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Basic matrix perturbation theory

Benjamin TEXIER

Abstract. In this expository note, we give proofs of several results in finite-dimensional matrix perturbation theory: continuity of the spectrum, regularity of the total eigenprojectors, existence and computation of one-sided directional derivatives of semi-simple eigenvalues, and Puiseux expansions of coalescing eigenvalues. These results are all classical, at least in the case of one-dimensional, analytical perturbations; a standard reference is the treatise of T. Kato, *Perturbation theory for linear operators* (Springer, 1980). In contrast with Kato, we consider perturbations which are not necessarily smooth, in arbitrary finite dimension, and for coalescing eigenvalues we do not use the notion of multi-valued function. The proofs use Rouché's theorem, representations of projectors as contour integrals, and the description of conjugacy classes of connected covering maps of the punctured disk.

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Consider a family of matrices M defined over an open set $\Omega \subset \mathbb{R}^d$:

$$M : \quad x \in \Omega \to M(x) \in \mathbb{C}^{N\times N}. \tag{1}$$

We denote $\operatorname{sp} M(x)$ the spectrum of matrix $M(x)$. The eigenspace associated with $\lambda \in \operatorname{sp} M(x)$ is the non-trivial kernel $\ker M(x) - \lambda \mathrm{Id}$. The generalized eigenspace associated with $\lambda \in \operatorname{sp} M(x)$ is the largest space in the (strictly increasing until stationary) sequence $\ker(M(x) - \lambda\mathrm{Id})^k$, $k \geq 1$. The index of an eigenvalue λ of $M(x)$ is the smallest k such that $\ker(M(x) - \lambda\mathrm{Id})^k$ is maximal. An eigenvalue is said to be semi-simple if the generalized eigenspace coincides with the eigenspace. In particular, the index of a semi-simple eigenvalue is equal to 1.

1. Continuity of the eigenvalues

Proposition 1.1. *If $x \to M(x)$ is continuous, then the spectrum of M is continuous, in the following sense: given $x_0 \in \Omega$, given $\lambda_0 \in \operatorname{sp} M(x_0)$ with multiplicity m as a root of the characteristic polynomial of $M(x_0)$, for any small enough $r > 0$, there exists a neighborhood U of x_0 in Ω, such that for all $x \in U$, the matrix $M(x)$ has m eigenvalues (counting multiplicities) in $B(\lambda_0, r)$.*

Proof. This is a consequence of Rouché's theorem (see for instance the Corollary to Theorem 20 in Chapter 4 of [Ahl]), which states that if f, g are holomorphic in $\bar{B}(\lambda_0, r) \subset \mathbb{C}$, and if $|f - g| < |g|$ in $\partial B(\lambda_0, r)$, then f and g have the same number of zeros (counting multiplicities) in the open ball $B(\lambda_0, r)$.

Let $\Pi(\lambda, x) = \det(\lambda - M(x))$, holomorphic in λ and continuous in x. By finiteness of the spectrum, if $r > 0$ is small enough, then $\Pi(\cdot, x_0)$ has only one zero in the closed ball $\bar{B}(\lambda_0, r)$, with multiplicity m. In particular, $|\Pi(\lambda, x_0)| > 0$ on the boundary $\partial B(\lambda_0, r)$, and the inequality

$$h(x) = \max_{\partial B(\lambda_0, r)} |\Pi(\cdot, x) - \Pi(\cdot, x_0)| - |\Pi(\cdot, x_0)| < 0 \tag{2}$$

holds at $x = x_0$. Inequality (2) still holds in a neighborhood of x_0. Indeed, by continuity of Π in (λ, x), for all $\lambda \in \partial B(\lambda_0, r)$, there can be found $\alpha_\lambda > 0$, such that $|\Pi(\mu, x) - \Pi(\mu, x_0)| < |\Pi(\mu, x_0)|$ for $|x - x_0| < \alpha_\lambda$ and $|\lambda - \mu| < \alpha_\lambda$ with $\mu \in \partial B(\lambda_0, r)$. The family of open sets $\{\mu \in \partial B(\lambda_0, r),\ |\mu - \lambda| < \alpha_\lambda\}$, indexed by $\lambda \in \partial B(\lambda_0, r)$, covers the compact boundary $\partial B(\lambda_0, r)$. A finite subcover is indexed by $i \in I$. The minimum $\alpha = \min_i \alpha_{\lambda_i}$ is positive. Then, for all x such that $|x - x_0| < \alpha$, we have $h(x) < 0$. Thus, by Rouché's theorem, applied with $f = \Pi(\cdot, x)$ and $g = \Pi(\cdot, x_0)$, with x fixed in $U = B(x_0, \alpha)$, the function $\Pi(\cdot, x)$ has the same number of zeros as $\Pi(\cdot, x_0)$ in $B(\lambda_0, r)$, counting multiplicities. This means that $M(x)$ has exactly m eigenvalues in $B(\lambda_0, r)$, for any $x \in U$, which concludes the proof. □

We assume continuity of M in the following. In particular, Proposition 1.1 applies. Let

$$\mathcal{S} := \bigcup_{x \in \Omega} \operatorname{sp} M(x) \times \{x\} = \Big\{(\lambda, x) \in \mathbb{C} \times \Omega,\ \det\big(\lambda - M(x)\big) = 0\Big\}$$

be the *spectrum* of the family of matrices M, and let the projection

$$\pi : (\lambda, x) \in \mathcal{S} \longrightarrow x \in \Omega, \tag{3}$$

so that the spectrum of matrix $M(x)$ is the fiber $\pi^{-1}(\{x\})$.

