of x_1. Since $x_2(x_2 + h_2) = 0$, we have

$$(2a + b) z_{i_1} z + z^2 = 0.$$

Since $H^4(M,\mathbb{Z})$ is a free abelian group with generators $z_i z_j$ for $1 \le i < j \le n$, we conclude from the above equality that $2a + b = 0$ and $z^2 = 0$. So $2 \mid h_2$, $h_2^2 = 0$, and $z = x_2 + \frac{1}{2}h_2$ satisfies $z^2 = 0$, thus equals to $\varepsilon_2 z_{i_2}$ for some $i_2 \neq i_1$, and $\varepsilon_2 = \pm 1$.

Now assume that for a fixed $2 \le k < n$, we already have $2 \mid h_j$, $h_j^2 = 0$ for each $j \le k$, and $x_j' := x_j + \frac{1}{2}h_j = \varepsilon_j z_{i_j}$ where i_1, ..., i_k are all distinct in A and $\varepsilon_j = \pm 1$. Since h_{k+1} is a linear combination of $x_1', \ldots, x_k'$, we can write

$$h_{k+1} = b_1 z_{i_1} + \cdots + b_k z_{i_k}$$

Also, let us write $x_{k+1} = a_1 z_{i_1} + \cdots + a_k z_{i_k} + z$, where z is a combination of those z_j for j in $A \setminus \{i_1, \ldots, i_k\}$. Now by applying the Grothendieck equation, namely, $x_{k+1}(x_{k+1} + h_{k+1}) = 0$, we get the equation

$$\sum_{j=1}^{k}(2a_j + b_j) z_{i_j} z + z^2 + \sum_{j,l=1}^{k} a_l(a_j + b_j) z_{i_j} z_{i_l} = 0.$$

Since z cannot be zero, we know that $b_j = -2a_j$ for each $j \le k$, so $2 \mid h_{k+1}$ and $h_{k+1}^2 = 0$. Furthermore, $x_{k+1}' = z$ is a square zero primitive element, thus must be of the form $\pm z_{i_{k+1}}$ for some i_{k+1} in $A \setminus \{i_1, \ldots, i_k\}$.

To summarize, we have established the following (also independently obtained by J. H. Kim [Kim]):

Lemma 3.1. *If M^n is a Bott manifold and ϕ is an isomorphism between the integral cohomology rings of $P = (\mathbb{P}^1)^n$ and M, then for any Bott tower (2.1) with $M = B_n$, we have $2 \mid h_j$ and $h_j^2 = 0$ for each j, and $\phi(c(P)) = c(M)$.*

Note that for any holomorphic line bundle Q on B, the projectivizations $\mathbb{P}(E)$ and $\mathbb{P}(E \otimes Q)$ are isomorphic to each other. In particular, for $B_j = \mathbb{P}(\mathcal{O} \oplus S_j)$ over B_{j-1}, one can replace S_j by its dual S_j^*, as

$$\mathcal{O} \oplus S_j \cong (S_j^* \oplus \mathcal{O}) \otimes S_j.$$

This replacement will not change B_j, but will affect the choice of sections L_j thus affecting x_j, while h_j is replaced by $-h_j$.

By the proof of the lemma above, we know that for any $\mathbb{Z}$-trivial Bott manifold M^n and any Bott tower (2.1) on M, if we write $z_j = x_j + \frac{1}{2}h_j$, then $\{z_1, \ldots, z_n\}$ is a set of generators for the cohomology ring, with $z_j^2 = 0$ for each j. For any $2 \le j \le n$, since $h_j^2 = 0$, we know that either $h_j = 0$, or $h_j = 2q_j z_{\sigma(j)}$ for some positive integer q_j and $\sigma(j) < j$. Here we used the fact that we can replace S_j by S_j^* without changing the Bott tower to ensure that these q_j be positive (compare with the Example in Section 2.1). From now on, we will make

these choices, so $q_j > 0$ whenever $h_j \neq 0$. That is, under our choices of these S_j, each x_j is represented by the central sections of π_j, and each $z_j = x_j + \frac{1}{2}h_j$ is represented by an effective divisor.

