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The critical Ising model on amenable graphs of exponential growth

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Abstract. The purpose of this note is to point out that the proof of the recent result of Hutchcroft [Hut] concerning continuity of the phase transition in Bernoulli percolation is applicable to the setting of the Ising model with free boundary conditions. This observation, together with a recent result of Aizenman, Duminil-Copin, and Sidoravicius [ADS] implies that the Ising model on any amenable transitive graph with exponential growth undergoes a continuous phase transition.

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1. Introduction

Almost a century has passed since the introduction of the Ising model. The Ising model is the archetypal example of a model undergoing an order/disorder phase transition and was introduced to explain the ferromagnetic/paramagentic phase transition in magnets. This concerns the physical phenomenon that certain materials, like iron, are paramagnetic at high temperatures: the material is magnetic in the presence of a magnetic field and loses its magnetic properties when the field vanishes. However, below a specific temperature T_c the same material is ferromagnetic: it is a magnet even after the applied magnetic field is vanished. The temperature T_c is called Curie's temperature after Pierre Curie who discovered this phenomenon in his doctoral thesis in 1895. For an introduction to the Ising model see [Grim, DC].

Let $G = \big(V(G), E(G)\big)$ be a connected countable locally finite transitive graph. Let $(J_{xy})_{x,y \in V(G)}$ be a family of nonnegative real numbers that are invariant under automorphisms of G, that is for any automorphism τ of G, $J_{\tau(x)\tau(y)} = J_{xy}$. For $h \in \mathbb{R}$, the *ferromagnetic Ising model* on a finite subset $\Lambda \subset V(G)$ is defined by the Hamiltonian

$$H_{\Lambda,h}(\sigma) = -\sum_{x,y\in\Lambda} J_{xy}\sigma_x\sigma_y - h\sum_{x\in\Lambda}\sigma_x$$

for any $\sigma \in \{-1,+1\}^\Lambda$.

For $\beta \in [0,\infty)$, define the *Ising measure* on Λ with magnetic field h at inverse temperature $\beta = \frac{1}{T}$ to be the measure $\mu_{\Lambda,\beta,h}$ defined on configurations $\sigma \in \{-1,+1\}^\Lambda$ by

$$\mu_{\Lambda,\beta,h}(\sigma) = \frac{\exp\big(-\beta H_{\Lambda,h}(\sigma)\big)}{Z(\Lambda,\beta,h)}, \tag{1.1}$$

where $Z(\Lambda,\beta,h)$ is a normalizing constant defined in such a way that the total mass of the measure is equal to one.

Let us contemplate for a moment on the measure and the physical meaning behind it. The vertices of the lattice are modeling the atoms of the magnetic material and the random variables $\{\sigma_x\}_{x\in\Lambda}$ denote the spin of the vertices. In a magnetic material, the spins of atoms interact with each other and the energy of the system is lower if the atoms in the vicinity of each other have the same

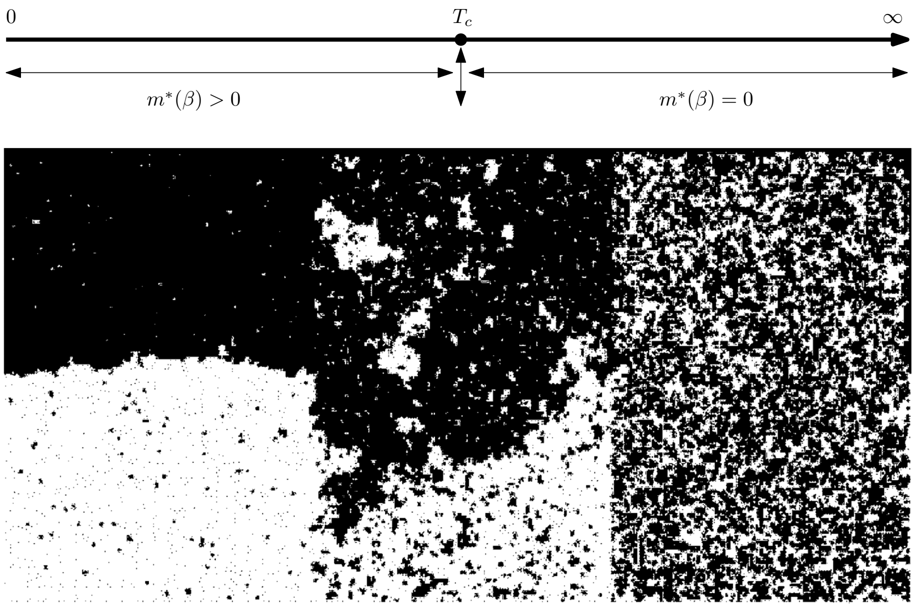

Figure 1

The three phases of the Ising models on the lattice $\mathbb{Z}^2$. In all of the three pictures the top half of the boundary of the box is forced to have a spin $+1$ (the black colors) and the bottom half of the boundary to have then spin -1 (the white colors). The simulation are due to S. Smirnov.