The *multiplicity* of a point $(\lambda, x) \in \mathcal{S}$ is the algebraic multiplicity of λ in $\operatorname{sp} M(x)$, that is the order of λ as a root of the characteristic polynomial of $M(x)$.

A point $(\lambda_0, x_0) \in \mathcal{S}$ is said to have *constant multiplicity* if locally around (λ_0, x_0), there exists only one eigenvalue of $M(x)$, not counting multiplicity.

Corollary 1.2. *Around a point of constant multiplicity, the projection π is a local homeomorphism. If the whole spectrum of $M(x_0)$ has constant multiplicity, then π is a covering map at x_0, and the number of sheets is equal to the number of distinct eigenvalues around x_0.*

Proof. If (λ_0, x_0) has constant multiplicity, the continuous branch of eigenvalues λ given by Proposition 1.1 is a continuous section of the projection π, such that $\lambda(x_0) = \lambda_0$. Thus in restriction to a neighborhood of (λ_0, x_0), the projection π is a homeomorphism. If the whole spectrum $\{\lambda_1, \dots, \lambda_p\}$ of $M(x_0)$ has constant multiplicity, then in addition the fibers have constant cardinality, equal to p, around x_0. Thus π is a covering map. $\square$

If a point in $\mathcal{S}$ does not have constant multiplicity, it is said to be a *coalescing* point in the spectrum. The associated multiplicity is strictly greater than one.

Coalescing points in the spectrum are not necessarily isolated, even if M is smooth. Consider for instance the case $\Omega = \mathbb{R}$, and let F be a closed set in $\mathbb{R}$. There exists a smooth $a \geq 0$ such that $F = a^{-1}(\{0\})$. Then for

$$\begin{pmatrix} 0 & 1 \\ a(x) & 0 \end{pmatrix}$$

every point in $\{0\} \times F$ is a coalescing point in the spectrum.

Proposition 1.3. *If $\Omega \subset \mathbb{R}$, or if Ω is an open subset of $\mathbb{C}$, and if $M(x)$ is a polynomial in $x \in \Omega$, then the spectrum has a finite number of coalescing points.*

Proof. We may work with irreducible components Π_j of the characteristic polynomial Π (a polynomial in two variables, λ and x). For every such component, Π_j and $\partial_\lambda \Pi_j$ are relatively prime. In particular (see for instance Theorem 3 in chapter 8 of [Ahl]), there are a finite number of x such that $\Pi_j(\cdot, x)$ and $\partial_\lambda \Pi_j(\cdot, x)$ have a common root $\lambda(x)$. These common roots $\big(x, \lambda(x)\big)$ are precisely the coalescing points in the spectrum. $\square$

We say that (λ, x) is a *isolated coalescing point* in the spectrum (of the family of matrices M introduced in (1)) there exists a neighborhood $\mathcal{U}$ of (λ, x) in $\mathbb{C} \times \Omega$ such that $\big(\mathcal{U} \setminus \{(\lambda, x)\}\big) \cap \mathcal{S}$ comprises only points of constant multiplicity.

Corollary 1.4. *If (λ_0, x_0) is an isolated coalescing point in the spectrum, then if $\varepsilon > 0$ is small enough, the restriction of the projection $\pi : \mathcal{S} \cap \pi^{-1}(B(x_0, \varepsilon)^*) \to B(x_0, \varepsilon)^*$ is a covering map. Here $\pi^{-1}(B(x_0, \varepsilon)^*)$ is the inverse image of the punctured ball $B(x_0, \varepsilon)^*$.*

Proof. Identical to the proof of Corollary 1.2, since the fibers above the (connected) punctured ball have constant cardinality. □

At a coalescing point in the spectrum, eigenvalues may fail to be differentiable, even if M is smooth. The canonical example is

$$\begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix}, \qquad x \geq 0. \tag{4}$$

Regularity issues for the eigenvalues are examined in Sections 3 and 4.

2. Cauchy formulas

We use notation $\mathcal{S}$ for the spectrum of the continuous family of matrices M, as defined in Section 1.

Proposition 2.1 (Cauchy formula for total eigenprojectors). *Let* $(\lambda_0, x_0) \in \mathcal{S}$*, and* γ *a closed, positively oriented curve in* $\mathbb{C}$*, which does not intersect* $\operatorname{sp} M(x_0)$*, and the interior of which intersects* $\operatorname{sp} M(x_0)$ *at* λ_0 *only. Then for* x *close to* x_0*,*

$$P(x) = \frac{1}{2i\pi} \int_\gamma \big(\lambda - M(x)\big)^{-1} \, d\lambda \tag{5}$$

is the sum of the projectors onto the generalized eigenspaces associated with eigenvalues of $M(x)$ *which lie in the interior of* γ*. In particular, the projector* P *is as regular as* M*.*

Above and below, *projectors onto generalized eigenspaces* (equivalently, generalized eigenprojectors) are implicitly *parallel to* the direct sum of the other generalized eigenspaces.

