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Local coefficients revisited

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Abstract. Two simple "simplicial approximation" tricks are invoked to prove basic results involving (co)-homology with local coefficients.

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1. Introduction

Homology and cohomology with local coefficients have applications in a variety of topics including Obstruction Theory, Spectral Sequences, Generalized Poincaré Duality and more. These homology and cohomology theories possess many properties analogous to those of homology and cohomology with constant coefficients (some of these properties will be presented in Section 2), yet some of the corresponding properties are missing. In particular, there is no Universal Coefficient Theorem linking homology with local coefficients with cohomology (there is a version in [Spa1, p. 283], though its application is limited). This among others poses extra difficulties in proving many familiar and useful properties of these theories. More than often, it takes a completely different proof to generalize a theorem involving (co)-homology with constant coefficients to one involving local coefficients.

The purpose of this paper is to establish three basic properties of (co)-homology with local coefficients.

The First Property (Theorem 3.1). *A weak homotopy equivalence induces an isomorphism on homology and cohomology with local coefficients.*

The Second Property (Theorem 4.1). *For a CW-pair the long exact sequences for singular and cellular homology and cohomology group with local coefficients are equivalent.*

Despite being very basic, no written account of these properties could be located. Quite likely this was due to the suspicion that the technical difficulties in rigorous proofs presented a price too high to pay for the final product. Our proofs of these properties are based on a simple "simplicial approximation" trick which avoids most of the technicalities.

The Third Property (Theorem 5.1). *The Poincaré Duality with local coefficients for closed orientable topological manifolds.*

In this case written (and complete) accounts of this property do exist. In fact, we are aware of two such accounts (cf. [Spa2, Sun]). In this paper we present yet another alternative proof based on a "simplicial approximation" trick. With certain topological input, this trick reduces the proof to the more elementary case of triangulated manifolds.

More comments of historical, motivational and mathematical nature are contained in corresponding sections dealing with the proofs.

2. Local coefficient systems

For more detail on this topic, the reader is referred to [Whi, Chapter VI, Section 1–4].

Let G be a bundle of Abelian groups (local coefficient system) on a topological space X, that is, G is a covariant functor from the fundamental groupoid of X to the category of Abelian groups. Recall that the *singular chain complex of X with coefficient in G* is defined as

$$S_k(X;G) = \bigoplus_{\sigma:\Delta^k \to X} G\big(\sigma(e_0)\big)$$

$$\partial(g\cdot\sigma) = G(\bar{\sigma})(g)\cdot\sigma_0 + \sum_{i=1}^{k}(-1)^i g\cdot\sigma_i \tag{1}$$

where $\Delta^k = \langle e_0, e_1, \cdots, e_k\rangle$ is the standard k-simplex, $g\cdot\sigma \in S_k(X;G)$ is the element that is g on the $G(\sigma(e_0))$ factor and 0 otherwise, $\bar{\sigma}$ is the homotopy class of the path obtained by precomposing σ with the linear path from e_1 to e_0 in Δ^k, σ_i is the restriction of σ to the i-th face.

Note that the notations we are using are slightly different from the one used in Whitehead's book.

It can be shown that $\{S_*(X;G), \partial\}$ is a chain complex, and its homology is called the *singular homology of X with coefficient in G*, denoted as $H_*(X;G)$.

In a similar way, one could define the *singular cochain complex $S^*(X;G)$*

$$S^k(X;G) = \prod_{\tau:\Delta^k\to X} G\big(\tau(e_0)\big)$$

(2) $$(-1)^k(\delta c)(\sigma) = G(\bar{\sigma})^{-1}c(\sigma_0) + \sum_{i=1}^{n+1}(-1)^i c(\sigma_i)$$

where $c \in S^k(X;G), \sigma : \Delta^{k+1} \to X$.

It turns out that $\{S^*(X;G), \delta\}$ forms a cochain complex and its homology is called the *singular cohomology of X with coefficient in G*, denoted as $H^*(X;G)$.

There is another way to define (co)homology groups with local coefficients that is equivalent to the one above. In the literature it sometimes carries the name "homology with twisted coefficients". We shall introduce this approach in Section 2.8.

Long exact sequence of pairs, homotopy invariance, excision and additivity (with respect to disjoint union) are still valid as in the case of constant coefficients (cf. [Wal, I. 2]). The equivalence between singular and cellular homology/cohomology (cf. [Wal, VI. 4]) is valid as well. The remainder of this section will be devoted to a brief review of these facts. Readers familiar with the material of Chapter VI of [Wal] or any equivalent exposition could skip to the next section.

We shall present these facts without their proofs. We will concentrate on cohomology, since in the remaining sections we shall deal mainly with cohomology groups. The results for homology are analogous.

In what follows, all spaces are topological spaces and maps between these spaces are continuous maps.

2.1. Relative (co)homology groups and long exact sequence. Let (X, A) be a pair of spaces and G a local coefficient system on X. Denote by i the inclusion $A \hookrightarrow X$. The restriction $G_{|A}$ is a local coefficient system on A and the restriction $i^\# : S^*(X;G) \to S^*(A;G_{|A})$ is naturally defined and surjective. The kernel of $i^\#$ is defined as $S^*(X,A;G)$. This is a cochain complex, whose homology is the relative cohomology group $H^*(X,A;G)$. The short exact sequence of cochain complexes

$$0 \to S^*(X,A;G) \to S^*(X;G) \to S^*(A;G_{|A}) \to 0$$

induces a long exact sequence

$$\cdots \longrightarrow H^{k-1}(A;G_{|A}) \xrightarrow{\delta} H^k(X,A;G) \longrightarrow H^k(X;G) \longrightarrow H^k(A;G_{|A}) \longrightarrow \cdots$$

Dually, $S_*(X,A;G)$ is defined as the cokernel of $i_\# : S_*(A;G_{|A}) \to S_*(X;G)$, and its homology group $H_*(X,A;G)$ fits into a long exact sequence

$$\cdots \longrightarrow H_k(A;G_{|A}) \xrightarrow{\delta} H_k(X;G) \longrightarrow H_k(X,A;G) \longrightarrow H_{k-1}(A;G_{|A}) \longrightarrow \cdots$$

Long exact sequences of triples are defined in a similar way.

In what follows, we shall abbreviate $G_{|A}$ as G.

2.2. Functoriality. Let G, H be bundles of Abelian groups over a space X. A homomorphism $\varphi : G \to H$ is a natural transformation from G to H. If $\varphi(x) : G_x \to H_x$ is an isomorphism of Abelian groups for each $x \in X$, we say φ is an isomorphism. A homomorphism (resp. isomorphism) φ between bundles induces a chain map (resp. isomorphism) $S^*(X;\varphi) : S^*(X;G) \to S^*(X;H)$, defined by

$$S^k(X;\varphi)(c)(\sigma) = \varphi\big(c(\sigma)\big), c \in S^k(X;G), \sigma : \Delta^k \to X$$

Denote the induced homomorphism on cohomology as $H^*(X;\varphi)$. There are, of course, $S_*(X;\varphi)$ and $H_*(X;\varphi)$ defined for homology.