spins. The system prefers lower energies, but the magnitude of this preference decreases with the temperature: the lower the temperature the more atoms prefer to have the same spins. This is modeled by the multiplication of the Hamiltonian by the inverse temperature β. We refer the enthusiastic reader to [Tou] for more rigorous discussions on the rationale behind the measure (1.1).

Statistical physics focuses on systems with a very large number of particles, and hence it is natural to define the Ising model on the infinite graph G. Let $(\Lambda_n)_{n\geq 1}$ be a sequence of nested finite subgraphs exhausting G. Define the Ising measure on G at inverse temperature β with external field h to be the weak limit of the measures $\mu_{\Lambda_n,\beta,h}$, and denote this measure by $\mu_{G,\beta,h}$. When $h=0$ we denote the measure by $\mu^0_{G,\beta}$. Let $\mu^+_{G,\beta}$ denote the measure which is the weak limit of the measures $\mu_{G,\beta,h}$ as $h\to 0^+$. The measure $\mu^0_{G,\beta}$ (resp. $\mu^+_{G,\beta}$) is called Ising measure with free (resp. plus) boundary conditions. Due to the monotonicity properties of the model, these limits exist and do not depend on the choice of $(\Lambda_n)_{n\geq 1}$. We refer the reader to [Grim] for more details.

Historically, the Ising model has been mostly studied on the graphs $G=\mathbb{Z}^d$. The reason behind this is that the lattice $\mathbb{Z}^d$ is a practical approximation of the physical world – at least in the dimension three. It was in the nineties that Benjamini and Schramm [BSc] started the systematic study of percolation and other statistical physics models on graphs other than $\mathbb{Z}^d$. A main motivation to study lattice models on general graphs is that there exists a rich interplay between the geometric properties of the graph and the behavior of statistical physics models on them. General graphs provide opportunities for a better understanding of the geometry-probability interaction. Connections to geometric group theory is a further motivation to study Ising model and other lattice models on general graphs. Cayley graphs of finitely generated groups are rich sources of examples for the transitive graphs. When G is a Cayley graph, the properties of the model is related to algebraic properties of the group. We refer the reader to the book [LP] for a detailed exposition on this subject.

Fix a vertex $x\in V(G)$. In the presence of the magnetic field $h>0$ the density of the positive spins is equal to $\mu_{G,\beta,h}(\sigma_x)+\frac{1}{2}$. When $h\to 0^+$, the limiting quantity $\mu^+_{G,\beta}(\sigma_x)$ is of special interest and is called the *spontaneous magnetization*. Positivity of $\mu^+_{G,\beta}(\sigma_x)$ indicates that the density of the positive spins is more than one-half, and the material is magnetic, although the applied magnetic field has vanished. While $\mu^+_{G,\beta}(\sigma_x)=0$ indicates that the model is in the paramagnetic phase. We say the model undergoes a *phase transition*, if the spontaneous magnetization is positive for some values of β, but not all. Define the critical parameter

$$\beta_c := \inf\{\beta : \mu^+_{G,\beta}(\sigma_x)>0\}.$$

The phase transition occurs if $\beta_c \in (0, \infty)$. For $G = \mathbb{Z}$ one finds $\beta_c = \infty$ and there is no phase transition. For $G = \mathbb{Z}^d$ with $d \geq 2$, an argument due to Peierls – arguably the most classical argument in the theory of the phase transition – implies that $\beta_c \in (0, \infty)$, or in other words, a phase transition occurs.

Whether the spontaneous magnetization is continuous in β or not is a natural mathematical and physical question. The continuity of the $\mu^+_{G,\beta}(\sigma_x)$ at β_c signifies that the magnetism appears continuously with respect to the temperature. If $\mu^+_{G,\beta}(\sigma_x)$ has a discontinuity at some β, then the system has the capacity to absorb a large amount of energy and change its macroscopic state while remaining at the same temperature.