Proof. If (λ_0, x_0) has constant multiplicity, or if it is an isolated coalescing point in the spectrum, then there is a constant number of distinct eigenvalues near λ_0 for x close to x_0. In general, however, for x close to x_0, the number of distinct eigenvalues of $M(x)$ near λ_0 may depend on x. Let $j(x)$ be this number, and $J(x)$ be the total number of distinct eigenvalues of $M(x)$. Thus for x close to x_0, the eigenvalues $\lambda_1(x), \ldots, \lambda_{j(x)}(x)$ belong to the interior of γ, while the other eigenvalues $\lambda_{j(x)+1}(x), \ldots, \lambda_{J(x)}(x)$ do not.

The spectral decomposition of $M(x)$ is

$$M(x) = \sum_{1 \leq j \leq J(x)} \big(\lambda_j(x) + N_j(x)\big) P_j(x), \tag{6}$$

where the P_j are projectors onto generalized eigenspaces, such that

$$\mathrm{Id} = \sum_{1 \le j \le J(x)} P_j(x), \qquad P_i(x) P_j(x) = 0 \quad \text{if} \quad i \neq j, \tag{7}$$

and the N_j are the associated nilpotent components, such that $N_j(x) P_j(x) = P_j(x) N_j(x)$, and $N_i(x) P_j(x) = 0$ if $i \neq j$.

By (6) and (7), for $x \in U$ and $\lambda \notin \mathrm{sp}\, M(x)$, we have

$$\big(\lambda - M(x)\big)^{-1} = \sum_{1 \le j \le J(x)} \big(\lambda - \lambda_j(x) - N_j(x)\big)^{-1} P_j(x), \tag{8}$$

which we may rewrite, the matrix $\mathrm{Id} - \mu N_j$ being invertible for all μ :

$$\big(\lambda - M(x)\big)^{-1} = \sum_{1 \le j \le J(x)} \big(\lambda - \lambda_j(x)\big)^{-1} \Big(\mathrm{Id} - \big(\lambda - \lambda_j(x)\big)^{-1} N_j(x)\Big)^{-1} P_j(x),$$

and, expanding in inverse powers of $\lambda - \lambda_j(x)$,

$$\begin{aligned} &\big(\lambda - M(x)\big)^{-1} \\ &\quad = \sum_{1 \le j \le J(x)} \Big(\big(\lambda - \lambda_j(x)\big)^{-1} + \sum_{1 \le k \le r_j(x) - 1} \big(\lambda - \lambda_j(x)\big)^{-(k+1)} N_j(x)^k \Big) P_j(x), \end{aligned} \tag{9}$$

where $r_j(x) \ge 2$ is the index of the nilpotent matrix $N_j(x)$, that is the smallest integer k such that $N_j(x)^k = 0$. We now compute residues:

$$\frac{1}{2i\pi} \int_\gamma \big(\lambda - \lambda_j(x)\big)^{-1} P_j(x)\, d\lambda = P_j(x), \quad 1 \le j \le j(x),$$

$$\int_\gamma \big(\lambda - \lambda_j(x)\big)^{-1} P_j(x)\, d\lambda = 0, \quad j(x) + 1 \le j \le J(x),$$

$$\int_\gamma \big(\lambda - \lambda_j(x)\big)^{-(k+1)} N_j(x)^k P_j(x)\, d\lambda = 0, \quad \text{for all } j \text{ and all } k \ge 1.$$

Thus $P(x) = \sum_{1 \le j \le j(x)} P_j(x)$ satisfies representation (5) for x close to, and different from, x_0. The above also shows that at $x = x_0$, the right-hand side of (5) is the eigenprojector onto the generalized eigenspace associated with λ_0. □

Corollary 2.2. *Around a point (λ_0, x_0) of constant multiplicity in the spectrum, the associated eigenvalue and generalized eigenprojector are as regular as M, and we have*

$$\big(\lambda(x) + N(x)\big) P(x) = \frac{1}{2i\pi} \int_\gamma \lambda \big(\lambda - M(x)\big)^{-1} d\lambda, \tag{10}$$

where $x \to \lambda(x)$ is the local branch of eigenvalues such that $\lambda(x_0) = \lambda_0$, P is the associated projector, and N the associated nilpotent.

Note that in the case of simple roots of the characteristic polynomial of M, the regularity of the eigenvalues follows directly from the implicit function theorem.

Proof. The constant multiplicity hypothesis implies that the total eigenprojector $P(x)$ from Proposition 2.1 is the generalized eigenprojector onto the unique eigenvalue $\lambda(x)$ of $M(x)$ near λ_0. Thus, by representation (5), the eigenprojector P is as regular as M.

Next we use a spectral decomposition of $M(x)$ in order to express $\lambda(\lambda - M(x))^{-1}$, for $\lambda \in \mathbb{C}$, as a sum of projectors, as we did for $(\lambda - M(x))^{-1}$ in (8) in the proof of Proposition 2.1:

$$\lambda\big(\lambda - M(x)\big)^{-1} = \lambda\big(\lambda - \lambda(x) - N(x)\big)^{-1} P(x) + \sum_{2 \le j \le J(x)} \lambda\big(\lambda - \lambda_j(x) - N_j(x)\big)^{-1} P_j(x),$$

where $\lambda(x)$ is the eigenvalue of $M(x)$ which is equal to λ_0 at x_0, and the $\lambda_j(x)$, for $2 \le j \le J(x)$ are the other eigenvalues of $M(x)$. For x close to x_0, the eigenvalues $\lambda_j(x)$ are far from λ_0. Computing residues as in the proof of Proposition 2.1, we find that if the interior of γ contains λ_0 and is small enough:

$$\frac{1}{2i\pi}\int_\gamma \lambda\big(\lambda - M(x)\big)^{-1} d\lambda = \frac{1}{2i\pi}\int_\gamma \lambda\big(\lambda - \lambda(x) - N(x)\big)^{-1} P(x)\, d\lambda. \tag{11}$$