Let G be a local coefficient system on Y, and $f : X \to Y$ a map. Then the pull-back $f^*G = G \circ f_\#$, where $f_\#$ is the induced functor between the fundamental groupoid of X and that of Y, is a local coefficient system on X. If $g : Z \to X$ is a map, then $(f \circ g)^*G = g^* f^* G$. For a homomorphism $\varphi : G \to G'$ between bundles of Abelian groups over Y, the pull back $f^*\varphi : f^*G \to f^*G'$ defined as $f^*\varphi(x) = \varphi(f(x))$ is a homomorphism between bundles over X.

We will define a category $\mathfrak{L}^*$ whose objects are of the form $(X, A; G)$ where (X, A) is a pair of spaces and G is a bundle of Abelian groups on X. A morphism ϕ from $(X, A; G)$ to $(Y, B; H)$ is a pair (ϕ_1, ϕ_2) where $\phi_1 : (X, A) \to (Y, B)$ is a map and $\phi_2 : \phi_1^* H \to G$ is a homomorphism. Suppose $\phi = (\phi_1, \phi_2) : (X, A; G) \to (Y, B; H)$ and $\psi = (\psi_1, \psi_2) : (Y, B; H) \to (Z, C; K)$ are morphisms in $\mathfrak{L}^*$. Then their composition $\omega = \psi \circ \phi$ is defined by

$$\omega_1 = \psi_1 \circ \phi_1 : (X, A) \to (Z, C), \omega_2 = \phi_2 \circ (\phi_1^* \psi_2) : \phi_1^* \psi_1^* K \to G$$

Note that a map $f : (X, A) \to (Y, B)$ and a local coefficient system G on Y induce a morphism $\bar{f} : (X, A; f^*G) \to (Y, B; G)$, where $\overline{f_1} = f, \overline{f_2} : f^*G \to f^*G$ is the identity. By abuse of notations, we will denote $\bar{f}$ simply by f.

A morphism $\phi : (X, A; G) \to (Y, B; H)$ induces a cochain map

$$\phi^\# : S^*(Y, B; H) \to S^*(X, A; G), \phi^\#(c)(\sigma) = \phi_2\big(\sigma(e_0)\big)\big(c(\phi_1 \circ \sigma)\big)$$

where $c \in S^k(Y, B; H)$ and $\sigma : \Delta^k \to X$. Thus ϕ induces a homomorphism $\phi^* = H^*(\phi) : H^*(Y, B; H) \to H^*(X, A; G)$. In this way H^k's become contravariant functors from $\mathfrak{L}^*$ to the category of Abelian groups.

There is a category $\mathfrak{L}_*$ dual to $\mathfrak{L}^*$. The two categories share the same class of objects, whereas a morphism in $\mathfrak{L}_*$ from $(X, A; G)$ to $(Y, B; H)$ is a pair $\psi = (\psi_1, \psi_2)$ where $\psi_1 : (X, A) \to (Y, B)$ is a map and $\psi_2 : G \to \psi_1^* H$ is a homomorphism. It is obvious how to define the induced $\psi_\#$ and $\psi_* = H_*(\psi)$.

Remark. It is easy to see that a morphism $\phi : (X, A; G) \to (Y, B; H)$ in either $\mathfrak{L}_*$ or $\mathfrak{L}^*$ actually induces a chain map between the corresponding long exact sequences of homology/cohomology. We shall need this fact in later sections.

2.3. Homotopy invariance in $\mathfrak{L}^*$ and $\mathfrak{L}_*$. The *prism* over an object $(X, A; G)$ of $\mathfrak{L}^*$ is the object $(X \times I, A \times I; p^*G)$ of $\mathfrak{L}^*$ where $p : X \times I \to X$ is the projection to the first factor and $I = [0, 1]$. The morphism $i^0 : (X, A; G) \to (X \times I, A \times I; p^*G)$ is defined by

$$i_1^0(x) = (x, 0), i_2^0 = \text{id} : i_1^{0*} p^*G = G \to G$$

Similarly we can define the morphism $i^1 : (X, A; G) \to (X \times I, A \times I; p^*G)$.

Let $\phi, \psi : (X, A; G) \to (Y, B; H)$ be two morphisms. A homotopy from ϕ to ψ is a morphism $\lambda : (X \times I, A \times I; p^*G) \to (Y, B; H)$ such that $\lambda \circ i^0 = \phi, \lambda \circ i^1 = \psi$. In this case we write $\phi \simeq \psi$, as usual.

The prism of $(X, A; G)$ in $\mathfrak{L}_*$ is the same as that in $\mathfrak{L}^*$, while the morphism $i^0 : (X, A; G) \to (X \times I, A \times I; p^*G)$ is defined by

$$i_1^0(x) = (x, 0), i_2^0 = \text{id} : G \to i_1^{0*} p^*G = G$$

The definition of i^1 is similar. Now the definition of homotopy in $\mathfrak{L}^*$ can be carried verbatim to $\mathfrak{L}_*$.

Homotopy equivalence between objects of $\mathfrak{L}^*$ or $\mathfrak{L}_*$ is defined in the obvious way. It is an equivalence relation, and we have:

Theorem 2.1 (Homotopy Invariance). *If $\phi, \psi : (X, A; G) \to (Y, B; H)$ are homotopic maps in $\mathfrak{L}^*$ (resp. $\mathfrak{L}_*$), then $H^*(\phi) = H^*(\psi)$ (resp. $H_*(\phi) = H_*(\psi)$).*

Of course this implies that a homotopy equivalence in $\mathfrak{L}^*$ or $\mathfrak{L}_*$ induces an isomorphism on (co)homology groups.

2.4. Homotopy invariance. As far as we are aware of, the material of this subsection has not appeared explicitly in the literature.

In practice, homotopy equivalences between topological spaces arise more often and more naturally than homotopy equivalence in $\mathfrak{L}^*$ and $\mathfrak{L}_*$. Fortunately, we have the following:

Theorem 2.2. *Suppose $f : (X, A) \to (Y, B)$ is a homotopy equivalence (between pairs of topological spaces) and G is a local coefficient system on Y, then $f^* : H^*(Y, B; G) \to H^*(X, A; f^*G)$ and $f_* : H_*(X, A; f^*G) \to H_*(Y, B; G)$ are isomorphisms.*

We shall prove the claim for cohomology, and note that the proof of the case of homology is entirely dual. The conclusion of Theorem 2.2 should be expected. Yet the same line of proof as with ordinary coefficients would immediately run into trouble because the coefficients change under the pull-back. The difficulties can be overcome by the following lemma.

