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Planar tropical cubic curves of any genus, and higher dimensional generalisations

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À la mémoire de notre ami Jean-Jacques Risler,
à qui nous n'avons pas eu le temps de raconter ces incongruités.

Abstract. We study the maximal values of Betti numbers of tropical subvarieties of a given dimension and degree in $\mathbb{T}P^n$. We provide a lower estimate for the maximal value of the top Betti number, which naturally depends on the dimension and degree, but also on the codimension. In particular, when the codimension is large enough, this lower estimate is larger than the maximal value of the corresponding Hodge number of complex algebraic projective varieties of the given dimension and degree. In the case of surfaces, we extend our study to all tropical homology groups. As a special case, we prove that there exist planar tropical cubic curves of genus g for any non-negative integer g.

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Contents

Throughout the text, we fix a field $\mathbb{K}$. The j^{th} Betti number $b_j(X)$ of a topological space X is the dimension of the j^{th} homology group $H_j(X;\mathbb{K})$ of X with coefficients in $\mathbb{K}$. Otherwise stated, we refer to [BIMS] for precise definitions of notions from tropical geometry needed in this text.

1. Introduction

1.1. Curves. A tropical curve C in $\mathbb{R}^n$ is a piecewise linear graph with finitely many vertices such that (see for example [BIMS, MR]):

- each edge e of C is equipped with an integer weight $w_e \in \mathbb{Z}_{>0}$, and has a directing vector in $\mathbb{Z}^n$;
- at each vertex v of C, adjacent to the edges $e_1, \cdots, e_l$, the following *balancing condition* is satisfied:

$$\sum_{i=1}^{l} w_{e_i} u_{e_i} = 0,$$

  where u_{e_i} is the primitive integer directing vector of e_i pointing away from v.

Some examples of tropical curves in $\mathbb{R}^2$ are depicted in Figure 1. A tropical curve is said to be of *degree* d if

$$d = \sum_e w_e \ \max_{j=1}^n\{0, u_{e,j}\},$$

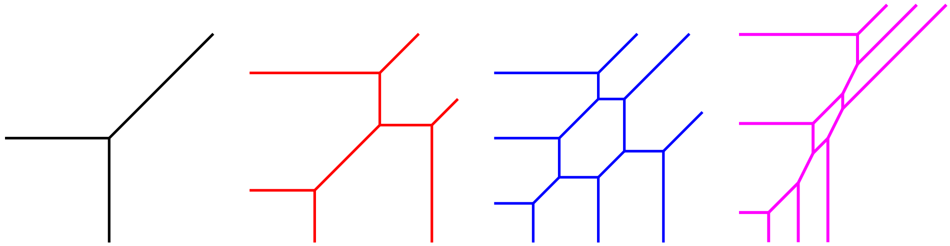


Figure 1
Examples of tropical curves in $\mathbb{R}^2$. All unbounded edges have integer direction $(-1,0)$, $(0,-1)$, or $(1,1)$ toward infinity, and all edges have weight 1.

where the sum ranges over all unbounded edges e of C, and $u_e = (u_{e,1}, \cdots, u_{e,n})$ is a primitive integer directing vector of e pointing toward infinity, see [BIMS]. Tropical curves appeared in several mathematical and physical contexts [AH, Ber, BG, Vir, Mik3], in particular in relation with complex and non-Archimedean amoebas [GKZ, Mik1, EKL].

Figure 1 suggests a relation between the topology of tropical curves and of plane algebraic curves. Indeed by [Mik2, Proposition 2.10], the first Betti number of a tropical curve in $\mathbb{R}^2$ of degree d is at most

$$\frac{(d-1)\cdot(d-2)}{2},$$

and equality holds in the case of so-called *non-singular* tropical curves (i.e. tropical curves in $\mathbb{R}^2$ of degree d with exactly d^2 vertices). It is standard that the same is true regarding the geometric genus of an algebraic curve of degree d in the projective plane, see for example [Sha, Chapter III 6.4]. Such similarity led to use the expression "genus of a tropical curve" in place of "first Betti number of a tropical curve".

Using linear projections, one easily sees that the above upper bound for the geometric genus of an algebraic curve in the projective plane is also an upper bound for the geometric genus of an algebraic curve of degree d in *any* projective space. The starting observation of this paper is that analogous statement does not hold in tropical geometry: there exist tropical curves of degree d in $\mathbb{R}^n$, with $n \geq 3$, with genus greater than the upper bound for tropical curves in $\mathbb{R}^2$. The first example is the tropical cubic curve of genus 2 in $\mathbb{R}^3$ depicted in Figure 2. Moreover, this curve is contained in a polyhedral complex L of dimension 2: one vertex from which emanate fours rays in the directions $(-1,0,0)$, $(0,-1,0)$, $(0,0,-1)$, and $(1,1,1)$, and six faces of dimension two generated by each pair of rays. It turns out that L is a *tropical plane* in $\mathbb{R}^3$, i.e., a tropical surface of

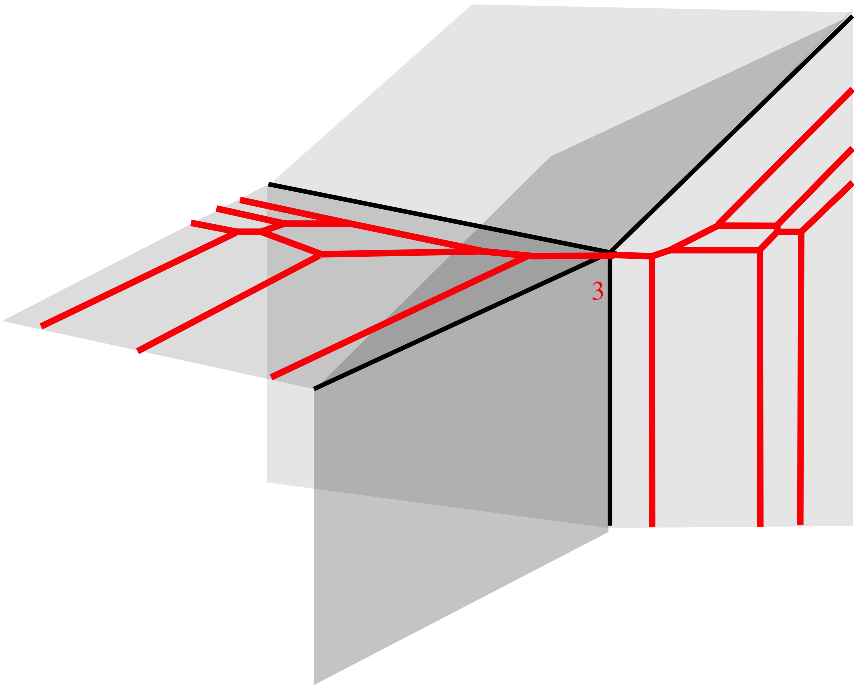


Figure 2
A tropical cubic curve of genus 2 in $\mathbb{R}^3$. All unbounded edges have integer direction $(-1,0,0)$, $(0,-1,0)$, $(0,0,-1)$, or $(1,1,1)$ toward infinity; all edges have weight 1, except the one with weight 3 indicated close to it.

degree 1 (see below). Hence Figure 2 exhibits a rather surprising (to us) example of a genus 2 tropical cubic in a tropical plane. We generalise this observation in next Theorem, where a tropical curve in $\mathbb{R}^n$ is called *planar* if it is contained in a tropical plane.

Theorem 1.1. *For any integers $d \geq 1$ and $n \geq 2$, there exists a planar tropical curve of degree d in $\mathbb{R}^n$ with genus*

$$(n-1)\cdot\frac{(d-1)\cdot(d-2)}{2}.$$

Generalising Figure 2, there exists therefore a planar tropical cubic curve of any given genus $g \geq 0$. Note that Theorem 1.1 disproves in particular [Yu, Conjecture 4.5].

T. Yu proved in [Yu, Proposition 4.1] that a tropical curve of degree d in $\mathbb{R}^n$ has no more than $2d^2\cdot(n-1)^2$ vertices, which implies that the genus of such tropical curve is bounded from above by a constant depending only on d and n. Nevertheless, to our knowledge the following question remains open in general.

Problem. What is the maximal possible genus of a tropical curve of degree d in $\mathbb{R}^n$?

In the case of planar tropical curves in $\mathbb{R}^3$, we can "almost" prove that Theorem 1.1 is optimal.

Theorem 1.2. *If $C \subset \mathbb{R}^3$ is a planar tropical curve of degree d with $4d$ unbounded edges, then C has genus at most $(d-1)\cdot(d-2)$.*

Note that a tropical curve C of degree d in $\mathbb{R}^3$ has at most $4d$ unbounded edges, and that there is equality if and only if C has exactly d unbounded edges of weight 1 in each of the outgoing direction

$$(-1,0,0),\ (0,-1,0),\ (0,0,-1),\ (1,1,1).$$

We believe that Theorem 1.2 still holds without the assumption on unbounded edges of C, and that a (quite technical) adjustment of our proof should work. It is nevertheless not so clear to us how to generalise our proof in higher dimensions.

1.2. Higher dimensions. Tropical curves generalise to tropical varieties in $\mathbb{R}^n$ of any dimension. These are finite polyhedral complexes in $\mathbb{R}^n$ such that all faces have a direction defined over $\mathbb{Z}$, all facets (i.e., faces of maximal dimension) are equipped with a positive integer weight, and which satisfy a balancing condition at each face of codimension 1. We refer to [BIMS, Section 5] for a precise definition of tropical subvarieties of $\mathbb{R}^n$. By convention, a tropical variety will always be of pure dimension: every face is contained in a facet.

There is also a notion of degree of a tropical variety X in $\mathbb{R}^n$, based on stable intersections defined in [RGST, Mik4]. Recall that a standard fan tropical linear space of dimension k in $\mathbb{R}^n$ is a polyhedral fan with a vertex from which emanate $n+1$ rays in the directions

$$(-1,0,\cdots,0),(0,-1,0,\cdots,0),\cdots,(0,\cdots,0,-1),(1,1,\cdots,1),$$

and having $\binom{n+1}{l}$ additional faces of dimension $l \in \{2,\cdots,l\}$ generated by each subset of l of the $n+1$ rays. The degree of a tropical variety X of codimension k in $\mathbb{R}^n$ is defined as the stable intersection number of X with a generic standard fan tropical linear space of dimension k.

The aim of this paper is to study the topology of tropical varieties. To this purpose, it is more convenient to deal with compact tropical varieties, and to consider *projective* tropical varieties, i.e. tropical subvarieties of the tropical projective space $\mathbb{T}P^n$. This latter is defined as the quotient of $([-\infty;+\infty[)^{n+1}\setminus\{(-\infty,\cdots,-\infty)\}$ by the equivalence relation

$$(x_0,\cdots,x_n)\sim(x_0+\lambda,\cdots,x_n+\lambda)\qquad \lambda\in\mathbb{R},$$

see for example [MR, Section 3.3]. The tropical projective space $\mathbb{T}P^n$ is the union of finitely many copies of $\mathbb{R}^k$ with $k\in\{0,\cdots,n\}$ defined by

$$\mathbb{R}_I=\{[x_0:\cdots:x_n]\mid x_i=-\infty \text{ if and only if } i\in I\}$$

where $I\subsetneq\{0,\cdots,n\}$. A tropical variety in $\mathbb{T}P^n$ is the union of the topological closure of finitely many tropical varieties contained in some $\mathbb{R}_I$. The notion of degree of a tropical variety extends to projective tropical varieties, see [MR, Section 5.2].

Now we are ready to state the main problem studied in this paper, as well as our main results. We define the numbers

$$B_j(m,k,d)=\sup_X\{b_j(X)\}\in\mathbb{N}\cup\{+\infty\},$$

where X ranges over all tropical subvarieties of dimension m and degree d in $\mathbb{T}P^{m+k}$.

Problem. Estimate the numbers $B_j(m,k,d)$.

Generalising what we saw in the case of curves, the values of the numbers $B_j(m,1,d)$ are well known by [Mik2, Proposition 2.10]: for $m=0$, $B_0(0,1,d)=d$ and for $m\geq 1$ and $d\geq 1$,

$$B_0(m,1,d)=1,\qquad B_1(m,1,d)=\cdots=B_{m-1}(m,1,d)=0,\qquad \text{and}$$
$$B_m(m,1,d)=\binom{d-1}{m+1}.$$

This follows from the existence of the dual subdivision of a tropical hypersurface. Determining the exact value of $B_j(m,k,d)$ for $k>1$ seems more difficult, and it is even not clear a priori that this number is finite. Our main result is the following.