We now expand in powers of $(\lambda - \lambda(x))^{-1}$:

$$\lambda\big(\lambda - \lambda(x) - N(x)\big)^{-1} = \lambda\big(\lambda - \lambda(x)\big)^{-1} + \sum_{1 \le k \le r(x)-1} \lambda\big(\lambda - \lambda(x)\big)^{-(k+1)} N(x)^k,$$

where $r(x)$ is the (possibly x-dependent) index of $N(x)$, for x close to x_0, and then again compute residues:

$$\frac{1}{2i\pi}\int_\gamma \lambda\big(\lambda - \lambda(x)\big)^{-1} P(x)\, d\lambda = \lambda(x)P(x),$$

$$\frac{1}{2i\pi}\int_\gamma \lambda\big(\lambda - \lambda(x)\big)^{-(k+1)} N(x)^k P(x)\, d\lambda = N(x)P(x), \quad k \ge 1.$$

With (11), this implies representation (10), from which we deduce that the map $x \to \big(\lambda(x) + N(x)\big)P(x)$ is as regular as M. Taking the trace, we find that $x \to m\lambda(x)$ is as regular as M, where $m \ge 1$ is the multiplicity of λ. $\square$

Corollary 2.3. *If (λ_0, x_0) is an isolated coalescing point in the spectrum, with multiplicity $m > 1$, we have*

$$\sum_{1 \le j \le m'} \big(\lambda_j(x) + N_j(x)\big) P_j(x) = \frac{1}{2i\pi}\int_\gamma \lambda\big(\lambda - M(x)\big)^{-1} d\lambda, \tag{12}$$

where $x \to \lambda_j(x)$, for $1 \le j \le m'$, are the distinct branches of eigenvalues such that $\lambda_j(x_0) = \lambda_0$, for some $m' \le m$, and the matrices P_j are the associated projectors, and N_j the associated nilpotents.

Proof. For all $x \in U \setminus \{x_0\}$, where U is some neighborhood of x_0, the matrix $M(x)$ has the same number of distinct eigenvalues in a neighborhood of λ_0. Let m' be this number, less than or equal to m, the multiplicity of λ_0. Let $\lambda_1, \dots, \lambda_{m'}$ be these eigenvalues. It suffices to reproduce the computations of the proof of Corollary 2.2, where each λ_j plays the same role as λ in the proof of Corollary 2.2, to arrive at (12). □

3. Hölder estimates

Proposition 3.1. *If M is differentiable at x_0, then for any local branch λ of eigenvalues of M around x_0, we have the bound*

$$|\lambda(x) - \lambda(x_0)| \le C(M)|x - x_0|^{1/m}, \tag{13}$$

locally around x_0, with $C(M) > 0$, where m is the index of $(\lambda(x_0), x_0)$, as defined in the introduction.

If $(\lambda(x_0), x_0)$ has constant multiplicity and M is locally Lipschitz, then by Corollary 2.2 the eigenvalues are actually Lipschitz, locally around x_0, which of course is much better than (13) in the case $m > 1$. Estimate (13) however accurately describes the eigenvalue behavior in the canonical coalescing case (4), for which $m = 2$.

Proof. Let γ be a path around $\lambda(x_0)$ and P be the associated total eigenprojector, as in Proposition 2.1. Then P is differentiable at x_0, just like M, by Proposition 2.1. For x close to x_0, let $u(x)$ be a unitary eigenvector associated with $\lambda(x)$. We have no information on the regularity of u. For x close to x_0, we have

$$\big(M(x) - \lambda(x_0)\big)^m P(x)u(x) = \big(\lambda(x) - \lambda(x_0)\big)^m u(x).$$

Taking norms, this gives

$$\big|\lambda(x) - \lambda(x_0)\big|^m = \big|\big(M(x) - \lambda(x_0)\big)^m P(x)\big|.$$

Since m is the index of $(\lambda(x_0), x_0)$, we have $(M(x_0) - \lambda(x_0)^m P(x_0) = 0$. Thus we may write the above as

$$\big|\lambda(x) - \lambda(x_0)\big|^m = \big|\big(M(x) - \lambda(x_0)\big)^m P(x) - \big(M(x_0) - \lambda(x_0)\big)^m P(x_0)\big|,$$

and we conclude by differentiability of $x \to (M(x) - \lambda\big(x_0)\big)^m P(x)$. □