Lemma 2.3. *Let $H : (X, A) \times I \to (Y, B)$ be a homotopy between f and g and G be a bundle of Abelian groups over Y. For $x \in X$, define H_x as the homotopy class of the path $t \mapsto H(x, 1-t)$. Then $x \mapsto G(H_x)$ defines a bundle isomorphism $\tilde{H} : f^*G \to g^*G$. Furthermore we have the equality $H^*(X, A; \tilde{H}) \circ f^* = g^* : H^*(Y, B; G) \to H^*(X, A; g^*G)$. Here $f^* : H^*(Y, B; G) \to H^*(X, A; f^*G), g^* : H^*(Y, B; G) \to H^*(X, A; g^*G)$ are induced by f, g viewed as morphisms in $\mathfrak{L}^*$.*

Proof. To prove $\tilde{H}$ is a natural transformation, consider, for a given path $u : I \to X$, the map $H \circ (u \times 1_I) : I \times I \to Y$. The two paths joining $(0,0)$ and $(1,1)$ along the boundary are homotopic, and naturality follows. For the second claim, we can repeat the prism construction as in the case of constant coefficients (though in most textbooks this construction is done for homology only, because of the Universal Coefficient Theorem). For any n, consider the prism $\Delta^n \times I$. Let v_i (resp. w_i) be the i-th vertex of $\Delta^n \times \{0\}$ (resp. $\Delta^n \times \{1\}$). Denote by ι_i the linear embedding $\Delta^{n+1} \to \Delta^n \times I$ sending $\Delta^{n+1} = \langle e_0, \cdots, e_{n+1} \rangle$ onto the simplex $\langle v_0, \cdots, v_i, w_i, \cdots, \omega_n \rangle$ preserving order of vertices.

For each n, define a homomorphism $P : S^{n+1}(Y, B; G) \to S^n(X, A; f^*G)$ by

$$P(c)(\sigma) = \sum_{i=0}^{n} (-1)^i c\big(H \circ (\sigma \times 1_I) \circ \iota_i\big), c \in S^{n+1}(Y; G), \sigma : \Delta^n \to X$$

One can easily check that $\delta P + P\delta = S^n(X; \tilde{H}) \circ (f^\# - g^\#)$, where δ is the coboundary homomorphisms of the chains $S^*(X, A; f^*G)$, $S^*(Y, B; G)$ and $f^\# : S^n(Y, B; G) \to S^n(X, A; f^*G), g^\# : S^n(Y, B; G) \to S^n(X, A; g^*G)$ are induced by f, g as morphisms in $\mathfrak{L}^*$. Consequently $g^* \circ H(X, A; \tilde{H}) = f^*$. □

Proof of Theorem 2.2. Let $g : (Y, B) \to (X, A)$ be the homotopy inverse of f. Thus we have homotopies $f \circ g \overset{H}{\simeq} 1_Y, g \circ f \overset{H'}{\simeq} 1_X$. Apply the above lemma to the first homotopy with bundle G over Y and to the second homotopy with f^*G over X, we have $(f \circ g)^* \circ H(Y, B; \tilde{H}) = 1_Y^*$ and $(g \circ f)^* \circ H(X, A; \widetilde{H'}) = 1_X^*$. In particular, $(f \circ g)^* = g^* \circ f^* : H^*(Y, B; G) \to H^*(Y, B; g^* f^* G)$ and $(g \circ f)^* = f^* \circ g^* : H^*(X, A; f^*G) \to H^*(X, A; f^* g^* f^* G)$ are isomorphisms. But this implies that $g^* : H^*(X, A; f^*G) \to H^*(Y, B; g^* f^* G)$ is both an injection and a surjection, thus a bijection. Hence $f^* : H^*(Y, B; G) \to H^*(X, A; f^*G)$ is an isomorphism. □

In Section 3, we will sharpen Theorem 2.2 by replacing “homotopy equivalence” in the statement with “weak homotopy equivalence”.

2.5. Excision, additivity and dimension axioms. Although the following properties will not be used in our paper, we list them for completeness sake.

Theorem 2.4. *Let $(X; X_1, X_2)$ be a triad of spaces such that $X = \operatorname{Int} X_1 \cup \operatorname{Int} X_2$, and let G be a bundle of Abelian groups over X. Then the injection $(X_1, X_1 \cap X_2) \hookrightarrow (X, X_2)$ induces isomorphisms $H^*(X, X_2; G) \to H^*(X_1, X_1 \cap X_2; G_{|X_1})$ and $H_*(X_1, X_1 \cap X_2; G_{|X_1}) \to H_*(X, X_2; G)$.*

Theorem 2.5. *Let X be a disjoint union of X_α's, G be a system of local coefficients over X and $G_\alpha = G_{|X_\alpha}$, $A \subseteq X$ be a subspace and $A_\alpha = A \cap X_\alpha$. Then $H^*(X, A; G) \to H^*(X_\alpha, A_\alpha; G_\alpha)$ and $H_*(X_\alpha, A_\alpha; G_\alpha) \to H_*(X, A; G)$ induced by inclusion for all α represent $H^*(X, A; G)$ as a direct product and $H_*(X, A; G)$ as a direct sum.*

Theorem 2.6. *If $X = \{*\}$ is a one point space, then $H^0(X; G) \cong G \cong H_0(X; G)$ and $H^k(X; G) = 0 = H_k(X; G)$ for $k \neq 0$.*

2.6. Cellular homology. The construction of cellular (co)homology carries over verbatim as the case of constant coefficients. Let (X, A) be a CW-pair and G be a system of local coefficients over a CW-pair X. Denote the k-th skeleton of X by X_k. Define the cellular cochain (resp. chain) complex $\Gamma^*(X, A; G)$ (resp. $\Gamma_*(X, A; G)$) of (X, A) with coefficients in G by $\Gamma^k(X, A; G) = H^k(X_k \cup A, X_{k-1} \cup A; G)$ (resp. $\Gamma_k(X, A; G) = H_k(X_k \cup A, X_{k-1} \cup A; G)$). The coboundary map $\delta : \Gamma^k(X, A; G) \to \Gamma^{k+1}(X, A; G)$ is defined as the connecting homomorphism of the cohomological long exact sequence of the triple $(X_{k+1} \cup A, X_k \cup A, X_{k-1} \cup A)$, while the boundary map $\partial : \Gamma_{k+1}(X, A; G) \to \Gamma_k(X, A; G)$ is defined using the homological long exact sequence of $(X_{k+1} \cup A, X_k \cup A, X_{k-1} \cup A)$. As in the case of constant coefficients, we have ([Whi, VI.4.1 and VI.4.1*]):

Theorem 2.7. *Let $h_\alpha : (\Delta^n, \partial\Delta^n) \to (X_n, X_{n-1}), \alpha \in \Lambda$ be characteristic maps of X of dimension n that are not contained in A. Denote $h_\alpha(e_0)$ as z_α. Then the homomorphisms (as α ranges through Λ) $G(z_\alpha) \to \Gamma_n(X, A; G)$ sending $g \in G(z_\alpha)$ to the homology class represented by $g \cdot h_\alpha$ represent $\Gamma_n(X, A; G)$ as a direct sum. Dually, the homomorphisms $\Gamma^n(X, A; G) \to G(z_\alpha)$ sending an element of $\Gamma^n(X, A; G)$ represented by $c \in S^n(X_n \cup A, X_{n-1} \cup A; G)$ to $c(h_\alpha)$ represent $\Gamma^n(X, A; G)$ as a direct product. If $k \neq n$, then $H_k(X_n \cup A, X_{n-1} \cup A; G) = 0 = H^k(X_n \cup A, X_{n-1} \cup A; G)$.*

There is an isomorphism between singular and cellular (co)homology groups with local coefficients for a CW-pair (X, A). The definition of this isomorphism is (again) carried verbatim from the case of constant coefficients, the references for which can be found in almost every textbook on algebraic topology (e.g., [Hat, Section 2.2 and Section 3.1]). In the remainder of this paper, we shall simply refer to this isomorphism as the (natural) isomorphism between singular and cellular (co)homology.