Theorem 1.3. *Let d,m and, k be three positive integers. Then the number $B_j(m,k,d)$ is finite for any j, and one has*

$$B_m(m,k,d)\geq k\cdot B_m(m,1,d).$$

Corollary 1.4. *For any integers $m \geq 1$ and $d \geq m+2$, we have*

$$\lim_{k \to +\infty} B_m(m,k,d) = +\infty.$$

From our proof that $B_j(m,k,d)$ is finite, it is possible to extract explicit upper bounds. Nevertheless these bounds seem far from being sharp (for example we did not succeed to obtain a better upper bound than T. Yu in the case of curves). The lower bound in Theorem 1.3 is obtained by constructing explicit examples. To do so, we use a method of construction of tropical varieties that we call floor composition (see Section 3), and which originates in the floor decomposition technique introduced by Brugallé and Mikhalkin ([BM2, BM3, BM1]), and in the tropical modifications introduced by Mikhalkin in [Mik4]. It is worth noting that the floor composed varieties we construct are actually projective hypersurfaces, thus generalising Theorem 1.1.

Theorem 1.5. *Let d, m and k be three positive integers. Then there exist a tropical linear space L of dimension $m+1$ in $\mathbb{T}P^{m+k}$, and a tropical hypersurface X of degree d in L such that*

$$b_m(X) \geq k \cdot B_m(m,1,d).$$

In connection to algebraic geometry, it seems also interesting to determine the maximal value of Betti numbers of tropical hypersurfaces of degree d of a given tropical linear space. At this time, we are not aware of any generalisation of Theorem 1.2 to tropical varieties of higher dimension.

Remark 1.6. All tropical varieties we construct in our proof of Theorem 1.5 are singular as soon as $k \geq 2$. It may be interesting to study bounds on Betti numbers, and more generally on tropical Hodge numbers, of non-singular tropical projective varieties of a given dimension, codimension, and degree. In particular, we do not know if there exist universal finite upper bounds which do not depend on the codimension. For example, it follows from the tropical adjunction formula [Sha3, Theorem 6] that the upper bound given by Theorem 1.2 can be refined to the classical bound $\frac{1}{2}(d-1)\cdot(d-2)$ under the additional assumption that C is locally of degree 1 in L (i.e. C is a non-singular tropical subvariety of L).

Homology groups of a tropical variety X are special instances of its tropical homology groups (we refer to [MZ2, BIMS, KSW] for the definition of tropical homology for locally finite polyhedral complexes in $\mathbb{T}P^n$). More precisely, the group $H_j(X;\mathbb{R})$ is canonically isomorphic to the tropical homology group $H_{0,j}(X;\mathbb{R})$. Our proof of finiteness of the numbers $B_j(m,k,d)$ in Theorem 1.3 also implies finiteness of the numbers

$$\sup_X \left\{ \dim H_{p,q}(X, \mathbb{R}) \right\} \in \mathbb{N} \cup \{+\infty\},$$

where X ranges over all tropical subvarieties of dimension m and degree d in $\mathbb{T}P^{m+k}$. In the case of surfaces, we compute all tropical homology groups of the tropical surfaces constructed in the proof of Theorem 1.5. Let us denote by $h^{\mathbb{C}}_{p,q}(d,m)$ the dimension of the (p,q)-tropical homology group of a non-singular tropical hypersurface of degree d in $\mathbb{T}P^{m+1}$. By [IKMZ, Corollary 2], this number does not depend on a particular choice of a tropical hypersurface, and is equal to the (p,q)-Hodge number of a non-singular complex algebraic hypersurface of degree d in $\mathbb{C}P^{m+1}$. In particular we have

$$h^{\mathbb{C}}_{2,0}(d,2) = \frac{(d-1)\cdot(d-2)\cdot(d-3)}{6} \quad \text{and} \quad h^{\mathbb{C}}_{1,1}(d,2) = \frac{4d^3 - 12d^2 + 14d}{6}.$$

A tropical surface in $\mathbb{T}P^n$ is called *spatial* if it is contained in a tropical linear space L of dimension 3.

Theorem 1.7. *Let k and d be two positive integers. Then there exist a spatial tropical surface X of degree d in $\mathbb{T}P^{2+k}$ with the following tropical Hodge diamond*

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ k\cdot h^{\mathbb{C}}_{2,0}(d,2) & & h^{\mathbb{C}}_{1,1}(d,2) + \frac{(k-1)\cdot(d-1)\cdot(2d^2-7d+9)}{3} & & k\cdot h^{\mathbb{C}}_{2,0}(d,2) \\ & (k-1)\cdot(d-1) & & 0 & \\ & & 1 & & \end{array}$$

where we use the convention that $h_{0,0}$ is the topmost number and $h_{2,0}$ the leftmost one.

Hence as soon as $d \geq 2$, the quantities $h_{1,1}(X)$ and $h_{2,1}(X)$ are not bounded from above among spatial tropical surfaces of degree d. Our proof of Theorem 1.7 generalises the computation by K. Shaw of tropical homology groups of floor composed surfaces in $\mathbb{T}P^3$ [Sha1]. We point out that the technique developed to prove Theorem 1.7 also applies to study tropical Hodge numbers of floor composed tropical varieties of any dimension. Nevertheless computations become a bit tedious starting from dimension 3, so we restricted ourselves to the case of surfaces.

1.3. Comparison with algebraic geometry. To a great extent, the tremendous development of tropical geometry the last fifteen years has been motivated by its deep relations to algebraic geometry. There exists several procedures that associate a tropical variety X to a family of projective complex algebraic varieties $(\mathcal{X}_t)$. For such a *realisable* tropical variety, the tropical Hodge numbers may be bounded from above in terms of the Hodge numbers of a general member of the family $(\mathcal{X}_t)$, see for example [HK, Corollary 5.8], [KS, Corollary 5.3], and [IKMZ, Corollary 2].

Hence it is reasonable to compare our main results stated above to what is known about Hodge numbers of projective complex algebraic varieties. As usual, in the case of hypersurfaces (and more generally of complete intersections) in $\mathbb{T}P^n$, both series of geometric invariants coincide: it follows from [IKMZ, Corollary 2] that the tropical Hodge numbers of a non-singular tropical hypersurface equal the Hodge numbers of a non-singular complex algebraic hypersurface of the same dimension and degree[1].

Given two positive integers m and d, the Hodge number $h^{p,q}(\mathcal{X})$ of a projective complex algebraic variety $\mathcal{X}$ of degree d and dimension m is bounded from above by some constant that only depends on m and d, see [Mil, Har]. For example, it is well known that a cubic curve in $\mathbb{C}P^n$ has genus at most 1, whatever the value of n is. Corollary 1.4 and Figure 2 show that the situation is drastically different in tropical geometry, where such an upper bound independent on the codimension does not exist. In particular, for k large enough with respect to some fixed m and d, the tropical varieties whose existence is attested by Theorem 1.5 are not the tropicalisation of any family of projective varieties of the same dimension and degree.

In a somewhat similar direction, Davidow and Grigoriev studied in [DG] the possible numbers of connected components of intersections of tropical varieties. They proved in particular that this number can also be much larger than the bound in algebraic geometry given by Bézout Theorem.

Organisation of the paper. Section 2 is devoted to showing the finiteness of $B_j(m,k,d)$ and proving Theorem 1.2. In Section 3 the floor composition method is introduced and we explain how to compute Betti numbers of the obtained varieties. In Section 4, we first prove Theorem 4.3 which contains Theorem 1.1. We then give lower estimates of $B_j(m,k,d)$ in general using Theorem 4.3 as induction basis, and floor composition to recursively construct the varieties of Theorem 1.5. Section 5 is dedicated to the computation of tropical Hodge numbers of floor composed tropical surfaces and to the proof of Theorem 1.7.

[1] Note however that this correspondence only concerns dimension of the corresponding vector spaces. There is no canonical isomorphism between tropical homology groups and Hodge groups in general.

2. Upper estimates

In this section, we prove the finiteness of the numbers $B_j(m,k,d)$, and Theorem 1.2. The main ingredient is tropical intersection theory, for which we refer to [AR, Sha2, BS] for more details.

2.1. Finiteness of $B_j(m,k,d)$. Our strategy to prove the finiteness of $B_j(m,k,d)$ is to reduce to the case of hypersurfaces by a suitable projection universal for all tropical subvarieties of dimension m and degree d in $\mathbb{T}P^{m+k}$. We denote by $Gr(m,\mathbb{Z}^{m+k}) \subset Gr(m,\mathbb{R}^{m+k})$ the space of subvector spaces of dimension m of $\mathbb{R}^{m+k}$ that are defined over $\mathbb{Z}$.

Lemma 2.1. *Let $\mathcal{V}(d,m,k)$ be the set of elements of $Gr(m,\mathbb{Z}^{m+k})$ that are the direction of a facet of a tropical variety of dimension m and degree d in $\mathbb{T}P^{m+k}$. Then $\mathcal{V}(d,m,k)$ is a finite set.*

Proof. The usual Plücker embedding of $Gr(m,\mathbb{R}^{m+k})$ lifts to an injection

$$\begin{array}{rccc} \phi: & Gr(m,\mathbb{Z}^{m+k}) & \longrightarrow & \Lambda^m\left(\mathbb{Z}^{m+k}\right)/\{\pm 1\} \\ & \mathrm{Span}(v_1,\cdots,v_m) & \longmapsto & v_1\wedge\cdots\wedge v_m \end{array},$$

where $(v_1,\cdots,v_m)\in(\mathbb{Z}^{m+k})^m$ is a basis of the lattice $\mathrm{Span}(v_1,\cdots,v_m)\cap\mathbb{Z}^{m+k}$. In the standard coordinates of $\Lambda^m\left(\mathbb{Z}^{m+k}\right)$, the coordinates of $\phi\big(\mathrm{Span}(v_1,\cdots,v_m)\big)$ are given by all $m\times m$ minors of the matrix $(v_1,\cdots,v_m)$.

Suppose now that $V\in\mathcal{V}(d,m,k)$, and choose a basis $(v_1,\cdots,v_m)$ of $V\cap\mathbb{Z}^{m+k}$. Let $(u_1,\cdots,u_{m+k})$ denote the canonical basis of $\mathbb{R}^{m+k}$. Then by the tropical Bézout Theorem, one has

$$\left|\det(u_{i_1},\cdots,u_{i_k},v_1,\cdots,v_m)\right|\le d$$

for any subset $\{i_1,\cdots,i_k\}\subset\{1,\cdots,m+k\}$. All these determinants are precisely the $m\times m$ minors of the matrix $(v_1,\cdots,v_m)$. Hence we deduce that $\phi\big(\mathcal{V}(d,m,k)\big)$ is a finite set, and so is $\mathcal{V}(d,m,k)$. □

Proposition 2.2. *Given any integers $m,k,d>0$ and $j\ge 0$, the number $B_j(m,k,d)$ is finite.*

Proof. Without loss of generality, we only consider tropical subvarieties of $\mathbb{T}P^n$ with no irreducible component contained in $\mathbb{T}P^n\setminus\mathbb{R}^n$. Then the number of faces of dimension j of a tropical hypersurface in $\mathbb{R}^n$ is equal to the number of faces of dimension $n-j$ in its dual subdivision. Any tropical hypersurface of degree d in $\mathbb{T}P^n$ has a Newton polytope included in the simplex with vertices

$$(0,\cdots,0),\ (d,0,0,\cdots,0),\ (0,d,0,\cdots,0),\ \cdots,\ (0,\cdots,0,d,0),\ (0,\cdots,0,0,d).$$

Hence the proposition holds true in the case of hypersurfaces, i.e., when $k=1$.