Next let us notice that x_j as a hypersurface is represented by an $(n-1)$-dimensional $\mathbb{Z}$-trivial Bott manifold. Indeed, we have that x_j is represented by the hypersurface $f^{-1}\xi_j$, the preimage of the central section ξ_j of $\pi_j : B_j \to B_{j-1}$, via the composition $f = \pi_n \circ \cdots \circ \pi_{j+1}$. We thus obtain a Bott tower $B'_{n-1} \to \ldots \to B'_0$ for this hypersurface, by setting

(1) $B'_i = B_i$ for $i < j-1$,

(2) $B'_{j-1} = \xi_j \cong B_{j-1}$, and

(3) $B'_k = \pi_{k+1}^{-1}(B'_{k-1}) \subset B_{k+1}$ for $j \leq k \leq n-1$.

To see this Bott tower is $\mathbb{Z}$-trivial, we recall from our discussion above that this property is characterized in terms of the elements h'_i associated to the Bott tower – it is necessary and sufficient that $2 \mid h'_i$ and $(h'_i)^2 = 0$ (for all i). When $i \leq j-1$, the elements h'_i for this Bott tower B' coincide with the elements h_i of the original Bott tower B. On the other hand, when $k \geq j$, the elements h'_k for the Bott tower C are the pull-backs, via the inclusion maps $B'_k \subset B_{k+1}$, of the elements h_{k+1} of the original B. In either case, the condition $2 \mid h'_i$ and $(h'_i)^2$ is inherited by the Bott tower B', giving us the desired $\mathbb{Z}$-triviality. We summarize our discussion so far in the following Lemma.

Lemma 3.2. *For any Bott tower* (2.1) *we can choose the generator sets* $\{x_1, \ldots, x_n\}$ *and* $\{z_1, \ldots, z_n\}$ *so that* (*i*) *each* $z_j^2 = 0$, (*ii*) *each* z_j *is represented by an effective divisor, and* (*iii*) *each* x_j *is represented by a smooth hypersurface, which is itself a* $\mathbb{Z}$*-trivial Bott manifold of dimension* $n-1$.

Obviously, for a given Bott manifold M^n, there are many Bott towers on it. So to sort out all distinct complex structures on $P = (S^2)^n$ given by the Bott manifolds, we need to find canonical representatives for the $\mathbb{Z}$-trivial Bott manifolds. This is the goal of the next section.

3.2. Bott diagrams for $\mathbb{Z}$-trivial Bott manifolds. Let us denote by $A = \{1, 2, \ldots, n\}$ and write $A_0 = \{j \in A \mid h_j = 0\}$. When $A_0 \neq A$, we have a map $\sigma : A \setminus A_0 \to A$ satisfying $\sigma(j) < j$, given by the equation $h_j = 2q_j z_{\sigma(j)}$. Let us denote by $A_1 = \sigma^{-1}(A_0)$, $A_2 = \sigma^{-1}(A_1)$, and so on. It is easy to see that there exists some positive integer r such that A is the disjoint union of non-empty sets A_0, A_1, …, A_r.

We will say that the *level* of $j \in A$ is k if $j \in A_k$. It takes σ exactly k times to send a level k element into A_0.

Definition 3.3. For a given Bott tower (2.1), we define its *Bott diagram* to be the following data: each element of A gives a vertex, each $j \in A \setminus A_0$ gives a vertical edge from j to $\sigma(j)$, marked with a positive integer q_j.

In other words, a Bott diagram G in dimension n is a disjoint union $A = A_0 \cup A_1 \cup \cdots \cup A_r$ into $r+1$ nonempty subsets, along with maps $A_r \to A_{r-1} \to \cdots \to A_1 \to A_0$ and a map $q : A \setminus A_0 \to \mathbb{Z}^+$. Here A is the set of n elements and $r \geq 0$.