Remark 3.2. Without appealing to the Cauchy formula of Proposition 2.1, we can show that λ satisfies $|\lambda(x)-\lambda(x_0)| \leq C(M)|x-x_0|^{1/p}$, where p is the multiplicity of $(\lambda(x_0), x_0)$, as follows. We denote $\lambda_0 = \lambda(x_0)$. The characteristic polynomial $\Pi(\lambda, x) = \det\big(\lambda - M(x)\big)$ factorizes into $\Pi = \Pi_0\Pi_1$, where $\Pi_1(\lambda_0, x_0) \neq 0$, and $\Pi_0(\lambda, x_0) = (\lambda - \lambda_0)^p$. The degree of Π_0 is equal to p, the multiplicity of (λ_0, x_0), and Π_0 is unitary. We may focus on Π_0 in the following. Let λ be a branch of eigenvalues such that $\lambda(x_0) = \lambda_0$. Expanding Π_0 in powers of $\lambda(x) - \lambda_0$, we find, since $\partial_\lambda^j \Pi_0(\lambda_0, x_0) = 0$ for $0 \leq j \leq p-1$:

$$\Pi_0\big(\lambda(x), x_0\big) = (p!)^{-1}\big(\lambda(x) - \lambda_0\big)^m + O\big(|\lambda(x) - \lambda_0|\big)^{p+1}.$$

Besides, the matrices M being differentiable at x_0, the characteristic polynomial Π is differentiable in x at x_0, and so is Π_0 :

$$\Pi_0\big(\lambda(x), x\big) = \Pi_0\big(\lambda(x), x_0\big) + O\big(|x - x_0|\big) \equiv 0.$$

Thus

$$(p!)^{-1}\big(\lambda(x) - \lambda_0\big)^p + O\big(|\lambda(x) - \lambda_0|\big)^{p+1} = O\big(|x - x_0|\big),$$

which implies (13), with p instead of m. We have $m \leq p$, and the inequality may of course be strict, so that the bound of Proposition 3.1 is stronger than the one proved here in this Remark.

The estimate of Proposition 3.1 is much improved in the semi-simple case:

Proposition 3.3. *If M is differentiable at x_0, and if (λ_0, x_0) is an isolated coalescing point such that λ_0 is a semi-simple eigenvalue of $M(x_0)$, any local branch λ of eigenvalues of M such that $\lambda(x_0) = \lambda_0$ has a one-sided directional derivative in every direction, and, for all $\vec{e} \in \mathbb{R}^d$,*

$$\lim_{\substack{t \to 0 \\ t > 0}} \frac{\lambda(x_0 + t\vec{e}\,) - \lambda(x_0)}{t} \in \operatorname{sp} P(\lambda_0, x_0) M'(x_0) \cdot \vec{e}\, P(\lambda_0, x_0),$$

where $P(\lambda_0, x_0)$ is the generalized eigenprojector onto the generalized eigenspace at (λ_0, x_0), and parallel to the direct sum of the other generalized eigenspaces. In particular, the eigenvalues are Lipschitz:

$$\big|\lambda(x) - \lambda(x_0)\big| \leq C(M)|x - x_0|,$$

locally around x_0, with $C(M) > 0$.

See Corollary 3.6 below for an improvement on Proposition 3.3.

Proof. Let m be the multiplicity of λ_0, and $\lambda_1, \dots, \lambda_{m'}$, $2 \le m' \le m$, the distinct eigenvalues that coalesce at x_0 with value λ_0. By Corollary 2.3,

$$\int_\gamma \lambda\big(\lambda - M(x_0+h)\big)^{-1}\, d\lambda = \sum_{1\le j\le m'} \big(\lambda_j(x_0+h) + N_j(x_0+h)\big) P_j(x_0+h),$$

where $h \in \mathbb{R}^d$ is small and γ is a suitable curve in $\mathbb{C}$. Above, N_j and P_j are the nilpotent and projector associated with λ_j. By Proposition 2.1,

$$\int_\gamma \big(\lambda - M(x_0+h)\big)^{-1}\, d\lambda = P(x_0+h) := \sum_{1\le j\le m'} P_j(x_0+h).$$

Thus

$$(14)\quad \int_\gamma (\lambda-\lambda_0)\big(\lambda - M(x_0+h)\big)^{-1}\, d\lambda \\ = \sum_{1\le j\le m'} \big(\lambda_j(x_0+h) - \lambda_0 + N_j(x_0+h)\big) P_j(x_0+h),$$

By differentiability of M at x_0:

$$\big(\lambda - M(x_0+h)\big)^{-1} = \big(\lambda - M(x_0)\big)^{-1} + \big(\lambda - M(x_0)\big)^{-1} M'(x_0)\cdot h\big(\lambda - M(x_0)\big)^{-1} + o(h).$$

Since λ_0 is semi-simple, the spectral decomposition at x_0 is

$$M(x_0) = \lambda_0 P(\lambda_0, x_0) + M(x_0)\big(\mathrm{Id} - P(\lambda_0, x_0)\big),$$

where $P(\lambda_0, x_0)$ is the generalized eigenprojector. Thus

$$\big(\lambda - M(x_0)\big)^{-1} = (\lambda-\lambda_0)^{-1} P(\lambda_0, x_0) + \big(\lambda - M(x_0)\big)^{-1}\big(\mathrm{Id} - P(\lambda_0, x_0)\big),$$

so that

$$\begin{aligned}
&(\lambda-\lambda_0)\big(\lambda - M(x_0+h)\big)^{-1} \\
&\quad = P(\lambda_0, x_0) \\
&\qquad + (\lambda-\lambda_0)\big(\lambda - M(x_0)\big)^{-1}\big(\mathrm{Id} - P(\lambda_0, x_0)\big) \\
&\qquad + P(\lambda_0, x_0) M'(x_0)\cdot h(\lambda - M(x_0))^{-1} \\
&\qquad + (\lambda-\lambda_0)\big(\lambda - M(x_0)\big)^{-1}\big(\mathrm{Id} - P(\lambda_0, x_0)\big) M'(x_0)\cdot h\big(\lambda - M(x_0)\big)^{-1} \\
&\qquad + o(h).
\end{aligned}$$