In the case when $k\geq 2$, we prove the proposition by induction on m, the case $m=0$ holding trivially. Note that a tropical subvariety of $\mathbb{T}P^n$ carries a canonical polyhedral decomposition when it is either a curve or a hypersurface, however this is no longer the case in higher dimensions and codimensions (think for example of the union of the two 2-planes with equations $x_1=x_2=0$ and $x_3=x_4=0$ in $\mathbb{R}^4$). Given any couple (V,V') of distinct elements of $\mathcal{V}(d,m,k)$, we fix a vector $u_{V,V'}\in V'\setminus V$, and we define

$$\mathcal{W}=\big\{V\oplus\mathbb{R}u_{V,V'}\mid (V,V')\in\mathcal{V}^2(d,m,k)\text{ and } V\neq V'\big\}.$$

By Lemma 2.1, both sets $\mathcal{V}(d,m,k)$ and $\mathcal{W}$ are finite, and there exists a vector space $W\in Gr(k-1,\mathbb{Z}^{m+k})$ such that

$$W\cap V=\{0\}\qquad \forall V\in\mathcal{V}(d,m,k)\cup\mathcal{W}.$$

Let $\pi:\mathbb{T}P^{m+k}\to\mathbb{T}P^{m+1}$ be the tropical map induced by the linear projection along W in $\mathbb{R}^{m+k}$. Note that for any tropical variety X of dimension m and degree d in $\mathbb{T}P^{m+k}$, the degree of $\pi(X)$ in $\mathbb{T}P^{m+1}$ is bounded from above by (and generically is equal to) a constant $D(d,W)$ that only depends on d and W. Since $W\cap V=\{0\}$ for any $V\in\mathcal{V}(d,m,k)$, the dimension of $\pi(F)$ equals the one of F for any facet F of X. The condition that $W\cap V=\{0\}$ for any $V\in\mathcal{W}$ guaranties that different elements of $\mathcal{V}(d,m,k)$ have distinct images under π. From now on, we consider the lift to X of the canonical polyhedral decomposition of $\pi(X)$. By construction, the preimage $\pi^{-1}(F)$ of any open facet F of $\pi(X)$ is the disjoint union of open facets of X.

Let W' be any element of $Gr(k,\mathbb{Z}^{m+k})$ which contains W and such that $W'\cap V=\{0\}$ for any $V\in\mathcal{V}(d,m,k)$. The tropical map $\pi':X\to\mathbb{T}P^m$ induced by the linear projection along W' is finite. Furthermore the tropical degree of π' is bounded from above by (and generically is equal to) a constant $D'(d,W')$ that only depends on d and W'. By the tropical Bézout Theorem, the fibre $\pi'^{-1}(x)$ contains at most $D'(d,W')$ points for any point x in $\mathbb{T}P^m$. Since each fibre of $\pi_{|X}$ is contained in a fibre of π', there are at most $D'(d,W')$ points in $\pi_{|X}^{-1}(x)$ for any $x\in\pi(X)$. Hence, we deduce that the number of facets of X, and so the number $B_m(m,k,d)$, is at most

$$D'(d,W')\cdot K\big(m,D(d,W)\big)<+\infty,$$

where $K\big(m,D(d,W)\big)$ is the maximal number of facets of a tropical hypersurface of degree $D(d,W)$ in $\mathbb{R}^{m+1}$. This proves the proposition when $j=m$.

We prove the results for $j < m$ by induction on m. Let us denote respectively by $Sk^{m-1}(X)$ and $Sk^{m-1}(\pi(X))$ the closure in $\mathbb{T}P^{m+k}$ and $\mathbb{T}P^{m+1}$ of the $(m-1)$-skeleton of $X \cap \mathbb{R}^{m+k}$ and $\pi(X) \cap \mathbb{R}^{m+1}$. The stable self-intersection of $\pi(X) \cap \mathbb{R}^{m+1}$ in $\mathbb{R}^{m+1}$ provides positive integer weights on $Sk^{m-1}(\pi(X))$, turning this latter into a tropical subvariety of $\mathbb{T}P^{m+1}$ of degree at most $D(d, W)^2$. In its turn, the stable intersection of $X \cap \mathbb{R}^{m+k}$ with $\pi^{-1}(X) \cap \mathbb{R}^{m+k}$ provides positive integer weights on $Sk^{m-1}(X)$, turning it into a tropical subvariety of $\mathbb{T}P^{m+k}$ whose degree is bounded by a number $\tilde{D}(d, W)$ which only depends on d and W. Since X is obtained from $Sk^{m-1}(X)$ by attaching m-cells, we have

$$B_j(m, k, d) \leq B_j\big(m-1, k+1, \tilde{D}(d, W)\big).$$

Since by assumption the number $B_j\big(m-1, k+1, \tilde{D}(d, W)\big)$ is finite, the proposition is proved. □

2.2. Auxiliary statements. The proof of Theorem 1.2 requires the following several auxiliary lemmas that will be combined in Section 2.3. Given a tropical plane L in $\mathbb{T}P^3$, we denote by $Sk^i(L)$ the closure in $\mathbb{T}P^3$ of the union of all faces of dimension i of $L \cap \mathbb{R}^3$.

Until the end of this section, we denote by L_0 the tropical plane in $\mathbb{T}P^3$ defined by the tropical polynomial "$x + y + z + 0$".

Lemma 2.3. *Theorem 1.2 holds if every edge of C is either disjoint from $Sk^1(L) \setminus Sk^0(L)$ or contained in $Sk^1(L)$.*

Proof. Denote by C_F the intersection of C with a facet F of L. By assumption, one has

$$b_1(C) = \sum_F b_1(C_F),$$

where the sum runs all over the facets of L. Let σ be the number of the directions $(0, 0, -1)$, $(0, -1, 0)$, $(-1, 0, 0)$, and $(1, 1, 1)$ along which L is not a cylinder, and let s be one of these four directions. The projection along s defines a degree one tropical map $\pi_s : L \to \mathbb{T}P^2$, and by assumption, one has

$$\sum_F b_1(C_F) = b_1\big(\pi_s(C)\big) \leq \frac{(d-1)\cdot(d-2)}{2},$$

where the sum runs over all facets of L not containing the direction s. Considering all possible directions s, each C_F contributes to the first Betti number of exactly two projections $\pi_s(C)$, so we get

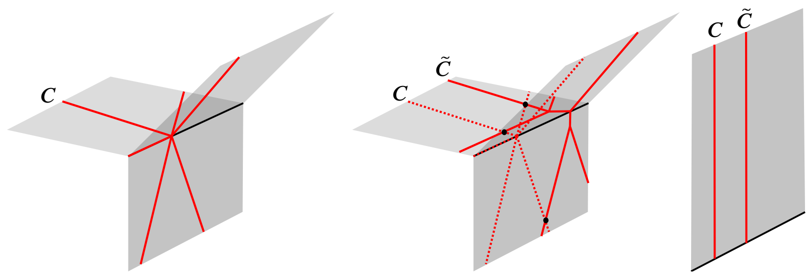
$$2g(C) \leq \sigma \cdot \frac{(d-1)\cdot(d-2)}{2} \leq 4 \cdot \frac{(d-1)\cdot(d-2)}{2},$$

which is the desired result. □

Given a tropical curve C in $\mathbb{T}P^n$ and $p \in C$, we denote by $\mathrm{val}_p(C)$ the valency of C at p, and by $\mathrm{Edge}_p(C)$ the set of edges of C adjacent to p (viewed as a vertex of C). We also define $C^0 = C \cap \mathbb{R}^n$ and $C^\infty = C \cap (\mathbb{T}P^n \setminus \mathbb{R}^n)$. If C is furthermore contained in a tropical plane L, we denote by $\mathrm{Edge}^2(C)$ the set of edges of C that are not contained in $Sk^1(L)$. The tropical curve C is called a *fan* tropical curve with vertex v if the support of C is the closure in $\mathbb{T}P^3$ of rays in $\mathbb{R}^3$ all emanating from v.

The proofs of next Lemmas and of Theorem 1.2 extensively use tropical intersection theory of tropical curves in tropical surfaces, for which we use the presentation given in [BS, Section 3] and [BIMS, Section 6.2]. For the reader convenience, we recall informally the definition of the local self-intersection C_p^2 at a point p of a tropical curve C in L_0 with no irreducible components in $\mathbb{T}P^3 \setminus \mathbb{R}^3$. Recall that each facet F of L_0 contains a unique corner point (i.e. with coordinates $(-\infty, -\infty)$ in an affine tropical chart $[-\infty; +\infty[^2$) that we denote by q_F. There are several cases to consider to define C_p^2, depending on the location of the point p in L_0:

- p is contained in $C^0 \setminus Sk^1(L_0)$: the tropical plane L_0 is then locally given at p by the affine tropical chart $\mathbb{R}^2$, and C_p^2 is defined as the stable intersection of C at p in L_0 [RGST];
- p is contained in $Sk^1(L_0) \setminus Sk^0(L_0)$: the curve C can be deformed in L_0 locally at p into a tropical curve $\tilde{C}$ intersecting C in $L_0 \setminus Sk^1(L_0)$, see Figure 3; we define C_p^2 as the sum of the tropical intersection multiplicity of intersection points of C and $\tilde{C}$ that are close to p (depicted in black dot points in Figure 3b, note that $C_p^2 = 0$ if $p \in C^\infty$);



(a) $p \in C^0 \cap Sk^1(L_0) \setminus Sk^0(L_0)$

(b)

(c) $p \in C^\infty$

FIGURE 3

- $p = q_F$ for some facet F of L_0: in an affine tropical chart of L_0 at q_F, the tropical curve C is defined by a tropical polynomial $P(x,y)$; denoting by $\Delta_{q_F}(C)$ the Newton polygon of $P(x,y)$, by $\overline{\Delta}_{q_F}(C)$ the convex hull of $\Delta_{q_F}(C) \cup \{(0,0)\}$, and by $\Gamma_{q_F}(C) = \overline{\Delta}_{q_F}(C) \setminus \Delta_{q_F}(C)$, we define
$$C^2_{q_F} = Area\big(\Gamma_{q_F}(C)\big),$$
where $Area$ stands for twice the Euclidean area;
- p is the origin in $\mathbb{R}^3$: denote by $\tilde{C}$ the fan tropical curve that coincide with C in a neighborhood of p, and by d the degree of $\tilde{C}$; we define
$$C^2_p = d^2 - \sum_{F \text{ facet of } L_0} \tilde{C}^2_{q_F}.$$

Note that we have $C^2_p \geq 0$ whenever p is not the origin. Next lemma provides a lower bound for C^2_0. Recall that each edge e of a tropical curve is equipped with a weight $w_e \in \mathbb{Z}_{>0}$.

Lemma 2.4. *Let $C \subset L_0$ be a fan tropical curve of degree d with vertex the origin. Then one has*
$$C^2_0 + \sum_{e \in Edge^2(C)} (w_e - 1) - val_0(C) \geq -d^2 + 2d - 4.$$

Proof. Let $\tilde{C}$ be a perturbation of C outside a neighbourhood of the origin into a tropical curve of degree d such that

- $\tilde{C}$ is still contained in L_0;
- any vertex v of $\tilde{C}^0$ distinct from and not adjacent to the origin (resp. connected to the origin by an edge e) is trivalent and
$$\tilde{C}^2_v = 1 \qquad (\text{resp. } \tilde{C}^2_v = w_e).$$
- $\tilde{C} \cap \mathbb{R}^3$ has unbounded edges only in the standard directions $(0,0,-1)$, $(0,-1,0)$, $(-1,0,0)$, and $(1,1,1)$.