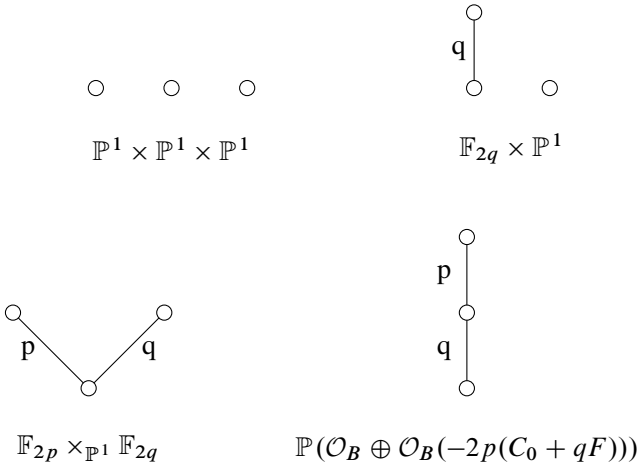
Two Bott diagrams are considered isomorphic, if there is a bijection from A to A which commutes with the partition of A and the maps.

We can arrange all the dots in A_k at the same height, and will refer to that (imagined) horizontal line the level k line (when $k = 0$ we will also call it the base line). The diagram is a graph, with finitely many connected components which are *trees*. Each tree has a distinguished vertex, lying in A_0, which is the *root* of the tree. Thus from a combinatorial viewpoint, a Bott diagram is a *rooted forest*. Clearly, the Bott manifold is a product of lower dimensional ones, with each factor corresponding to a connected component of the Bott diagram. So M^n is irreducible (in the sense that it is not the product of lower dimensional Bott manifolds) if and only if the Bott diagram is connected, which occurs if and only if A_0 contains only one element.

Example. To illustrate how these diagrams work, let us first consider the case $n = 2$. In this case we have only two possibilities for the Bott diagram: the first one just has two dots lying horizontally, with no edges, representing the surface $\mathbb{P}^1 \times \mathbb{P}^1$; and the second one is two dots with a vertical edge connecting them, marked by a positive integer q. This corresponds to the Hirzebruch surface $\mathbb{F}_{2q}$.

Example. For $n = 3$, we have three horizontal dots, corresponding to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$; two dots on the base line, the third dot on top of the left one with a vertical edge marked q, corresponding to $\mathbb{F}_{2q} \times \mathbb{P}^1$; one dot on the base line, two dots on the level 1 line joining the base point by edges marked with q and p, which corresponds to the fiber product $\mathbb{F}_{2p} \times_{\mathbb{P}^1} \mathbb{F}_{2q}$; and finally, we have three dots lined up in a vertical line, with two edges marked with p and q (with p on top). In this case the threefold is $M^3 = \mathbb{P}(\mathcal{O}_B \oplus \mathcal{O}_B(-2p(C_0 + qF)))$, where $B = \mathbb{F}_{2q}$ is the Hirzebruch surface with F the ruling and C_0 the central section (so $C_0^2 = -2q$). See Figure 1 for an illustration of the possible Bott diagrams.

3.3. Determining biholomorphism type of $\mathbb{Z}$-trivial Bott manifolds. Our goal here is to complete the proof of our main theorem, by showing the following:



$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ $\quad$ $\mathbb{F}_{2q} \times \mathbb{P}^1$

$\mathbb{F}_{2p} \times_{\mathbb{P}^1} \mathbb{F}_{2q}$ $\quad$ $\mathbb{P}(\mathcal{O}_B \oplus \mathcal{O}_B(-2p(C_0 + qF)))$

FIGURE 1
Bott diagrams for 3-dimensional $\mathbb{Z}$-trivial Bott manifolds

Theorem 3.4. *Two $\mathbb{Z}$-trivial Bott manifolds of dimension n are biholomorphic to each other if and only if they have isomorphic Bott diagrams.*

Since one can build up a Bott tower from the data of a Bott diagram, we just need to prove the "only if" part of the statement, namely, if $f : M' \to M$ is a biholomorphism between $\mathbb{Z}$-trivial Bott manifolds, then M' and M must have isomorphic Bott diagrams.