2.7. Cap product. We start by defining the tensor product of bundles of modules. Let R be a principal ideal domain. A bundle G of R-modules over a space X is a covariant functor from the fundamental groupoid of X into the category of R-modules. Let G, G' be bundles of R-modules over X. The tensor product $G \otimes_R G'$ is defined by $G \otimes_R G'(x) = G(x) \otimes_R G'(x)$ for any $x \in X$ and $G \otimes_R G'([u]) = G([u]) \otimes_R G'([u])$ for any $u : I \to X$, where $[u]$ is the homotopy class of u.

Denote the vertices of Δ^n as $e_0, e_1, \cdots, e_n$. Let $\sigma : \Delta^n \to X$ be a continuous map. For $0 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$, let $\sigma_{[i_1, i_2, \cdots, i_k]}$ denote σ restricted (preserving order) to the simplex $\langle e_{i_1} e_{i_2} \cdots e_{i_k} \rangle$.

For $\sigma : \Delta^n \to X$ and $g \in G(\sigma(e_0))$, let

$$g\sigma \in S_n(X; G) = \bigoplus_{\eta : \Delta^n \to X} G\big(\eta(e_0)\big)$$

denote the element which has value g on the σ-coordinate and 0 otherwise.

Now one is able to define the cap product on (absolute) chains and cochains. Assume that G, G' are bundles of R-modules over a space X, the cap product is defined as

$$\begin{aligned} S^k(X; G) \otimes_R S_n(X; G') &\overset{\frown}{\longrightarrow} S_{n-k}(X; G \otimes_R G') \\ c \otimes g\sigma &\longrightarrow (G(\sigma_{[0,n-k]})(c(\sigma_{[n-k,\cdots,n]})) \otimes g)\sigma_{[0,\cdots,n-k]} \end{aligned}$$

where $c \in S^k(X; G)$, $g \in G'(\sigma(e_0))$.

If A_1, A_2 are subspaces of X, define $S_n(A_1 + A_2; G') = S_n(A_1; G') + S_n(A_2; G') \subseteq S_n(X; G')$ and define $S_n(X, A_1 + A_2; G') = S_n(X; G')/S_n(A_1 + A_2; G')$. The above absolute cap product induces a relative product

$$S^k(X, A_1; G) \otimes_R S_n(X, A_1 + A_2; G') \overset{\frown}{\longrightarrow} S_{n-k}(X, A_2; G \otimes_R G')$$

The cap product satisfies the identity

$$\partial(c \frown \alpha) = c \frown (\partial\alpha) - (\delta c) \frown \alpha, c \in C^k(X; G), \alpha \in C_n(X; G')$$

Note that the sign appearing in the above equation is a result of our adopting the definitions (and thus the sign conventions) in [Whi].

Consequently there is an induced cap product on (co)homology

$$H^k(X, A_1; G) \otimes_R H_n(X, A_1 + A_2; G') \xrightarrow{\frown} H_{n-k}(X, A_2; G \otimes_R G')$$

If $\mathrm{Int}_{A_1 \cup A_2} A_1 \cup \mathrm{Int}_{A_1 \cup A_2} A_2 = A_1 \cup A_2$, then the inclusion induces an isomorphism $H_n(X, A_1 + A_2; G') \cong H_n(X, A_1 \cup A_2; G')$ and we have a cap product

$$H^k(X, A_1; G) \otimes_R H_n(X, A_1 \cup A_2; G') \xrightarrow{\frown} H_{n-k}(X, A_2; G \otimes_R G')$$

2.8. Homology with twisted coefficients. Suppose X is a path-connected space. If G is a local coefficient system on X, and G_0 is the group (fiber) over a chosen base point x_0. Then $\pi_1(X, x_0)$ acts on G_0 from the left by definition of G. Conversely, a (left) action of $\pi_1(X, x_0)$ on an Abelian group G_0 induces a bundle of Abelian groups G over X which is unique up to isomorphism, such that the fiber over x_0 is G_0 and the induced action on G_0 is the given action (cf. [Whi, p. 263]). Thus for connected spaces, bundles of Abelian groups are essentially equivalent to actions of fundamental groups. The data in the form of fundamental group actions has the advantage of being simpler and more explicit, and thus is preferred in the field of low dimensional topology.

The homology and cohomology groups with local coefficients we have defined so far could be expressed in terms of fundamental group actions, in the following way. Suppose our connected space X assumes a universal cover $\tilde{X}$. Choose a base point $\tilde{x_0}$ of $\tilde{X}$ covering x_0, we may identify $\pi_1(X, x_0)$ with the group Π of covering translations (deck transformations). Given an (left) action of $\pi_1(X, x_0)$ on an Abelian group G_0, we define a right action by requiring $g \cdot [u] = [u]^{-1} \cdot g$ for any $g \in G_0, [u] \in \pi_1(X)$.

On the other hand, Π acts on $\tilde{X}$ from the left. Hence there is an induced left action on $S_*(\tilde{X})$, the group of integral singular chains.

Let $\mathbb{Z}[\Pi]$ be the group ring of Π over $\mathbb{Z}$, one can form the chain complex $G_0 \otimes_{\mathbb{Z}[\Pi]} S_*(\tilde{X})$. It can be shown (see [Hat, Proposition 3H.4] or [Whi, Theorem VI.3.4]) that the homology groups of this chain complex (sometimes called homology with twisted coefficients) is isomorphic to $H_*(X; G)$, where G is the bundle of groups induced by the action of $\pi_1(X, x_0)$ on G_0. The dual results for cohomology is also true ([Whi, Theorem VI.3.4*]). Homology with twisted coefficients is usually easier to handle in explicit computations.