Such perturbation $\tilde{C}$ exists: it suffices to perturb C in a neighborhood of each point q_F according to any convex triangulation of $\Gamma_{q_F}(C)$ such that each edge of $\Gamma_{q_F}(C) \cap \Delta_{q_F}(C)$ is the edge of a triangle, and containing the maximal number of triangles among all triangulations satisfying this condition. An elementary Euler characteristic computation gives

$$2g(\tilde{C}) = \sum_{v \in \tilde{C}^0} (\mathrm{val}_v(\tilde{C}) - 2) + 2 - \left|\tilde{C}^\infty\right|. \tag{1}$$

Next, by [BS, Definition 3.6] we have

$$C_0^2 = d^2 - \sum_{v\in\tilde{C}^0\setminus\{0\}} \tilde{C}_v^2$$
$$= d^2 - \sum_{v\in\tilde{C}^0\setminus\{0\}} (\mathrm{val}_v(\tilde{C})-2) - \sum_{e\in\mathrm{Edge}^2(C)} (w_e-1),$$

from which we deduce that

$$\sum_{v\in\tilde{C}^0} \big(\mathrm{val}_v(\tilde{C})-2\big) = d^2 - C_0^2 - \sum_{e\in\mathrm{Edge}^2(C)} (w_e-1) + \mathrm{val}_0(C) - 2.$$

Just as C, the tropical curve $\tilde{C}$ satisfies to the hypothesis of Lemma 2.3. Hence combining this latter identity together with Lemma 2.3 and equation (1), we obtain

$$d^2 - C_0^2 - \sum_{e\in\mathrm{Edge}^2(C)} (w_e-1) + \mathrm{val}_0(C) - |\tilde{C}^\infty| \le 2d^2 - 6d + 4.$$

Now the result follows from the fact that $|\tilde{C}^\infty| \le 4d$. □

Lemma 2.5. *Let $L \subset \mathbb{T}P^3$ be a tropical plane, and let $C \subset L$ be a fan tropical curve with vertex v_0 contained in $Sk^1(L)\setminus Sk^0(L)$. Then one has*

$$C_{v_0}^2 \ge val_{v_0}(C) - 3.$$

Proof. The lemma is true if v_0 is a trivalent vertex of C since in this case $C_{v_0}^2 = 0$. If v_0 is not a trivalent vertex of C, then we perturb C as depicted in Figure 3a and b, into a tropical curve $\tilde{C}$ such that

- $\tilde{C}$ is contained in L;
- C and $\tilde{C}$ have the same directions of unbounded edges;
- $\tilde{C}$ intersects $Sk^1(L)$ in a single trivalent vertex $\tilde{v}_0$.

We have

$$C_{v_0}^2 = \sum_{v\in\tilde{C}^0} \tilde{C}_v^2.$$

Furthermore if $v \ne v_0$ and v is not a 2-valent point of $\tilde{C}$, it follows from Pick Formula that

$$\tilde{C}_v^2 \ge \sum_{e\in\mathrm{Edge}_v(\tilde{C})} w_e - 2.$$

Furthermore we have

$$\tilde{C}_{\tilde{v}_0}^2 = 0 = \mathrm{val}_{\tilde{v}_0}(\tilde{C}) - 3,$$

and the result follows. □

2.3. Proof of Theorem 1.2. Let us denote by $L \subset \mathbb{T}P^3$ a tropical plane containing C, and by a the number of vertices of C which are contained in $Sk^1(L) \setminus Sk^0(L)$. We claim that

$$\sum_{v \in C^0} \big(\mathrm{val}_v(C) - 2\big) \leq 2d^2 - 2d + 2. \tag{2}$$

Assuming that this inequality holds, we have

$$\begin{aligned} 2g(C) &= \sum_{v \in C^0} \big(\mathrm{val}_v(C) - 2\big) + 2 - |C^\infty| \\ &\leq 2d^2 - 6d + 4 \\ &\leq 2(d-1)\cdot(d-2). \end{aligned}$$

Hence it remains to prove Inequality (2). Suppose first that C does not pass through $Sk^0(L)$.

The self-intersection of C in L is equal to d^2, hence it follows from Pick Formula and Lemma 2.5 that

$$\begin{aligned} d^2 &= \sum_{v \in C^0} C_v^2 \\ &\geq \sum_{v \in C \setminus Sk^1(L)} \Big(\sum_{e \in \mathrm{Edge}_v(C)} w_e - 2\Big) + \sum_{v \in C^0 \cap Sk^1(L)} \big(\mathrm{val}_v(C) - 3\big) \\ &\geq \sum_{v \in C^0} \big(\mathrm{val}_v(C) - 2\big) - a. \end{aligned}$$

Since $a \leq d$, we have $d^2 + a \leq 2d^2 - 2d + 2$ and Inequality (2) holds.

Suppose now that C passes through $Sk^0(L)$. Again, it follows from Pick Formula and Lemma 2.5 that

$$\begin{aligned} d^2 &= \sum_{v \in C^0 \setminus Sk^1(L)} C_v^2 + \sum_{v \in C^0 \cap Sk^1(L) \setminus \{0\}} C_v^2 + C_0^2 \\ &\geq \sum_{v \in C^0 \setminus Sk^1(L)} \Big(\sum_{e \in \mathrm{Edge}_v(C)} w_e - 2\Big) + \sum_{v \in C^0 \cap Sk^1(L) \setminus \{0\}} \big(\mathrm{val}_v(C) - 3\big) + C_0^2 \\ &\geq \sum_{v \in C^0 \setminus \{0\}} \big(\mathrm{val}_v(C) - 2\big) - a + \sum_{e \in \mathrm{Edge}_0^2(C)} (w_e - 1) + C_0^2 \\ &\geq \sum_{v \in C^0} \big(\mathrm{val}_v(C) - 2\big) - a + \sum_{e \in \mathrm{Edge}_0^2(C)} (w_e - 1) + C_0^2 - \mathrm{val}_0(C) + 2. \end{aligned}$$

Denoting by d_0 the local intersection number of C and $Sk^1(L)$ at the origin, it follows from Lemma 2.4 that

$$\sum_{v\in C^0}\bigl(\mathrm{val}_v(C)-2\bigr)\le d^2+d_0^2-2d_0+2+a.$$

Since the total intersection number of C and $Sk^1(L)$ is equal to d, and that each local intersection multiplicity on $Sk^1(L)\setminus\{0\}$ is positive, we deduce that $d_0\le d-a$ and $a\le d-1$. In particular we have $d_0^2-2d_0\le(d-a)^2-2(d-a)$, and

$$\begin{aligned}\sum_{v\in C^0}\bigl(\mathrm{val}_v(C)-2\bigr)&\le 2d^2-2d+2+a\cdot(a-2d+3)\\&\le 2d^2-2d+2,\end{aligned}$$

i.e., Inequality (2) holds in this case as well. □

3. Floor composition

We describe a method of construction of tropical varieties which we will use in Section 4 to exhibit tropical varieties with large top Betti numbers. This method originates in the floor decomposition technique introduced by Brugallé and Mikhalkin ([BM2, BM3, BM1]), whose roots can in their turn be traced back to earlier ideas by Mikhalkin. A floor composed tropical variety is a m-dimensional tropical variety in $\mathbb{R}^{n+1}$ which is built out of the data of a collection of m-dimensional varieties in $\mathbb{R}^n$ together with some effective divisors on elements of this collection. In the case when the varieties and the divisors involved are homology bouquets of spheres, we express the Betti numbers of the floor composed variety in term of those of the construction's data.

3.1. Tropical birational modifications. Here we slightly generalise the notion of *tropical modifications* introduced in [Mik4] and further developed in [BLdM, Sha1, Sha2, BMa, CM16]. Let X be a tropical subvariety in $\mathbb{R}^n$. Recall (see for example [BIMS, Section 5.6] or [MR, Section 4.4]) that to a tropical rational function $f : X\to\mathbb{R}$ corresponds its divisor $\mathrm{div}_X(f)$ which is a codimension one tropical cycle on X. We denote $\Gamma_f(X)\subset X\times\mathbb{R}\subset\mathbb{R}^n\times\mathbb{R}$ the graph of f with weights inherited from X. Given a closed polyhedron F in $\mathbb{R}^n\times\mathbb{R}$ equipped with a weight w_F, we denote by F^- (resp. F^+) the polyhedral cell $F-\mathbb{R}_{\ge 0}(0,\cdots,0,1)$ (resp. $F+\mathbb{R}_{\ge 0}(0,\cdots,0,1)$) equipped with the weight w_F.

Definition 3.1. *Let X be a tropical variety in $\mathbb{R}^n$, and $f : X\to\mathbb{R}$ be a tropical rational function. Suppose that there exist two effective tropical divisors D_+ and D_- on X such that $\mathrm{div}_X(f)=D_+-D_-$. The tropical variety $\tilde X$ in $\mathbb{R}^{n+1}$ defined by*

$$\tilde X=\Gamma_f(X)\bigcup\Gamma_f(D_+)^-\bigcup\Gamma_f(D_-)^+$$

is called a ***birational tropical modification*** *of* X *along the divisor* $D_+ - D_-$*. If* $D_- = \emptyset$*, then* $\tilde{X}$ *is called a* ***tropical modification*** *of* X *along* D_+*.*

Our definition of tropical modification coincides with the definition from [Mik4, Sha1, Sha2, BIMS].

Example 3.2. The tropical line L in $\mathbb{R}^2$ defined by the tropical polynomial "$x + y + 0$" is a tropical modification of $\mathbb{R}$ along 0. The tropical plane in $\mathbb{R}^3$ defined by the tropical polynomial "$x + y + z + 0$" is a tropical modification of $\mathbb{R}^2$ along the line L. More generally, any tropical linear space of dimension m in $\mathbb{R}^n$ can be obtained from $\mathbb{R}^m$ by a sequence of tropical modifications along tropical linear spaces of dimension $m-1$.

Example 3.3. The tropical surface in $\mathbb{R}^3$ defined by the tropical polynomial "$(y + 0)z + x + 0$" is a birational tropical modification of $\mathbb{R}^2$ along $L_+ - L_-$, where L_+ (resp. $L-$) is the tropical line in $\mathbb{R}^2$ defined by the tropical polynomial "$x + 0$" (resp. "$y + 0$"), see Figure 4. This surface may be thought as an open part of the blow-up of $\mathbb{T}P^2$ at the point $(0,0)$, the line $(0,0)\times\mathbb{R}$ corresponding to an open part of the exceptional divisor.

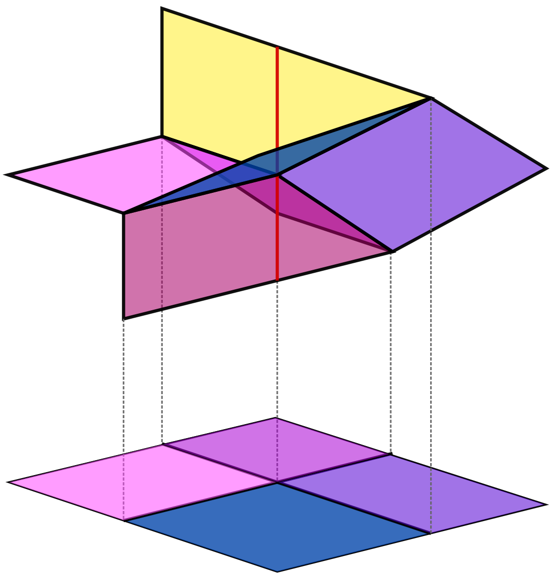


Figure 4

The tropical birational modification of $\mathbb{R}^2$ along the divisor $\mathrm{div}_{\text{“}x+0\text{”}}(\mathbb{R}^2) - \mathrm{div}_{\text{“}y+0\text{”}}(\mathbb{R}^2)$. The line $(0,0) \times \mathbb{R}$ is contained in the tropical surface.

Given a tropical variety X in $\mathbb{R}^n$ and a divisor D on X, it is not true in general that there exists a tropical rational function $f : X \to \mathbb{R}$ such that $\mathrm{div}_X(f) = D$. Nevertheless the following proposition shows that this is true when X is a tropical linear space. This is an immediate generalisation of [Sha2, Lemma 2.23] which treats the case of fan tropical linear spaces. The proof from [Sha2, Lemma 2.23] is based on the following two facts:

- any fan tropical linear space of dimension m in $\mathbb{R}^n$ is obtained from $\mathbb{R}^m$ by a sequence of tropical modifications along fan tropical linear spaces of dimension $m-1$;
- any tropical divisor in $\mathbb{R}^m$ is the divisor of a tropical rational function.

Since the first point extends to tropical linear spaces which are not necessarily fans, the proof of [Sha2, Lemma 2.23] extends immediately as well.

Proposition 3.4. *Let L be a tropical linear space in $\mathbb{R}^n$. Then any tropical divisor D in L is the divisor of some tropical rational function $f : L \to \mathbb{R}$.*

3.2. Floor composed varieties. A *construction pattern* is a set $K = \{X_1, \cdots, X_d, D_0, \cdots, D_d, f_1, \cdots, f_d\}$ where

- X_i is a m-dimensional connected tropical variety in $\mathbb{R}^n$;
- D_{i-1} and D_i are effective tropical divisors on X_i, and $f_i : X_i \to \mathbb{R}$ is a tropical rational function such that $\mathrm{div}_{X_i}(f_i) = D_i - D_{i-1}$;
- D_i is non-empty for $i \in \{1, \cdots, d-1\}$;
- $f_i(p) > f_{i+1}(p)$ for any $p \in D_i$.