We now compute residues. First, by choice of γ, definition of $P(\lambda_0, x_0)$ and Proposition 2.1,

$$\frac{1}{2i\pi}\int_\gamma \big(\lambda - M(x_0)\big)^{-1}\, d\lambda = P(\lambda_0, x_0).$$

Second,

$$\int_\gamma (\lambda - \lambda_0)\big(\lambda - M(x_0)\big)^{-1}\, d\lambda = 0,$$

and

$$\int_\gamma (\lambda - \lambda_0)\big(\lambda - M(x_0)\big)^{-1}\big(\mathrm{Id} - P(\lambda_0, x_0)\big)\, d\lambda = 0,$$

and

$$\int_\gamma (\lambda - \lambda_0)\big(\lambda - M(x_0)\big)^{-1}\big(\mathrm{Id} - P(\lambda_0, x_0)\big) M'(x_0)\cdot h\big(\lambda - M(x_0)\big)^{-1}\, d\lambda = 0,$$

since in all three cases the integrands do not have poles in the interior of γ. From (14) and the above, we deduce

$$\sum_{1\le j\le m'} \left(\frac{\lambda_j(x_0+h) - \lambda_0 + N_j(x_0+h)}{|h|}\right) P_j(x_0+h) = P(\lambda_0, x_0) M'(x_0)\cdot \frac{h}{|h|} P(\lambda_0, x_0) + o(1). \tag{15}$$

Equating spectra, evaluating at $h = t\vec{e}$, for $t > 0$, and taking the limit $t \to 0$ (as we may by Proposition 1.1), we arrive at the result. $\square$

Remark 3.4. If (λ_0, x_0) has constant multiplicity, then by Corollary 2.2, the branch of eigenvalues λ and the associated eigenprojector P are as smooth as M. If M is differentiable, the proof of Proposition 3.3 shows that $\lambda'(x_0)\cdot hP(\lambda_0, x_0) = P(\lambda_0, x_0)M'(x_0)\cdot hP(\lambda_0, x_0)$. A shortcut here consists in differentiating the identity $M(x)P(x) = \lambda(x)P(x)$, for x close to x_0, which gives

$$M'(x)P(x) + M(x)P'(x) = \lambda'(x)P(x) + \lambda(x)P'(x),$$

and then, since $PP'P \equiv 0$ (simply because P is a projector), by applying P to the left and the right of the above identity, we find $PM'P = \lambda'P$.

Lemma 3.5. *Given (λ_0, x_0) in the spectrum of M, with index m, if M is $q \ge 1$ times differentiable at x_0, denote M_0 the Taylor expansion of M at x_0:*

$$M(x) = M_0(x) + |x - x_0|^q R(x_0, x), \tag{16}$$

where M_0 is a degree-q polynomial in $x - x_0$, and $R(x_0, x) \to 0$ as $x \to x_0$. Then, for any branch λ of eigenvalues of M such that $\lambda(x_0) = \lambda_0$, for some branch μ of eigenvalues of M_0, we have

$$\lambda(x) = \mu(x) + o(|x - x_0|^{q/m}). \tag{17}$$

Proof. Let

$$\mathbf{M}(x,y) = M_0(x) + y, \quad y \in \mathbb{C}^{N^2}.$$

Then, $\mathbf{M}(x_0,0) = M_0(x_0) = M(x_0)$. In particular, the point $(\lambda_0, x_0, 0)$ has multiplicity m in the spectrum of $\mathbf{M}$. Let $\boldsymbol{\lambda}$ be a local branch of eigenvalues of $\mathbf{M}$ such that $\boldsymbol{\lambda}(x_0,0) = \lambda_0$. By Proposition 3.1, where the variable $y \in \mathbb{C}^{N^2}$ is seen as a real variable $y \in \mathbb{R}^{2N^2}$, we have

$$\boldsymbol{\lambda}(x,y) - \boldsymbol{\lambda}(x,0) = O(|y|^{1/m}), \quad \text{for small} |y| \text{and} x \text{near} x_0. \tag{18}$$

Specializing to $y = |x-x_0|^q R(x_0,x)$ for x near x_0, we observe that, given λ a branch of eigenvalues of M such that $\lambda(x_0) = \lambda_0$, we have

$$\lambda(x) = \boldsymbol{\lambda}\big(x, |x-x_0|^q R(x_0,x)\big).$$

Since $\boldsymbol{\lambda}(\cdot,0)$ is a branch of eigenvalues of M_0, we deduce (17) from (18) and the fact that $R(x_0,x) \to 0$ as $x \to x_0$. □

With the help of Lemma 3.5, we may remove, in the statement of Proposition 3.3, the assumption that (λ_0,x_0) is an *isolated* coalescing point in the spectrum:

Corollary 3.6. *If M is differentiable at x_0, and if (λ_0,x_0) is a coalescing point such that λ_0 is a semi-simple eigenvalue of $M(x_0)$, then the conclusion of Proposition 3.3 holds. That is, the assumption that (λ_0,x_0) is an isolated coalescing point in the spectrum can be removed in Proposition 3.3.*