Let us fix a Bott tower on M. By our previous discussion, we know that each x_j is represented by a smooth hypersurface X_j in the sense that $x_j = c_1(X_j)$, where the divisor X_j is identified with the line bundle associated with it, and each z_j is represented by an effective divisor. To be more precise, for any $j \in A_0$ of level 0, $Z_j = X_j$ is irreducible. For any $j \in A_1$, $Z_j = X_j + q_j X_{\sigma(j)}$. For any $j \in A_2$, we have

$$Z_j = X_j + q_j(X_l + q_l X_{\sigma(l)}), \quad \text{where } l = \sigma(j) \in A_1.$$

Note that each X_j is itself a Bott manifold of dimension $n-1$, and the support of each Z_j is a normal crossing divisor.

In the case of a Bott tower, X_1 is a fiber of the composition map $\pi = \pi_2 \circ \pi_3 \circ \cdots \circ \pi_n$ from M^n to $\mathbb{P}^1$, and M is covered by the pencil of the divisor X_1. Given a Bott diagram, for any $j \in A_0$, we can choose a Bott tower on M so that j corresponds to the bottom layer, so we know that M is

covered by a pencil $|X_j|$ of the smooth hypersurface X_j, which we denote by Y_t for $t \in \mathbb{P}^1$. These Y_t do not intersect with each other.

We claim that, if D is any effective divisor in M homologous to X_j, then $D \in |X_j|$, namely, D is a member of the pencil. To see this, first consider the special case when D is irreducible. If D is not in $|X_j|$, we may choose a member Y in the pencil so that $D \cap Y \neq \phi$. This codimension 2 subvariety is homologous to 0, since $D \sim X_j$ and $x_j^2 = 0$, which is a contradiction since M is projective. The same argument works when D is just effective.

Now suppose $f : M' \to M$ is a biholomorphism. It induces a graded isomorphism $\phi = f^*$ between the cohomology rings. We want to show that f induces an isomorphism between the Bott diagrams as well. First we claim that f induces a bijection between A_0' and A_0, the set of level 0 vertices. Let $\{z_1, \ldots, z_n\}$ and $\{z_1', \ldots, z_n'\}$ be generators on M and M' as before. Then $\phi(z_j) = z'_{\tau(j)}$ for some permutation τ on A (the sign is positive since z_j, z_j' are all represented by non-trivial effective divisors). If $j \in A_0$, then M is covered by the pencil $X_j = Z_j$ of non-intersecting divisors. Consider the effective divisor $D = f(Z'_{\tau(j)})$ in M. D represents z_j, thus is homologous to Z_j. By the claim above, we know that D must be irreducible and is a member of the pencil $|X_j|$. This means that $\tau(j)$ lies in A_0'. So the level 0 sets of M and M' are bijective to each other.

Note that in the above argument, we furthermore obtained the fact that $f(X'_{\tau(j)}) = X_j$ for $j \in A_0$. For $j \in A_0$, the smooth hypersurface X_j is itself an $(n-1)$-dimensional Bott manifold. Its Bott diagram is obtained from that of M^n by deleting the vertex corresponding to j, and pulling down one level in the tree above this vertex, while keeping everything else unchanged. We will call this new Bott diagram the *card* at vertex j. Now since $f(X'_{\tau(j)}) = X_j$, the two Bott $(n-1)$-manifolds $X'_{\tau(j)}$ and X_j are biholomorphic, so by induction on the dimension of the Bott manifolds, we see that the card of the Bott diagram G' at vertex $\tau(j)$ must be isomorphic to the card of the Bott diagram G at the vertex j.

When A_0 has more than one elements, we have at least two cards, and we can use the set of cards to reconstruct the Bott diagram, see Proposition A.3 and Remark A.4. This implies that G and G' must be isomorphic to each other. When A_0 has only one element, the Bott diagrams agree as graphs, but we additionally need to show that the marking numbers q_j for $j \in A_1$ should match those on M' (see Remark A.4).