Note that the above varieties X_i are not disjoint since $D_i \subset X_i \cap X_{i+1}$. Given such a construction pattern K, we construct a tropical variety X_K of dimension m in $\mathbb{R}^{n+1}$ as follows. For any $i \in \{1, \cdots, d-1\}$, we define $\mathcal{W}_i$ as the polyhedral complex

$$\mathcal{W}_i = \Gamma_{f_i}(D_i)^- \cap \Gamma_{f_{i+1}}(D_i)^+$$

equipped with weight inherited from D_i. We also define

$$\mathcal{W}_d = \Gamma_{f_d}(D_d)^- \qquad \text{and} \qquad \mathcal{W}_0 = \Gamma_{f_1}(D_0)^+$$

equipped with with weight inherited from D_d and D_0 respectively. Finally we define X_K as follows

$$X_K = \mathcal{W}_0 \cup \bigcup_{i=1}^{d} \left(\Gamma_{f_i}(X_i) \cup \mathcal{W}_i\right).$$

Note that $X_K \subset \bigcup_{i=1}^d X_i \times \mathbb{R}$ by construction.

Definition 3.5. *The tropical variety X_K in $\mathbb{R}^{n+1}$ is called **the floor composed tropical variety** with pattern K.*

Example 3.6. A classical use of the above construction is with a construction pattern K where each X_i is $\mathbb{R}^m$, each divisor D_i is a hypersurface defined by a tropical polynomial P_i of degree i in $\mathbb{R}^m$, and $f_i =$ "P_i/P_{i-1}". In this case X_K is a tropical hypersurface of degree d in $\mathbb{R}^{m+1}$. An example of such a construction pattern and the corresponding floor composed tropical cubic curve in $\mathbb{R}^2$ is depicted in Figure 5.

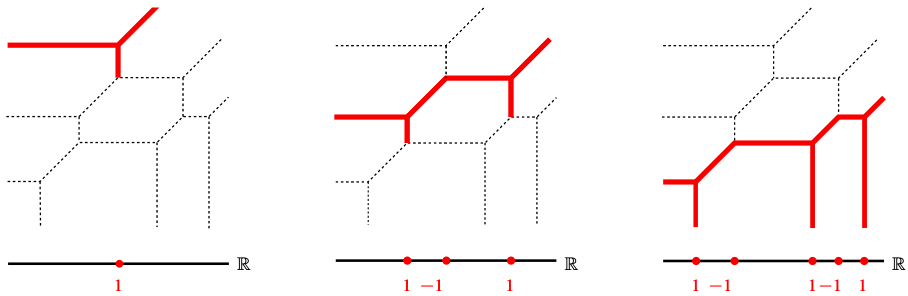


Figure 5
A floor composed cubic tropical curve in $\mathbb{R}^2$. For each $X_i = \mathbb{R}$, we depicted $D_i - D_{i-1}$ and $\Gamma_{f_i}(\mathbb{R}) \cup \mathcal{W}_i$.

Our main construction in Section 4.2 uses a generalisation of the previous example with an arbitrary tropical linear space in place of $\mathbb{R}^n$. Given a tropical linear space of dimension m in $\mathbb{R}^n$ and a surjective linear projection $\pi : L \to \mathbb{R}^m$ to a coordinate m-plane, we denote by $U_\pi \subset \mathbb{R}^m$ the set of points whose preimage by π consists of a single point (it is the complement in $\mathbb{R}^m$ of an arrangement of at most $n-m$ tropical hyperplanes).

Definition 3.7. *Let L be a tropical linear space of dimension m in $\mathbb{R}^n$ and $f : L \to \mathbb{R}$ a tropical rational function. The function f is said to have **degree at most** d if for any surjective linear projection $\pi : L \to \mathbb{R}^m$ to a coordinate m-plane, the function $f \circ \pi^{-1} : U_\pi \to \mathbb{R}$ is the restriction to U_π of a tropical polynomial P_π of degree at most d in $\mathbb{R}^m$. It is of **degree** d if it is of degree at most d, and not at most $d-1$ (i.e. when at least one of these polynomials has degree d).*

Note that with the above definition, not any tropical rational function $f : L \to \mathbb{R}$ has a degree. This is the case if and only if f restricts to a tropical polynomial on every facet of L.

Example 3.8. Let L be the tropical hyperplane in $\mathbb{R}^3$ defined by the tropical polynomial "$x + y + z + 0$". The tropical rational function

$$f(x, y, z) = \text{“}\ \frac{(x + y) \cdot (z + 0)}{x + y + z + 0}\ \text{”}$$

has degree 1 on L. Indeed, by symmetry it is enough to consider the projection $\pi(x, y, z) = (x, y)$, and in this case $f \circ \pi^{-1}(x, y) =$ "$x + y$". Note that $\mathrm{div}_L(f)$ is the line $\mathbb{R}(1, 1, 0)$, see Figure 6.

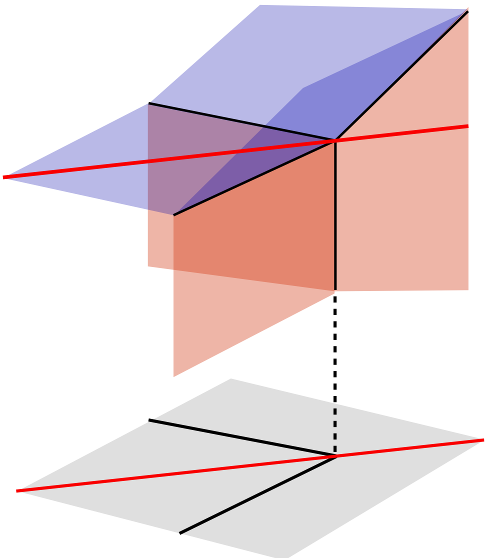


FIGURE 6

Lemma 3.9. *Let L be a tropical linear space in $\mathbb{R}^n$, and X a tropical subvariety in L of codimension 1 and degree d. Then there exists a tropical rational function $f : L \to \mathbb{R}$ of degree d such that $X = \mathrm{div}_L(f)$.*

Proof. Denote by m the dimension of L, and let $\pi : L \to \mathbb{R}^m$ be a surjective linear projection to a coordinate m-plane. For any direction x_i which is not contracted by π, we have

$$\text{“}\ \frac{f}{x_i}\ \text{”} \circ \pi^{-1} = \text{“}\ \frac{f \circ \pi^{-1}}{x_i}\ \text{”}. \tag{3}$$

Hence we may assume without loss of generality that the tropical rational function f has a well defined degree but that "f/x_i" does not for any index i.

Suppose now that there exists a projection π_0 as above such that the tropical polynomial $f \circ \pi_0^{-1}$ has degree at least $d+1$. Since $\pi_0(X)$ has degree d, it follows that there exists a direction x_i which is not contracted by π_0 and such that "$f \circ \pi_0^{-1}/x_i$" is still a tropical polynomial. But then it follows from (3) that "$f \circ \pi^{-1}/x_i$" is a tropical polynomial for any projection π that does not contract the direction x_i. Hence the tropical rational function "f/x_i" has a well defined degree in contradiction with our assumptions. □

Next proposition generalises Example 3.6.

Proposition 3.10. *Let L be a tropical linear space in $\mathbb{R}^n$, let $h_0, h_1, \cdots, h_d$ be tropical rational functions on L such that h_i is of degree i, and let $f_i =$ "h_i/h_{i-1}". If $f_i(p) > f_{i+1}(p)$ for any $p \in \mathrm{div}_L(h_i)$, then the tropical floor composed variety X_K with pattern $K = \{L, \cdots, L, \mathrm{div}_L(h_0), \cdots, \mathrm{div}_L(h_d), f_1, \cdots, f_d\}$ is of degree d in $\mathbb{R}^{n+1}$, and is contained in the tropical linear space $L \times \mathbb{R}$.*

Proof. The only thing we have to prove is that X_K is of degree d. We denote by m the dimension of L. Let Π be a tropical linear space of dimension $n-m$ in $\mathbb{R}^n$ which intersects L in a single point and away from $\bigcup_{i=1}^{d} \mathrm{div}_L(h_i)$. Hence $\Pi \times \mathbb{R}$ is a tropical linear space in $\mathbb{R}^{n+1}$ which intersects X_K in exactly d points, all of them of tropical multiplicity 1. The condition that h_i and h_{i+1} have degree differing by 1 ensures that the closures of X_K and $\Pi \times \mathbb{R}$ in $\mathbb{T}P^{n+1}$ do not intersect in $\mathbb{T}P^n \setminus \mathbb{R}^n$, and the proposition is proved. □

An m-dimensional tropical variety X is called a *homology bouquet of spheres* if

$$b_0(X) = 1 \qquad \text{and} \qquad b_j(X) = 0 \quad \forall j \in \{1, \cdots, m-1\}.$$

Note that any connected tropical curve is a bouquet of sphere.

Proposition 3.11. *Let $K = \{X_1, \cdots, X_d, D_0, \cdots, D_d, f_1, \cdots, f_d\}$ be a construction pattern where the tropical varieties $X_1, \cdots, X_d$ are homology bouquets of spheres of dimension m. Suppose that $f_i(p) > f_j(p)$ for any $i < j$ and $p \in X_i \cap X_j$.*

If $m = 1$, then we have

$$b_1(X_K) = \sum_{i=1}^{d} b_1(X_i) + \sum_{i=1}^{d-1} \big(b_0(D_i) - 1\big).$$

If $m \geq 2$ and if the tropical varieties $D_0, \cdots, D_d$ are homology bouquets of spheres, then the floor composed tropical variety X_K is also a bouquet of spheres and

$$b_m(X_K) = \sum_{i=1}^{d} b_m(X_i) + \sum_{i=1}^{d-1} b_{m-1}(D_i).$$

Proof. This is an elementary application of the Mayer–Vietoris long exact sequence. We prove the proposition by induction on d. The case $d = 1$ is clear since in this case X_1 is a deformation retract of X_K. Let $K' = \{X_1, \cdots, X_{d-1}, D_0, \cdots, D_{d-1}, f_1, \cdots, f_{d-1}\}$ and let us assume that the proposition holds for $X_{K'}$.

Defining

$$F_d = \mathcal{W}_d \cup \Gamma_{f_d}(X_d) \cup \mathcal{W}_{d-1} \qquad \text{and} \qquad X^o_{K'} = X_{K'} \setminus \Gamma_{f_d}(D_{d-1})^-,$$

we have

$$X_K = F_d \cup X^o_{K'} \qquad \text{and} \qquad \mathcal{W}_{d-1} = F_d \cap X^o_{K'}.$$

Figure 7 illustrates the above sets on an example. Since D_{d-1} (resp. X_d, $X^o_{K'}$) is a deformation retract of W_{d-1} (resp. F_d, $X_{K'}$), the Mayer–Vietoris long exact sequence applied to the decomposition $X_K = F_d \cup X^o_{K'}$ gives

$$(4) \qquad \cdots \longrightarrow H_j(D_{d-1}) \longrightarrow H_j(X_d) \oplus H_j(X_{K'}) \longrightarrow H_j(X_K) \longrightarrow$$
$$\longrightarrow H_{j-1}(D_{d-1}) \longrightarrow \cdots$$

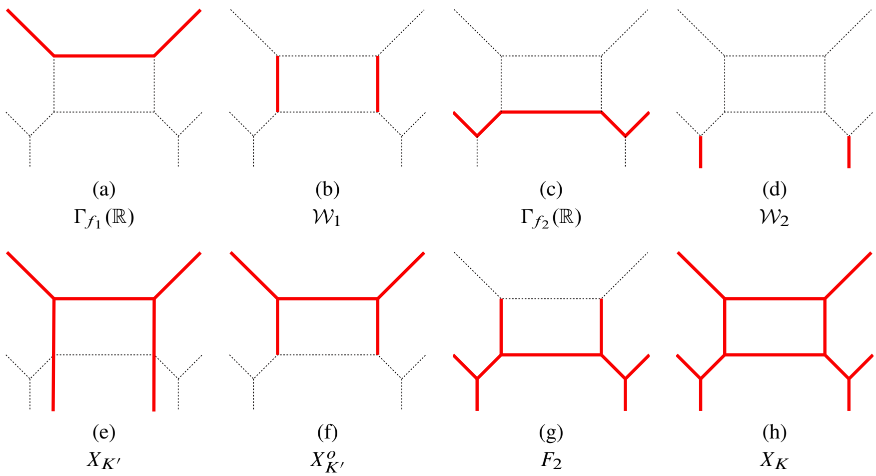


FIGURE 7
Examples of sets defined in the proof of Theorem 3.11 with
$K = \{\mathbb{R}, \mathbb{R}, \varnothing, \{-1, 1\}, \{-2, 2\}, \text{“}1 + x + x^{-1}\text{”}, \text{“}\frac{-2x-2x^{-1}}{-1+x+x^{-1}}\text{”}\}$
and $K' = \{\mathbb{R}, \varnothing, \{-1, 1\}, \text{“}1 + x + x^{-1}\text{”}\}$.