The question of continuity of the spontaneous magnetization can be seen in a broader context. One can ask: where are the singularities of any observable of the system (when considered as a function of the inverse temperature) and what are their orders? What is the relationship between the phase transition and the singularities of the observables? In general, one expects that many natural observables of the system have singularities only at the critical point.

Determining the continuity of the magnetization on $\mathbb{Z}^d$ has a long-standing history. The right continuity of the spontaneous magnetization could be easily shown by simple semi-continuity arguments. (See Claim 3 of Proposition 3 below for an example of this type of arguments.) Therefore, the main question is the left continuity of the spontaneous magnetization.

In dimension $d = 2$, Yang [Yan], inspired by works of Onsager [Ons] and Kaufmann [Kau], computed the spontaneous magnetization for the nearest neighbor case and the continuity for all β was established. In dimension $d \geq 4$ the continuity at β_c was proved in [AF] at criticality (see also [Sak]). Finally, in [ADS] the continuity was established in $d = 3$ at criticality. Apart from $\mathbb{Z}^d$, continuity at β_c has been proved for the regular trees in [Häg] (see also [BRZ] and the references therein). Previous to the present work, not much has been known about the continuity of the magnetization at β_c for graphs other than $\mathbb{Z}^d$ and trees.

In [Bod], the continuity of magnetization was settled for any $\beta > \beta_c$ in dimension $d \geq 3$. Recently the same result has been proved in [Rao] for all the amenable graphs.

In this paper, we focus on the continuity of the magnetization on the amenable graphs of exponential growth. Let us recall the definitions of amenability and exponential growth for graphs. Let G be a countable locally finite graph. Let $d(.,.)$ denote the graph distance on G. For a vertex $x \in V(G)$, define $\Lambda_n(x) := \{y \in V(G) : d(x, y) \leq n\}$. Fix a vertex $x \in V(G)$. We say G has *exponential growth* if

$$\liminf_{n\to\infty} |\Lambda_n(x)|^{1/n} > 1.$$

It is easy to verify that the above definition is independent of the choice of the vertex x. For a subset $A \subset V(G)$ define $\partial A := \{x \in A : \exists y \in G \setminus A, \{x,y\} \in E(G)\}$. We say G is *amenable* if

$$\inf_{A \subset G, |A| < \infty} \frac{|\partial A|}{|A|} = 0.$$

Apart from the invariance of the coupling constants under the automorphisms of the graph, assume that they are irreducible and locally finite. Irreducible coupling constants are such that for any $x, y \in G$, there exist $v_1, v_2, \ldots, v_k \in G$ satisfying

$$J_{xv_1} J_{v_1 v_2} \ldots J_{v_{k-1} v_k} J_{v_k y} > 0.$$

Locally finite coupling constants are such that for any $x \in G$, $\sum_{y \in G} J_{xy} < \infty$.

Theorem 1. *Let G be amenable transitive graph with exponential growth. The magnetization is continuous in β at β_c, i.e., for any $x \in G$,*

$$\mu^+_{G,\beta_c}(\sigma_x) = 0.$$

Let us give two examples of transitive amenable graphs with an exponential growth. The first example is every Cayley graph of the lamplighter group on $\mathbb{Z}^d$ [Pet]. Another example is every Cayley graph of the Baumslag–Solitar group $BS(1,n)$ [BS]. In fact there is a classical theorem of Milnor [Mil] and of Wolf [Wol], that states any finitely generated solvable group that is not virtually nilpotent has exponential growth. Since solvable groups are amenable, any Cayley graph of a not virtually nilpotent solvable group is an example of the graphs we are considering in this article.

The result is expected to be true for any transitive graphs with the nearest neighbor coupling constants. On the other hand, when $G = \mathbb{Z}$ and $J_{xy} = |x-y|^{-2}$ the phase transition is not continuous, that is the magnetization is positive at β_c [ACCN]. We refer the reader to [Rao] for the case $\beta \neq \beta_c$.

Notation. From now on, we fix $G = \big(V(G), E(G)\big)$ a connected countable locally finite transitive amenable graph with exponential growth. We drop G from the notations.

2. Proof of Theorem 1

We first rewrite the beautiful proof of Hutchcroft [Hut] in the case of Ising model with free boundary conditions. Before starting, let us mention an important ingredient of the proof.