Without loss of generality, let us assume that $A_0 = \{1\} = A_0'$. We already know that $f(X_1') = X_1$, and the card of the Bott graph G at vertex 1 is isomorphic to the card of the Bott graph G' at vertex 1. So f gives a bijection between A_1

and A_1'. Again without loss of generality, let us assume that $\phi(z_2) = z_2'$, where $2 \in A_1$ and $2 \in A_1'$. We have

$$Z_2 = X_2 + q_2 X_1, \quad Z_2' = X_2' + q_2' X_1'.$$

Consider the irreducible divisor $D = f(X_2')$. We have $D + q_2' X_1 \sim X_2 + q_2 X_1$. If $D \neq X_2$, then $D \cap X_2$ is an effective cycle of codimension 2 (could be trivial), and the intersection

$$DX_2 + q_2' X_1 X_2 \sim Z_2 X_2 \sim Z_2(Z_2 - q_2 X_1) \sim -q_2 Z_2 X_1 \sim -q_2 X_2 X_1,$$

so the non-trivial effective cycle $D \cap X_2 + (q_2 + q_2') X_1 \cap X_2$ would be homologous to 0 – which is impossible since M is projective. So D must be equal to X_2, forcing $q_2' = q_2$. So the Bott diagrams of M and M' are indeed isomorphic to each other. This completes the proof of the theorem.

4. Concluding remarks

Recall that the rational Pontrjagin classes of a smooth manifold are defined using the smooth structure. A celebrated result of Novikov [Nov] shows however that these classes in fact only depend on the underlying topological structure (other proofs were given in [Gro], [Ran], [RY], and [RW]). More precisely, if one has a pair of homeomorphic smooth manifolds, then the homeomorphism can be chosen to take the total rational Pontrjagin class to the total rational Pontrjagin class. In the complex setting, the natural analogue to ask is the following:

Question. Which rational polynomials in the Chern classes are smooth invariants on the class of n-dimensional closed complex manifolds?

Note that we are not assuming the degree of the polynomial equals the dimension of the manifold. Indeed, in the top degree case, this question was a well-known conjecture of Hirzebruch – that the only rational linear combinations of the Chern numbers that are (oriented) smooth invariants are the linear combinations of the Euler number and Pontrjagin numbers. This conjecture was confirmed by Kotschick (see [Kot1], [Kot2]). Kotschick's result was recently extended by Schreider and Tasin [ST], who analyzed, within the class of smooth projective varieties, which Chern numbers are determined up to finite ambiguity by the underlying smooth manifold.

The situation for lower degrees seems a lot less clear. In lower degree, the diffeomorphism group of the manifold can act non-trivially on cohomology, so invariance means up to the action of $\mathrm{Diff}(M)$ on $H^*(M)$. Borel and Hirzebruch

[BH, §13.9] (see also [Hirl]) constructed examples of 5-dimensional complex flag manifolds that are diffeomorphic, but have distinct c_1. One can also use Kotschick's result to argue that certain homogenous polynomials, whose degree divides the dimension of the manifolds, cannot give smooth invariants.

Part of the difficulty in addressing this question is the lack of examples. Indeed, several classes of manifolds are known to support unique Kähler structures – see, for instance, the rigidity results of Hirzebruch-Kodaira [HK], Yau [Yau] and Siu [Siu]. In contrast, there are relatively few smooth manifolds that are known to support multiple distinct Kähler structures. This motivated our interest in distinguishing Kähler structures on products of S^2, leading us to complex Bott manifolds. Note that the space $\mathcal{K}$ of Kähler structures on $(S^2)^n = S^2 \times \cdots \times S^2$ remains rather mysterious when $n \geq 3$. For example, we do not know whether the Kähler structures discussed in this paper lie in distinct connected components of $\mathcal{K}$ (since they all have the same total Chern class).

A. Appendix: Reconstruction of rooted forests

A famous open problem in graph theory is the *reconstruction conjecture*. This conjecture asserts that finite graphs with at least three vertices are completely determined by their collection of vertex deleted graphs, see [Har], [BH], [Man]. In this short appendix, we formulate and prove the analogous conjecture for rooted forests.