Since X_d and $X_{K'}$ are connected, and D_{d-1} is non-empty, we deduce that the map $H_0(D_{d-1}) \to H_0(X_d) \oplus H_0(X_{K'})$ has rank one. This proves the result if $m = 1$.

If $m \geq 2$, since D_{d-1} and X_d are homology bouquet of spheres, as well as $X_{K'}$ by induction hypothesis, the long exact sequences (4) gives

$$H_0(X_K) \simeq \mathbb{Z}, \qquad H_1(X_K) = \cdots = H_{m-1}(X_K) = 0,$$

and

$$0 \longrightarrow H_m(X_d) \oplus H_m(X_{K'}) \longrightarrow H_m(X_K) \longrightarrow H_{m-1}(D_{d-1}) \longrightarrow 0.$$

So the proposition follows by induction on d. □

4. Lower estimates

The main goal of this section is to prove Theorem 1.5. We first study subvarieties in $\mathbb{R}^n$, the case of curves in Section 4.1, from which we deduce a construction of higher dimensional tropical varieties by floor composition in Section 4.2. Then we prove Theorem 1.5 in Section 4.3.

Recall that the *recession cone* $R(X)$ of a tropical cycle X in $\mathbb{R}^n$ is the tropical fan defined by

$$R(X) = \lim_{t \to 0} t \cdot X.$$

4.1. Curves in $\mathbb{R}^n$. Theorem 1.1 is contained in Theorem 4.3 below. In the proof of this latter, we will need the auxiliary families of curves constructed in the next two lemmas. The conditions regarding intersections in Lemmas 4.1 and 4.2 will be used in Section 5 in the proof of Theorem 1.7.

The *multiplicity* of a vertex of a tropical curve in $\mathbb{R}^2$ is twice the Euclidean area of the polygon dual to this vertex. Such a vertex is said to be *non-singular* if it has multiplicity 1. An intersection point p of two tropical curves C_1 and C_2 in $\mathbb{R}^2$ is said to be tropically transverse if p is a vertex of multiplicity 2 of $C_1 \cup C_2$. Here we denote by L_0 the tropical line in $\mathbb{R}^2$ defined by the tropical polynomial "$x + y + 0$".

Lemma 4.1. *There exists a family of tropical curves $(\tilde{C}_d)_{d \geq 1}$ in $\mathbb{R}^2$ satisfying the following properties (see Figure 8 for $d = 2, 3$):*

- *$\tilde{C}_1 = L_0$;*
- *$\tilde{C}_d$ is of degree d and genus $\frac{(d-1)\cdot(d-2)}{2}$;*
- *$\tilde{C}_d$ has an infinite edge e_∞ of weight d in the direction $(-1, 0)$, which is contained in the line $\{y = 0\}$;*

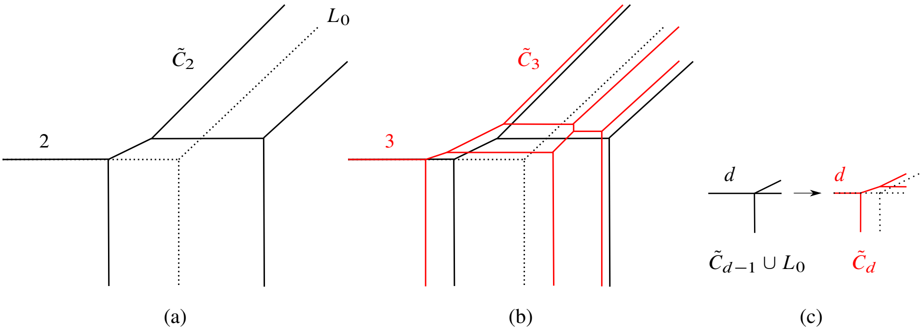


FIGURE 8

- *each vertex of $\tilde{C}_d$ not adjacent to e_∞ is non-singular;*
- *$\tilde{C}_d$ and $\tilde{C}_{d-1}$ intersect in exactly 1 unbounded segment and $(d-1)^2$ points, all of them being tropically transverse intersection points;*
- *$\tilde{C}_d$ and L_0 intersect in exactly 1 unbounded segment and $d-1$ points, all of them being tropically transverse intersection points.*
- *$\bigcap_{d \geq 1} \tilde{C}_d$ contains one unbounded segments in the direction $(-1,0)$;*
- *$R(\tilde{C}_d) = d \cdot L_0$.*

Proof. The proof is by induction on d. For $\tilde{C}_2$, we choose the tropical conic depicted in Figure 8a. To construct the curve $\tilde{C}_d$, we perturb the union of $\tilde{C}_{d-1}$ with L_0, keeping an edge of multiplicity d. Each non-singular vertex of $\tilde{C}_{d-1}$ gives rise to a transverse intersection point of $\tilde{C}_d$ and $\tilde{C}_{d-1}$. This gives $(d-1)\cdot(d-2)$ such points. Similarly, each tropically transverse intersection point of $\tilde{C}_{d-1}$ and L_0 gives rise to a non-singular vertex of $\tilde{C}_d$, a transverse intersection point of $\tilde{C}_d$ and L_0, and a transverse intersection point of $\tilde{C}_d$ and $\tilde{C}_{d-1}$. In each case this gives $d-2$ such intersection points. The vertex of L_0 gives rise to a transverse intersection point of $\tilde{C}_d$ and L_0, hence we have $d-1$ tropically transverse intersection points of $\tilde{C}_d$ and L_0 as stated. The vertex adjacent to e_∞ is perturbed as depicted in Figure 8c, which adds one additional transverse intersection point of $\tilde{C}_d$ and $\tilde{C}_{d-1}$. The curve $\tilde{C}_3$ is depicted on Figure 8b.

To ensure the last condition, we choose $\tilde{C}_d$ such that the distance between the vertex of L_0 and every vertex of $\tilde{C}_d$ is bounded uniformly with respect to d. □

The proof of next lemma is similar to the proof of Lemma 4.1 and is left to the reader.

Lemma 4.2. *There exists a family of tropical curves* $(\overline{C}_d)_{d\geq 1}$ *in* $\mathbb{R}^2$ *satisfying the following properties (see Figure 9 for* $d=2,3$*):*

- $\overline{C}_1 = L_0$;
- $\overline{C}_d$ *is of degree* d *and genus* $\frac{(d-1)\cdot(d-2)}{2}$;
- $\overline{C}_d$ *has an infinite edge* e_∞ *of weight* d *in the direction* $(-1,0)$*, which is contained in the line* $\{y=0\}$;
- $\overline{C}_d$ *has an infinite edge* e'_∞ *of weight* d *in the direction* $(1,1)$*, which is contained in the line* $\{x=y\}$;
- *each vertex of* $\overline{C}_d$ *not adjacent to* e_∞ *or* e'_∞ *is non-singular;*
- $\overline{C}_d$ *and* $\overline{C}_{d-1}$ *intersect in* 2 *segments and* $(d-1)\cdot(d-2)$ *points, all them being tropically transverse intersection points;*
- $\overline{C}_d$ *and* L_0 *intersect in exactly* 2 *segments and* $d-2$ *points, all of them being tropically transverse intersection points.*
- $\bigcap_{d\geq 1}\overline{C}_d$ *contains* 2 *unbounded segments in directions* $-(1,0)$ *and* $(1,1)$;
- $R(\overline{C}_d) = d\cdot L_0$.

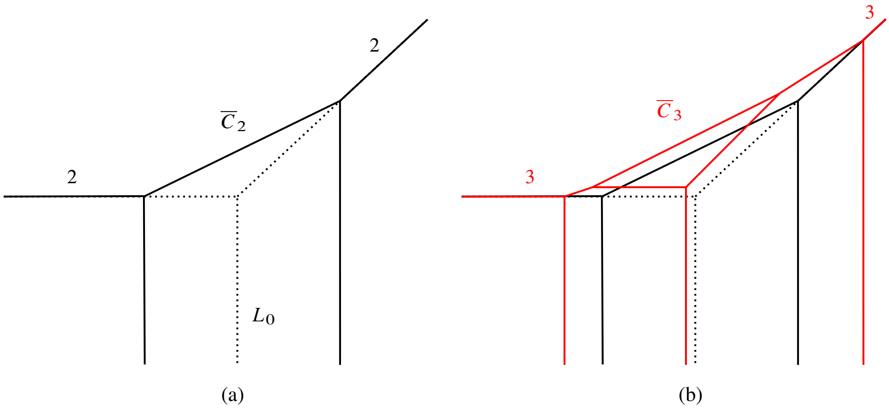


FIGURE 9

Theorem 4.3. *For any positive integer* k*, there exists a tropical plane* L_k *in* $\mathbb{R}^{k+1}$ *and a family of tropical curves* $(C_{k,d})_{d\geq 1}$ *in* L_k *such that (see Figure 2 for* $k=2$ *and* $d=3$*):*

- *$C_{k,d}$ is tropical curve of degree d and genus $k \cdot \frac{(d-1)\cdot(d-2)}{2}$;*
- *the intersection $C_{k,d} \cap C_{k,d-1}$ consists of exactly $(d-1)\cdot\big[2(d-1) + (k-2)\cdot(d-2)\big]$ transverse intersection points and $k-1$ segments;*
- *$R(C_{k,d})$ is d times the fan tropical line with one unbounded ray in each of the directions*

$$(-1,0,0,\cdots,0),(0,-1,0,\cdots,0),\cdots,(0,\cdots,0,-1),(1,\ldots,1).$$

Proof. The case $k = 1$ is well known, and can be proved for example by perturbing the curves constructed in Lemma 4.1. For $k \geq 2$, we use the following `Gluing` construction routine. We say that a tropical curve C of degree d in $\mathbb{R}^n$ is right-degenerate (resp. left-degenerate) if C has an unbounded edge of weight d in the direction $(1,1,\cdots,1)$ (resp. $(-1,0,0,\cdots,0)$) and passing through the origin. Finally, we denote by H_{n-1} the tropical hyperplane in $\mathbb{R}^n$ defined by the tropical polynomial "$x_1 + \cdots + x_n + 0$".

`Gluing` routine
INPUT

- a tropical linear plane L in $\mathbb{R}^n$;
- a right-degenerate tropical curve C_1 of degree d in L;
- a left-degenerate tropical curve C_2 of degree d in $\mathbb{R}^2$.

OUTPUT

- a tropical linear plane $\tilde{L}$ in $\mathbb{R}^{n+1}$;
- a tropical curve C of degree d in $\tilde{L}$.

DO

Let e_i be the edge of C_i passing through the origin. Since the multiplicity of intersection at the origin of H_{n-1} (resp H_1) and C_1 (resp. C_2) is d, we deduce that $C_1 \cap H_{n-1} \subset e_1$ (resp. $C_2 \cap H_1 \subset e_2$). We denote by $\widehat{C}_1$ (resp. $\widehat{C}_2$) the topological closure of $C_1 \setminus H_{n-1}$ (resp. $C_2 \setminus H_1$).

We embed $\widehat{C}_1$ and $\widehat{C}_2$ in $\mathbb{R}^{n+1}$ in such a way that the union of the images is a tropical curve. The embeddings are given by the two following linear maps:

$$\gamma_n(x_1,\cdots,x_n) = (x_1,\cdots,x_n,0) \in \mathbb{R}^{n+1}$$

and

$$\gamma(x,y) = (x,\cdots,x,y) \in \mathbb{R}^{n+1}.$$

We define C to be the union of the images of $\widehat{C}_1$ and $\widehat{C}_2$ by γ_k and γ respectively, equipped with the weights inherited from $\widehat{C}_1$ and $\widehat{C}_2$.

By construction C is a tropical curve of degree d contained in $\mathbb{R}^{n+1}$. Furthermore, the tropical curve C is contained in the tropical modification $\tilde{L}$ of the tropical plane L along the function "$x_1 + \cdots + x_n + 0$".