Theorem 2 ([ABF, DCT]). *Let G be a transitive graph and $x \in V(G)$. For any $\beta < \beta_c$,*

$$\sum_{y \in V(G)} \mu^0_\beta(\sigma_x \sigma_y) < \infty.$$

We also use the *Griffiths inequality* [Gri] in several places of the proof. This inequality states that for every $A, B \subset V(G)$

$$\mu^0_\beta(\sigma_A \sigma_B) \geq \mu^0_\beta(\sigma_A)\mu^0_\beta(\sigma_B), \tag{Griffiths}$$

where $\sigma_X = \prod_{x \in X} \sigma_x$ for $X \subset V(G)$.

Proposition 3. *Fix $x \in V(G)$. There exists $\rho < 1$ such that for all $n \geq 0$,*

$$\min \left\{\mu^0_{\beta_c}(\sigma_x \sigma_y) : y \in \Lambda_n(x)\right\} \leq \rho^n. \tag{2.1}$$

Proof. Define

$$\kappa_\beta(n) := \min \left\{\mu^0_\beta(\sigma_x \sigma_y) : y \in \Lambda_n(x)\right\}.$$

The proof consists of three claims.

Claim 1. The sequence $\left(\kappa_\beta(n)\right)_{n \geq 0}$ is supermultiplicative.

Proof of Claim 1. Let $y \in \Lambda_{m+n}(x)$, there exists a vertex $z \in \Lambda_n(x)$ such that $y \in \Lambda_m(z)$. Griffiths inequality implies

$$\mu^0_\beta(\sigma_x \sigma_y) = \mu^0_\beta(\sigma_x \sigma_z \sigma_z \sigma_y) \geq \mu^0_\beta(\sigma_x \sigma_z)\mu^0_\beta(\sigma_z \sigma_y) \geq \kappa_\beta(n)\kappa_\beta(m).$$

Thus,

$$\kappa_\beta(n+m) = \min \left\{\mu^0_\beta(\sigma_x \sigma_y), y \in \Lambda_{n+m}(x)\right\} \geq \kappa_\beta(n)\kappa_\beta(m).$$

□

Claim 2. There exists $\rho < 1$, such that for any $\beta < \beta_c$,

$$\sup_{n \geq 0} \left(\kappa_\beta(n)\right)^{1/n} < \rho.$$

Proof of Claim 2. Based on the definition of $\kappa_\beta(n)$,

$$\kappa_\beta(n) \cdot |\Lambda_n| \leq \sum_{y \in \Lambda_n(x)} \mu^0_\beta(\sigma_x \sigma_y) \leq \sum_{y \in V(G)} \mu^0_\beta(\sigma_x \sigma_y). \tag{2.2}$$

As $\kappa_\beta(n)$ is supermultiplicative, Fekete's lemma implies that $\lim_{n \to \infty} \left(\kappa_\beta(n)\right)^{1/n}$ exists and is equal to $\sup_n \left(\kappa_\beta(n)\right)^{1/n}$. Combining this fact with (2.2) gives us

$$\sup_n \left(\kappa_\beta(n)\right)^{1/n} = \lim_{n\to\infty} \left(\kappa_\beta(n)\right)^{1/n} \le \lim_{n\to\infty} \left(\frac{\sum_{y\in V(G)} \mu^0_\beta(\sigma_x\sigma_y)}{|\Lambda_n|}\right)^{1/n}$$
$$= \lim_{n\to\infty} \left(\frac{1}{|\Lambda_n|}\right)^{1/n}.$$

In the last equality we used Theorem 2. Since the graph has an exponential growth, $\lim_{n\to\infty} (\frac{1}{|\Lambda_n|})^{1/n} < 1$, and the claim follows. Note that this is the only place where exponential growth of the graph is used. □

Claim 3. The map $\beta \to \sup_n \left(\kappa_\beta(n)\right)^{1/n}$ is left continuous at $\beta \in [0,\infty)$.

Proof of Claim 3. By Griffiths inequality, $\mu^0_\beta(\sigma_x\sigma_y) = \sup_k \mu^0_{\Lambda_k,\beta,0}(\sigma_x\sigma_y)$. Since Λ_k is finite, and because of Griffiths inequality, $\beta \to \mu^0_{\Lambda_k,\beta,0}(\sigma_x\sigma_y)$ is continuous and increasing. The supremum of increasing continuous functions is left continuous. Hence for fixed $x, y \in V(G)$ the map $\beta \to \mu^0_\beta(\sigma_x\sigma_y)$ is left continuous.