3. Weak homotopy equivalence and cohomology

It is a standard result in homotopy theory that weak homotopy equivalences (continuous maps which induce isomorphisms of all homotopy groups with all choices of basepoints) induce isomorphisms on singular homology and cohomology. In this section, we will provide a proof of the following result:

Theorem 3.1. *Let $f : (X, A) \to (Y, B)$ be a weak homotopy equivalence of pairs (that is, both f and $f_{|A}$ are weak homotopy equivalences). Let G be a local coefficient system on Y. Then $f_* : H_*(X, A; f^*G) \to H_*(Y, B; G)$ and $f^* : H^*(Y, B; G) \to H^*(X, A; f^*G)$ are isomorphisms, where f^*G is the pull-back of G via f.*

3.1. Background. Beside the natural expectation of the validity of the above result, Theorem 3.1 is also needed for the following definition in Obstruction Theory.

Suppose (K, L) is a relative CW-complex, $p : X \to B$ is a fibration with $(n-1)$-connected fiber F, and we are given a commutative diagram:

$$\begin{array}{ccc} L & \xrightarrow{f} & X \\ \big\downarrow & & \big\downarrow p \\ K & \xrightarrow{\phi} & B \end{array}$$

The diagram induces an element $\bar{\gamma}^{n+1}(f) \in H^{n+1}(K, L; \phi^*\pi_n(\mathcal{F}))$, called the *primary obstruction to extending f* ([Whi, p. 298]). The name comes from the fact that f can be extended to a partial lifting $f_{n+1} : K^{n+1} \to X$ of ϕ if and only if $\bar{\gamma}^{n+1}(f) = 0$.

Some times it is useful to have a primary obstruction defined when (K, L) is replaced by an arbitrary topological pair (P, Q). In particular, one needs such definition when defining the Whitney class of a vector bundle over an arbitrary base (not necessarily homotopy equivalent to a CW-complex), or constructing the Leray–Serre spectral sequences of a fibration over an arbitrary base.

To this end, we can take a CW-approximation $\varphi : (K, L) \to (P, Q)$, i.e., a map of pairs such that (K, L) is a CW-pair and φ is a weak homotopy equivalence of pairs. Thus we have a diagram:

$$\begin{array}{ccccc} L & \xrightarrow{\varphi} & Q & \xrightarrow{f} & X \\ \big\downarrow & & \big\downarrow & & \big\downarrow p \\ K & \xrightarrow{\varphi} & P & \xrightarrow{\phi} & B \end{array}$$

The element $\bar{\gamma}^{n+1}(f \circ \varphi) \in H^{n+1}(K, L; \varphi^*\phi^*\pi_n(\mathcal{F}))$ is well-defined. By Theorem 3.1,

$$\varphi^* : H^{n+1}\big(P, Q; \phi^*\pi_n(\mathcal{F})\big) \longrightarrow H^{n+1}\big(K, L; \varphi^*\phi^*\pi_n(\mathcal{F})\big)$$

is an isomorphism (in [Whi, p. 300] this is assumed without any explanation), therefore we could define $\bar{\gamma}^{n+1}(f) := \varphi^{*-1}\bar{\gamma}^{n+1}(f \circ \varphi) \in H^{n+1}(P, Q; \phi^* \pi_n(\mathcal{F}))$. An easy argument of naturality shows that this construction is independent of the CW-approximation φ. Hence we have a well-defined primary obstruction $\bar{\gamma}^{n+1}(f)$.

There are at least two ways to prove that a weak homotopy equivalence induces isomorphisms on homology (with constant coefficients) in the literature. One approach uses the Hurewicz Theorem ([Spal, 7.5.9, 7.6.25]). The other proof ([Hat, Proposition 4.21]) is by a construction that relies heavily on the finiteness of singular chains. The analogous result for cohomology with constant coefficients follows from this via the Universal Coefficient Theorem. To illustrate, we have:

Table 1

	Constant coefficients	Local coefficients
Homology	Hurewicz/Construction	Construction
Cohomology	Universal Coefficient Theorem	?

When it comes to local coefficients, the Hurewicz Theorem is no longer available. The constructive proof still works for homology with local coefficients. Yet due to the absence of the Universal Coefficient Theorem, the result for cohomology does not follow automatically. Our proof turns out to be quite different from those above. As far as we know, no alternative exists in the literature for cohomology.

3.2. Singular complex. We begin with the notion of singular complex, which is central in our proof.

Let X be a topological space. Consider the disjoint union of k-simplexes, one for each continuous map $\sigma : \Delta^k \to X$. Repeat this for all integer $k \geqslant 0$ and then glue the simplexes according to restriction of maps to faces. The resulted CW-complex is called the singular complex of X, denoted by SX. In fact, SX is a Δ-complex ([Hat, p. 164]).

A continuous map $f : X \to Y$ induces (with the obvious definition) a map $Sf : SX \to SY$.

Since the simplexes of SX corresponds to continuous maps $\Delta^k \to X$, there is a canonical map $I_X : SX \to X$ mapping each simplex via the map defining it. I_X is natural with respect to a continuous map $f : X \to Y$, i.e., the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle I_X}\uparrow & & \uparrow{\scriptstyle I_Y} \\ SX & \xrightarrow{Sf} & SY \end{array}$$

The following result will be used in our considerations:

Theorem 3.2. *For any topological space X, I_X is a weak homotopy equivalence.*

Proof. See [LW, Chapter III, Theorem 6.7] or [Gra, Theorem 16.43 on p. 149]. □

For a local coefficient system G on a Δ-complex K, there is a version of simplicial homology/cohomology. The definition of chains/cochains and boundary/coboundary maps are the same as the singular ones, except that the direct sum/product is now over all simplices of K. We denote the simplicial chains/cochains of K with coefficient in G by $\Delta_*(K;G)$ and $\Delta^*(K;G)$, and simplicial homology/cohomology by $H_*^{\Delta}(K;G)$ and $H^*_{\Delta}(K;G)$.

As in the case of constant coefficients, we still have:

Theorem 3.3. *The canonical injection $\Delta_*(K;G) \overset{j_\#}{\hookrightarrow} S_*(K;G)$ and projection $S^*(K;G) \overset{j^\#}{\to} \Delta^*(K;G)$ are chain maps that induce isomorphisms on homology/cohomology groups.*

Proof. For homology one could repeat the proof of [Hat, 2.27], except that one needs to quote Theorem 2.7 of this paper (since homology with local coefficients does not always behave well upon taking quotient of spaces). Although the Universal Coefficient Theorem is not available, one could easily adapt the above mentioned proof to the case of cohomology, with little change. □

Now suppose G is a local coefficient system on a topological space X. Let I_X^*G be the pullback of G via I_X. We have

$$S^*(X;G) \xrightarrow{I_X^\#} S^*(SX;I_X^*G) \xrightarrow{j^\#} \Delta^*(SX;I_X^*G)$$

It is easy to see that the composition $j^\# \circ I_X^\#$ identifies $S^k(X;G) = \prod_{\sigma:\Delta^k\to X} G(\sigma(e_0)) = \Delta^k(SX;I_X^*G)$. In particular, $j^\# \circ I_X^\#$ is a cochain isomorphism and $j^* \circ I_X^* : H^*(X;G) \to H^*_{\Delta}(SX;I_X^*G)$ is an isomorphism as well. By a similar argument one can show that $I_{X*} \circ j_* : H_*^{\Delta}(SX;I_X^*G) \to H_*(X;G)$ is an isomorphism. Combined with Theorem 3.3 we have shown:

Theorem 3.4. *The map I_X induces isomorphisms on homology and cohomology. To be more precise, for any local coefficient system G on X, $I_{X*} : H_*(SX;I_X^*G) \to H_*(X;G)$ and $I_X^* : H^*(X;G) \to H^*(SX;I_X^*G)$ are isomorphisms.*

3.3. Proof of Theorem 3.1.

Proof. By the naturality of homological long exact sequences of pairs (see the remark at the end of Section 2.2), we may reduce the theorem to the absolute case. Let $f : X \to Y$ be a weak homotopy equivalence. As noted above there's a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle I_X}\uparrow & & {\scriptstyle I_Y}\uparrow \\ SX & \xrightarrow{Sf} & SY \end{array}$$

in which I_X, I_Y are weak homotopy equivalences by Theorem 3.2.