END

Note that if C is the result of $\texttt{Gluing}(L, C_1, C_2)$, the genus of C is clearly the sum of the genera of C_1 and C_2.

Let $p = (x_p, 0) \in \mathbb{R}^2$ (resp. $q = (x_q, x_q) \in \mathbb{R}^2$) be a point that is contained in e_∞ (resp. e'_∞) of all tropical curves $\tilde{C}_d$ and $\overline{C}_d$ (resp. $\overline{C}_d$) from Lemmas 4.1 and 4.2. For $u \in \mathbb{R}^n$ we denote by τ_u the translation in $\mathbb{R}^n$ by the vector u. Given $d \geq 1$, we define the families of tropical linear spaces $(L'_k)_{k\geq 1}$ and of tropical curves $(C'_{k,d})_{k\geq 1}$ of degree d recursively as follows:

- Let $\tilde{C}'_d$ be the tropical curve which is the image of $\tilde{C}_d$ under the map $r : (x, y) \mapsto (-x, y - x)$, and translated so that $r(p)$ is mapped to the origin; set $L'_1 = \mathbb{R}^2$ and $C'_{1,d} = \tilde{C}'_d$;
- $(L'_{k+1}, C'_{k+1,d})$ is the translation of $\texttt{Gluing}\big(L'_k, C'_{k,d}, \tau_{-p}(\overline{C}_d)\big)$ by the vector $(x_p - x_q, \cdots, x_p - x_q, -x_q)$.

We define $(L_k, C_{k,d})$ as the output of $\texttt{Gluing}(L'_{k-1}, C'_{k-1,d}, \tau_{-p}\tilde{C}_d)$.

Since the tropical curves $\tilde{C}_d$ and $\overline{C_d}$ are of genus $\frac{(d-1)\cdot(d-2)}{2}$, the tropical curve $C_{k,d}$ is of genus $k \cdot \frac{(d-1)\cdot(d-2)}{2}$. Each call to $\texttt{Gluing}$ yields one (bounded) segment in $C_{k,d} \cap C_{k,d-1}$, which thus contains $k-1$ segments. All other intersections are tropically transverse. By Lemmas 4.1 and 4.2, the number of tropically transverse intersection points of $C_{k,d}$ and $C_{k,d-1}$ is equal to $(k-2)\cdot(d-1)\cdot(d-2) + 2(d-1)^2$. By construction, the recession fan $R(C_{k,d})$ is as stated. □

4.2. Higher dimensional tropical varieties in $\mathbb{R}^n$. We describe in this section an inductive construction of tropical varieties in $\mathbb{R}^n$ with large Betti numbers, using the curves whose existence is attested by Theorem 4.3 as the initial step. We first need the notion of recession cone of a rational tropical function on a tropical linear space. Note that if L is a tropical linear space in $\mathbb{R}^n$, then there is a canonical one to one correspondence $F \mapsto F^\infty$ between faces of $R(L)$ and unbounded faces of L.

Lemma 4.4. *Let L be a tropical linear space in $\mathbb{R}^n$, and $f : L \to \mathbb{R}$ be a tropical rational function. Let $u \in R(L)$, and denote by $S(u)$ the union of all faces of $R(L)$ containing u, by $S^\infty(u)$ the union of the corresponding unbounded faces of L, and by $S_0^\infty(u)$ the set of points p in $S^\infty(u)$ such that the half-line $p + \mathbb{R}_{\geq 0}u$ is contained in $S^\infty(u) \setminus \mathrm{div}_L(f)$. Then the function $p \mapsto df_p(u)$ is constant on $S_0^\infty(u)$.*

Proof. Let p_1 and p_2 be two points in $S_0^\infty(u)$. Hence there exists a path from p_1 to p_2 in $S_0^\infty(u)$ which crosses $\text{div}_L(f)$ only along its facets containing the direction u. By definition of $\text{div}_L(f)$, the value of $df_p(u)$ does not change when crossing such a facet. □

As a consequence, there is a well defined map

$$\begin{array}{rccc} R(f): & R(L) & \longrightarrow & \mathbb{R} \\ & u & \longmapsto & df_p(u) \end{array}$$

where p is any point in $S_0^\infty(u)$. The map $R(f)$ is called the *recession map* of f.

Theorem 4.5. *For any positive integers m and k, there exist a tropical linear space $L_{m,k}$ of dimension $m+1$ in $\mathbb{R}^{m+k}$, a tropical linear space $L'_{m,k}$ of dimension m in $\mathbb{R}^{m+k}$ and a family of tropical hypersurfaces $(X_{m,k,d})_{d\geq 1}$ in $L_{m,k}$ such that for any $d \geq 1$,*

- *$X_{m,k,d}$ is of degree d;*
- *$X_{m,k,d}$ is a homology bouquet of spheres and*

$$b_m(X_{m,k,d}) = k \cdot B_m(m,1,d);$$

- *$R(X_{m,k,d}) = d \cdot L'_{m,k}$.*

Proof. We fix k and we proceed by induction on m. The case $m = 1$ holds by Theorem 4.3.

Suppose now that $L_{m,k}$, $L'_{m,k}$, and the family $(X_{m,k,d})_{d\geq 0}$ have been constructed. By Lemma 3.9, for any $d \geq 0$, there exists a tropical rational function $h_d : L_{m,k} \to \mathbb{R}$ of degree d such that $\text{div}_{L_{m,k}}(h_d) = X_{m,k,d}$. The recession cone $R(X_{m,k,d} - X_{m,k,d-1}) = L'_{m,k}$ does not depend on d, hence the recession map of "h_d/h_{d-1}" is of degree 1 and does not depend on d. In particular, there exists a sequence $(\alpha_d)_{d\geq 0}$ of real numbers such that for any sequence $(a_d)_{d\geq 0}$ of real numbers satisfying $a_{d+1} < a_d - \alpha_d$, we have

$$\text{“}a_{d+1} \cdot h_{d+1}/h_d(p)\text{”} < \text{“}a_d \cdot h_d/h_{d-1}(p)\text{”} \qquad \forall p \in L_{m,k}.$$

Hence we obtain that the set

$$K_{m,k,d} = \{L_{k,m}, \cdots, L_{k,m}, X_{m,k,0}, X_{m,k,1}, \cdots, X_{m,k,d}, f_1, \cdots, f_d\}$$

is a construction pattern, where $f_d = \text{“}a_d \cdot h_d/h_{d-1}\text{”}$ with $(a_d)_{d\geq 0}$ as above. We denote by $X_{m+1,k,d}$ the floor composed tropical variety of dimension $m+1$

in $\mathbb{R}^{m+k+1}$ with pattern $K_{m,k,d}$. Recall that $B_m(m,1,d) = \binom{d-1}{m+1}$. By Proposition 3.11, the tropical variety $X_{m+1,k,d}$ is a homology bouquet of spheres and we have

$$\begin{aligned} b_{m+1}(X_{m+1,k,d}) &= \sum_{i=1}^{d-1} b_m(X_{m,k,i}) \\ &= \sum_{i=1}^{d-1} k \cdot \binom{i-1}{m+1} \\ &= k \cdot \binom{d-1}{m+2} \\ &= k \cdot B_{m+1}(m+1,1,d). \end{aligned}$$

Furthermore, by Proposition 3.10, $X_{m+1,k,d}$ has degree d and is contained in the tropical linear space $L_{m+1,k} = L_{m,k} \times \mathbb{R}$. Since the recession map of f_d is of degree one and does not depend of d, the recession fan $R(X_{m+1,k,d})$ is d times a fan tropical linear space $L'_{m+1,k}$ in $\mathbb{R}^{m+1}$ which does not depend on d. Hence the tropical linear spaces $L_{m+1,k}$ and $L'_{m+1,k}$, and the family $(X_{m+1,k,d})_{d\geq 0}$ have been constructed, and the Theorem is proved. □

4.3. Proof of Theorem 1.5. Let d, m and k be three positive integers. We choose L (resp. X) to be the closure in $\mathbb{T}P^n$ of the tropical linear space $L_{m,k}$ (resp. the tropical variety $X_{m,k,d}$) from Theorem 4.5. Since $X \setminus X_{m,k,d}$ is a polyhedral complex of dimension at most $m-1$, we have

$$b_m(X) \geq b_m(X_{m,k,d}),$$

and the theorem is proved. In the case $m = 1$, we furthermore have $b_1(X) = b_1(X_{1,k,d})$ since the recession fan $R(X)$ is d times the fan tropical line with unbounded edges in standard directions. □

Theorem 1.5 together with Proposition 2.2 prove Theorem 1.3 from the introduction.

5. Tropical homology of floor composed surfaces

In this section we explicitly compute tropical homology of the floor composed surfaces constructed in the proof of Theorem 1.5. We refer to [MZ2, BIMS, KSW] for the definition of tropical homology for locally finite polyhedral complexes in the standard projective space $\mathbb{T}P^n$. All tropical homology groups are considered with coefficients in $\mathbb{R}$. This section partially generalises results from [Sha1].

We first start by computing tropical homology of simple tropical bundles, and apply these results to floor composed surfaces. Recall that the Mayer–Vietoris Theorem holds for tropical homology [Sha1, Proposition 4.2], and that an irreducible compact trivalent[2] tropical curve of genus g has the following tropical Hodge diamond:

$$\begin{matrix} & 1 & \\ g & & g \\ & 1 & \end{matrix}$$

5.1. Tropical homology of tropical ruled varieties. We denote by Δ_n the standard unimodular simplex in $\mathbb{R}^n$, and by $\tilde{\Delta}_n^i$ the convex polytope in $\mathbb{R}^n$ which is the convex hull of the union of $i \cdot \Delta_{n-1} \times \{0\}$ and $\Delta_{n-1} \times \{1\}$. The corresponding algebraic toric variety is

$$\mathbb{P}\left(\mathcal{O}_{\mathbb{C}P^{n-1}}(i) \oplus \mathcal{O}_{\mathbb{C}P^{n-1}}\right) = \mathbb{P}\left(\mathcal{O}_{\mathbb{C}P^{n-1}}(-i) \oplus \mathcal{O}_{\mathbb{C}P^{n-1}}\right).$$

We denote by $\mathbb{T}\tilde{\Delta}_n^i$ the corresponding tropical toric variety. The faces $i \cdot \Delta_{n-1} \times \{0\}$ and $\Delta_{n-1} \times \{1\}$ of $\tilde{\Delta}_n^i$ correspond to two divisors of $\mathbb{T}\tilde{\Delta}_n^i$, respectively denoted by E_- and E_+, that are contained in the boundary of $\mathbb{T}\tilde{\Delta}_n^i$. Note that both E_- and E_+ are equal to $\mathbb{T}P^{n-1}$. Furthermore, there are two natural projections $\pi_\pm : \mathbb{T}\tilde{\Delta}_n^i \to E_\pm$, which are tropical morphisms, and whose fibre over any point is $\mathbb{T}P^1$.

Example 5.1. The standard tropical Hirzebruch surface $\mathbb{T}\mathbb{F}_i$ of degree i is defined as $\mathbb{T}\tilde{\Delta}_2^i$. Note that the divisor E_+ is tropically linearly equivalent (see for example [Mik4, Section 4.3] or [MR, Section 6.3]) to the divisor $E_- + iF$, where F is any fibre of $\pi_\pm$.

Definition 5.2. *Let X be a tropical variety in $\mathbb{T}P^n$ identified with $E_- \subset \mathbb{T}\tilde{\Delta}_n^i$.*

- *The cylinder $\Sigma = \pi_-^{-1}(X)$ over X in $\mathbb{T}\tilde{\Delta}_n^i$ is called a $\mathbb{T}P^1$**-bundle over** X. The intersection of Σ with $E_\pm$ is denoted by $X_\pm$.*
- *The tropical varieties $\Sigma_- = \Sigma \setminus X_+$ and $\Sigma_+ = \Sigma \setminus X_-$ are called **tropical line bundles over** X.*
- *The tropical variety $\Sigma^{oo} = \Sigma_- \cap \Sigma_+$ is called a $\mathbb{T}^\times$**-bundle over** X.*

[2] An irreducible tropical curve C in $\mathbb{T}P^n$ is said to be trivalent if $\mathrm{val}_p(C) \leq 3$ for every point $p \in C$.