Proof. Let (16) be the Taylor expansion of M at x_0, with $q = 1$. The eigenvalue λ_0 of $M(x_0)$ is also a semi-simple eigenvalue of $M_0(x_0)$. Consider one-dimensional perturbations $x = x_0 + t\vec{e}$, where $\vec{e}$ is given in $\mathbb{R}^d$, and $t \in \mathbb{R}$. Proposition 1.3 applies to the family of matrix polynomials in one variable $t \to M_0(x_0+t\vec{e})$. In particular, the coalescing point $(\lambda_0, 0)$ is isolated in the spectrum of $t \to M_0(x_0+t\vec{e})$. We may thus apply Proposition 3.3: for any branch $t \to \mu(t)$ of eigenvalues of $t \to M_0(x_0+t\vec{e})$, we have

$$\lim_{\substack{t\to 0\\ t>0}} \frac{\mu(t)-\mu(0)}{t} \in \operatorname{sp} P(\lambda_0,x_0)M'(x_0)\cdot\vec{e}\, P(\lambda_0,x_0). \tag{19}$$

Here we used $M(x_0) = M_0(x_0)$, so that the relevant generalized eigenprojector for M_0 at (λ_0,x_0) coincides with the projector for M, and $M'(x_0) = M_0'(x_0)$.

Now given λ a branch of eigenvalues of M such that $\lambda(x_0) = \lambda_0$, by Lemma 3.5 with $q = m = 1$ we have

$$\lambda(x_0+t\vec{e}) - \mu(t) = o(t),$$

for some branch μ of eigenvalues of $t \to M_0(x_0 + t\vec{e})$. Thus, with (19), we have

$$\lambda(x_0 + t\vec{e}) = \lambda(x_0) + \alpha t + o(t), \qquad t > 0,$$

where α is in the spectrum of $P(\lambda_0, x_0)M'(x_0) \cdot \vec{e}\, P(\lambda_0, x_0)$. This is precisely the conclusion of Proposition 3.3. $\square$

4. Puiseux expansions

We describe eigenvalues around a coalescing point, following the approach of [Tex].

Consider a point $(\lambda_0, x_0) \in \mathcal{S}$, and suppose that M is $q \geq 1$ times differentiable at x_0, so that the Taylor expansion (16) holds. We reproduce (16) here:

$$M(x) = M_0(x) + |x - x_0|^q R(x_0, x), \qquad R(x_0, x) \to 0 \text{ as } x \to x_0.$$

The entries of matrix M_0 are polynomials of degree q in $x - x_0 \in \mathbb{R}^d$. In particular, M_0 has an extension to $\mathbb{C}^d$. Let $\vec{e} \in \mathbb{R}^d$ be a fixed spatial direction, and consider

$$\mathcal{S}_0 := \Big\{(\lambda, z) \in \mathbb{C} \times B(0, \varepsilon),\ \det\big(M_0(x_0 + z\vec{e}) - \lambda \mathrm{Id}\big) = 0\Big\},$$

where $B(0, \varepsilon) \subset \mathbb{C}$ is the open disk centered at 0 and with radius $\varepsilon > 0$ in the complex plane. We denote π_0 the projection

$$\pi_0 :\ (\lambda, z) \in \mathcal{S}_0 \longrightarrow z \in B(0, \varepsilon).$$

By Proposition 1.3, if ε is small enough then $\mathcal{S}_0$ has only $(\lambda_0, 0)$ as a coalescing point. Thus by Corollary 1.4, the restriction of π_0 to $\mathcal{S}_0 \cap \pi_0^{-1}(B(0, \varepsilon)^*)$ is a covering of $B(0, \varepsilon)^*$ if ε is small enough. Let V be a connected component of $\mathcal{S}_0 \cap \pi_0^{-1}(B(0, \varepsilon)^*)$. Since $B(0, \varepsilon)^*$ is connected and locally path-connected, the restriction $\tilde{\pi}_0$ of π_0 to V is a covering map with base $B(0, \varepsilon)^*$:

$$\tilde{\pi}_0 : \quad (\lambda, z) \in V \longrightarrow z \in B(0, \varepsilon)^*.$$

Lemma 4.1. *The covering map $\tilde{\pi}_0$ is conjugated to the covering $p : z \to z^{m'}$ of $B(0, \varepsilon)^*$ for some $m' \in \mathbb{N}^*$ that is at most equal to the multiplicity of (λ_0, x_0). That is, there exists a homeomorphism ψ such that the following diagram is commutative:*

$$\begin{tikzcd} V \arrow[r, "\psi"] \arrow[d, "\tilde{\pi}_0"'] & p^{-1}\big(B(0,\varepsilon)^*\big) \arrow[dl, "p"] \\ B(0,\varepsilon)^* & \end{tikzcd}$$

Proof. Let $\lambda_1(x_0+z\vec{e}),\ldots,\lambda_{m'}(x_0+z\vec{e})$ be the distinct eigenvalues of M_0 which takes values in V for $z \in B(0,\varepsilon)^*$. The number of these eigenvalues is constant over $B(0,\varepsilon)^*$, and at most equal to the multiplicity of (λ_0, x_0). In particular, $\tilde{\pi}_0$ is an m'-sheeted covering of $B(0,\varepsilon)^*$. Connected coverings of a punctured ball in $\mathbb{C}$ are determined, up to isomorphism, by their numbers of sheets (see for instance [Mas, Chapter V, Theorem 6.6]). Thus $\tilde{\pi}_0$ is conjugated to p, by a homeomorphism ψ. □