Commutativity implies that Sf is also a weak homotopy equivalence. Since SX, SY are CW-complexes, the Whitehead Theorem (cf. [Hat]) implies that Sf is a homotopy equivalence, and hence induces isomorphisms on homology/cohomology with local coefficients.

Now apply Theorem 3.4 to the induced (commutative) diagram on homology/cohomology and the desired result follows. □

4. Identifying singular and cellular long exact sequences

4.1. Background. Let (K, L) be a CW-pair, G be a local coefficient system on K. There are long exact sequences (cf. Section 2.1)

$$\longrightarrow H_n(L; G) \longrightarrow H_n(K; G) \longrightarrow H_n(K, L; G) \longrightarrow \tag{3}$$

and

$$\longrightarrow H^n(K, L; G) \longrightarrow H^n(K; G) \longrightarrow H^n(L; G) \longrightarrow \tag{4}$$

There are also naturally defined short exact sequences of cellular chain/cochain complexes

$$0 \longrightarrow \Gamma_*(L; G) \longrightarrow \Gamma_*(K; G) \longrightarrow \Gamma_*(K, L; G) \longrightarrow 0$$

and

$$0 \longrightarrow \Gamma^*(K, L; G) \longrightarrow \Gamma^*(K; G) \longrightarrow \Gamma^*(L; G) \longrightarrow 0$$

which induce long exact sequences

$$\longrightarrow H_n\big(\Gamma_*(L); G\big) \longrightarrow H_n\big(\Gamma_*(K); G\big) \longrightarrow H_n\big(\Gamma_*(K, L); G\big) \longrightarrow \tag{5}$$

and

$$\longrightarrow H^n\big(\Gamma^*(K, L); G\big) \longrightarrow H^n\big(\Gamma^*(K); G\big) \longrightarrow H^n\big(\Gamma^*(L); G\big) \longrightarrow \tag{6}$$

The groups in (3) and (5) (resp. (4) and (6)) are term-wise isomorphic. It is natural to ask whether the long exact sequences (viewed as chain complexes) are chain isomorphic. This is used in the proof of [Wal, Theorem VI.6.9] (again this is used without any justification). It is natural to expect that this problem could be solved by diagram chasing, since the definition of cellular homology/cohomology can be described by chasing certain diagrams (cf. [Hat, p. 139 and p. 203]). Yet as far as we know, no such proof has been given. In fact the only relevant results in the literature are given in Schubert's book (cf. [Sch, p. 303]) and Luck's paper [Lüc]. In both references, an intermediate between the singular and cellular chain complex, called normal chain complex is used it to show (3) and (5) are chain isomorphic. The construction and proof (again!) depends heavily on the finiteness of singular chains, thus fails to prove the result for cohomology with local coefficients (though for constant coefficients one could still use the Universal Coefficient Theorem to dualize everything).

Our goal is to prove:

Theorem 4.1. *The long exact sequences* (3) *and* (5) (*resp.* (4) *and* (6)) *are chain isomorphic.*

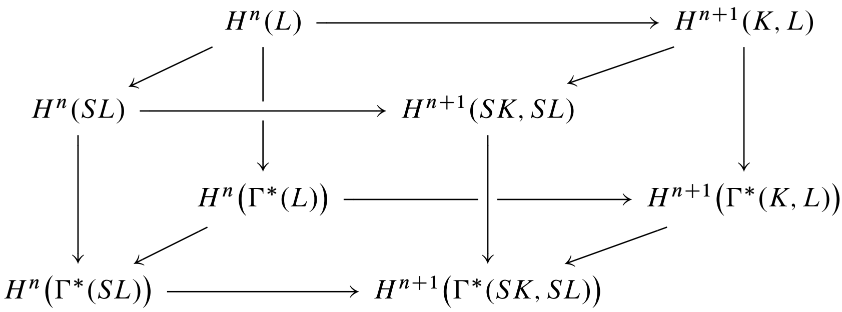
We shall prove the result for cohomology, the proof for homology is analogous.

4.2. Proof of Theorem 4.1.

Proof. Since the identification of singular and cellular homology/cohomology groups are natural with respect to cellular maps (whether the system of coefficients is constant or local), it suffices to check the commutativity of the diagram (coefficients omitted):

$$\begin{array}{ccc} H^n\big(\Gamma^*(L)\big) & \xrightarrow{\ \delta\ } & H^{n+1}\big(\Gamma^*(K,L)\big) \\ \updownarrow & & \updownarrow \\ H^n(L) & \xrightarrow{\ \delta\ } & H^{n+1}(K,L) \end{array}$$

Note that SL can be identified canonically with a subspace of SK and $I_K : SK \to K$ restricts to I_L on SL. Thus we have a weak homotopy equivalence $I_K : (SK, SL) \to (K, L)$. Since both the domain and codomain are CW-complexes, I_K is actually a homotopy equivalence. Homotope I_K to a cellular map J_K and consider the diagram:



in which vertical arrows are isomorphisms between singular and cellular cohomology groups, horizontal arrows are boundary maps in the corresponding long exact sequence and arrows going down left are induced by J_K. Coefficients are obvious and omitted.

The rectangle on top of the above diagram commutes since the map (in this case J_K) induces a chain map between singular long exact sequences (see the remark at the end of Section 2.2). Similarly, the cellular map J_K induces a chain map between cellular long exact sequences, hence the commutativity of the bottom rectangle. The rectangles on the left and right commute because of the naturality of the identification of singular cohomology with cellular cohomology under cellular maps.

All down left arrows are isomorphisms since J_K is a (cellular) homotopy equivalence. In particular the map $J_K^* : H^{n+1}(\Gamma^*(K,L)) \to H^{n+1}(\Gamma^*(SK,SL))$ is an isomorphism.

We intend to prove the commutativity of the rectangle in the back. As indicated by the above argument, it suffices to show that for the front rectangle. In other words, we have reduced the problem to the case where (K,L) is a pair of Δ-complexes. We shall assume this from now on.