Now fix $n \in \mathbb{N}$. Since Λ_n is finite, $\kappa_\beta(n)$ is the minimum of finitely many left continuous increasing functions, so is left continuous and increasing in β. Finally, the map $\beta \to \sup_n \left(\kappa_\beta(n)\right)^{1/n}$ is the supremum of left continuous increasing functions, so is left continuous. □

Claim 2 and Claim 3 together conclude the proof. Indeed, $\sup_n \left(\kappa_\beta(n)\right)^{1/n}$ is uniformly bounded above by some $\rho < 1$ when $\beta < \beta_c$, and since $\beta \to \sup_n \left(\kappa_\beta(n)\right)^{1/n}$ is left continuous, it follows that $\sup_n \left(\kappa_{\beta_c}(n)\right)^{1/n} \le \rho$. Hence for any $n \ge 1$, $\kappa_{\beta_c}(n) \le \rho^n$ which is the claim. □

Recently Aizenman, Duminil-Copin, and Sidoravicius proved the following theorem, which establish a connection between μ^+_β and μ^0_β. Their approach is based on the random current representation of the Ising model.

Theorem 4 ([ADS]). *Let G be an amenable transitive graph, and let $\beta \in [0,\infty)$. If*

$$\inf_{K\subset V(G),\,|K|<\infty} \frac{\sum_{x,y\in K} \mu^0_\beta(\sigma_x\sigma_y)}{|K|^2} = 0, \tag{2.3}$$

then for any $x \in V(G)$, $\mu^+_\beta(\sigma_x) = 0$.

Remark 5. Theorem 4 is used in [ADS] to prove $\mu^+_\beta(\sigma_x) = 0$ for the graphs $\mathbb{Z}^d$. There, the infra-red bound [Bis] is utilized to obtain (2.3). There is no proof of the infra-red bound for graphs other than the lattices $\mathbb{Z}^d$.

Remark 6. Theorem 4 is not stated in the above form in [ADS]. There, it is stated that on $\mathbb{Z}^d$ if the Long Range Order parameter vanishes, then spontaneous magnetization is also 0. However, their proof works for any amenable graph and the condition of vanishing magnetization could be weakened to 2.3. It is worth highlighting that amenability has a vital importance in the argument of [ADS] to obtain the uniqueness of the infinite cluster in the infinite-volume double random current via a Burton-Keane type argument. This is where the irreducibility and the locally finiteness of the coupling constants are required.

Proof of Theorem 1. Theorem 4 implies that in order to conclude the proof of Theorem 1 it is enough to construct a family $\{K_n\}_{n\geq 1}$ of finite subsets of $V(G)$, such that

$$\inf_n \frac{\sum_{x,y\in K_n} \mu^0_{\beta_c}(\sigma_x\sigma_y)}{|K_n|^2} = 0.$$

Let $c = \min_{\{x,y\}\in E(G)} \mu^0_{\beta_c}(\sigma_x\sigma_y)$. Griffiths inequality implies that that for any $x, y \in V(G)$, $\mu^0_{\beta_c}(\sigma_x\sigma_y) \geq c^{d(x,y)}$. Choose $k \geq 2$ an integer such that $c \geq \rho^k$, Where ρ is the same constant as of (2.1). Fix a vertex $x_1 \in V(G)$, and for each $n > 1$, define $x_n \in V(G)$ such that $d(x_n, x_1) = k^n$ and

$$\mu^0_{\beta_c}(\sigma_{x_1}\sigma_{x_n}) \leq \rho^{k^n},$$

Proposition 3 guarantees the existence of x_n. Let $K_n = \{x_i : 1 \leq i \leq n\}$. For $1 < i < j \leq n$, Griffiths inequality implies

$$\mu^0_{\beta_c}(\sigma_{x_i}\sigma_{x_j}) \leq \frac{\mu^0_{\beta_c}(\sigma_{x_1}\sigma_{x_j})}{\mu^0_{\beta_c}(\sigma_{x_1}\sigma_{x_i})} \leq \frac{\rho^{k^j}}{c^{k^i}} \leq \rho^{k^j - k^{i+1}} \leq \rho^{j-i},$$

for $j - i$ large enough. This implies that there exists a constant C independent of n such that,

$$\frac{\sum_{x,y\in K_n} \mu^0_{\beta_c}(\sigma_x\sigma_y)}{|K_n|^2} \leq \frac{C|K_n|}{|K_n|^2} \leq \frac{C}{n}.$$

Hence $\inf_{n\geq 1} \frac{\sum_{x,y\in K_n} \mu^0_{\beta}(\sigma_x\sigma_y)}{|K_n|^2} = 0$. □

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