Based on Lemma 4.1, we may give Puiseux expansions of eigenvalues around a coalescing point:

Proposition 4.2. *If (λ_0, x_0) is a coalescing point in the spectrum of M, with index m, and if M is $q \geq 1$ times differentiable at x_0, then for any local branch λ of eigenvalues of M which coalesce at x_0 with value λ_0, any $\vec{e} \in \mathbb{R}^d$, there exists a smooth map ϕ defined in $[0,t_0]$, for some $t_0 > 0$, and a positive integer m' that is at most equal to the multiplicity of (λ_0, x_0), such that*

(20) $$\lambda(x_0 + t\vec{e}\,) = \phi(t^{1/m'}) + o(t^{q/m}),$$

for $0 \leq t \leq t_0$.

By Proposition 3.1, we also know that $|\lambda(x_0 + t\vec{e}\,) - \lambda(x_0)| = O(t^{1/m})$. In particular, $\phi(0) = \lambda_0$, and, if $m' > m$, then the first derivative or derivatives of ϕ are equal to 0 at $t = 0$: $\phi^{(k)}(0) = 0$ for $0 < k < m'/m$.

Proof. Given $\varepsilon > 0$ and V as in the discussion preceding Lemma 4.1, let μ be a local section of $\tilde{\pi}_0$, that is a branch of eigenvalues of $M_0(x_0 + z\vec{e})$. We have $\tilde{\pi}_0(\mu) \equiv \mathrm{Id}$, hence, by Lemma 4.1, $p \circ \psi^{-1} \circ \mu \equiv \mathrm{Id}$. Thus $\psi^{-1} \circ \mu$ is a section of p, meaning an m'-th root of unity:

(21) $$\mu(z) = \phi(\omega z^{1/m'}),$$

where ϕ is the first component of ψ, and ω is a given m'-th root of unity.

We now specialize to a local section μ which is defined at some $t_0 > 0$, so that $\big(t_0, \mu(t_0)\big) \in V$. Then, the set $\{\big(t, \mu(t)\big), 0 < t \leq t_0\}$ is connected in $\mathcal{S}_0 \cap \pi_0^{-1}\big(B(0,\varepsilon)^*\big)$, by continuity of μ, hence included in the connected component V. Thus equality (21) holds for small enough $t > 0$. In particular,

$$\mu(t^{m'}) = \phi(\omega t), \quad \text{for } 0 < t \leq t_0,$$

implying that $t \to \phi(\omega t)$ is as regular as μ, hence analytical (by Corollary 2.2, since only 0 is a coalescing point and M_0 is analytical). Thus, $t \to \phi(\omega t)$, being analytical in $0 < t \leq t_0$ and bounded around $t = 0$, is analytical in $[0, t_0]$, so that (21) holds for all $0 \leq t \leq t_0$, with $\mu(0) = \phi(0)$.

Let finally λ be a branch of eigenvalues of M such that $\lambda(x_0) = \lambda_0$. By Lemma 3.5, for some branch μ of eigenvalues of M_0, we have

$$\lambda(x_0 + t\vec{e}\,) = \mu(t) + o(t^{q/m}).$$

Together with (21), this implies (20), with a slight change of notation for ϕ. □

Bibliographical note. The Cauchy formula of Proposition 2.1 is found in Equation (1.16), Paragraph 1.4, Chapter 2, in Kato [Kat]. The proof of Proposition 3.1 is borrowed from Saad ([Saa, Proposition 3.3 in Section 3.1.5]). The existence of directional derivatives (Proposition 3.3) is found in Theorem 2.3, Paragraph 2.3, Chapter 2, in [Kat]. Kato refers to Knopp [Kno], without proof, for details on Puiseux expansions (see [Kat, Chapter 2, Paragraph 1.2]). So do Reed and Simon ([RS, XII.1]). Knopp's discussion is limited to polynomials in two variables, the roots of which are described as multi-valued analytical functions; here eigenvalues around a coalescing point are seen as perturbations of sections of a ramified covering of a disk in the complex plane.

Remark 4.3 (On hyperbolic polynomials). If the spectrum of $M(x)$ is real for all $x \in \Omega$, then the family M is said to be hyperbolic. The eigenvalues are then locally Lipschitz; see Brohnstein [Bro], or Kurdyka and Paunescu [KP]. In one space dimension, Rellich's theorem [Rel] states that analytic families of Hermitian matrices have analytic eigenvalues and eigenvectors.

Remark 4.4 (On geometric optics). An important consequence of Proposition 3.3 is that the amplitude of a wave-packet is transported by a hyperbolic system at group velocity; this is a crucial step in the derivation of amplitude equations in geometric optics, see [Tex] and references therein.

Similar formulas exist for higher derivatives (see [Tex, Proposition 2.6 and Remark 2.7] and Kato [Kat, Paragraphs 2.1 and 2.2, Chapter 2]). The corresponding identity for second-order derivatives describes the Schrödinger correction to the transport along rays for distances of propagation equal to the inverse of the wavelength.

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Benjamin Texier, IMJ-PRG – UMR7586, Université Paris Diderot,
Bâtiment Sophie Germain, 8 Place Aurélie Nemours, 75205 Paris Cedex 13, France

e-mail: benjamin.texier@imj-prg.fr