For a Δ-complex pair (K,L) and a local coefficient system G on K, there is an isomorphism $\Phi : \Delta^n(K,L;G) \to \Gamma^n(K,L;G)$ defined by the identification

$$\Delta^n(K,L;G) \longleftrightarrow \prod_\sigma G(\sigma(e_0)) \longleftrightarrow \prod_\sigma H^n(\Delta^n, \partial\Delta^n; \sigma^*G) \longleftrightarrow \Gamma^n(K,L;G)$$

where the direct products are over all n-simplexes of $K-L$. It is easy to check (by diagram chasing) that Φ commutes with boundary maps of the two chain complexes and hence is a chain isomorphism.

Now that $j^\#$ and Φ induce the following commutative diagram joining the singular, simplicial, and cellular short exact sequences of (K,L) (coefficients omitted)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S^*(K,L) & \longrightarrow & S^*(K) & \longrightarrow & S^*(L) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Delta^*(K,L) & \longrightarrow & \Delta^*(K) & \longrightarrow & \Delta^*(L) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma^*(K,L) & \longrightarrow & \Gamma^*(K) & \longrightarrow & \Gamma^*(L) & \longrightarrow & 0
\end{array}$$

which induces a commutative diagram for the boundary homomorphisms in the corresponding long exact sequences

$$\begin{array}{ccc}
H^n(L) & \xrightarrow{\ \delta\ } & H^{n+1}(K,L)\\
\downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j^*}\\
H^n_\Delta(L) & \xrightarrow{\ \delta\ } & H^{n+1}_\Delta(K,L)\\
\downarrow{\scriptstyle \Phi^*} & & \downarrow{\scriptstyle \Phi^*}\\
H^n\big(\Gamma^*(L)\big) & \xrightarrow{\ \delta\ } & H^{n+1}\big(\Gamma^*(K,L)\big)
\end{array}$$

The theorem will then follow from the lemma below. □

Lemma 4.2. *The isomorphism $\Phi^* \circ j^*$ is exactly the natural identification between cellular and singular cohomology.*

Proof. We shall prove this for the absolute case (i.e., $L = \varnothing$). The proof of the relative case is similar. The local coefficient system G will be omitted in what follows.

Consider the following diagram:

$$\begin{array}{ccc}
H^n(K) & & \\
\downarrow{\scriptstyle \iota} & \searrow^{j^*} & \\
H^n(K^n) & & H^n_\Delta(K)\\
{\scriptstyle \kappa}\uparrow & \swarrow_{\Phi^*} & \\
H^n\big(\Gamma^*(K)\big) & &
\end{array}$$

In the above diagram, ι is induced by the inclusion $K^n \hookrightarrow K$ and κ is the homomorphism $\Gamma^n(K) = H^n(K^n, K^{n-1}) \to H^n(K^n)$ induced by identity.

For any $[b] \in H^n(K)$ where $b \in S^n(K)$, let $[\alpha] \in H^n(\Gamma(K))$ be the element corresponding to $[b]$ via the natural identification between singular and cellular cohomology. Then $\kappa([\alpha]) = \iota([b])$, hence $\alpha = [a]$ for some $a \in S^n(K^n, K^{n-1}) \subset S^n(K^n)$ such that $a - b_{|K^n} = \delta c$ for some $c \in S^{n-1}(K^n)$.

It suffices then to show that $\Phi^{*-1}([\alpha]) = j^*([b])$.

We know that

$$j^\#(a) - j^\#(b) = j^\#(a - b_{|K^n}) = j^\#(\delta c) = \delta j^\#(c) \in \Delta^n(K^n) = \Delta^n(K)$$

In other words, $j^\#(a), j^\#(b)$ are cohomologous.

Also, one can check that

$$\begin{aligned} & j^\#(a) \\ &= \Phi^{\#-1}([a]) \\ &= \Phi^{\#-1}(\alpha) \in \Delta^n(K) = \prod_\sigma G\big(\sigma(e_0)\big) \end{aligned}$$

by looking at their value on each n-simplex σ.

Thus $\Phi^{*-1}([\alpha]) = [\Phi^{\#-1}(\alpha)] = [j^\#(a)] = [j^\#(b)] = j^*([b])$. □

5. Simplicial approximation and Poincaré Duality

We now turn to another type of simplicial approximation. Let $\mathcal{M}$ be a closed n-dimensional topological manifold, R be a principal ideal domain and G a bundle of R-modules.

The Poincaré Duality Theorem (with local coefficients) states that

$$H^i_c(\mathcal{M}; G) \xrightarrow{\frown \mu_{\mathcal{M}}} H_{n-i}(\mathcal{M}; G \otimes_R \mathcal{M}_R), 0 \leqslant i \leqslant n$$

where H^*_c stands for singular cohomology with compact support, $\mu_{\mathcal{M}}$ is the (generalized) fundamental class and $\mathcal{M}_R$ is the orientation bundle of $\mathcal{M}$ with coefficient in R.

For relevant definitions and proof of the theorem, see [Spa2] or [Sun].

There is a version of this duality for compact triangulated manifolds with or without boundary (see [Lee]), which dates back much earlier, i.e., the original proof given by S. Lefschetz. The proof in [Lee] deals with the case of $\Lambda = \mathbb{Z}[\pi]$ coefficients. By an argument attributed to J. Milnor (cf. [Wal, Lemma 1.1]) the mentioned duality remains valid for every (left) Λ-module B. This in turn can be viewed (cf. [Hat, Proposition 3H.4, p. 331, and the bottom of p. 333]) as a duality with local coefficients as stated above. This proof is short and purely geometric (it uses the dual decomposition of the corresponding simplicial complex). Thus it would be nice if one could reduce the general case to the case of triangulated manifolds. This is when simplicial approximation comes into the picture.

Assume, for simplicity, that $\mathcal{M}$ is closed and orientable.

Theorem 5.1. *Let G be a local coefficient system on an n-dimensional closed oriented topological manifold $\mathcal{M}$. Denote by $\mu_{\mathcal{M}}$ the fundamental class of M. Then $H^i(\mathcal{M};G) \xrightarrow{\frown\mu_{\mathcal{M}}} H_{n-i}(\mathcal{M};G)$ is an isomorphism for all $0 \leqslant i \leqslant n$.*

Proof. By [KS, top of the page 301 (a)], there is a stable normal k-disk bundle E of $\mathcal{M}$ in $\mathbb{R}^{n+k}$. Also, E admits a triangulation. Obviously E is orientable as a manifold with boundary. Consider the map

$$H^{i+k}(E,\partial E;p^*G) \xrightarrow{\frown\mu_E} H_{n-i}(E;p^*G)$$

where μ_E is the fundamental class of E and p^*G is the pull-back bundle. Since E is triangulable, this is an isomorphism.

Next we prove that E is orientable as a disk bundle.