Definition A.1. A contractible connected graph is called a *tree*. If such a graph is marked with a distinguished vertex (the *root*), we call it a *rooted tree*. If every connected component of a graph is a (rooted) tree, then we call the graph a (rooted) forest.

Given a rooted tree T with n vertices, we can form an associated rooted forest with $n-1$ vertices by deleting the root v of T (and all incident edges). This leaves a forest with connected components $T_1, \ldots, T_k$, and we can pick a root on each tree T_i to be the unique vertex v_i of T_i that was incident to v. We denote this rooted forest by $\widehat{T}$, and call its individual trees the *children* of the original tree T.

Definition A.2. Let F be a rooted forest, with connected components the rooted trees $T_1, \ldots, T_k$. Given a component T_i, we define the associated *card* to be the rooted forest with components $T_1, \ldots, T_{i-1}, \widehat{T}_i, T_{i+1}, \ldots, T_k$, i.e. we replace the rooted tree T_i by its collection of children. The forest F has k associated cards, each of which is a rooted forest.

A graph is finite if it has finitely many vertices. We can now establish the reconstruction conjecture for rooted forests.

Proposition A.3. *Let F be a finite rooted forest. Then the set of cards of F uniquely determines the forest F.*

In other words, if one has a pair of forests F_1, F_2, and a bijection between the set of cards of F_1 and those of F_2, which has the property that corresponding cards are isomorphic (as rooted forests), then the original forests have to be isomorphic.

Proof. We prove the statement using mathematical induction on the number of cards. Note that the number of cards coincides with the number of roots (and hence the number of connected components) in the original rooted forest F.

Base case: When there is only one card, we know that there is only one tree T in the original forest F. The individual trees in the single card are the children of T. We can thus reconstruct T by taking a root vertex v, and for each of the children of T, connecting its root to the vertex v. The resulting rooted tree is the single tree in the forest F.

Inductive step: Let there be $n \geq 2$ cards in total. We run through the n cards and locate a maximal tree (i.e., with the maximal number of vertices) among all the trees appearing on all the cards. Of course there could be more than one such tree, but we pick one of them. Let us call this chosen maximal tree T. Our claim is that T must be a rooted tree present in the original forest. If not, then T appeared on the card after eliminating the root of one of the original trees T_i of the forest. This means that T is a proper subgraph of T_i, and that T_i contains more vertices than T (as the root v_i of T_i is not in T). Since $n \geq 2$, there is at least one other card, arising from the deletion of another root v_j. The corresponding card contains T_i as a rooted tree, contradicting the fact that T was a maximal tree from all the cards. Note that this argument also shows that T is not a child of any of the rooted trees in the original rooted forest.

Now that we have established T is one of the original trees in the forest we are trying to rebuild, let us try and identify the multiplicity with which it occurs in the forest. Assume the forest consists of n rooted trees, and that r of them are isomorphic to T (where $1 \leq r \leq k$). Then there are precisely r cards that contain $r-1$ copies of T, and $n-r$ cards that contain r copies of T. Thus, we may compute the integer r from the set of cards.

Let F' denote the forest obtained from the original forest F by removing the r copies of T. If we can reconstruct F', then by adding in r copies of T, we will have reconstructed F. But note that the cards of F' are easy to

identify: just take the $n-r$ cards of F that contain exactly r copies of T, and remove from each of these cards the r copies of T. The resulting $n-r$ rooted forests are the cards of F'. Since $r \geq 1$, the rooted forest F' has $n-r < n$ cards, so by the inductive hypothesis, F' can be reconstructed from its cards. Adding in r disjoint copies of T then produces F, and completes the proof of the Proposition. □

Remark A.4. Note that the proof of the proposition also holds for *labelled* rooted forests, where the cards are equipped with the natural induced labelling. In this setting, you need to additionally assume that the number of cards is $n \geq 2$ (i.e., this is the base case of the induction, and is argued exactly like the inductive step above). When $n = 1$, the only indeterminacy lies in the labels for the edges in the rooted tree which are connected to the root vertex. These are obviously not recoverable from the single corresponding card. This is the reason for the additional argument at the end of the proof of Theorem 3.4.

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