Claim 1. *There exists an element $U \in H^k(E,\partial E;\mathbb{Z})$ that restrict to a generator $U_{|x} \in H^k(E_x,\partial E_x;\mathbb{Z})$ for every $x \in \mathcal{M}$. Here $(E_x,\partial E_x)$ stands for the fiber over x.*

Proof. The composition

$$H^k(E,\partial E;\mathbb{Z}) \xrightarrow{\frown\mu_E} H_n(E;\mathbb{Z}) \xrightarrow{p_*} H_n(\mathcal{M};\mathbb{Z})$$

is an isomorphism because p is a homotopy equivalence. Define $U \in H^k(E,\partial E;\mathbb{Z})$ by $U \frown \mu_E = p_*^{-1}(\mu_{\mathcal{M}})$.

Note that $\mathcal{M}$ embeds in E by the zero-section. For $x \in \mathcal{M}$, consider the following diagram, where vertical maps are induced by inclusions

$$\begin{array}{ccccc}
H^k(E,\partial E;\mathbb{Z}) \otimes H_{n+k}(E,\partial E;\mathbb{Z}) & \xrightarrow{\ \frown\ } & H^n(E;\mathbb{Z}) & \xrightarrow{p_*} & H_n(\mathcal{M};\mathbb{Z}) \\
\downarrow i_1 \qquad\qquad i_2 \uparrow & & \downarrow i_3 & & \downarrow i_4 \\
H^k(E|\mathcal{M};\mathbb{Z}) \otimes H_{n+k}(E|(x,0);\mathbb{Z}) & \xrightarrow{\ \frown\ } & H^n(E|E_x;\mathbb{Z}) & \xrightarrow{p_*} & H_n(\mathcal{M}|x;\mathbb{Z})
\end{array}$$

Here $X|Y$ denotes the pair $(X, X-Y)$. The rightmost rectangle is easily seen to be commutative. □

Claim 2. *The homomorphism i_1 is actually an isomorphism.*

Proof. To justify Claim 2, it suffices to prove that the inclusion $\partial E \hookrightarrow E-\mathcal{M}$ is a weak homotopy equivalence (this would actually make the inclusion a homotopy equivalence, for both ∂E and $E-\mathcal{M}$ have the homotopy type of CW-complexes). Since $p : E \to \mathcal{M}$ is a D^k-bundle and $\mathcal{M}$ is disjoint from the boundary of E, the restriction of p to $E-\mathcal{M}$ is a fiber bundle with fiber $D^k - 0$. This can be seen by taking a local trivialization over suitably small open set of $\mathcal{M}$, treating IntD^k as $\mathbb{R}^k$ and translating points of $\mathcal{M}$ to 0 by subtraction. On the other hand, the restriction of p to ∂E is a bundle with fiber S^{k-1}. The inclusion of ∂E in $E - \mathcal{M}$ induces a map between the Serre exact sequences of homotopy groups of respective bundles. Now the Five Lemma shows that the inclusion $\partial E \hookrightarrow E - \mathcal{M}$ induces isomorphisms on homotopy groups. This finishes the proof of Claim 2. □

Let $U' = i_1^{-1}(U) \in H^k(E|\mathcal{M};\mathbb{Z})$. Define $\mu_{(x,0)} = i_2(\mu_E) \in H_{n+k}(E|(x,0);\mathbb{Z})$. Then by naturality of cap products, $U' \frown \mu_{(x,0)} = i_3 p_*^{-1}\mu_{\mathcal{M}}$. By commutativity and definition of the fundamental class $\mu_{\mathcal{M}}$, $p_* i_{3*} p_*^{-1}\mu_{\mathcal{M}} = i_4\mu_{\mathcal{M}}$ is a generator of $H_n(\mathcal{M}|x;\mathbb{Z})$. Since $p : (E|E_x) \to (\mathcal{M}|x)$ is a homotopy equivalence, $i_{3*}p_*^{-1}\mu_{\mathcal{M}}$ is also a generator.

Now choose an open disk neighborhood W of x in $\mathcal{M}$ upon which E admits a local trivialization $\Phi : p^{-1}(W) \to W \times D^k$. Let $\Phi_x : E_x \to D^k$ be the restriction of Φ. Consider the following diagram:

$$\begin{array}{ccc}
H^k(E|\mathcal{M};\mathbb{Z}) \otimes H_{n+k}(E|\Phi^{-1}(x,0);\mathbb{Z}) & \xrightarrow{\ \frown\ } & H^n(E|E_x;\mathbb{Z}) \\
\downarrow \Phi_1 \qquad \Phi_2 \uparrow & & \Phi_3 \uparrow \\
H^k(W \times D^k|W \times 0;\mathbb{Z}) \otimes H_{n+k}(W \times D^k|(x,0);\mathbb{Z}) & \xrightarrow{\ \frown\ } & H^n(W \times D^k|E_x;\mathbb{Z})
\end{array}$$

where vertical homomorphisms are induced by Φ.

By excision, Φ_2, Φ_3 are isomorphisms. Hence by naturality $\Phi_1(U') \frown \Phi_2^{-1}(\mu_{(x,0)}) = \Phi_3 i_{3*} p_*^{-1}\mu_{\mathcal{M}}$, which is a generator of $H_n(W \times D^k|E_x;\mathbb{Z}) \cong \mathbb{Z}$. Note that the cross product induces an isomorphism

$$H^k\big(W \times D^k|W \times 0;\mathbb{Z}\big) \xleftarrow{\cong} H^k\big(D^k|0;\mathbb{Z}\big) \otimes H^0(W;\mathbb{Z})$$

and $\Phi_1(U')$ corresponds to $\Phi_{x*}(U_{|x}) \times 1$. This forces $\Phi_{x*}(U_{|x})$ and thus $U_{|x}$ to be a generator. This proves Claim 1. Thus U is an orientation for the disk bundle E.

Since U is an orientation, we have a Thom homomorphism $H^i(\mathcal{M};G) \to H^{i+k}(E,\partial E;p^*G)$ sending α to $p^*(\alpha) \smile U$. This is an isomorphism (cf. [Spa1, p. 283]).

Now we have a commutative diagram

$$\begin{array}{ccc}
H^i(\mathcal{M};G) & \xrightarrow{\ \frown \mu_{\mathcal{M}}\ } & H_{n-i}(\mathcal{M};G) \\
\downarrow & & p_* \uparrow \\
H^{i+k}(E,\partial E;p^*G) & \xrightarrow{\ \frown \mu_E\ } & H_{n-i}(E;p^*G)
\end{array}$$

where the left vertical map is the Thom isomorphism.

Now:

$$
\begin{aligned}
& p_*\Big(((p^*\alpha) \smile U) \frown \mu_E\Big) \\
& = p_*(p^*\alpha \frown (U \frown \mu_E)) \\
\text{(naturality)} \quad & = \alpha \frown p_*(U \frown \mu_E) \\
\text{(definition of U)} \quad & = \alpha \frown \mu_{\mathcal{M}}
\end{aligned}
$$

The top arrow is thus an isomorphism since all others are isomorphisms. □

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