This is a rather restrictive notion of $\mathbb{T}P^1$/line/$\mathbb{T}^\times$ bundles, however it will be sufficient for our purposes. We refer for example to [MZ1, All] for a more general definition of tropical line bundles.

A $\mathbb{T}P^1$-bundle Σ over a projective tropical variety X comes naturally equipped with two natural tropical projections $\pi_\pm : \Sigma \to X_\pm$ with a section $\iota_\pm : X \to X_\pm \subset \Sigma$.

We compute, in the following lemmas, tropical homology groups of $\mathbb{T}P^1$, line and $\mathbb{T}^\times$ bundles.

Lemma 5.3. *Let $\Sigma_\pm$ be a tropical line bundle over a tropical variety X. Then for any pair (p,q), the inclusion $\iota_\pm$ induces an isomorphism*

$$\iota_{\pm *} : H_{p,q}(X) \simeq H_{p,q}(\Sigma_\pm).$$

Proof. The morphism $\iota_{\pm *}$ is injective since it is clearly a section of the morphism $H_{p,q}(\Sigma_\pm) \to H_{p,q}(X)$ induced by the projection $\pi_\pm$.

Equip $\Sigma_\pm$ with any locally finite polyhedral subdivision compatible with its tropical structure. Recall that the cellular tropical homology of $\Sigma_\pm$ is isomorphic to the singular tropical homology of $\Sigma_\pm$, and is thus independent of the chosen subdivision [MZ2, Proposition 2.2]. A (p,q)-cell σ in $\Sigma_\pm$ is called vertical if $\pi_\pm(\sigma)$ has dimension strictly less than q. A (p,q)-chain in $\Sigma_\pm$ is called vertical if every cell in its support is vertical. Any (p,q)-chain in $\Sigma_\pm$ is homologous to the sum of a (p,q)-chain with support in $X_\pm$ and a vertical (p,q)-chain. Since no vertical chain in $\Sigma_\pm$ can be closed, we obtain that any (p,q)-cycle in $\Sigma_\pm$ can be represented by a (p,q)-cycle in $X_\pm$. In other words, the map $\iota_{\pm *}$ is surjective and is thus an isomorphism. □

Let Σ be a $\mathbb{T}P^1$-bundle over a tropical variety X, and let u_- be the primitive integer vector generating the kernel of $d\pi_-$ and pointing away from X_- (there is a unique choice of such a vector in each tropical tangent space of Σ). To a $(p-1,q-1)$-cell $\sigma = \beta_Q \cdot Q$ in X, with Q a $(q-1)$-dimensional face of X and $\beta_Q \in \mathcal{F}_{p-1}(Q)$, we associate the (p,q)-cell $\kappa(\sigma) = (u_- \wedge \beta_Q) \cdot \pi_-^{-1}(\iota_-(Q))$ in Σ, where the orientation of $\pi_-^{-1}(\iota_-(Q))$ is induced by the orientation on $\iota_-(Q)$. This induces a linear map

$$\kappa : H_{p-1,q-1}(X) \to H_{p,q}(\Sigma),$$

that we call a tropical Gysin map. Note that the tropical Gysin map is the same if one defines it using the section ι_+ instead of ι_-. Furthermore it maps straight classes (i.e., classes induced by tropical cycles) of X to straight classes of Σ. The inclusion map $\iota_\pm : X \to \Sigma$ induces a linear map $H_{p,q}(X) \to H_{p,q}(\Sigma)$ that we still denote by $\iota_{\pm *}$ to avoid additional notations. This slight abuse of notation is justified in particular by next lemma.

Lemma 5.4. *For any* $\mathbb{T}P^1$*-bundle* Σ *over a tropical variety* X*, and for any pair* (p,q)*, the maps* ι_{-*} *and* κ *induce an isomorphism*

$$(\iota_{-*},\kappa):\ H_{p,q}(X)\times H_{p-1,q-1}(X)\simeq H_{p,q}(\Sigma).$$

Proof. The map ι_{-*} is injective since it is a section of π_{-*}. As X and Σ are both compact, we choose their polyhedral subdivision induced by the tropical structure on X. As in the proof of Lemma 5.3, any (p,q)-chain σ in Σ is homologous to the sum of a (p,q)-chain σ_- in X_- and a vertical (p,q)-chain σ_v.

Suppose that σ is a (p,q)-cycle in Σ. The cellular boundary of any vertical cell of Σ intersects X_+ which is disjoint from X_-. Hence the vector u_- divides the framing of each cell contained in the support of σ_v, that is to say $\sigma_v=\kappa(\sigma_0)$ with σ_0 a $(p-1,q-1)$-chain in X. In turn, this implies that the support of $\partial\sigma_v$ is disjoint from X_-, from which we deduce that

$$\partial\sigma_-=\partial\sigma_v=0.$$

This proves that the map $\iota_{-*}\times\kappa$ is surjective.

Conversely, suppose that σ' and σ'' are respectively (p,q) and $(p-1,q-1)$-cycles in X such that

$$\iota_{-*}(\sigma')+\kappa(\sigma'')=\partial\gamma.$$

As above, we have $\iota_{-*}(\sigma')=\partial\gamma_-$ and $\kappa(\sigma'')=\partial\gamma_v=\kappa(\partial\gamma_0)$, which further implies that both σ' and σ'' are null homologous. Hence the map $\iota_{-*}\times\kappa$ is injective, and the lemma is proved. □

The map κ does not depend on which section ι_- or ι_+ we choose to define it, however the inclusion $H_{p,q}(X)\to H_{p,q}(\Sigma)$ does. Let

$$\nu_{p,q}:H_{p,q}(X)\to H_{p-1,q-1}(X)$$

be the linear map obtained by the following compositions

$$H_{p,q}(X)\xrightarrow{\ \iota_{+*}\ }H_{p,q}(\Sigma)\xrightarrow{(\iota_{-*},\kappa)^{-1}}H_{p,q}(X)\times H_{p-1,q-1}(X)\longrightarrow$$
$$\longrightarrow H_{p-1,q-1}(X),$$

where the last map is the projection on the second factor. Note that $\nu_{p,q}$ is the zero map if and only if $\iota_{+*}=\iota_{-*}$. The image of $\nu_{\dim X,\dim X}$ is called *the first Chern class* of the tropical line bundle Σ_- (and so it is minus the first Chern class of the line bundle Σ_+).

Example 5.5. Consider the tropical Hirzebruch surface $\mathbb{T}\mathbb{F}_i$ of degree i. Recall that the divisor E_+ is tropically linearly equivalent to the divisor $E_- + iF$, where F is the divisor of $\mathbb{T}\mathbb{F}_i$ corresponding to the side $[(0,0);(0,1)]$ of $\tilde{\Delta}_2^i$. Hence the corresponding straight classes satisfy

$$[E_+] = [E_-] + i[F]$$

in $H_{1,1}(\mathbb{T}\mathbb{F}_i)$. In particular, the first Chern class of $\mathbb{T}\mathbb{F}_i \setminus E_+$ is i times the class of a point.

More generally, let $\Sigma \subset \tilde{\Delta}_n^i$ be a $\mathbb{T}P^1$-bundle over a compact tropical curve of degree d in E_-. It follows from the balancing condition that the first Chern class of Σ_- is equal to $i \cdot d$ times the class of a point.

Next we turn to tropical homology of $\mathbb{T}^\times$-bundles.

Corollary 5.6. *For any $\mathbb{T}P^1$-bundle Σ over a tropical variety X, and for any pair (p,q), one has the isomorphism*

$$H_{p,q}(\Sigma^{oo}) \simeq Ker\ \nu_{p,q} \times \left(H_{p-1,q}(X)/Im\ \nu_{p,q+1}\right).$$

Proof. The Mayer–Vietoris Theorem applied to the triple $(\Sigma, \Sigma_-, \Sigma_+)$ gives the long exact sequence

$$(5) \quad \dots \longrightarrow H_{p,q}(\Sigma^{oo}) \longrightarrow H_{p,q}(\Sigma_-) \times H_{p,q}(\Sigma_+) \longrightarrow H_{p,q}(\Sigma) \longrightarrow \\ \longrightarrow H_{p,q-1}(\Sigma^{oo}) \longrightarrow \dots$$

By Lemma 5.3, we have canonical isomorphisms $\iota_{\pm *} : H_{p,q}(X) \to H_{p,q}(\Sigma_\pm)$. By Lemma 5.4, we have an isomorphism $\iota_{-*} \times \kappa : H_{p,q}(X) \times H_{p-1,q-1}(X) \to H_{p,q}(\Sigma)$. With these identifications, the image of the map

$$H_{p,q}(\Sigma_-) \times H_{p,q}(\Sigma_+) \longrightarrow H_{p,q}(\Sigma)$$

is precisely $H_{p,q}(X) \times \mathrm{Im}\ \nu_{p,q}$. Hence the long exact sequence (5) splits into the short exact sequences

$$0 \longrightarrow H_{p-1,q}(X)/\mathrm{Im}\ \nu_{p,q+1} \longrightarrow H_{p,q}(\Sigma^{oo}) \longrightarrow \mathrm{Ker}\ \nu_{p,q} \longrightarrow 0,$$

and the result follows. □

Example 5.7. In the extremal cases when $p = \dim X + 1$, or $p = 0$, or $q = \dim X + 1$, Corollary 5.6 gives

$$H_{\dim \Sigma, q}(\Sigma^{oo}) = H_{\dim X, q}(X), \quad H_{p, \dim \Sigma}(\Sigma^{oo}) = 0, \quad \text{and} \quad H_{0,q}(\Sigma^{oo}) = H_{0,q}(X).$$

Example 5.8. Suppose that X is a compact trivalent tropical curve of genus g. Then Corollary 5.6 gives the following tropical Hodge diamond for Σ^{oo} (as in the introduction, by convention, $h_{0,0}$ is the topmost number and $h_{2,0}$ the leftmost):

$$\begin{array}{ccccc} & & 1 & & \\ & g+\varepsilon & & g & \\ g & & g+\varepsilon & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{array}$$

where $\varepsilon = 0$ if the first Chern class of Σ_- does not vanish, and $\varepsilon = 1$ if it does. Note that this example corrects a small mistake in [Sha1, Lemma 4.3ii)].

5.2. Tropical homology of birational tropical modifications. The method we used in Section 5.1 also allows the computation of tropical homology of a birational tropical modification of a tropical variety. Recall that $\pi_- : \mathbb{T}\tilde{\Delta}_n^i \to E_-$ is a $\mathbb{T}P^1$-bundle over $E_- = \mathbb{T}P^{n-1}$. As in Section 5.1, we denote by u_- the primitive integer vector generating the kernel of $d\pi_-$ and pointing away from E_-. If Y is a tropical variety in $\mathbb{T}\tilde{\Delta}_n^i$, we denote by $Y_\pm$ its intersection with the divisor $E_\pm$, and by Y_+^p the tropical variety $\pi_-(Y_+)$.

Definition 5.9. *A tropical variety Y in $\mathbb{T}\tilde{\Delta}_n^i$ is called a **birational tropical modification** of $X \subset E_-$ along the divisor $Y_- - Y_+^p$ if $Y \cap \mathbb{R}^n$ is a birational tropical modification of $X \cap \mathbb{R}^{n-1}$ along the divisor $(Y_- - Y_+^p) \cap \mathbb{R}^{n-1}$, and if Y is the topological closure of $Y \cap \mathbb{R}^n$ in $\mathbb{T}\tilde{\Delta}_n^i$.*

If $Y_+ = \varnothing$, then Y is called a tropical modification of X along the divisor Y_-.

Given such a birational tropical modification Y of X, we still denote by π_- the restriction of π_- to Y. We emphasise that in the next proposition, it is not assumed that the tropical prevariety $Y_- \cap Y_+^p$ is a tropical variety (recall that a tropical variety is defined as the set-theoretic intersection of some tropical varieties, see [RGST, Section 3]).

Lemma 5.10. *Let $Y \subset \mathbb{T}\tilde{\Delta}_n^i$ be a birational tropical modification of $X \subset E_-$ along the divisor $Y_- - Y_+^p$. Then for any pair (p,q), one has*

$$H_{p,q}(Y) \simeq H_{p,q}(X) \times H_{p-1,q-1}(Y_- \cap Y_+^p).$$

Proof. Since all tropical varieties involved are compact, we choose their polyhedral subdivision induced by their tropical structure. The map $\pi_- : Y \to X$ induces a map on the chain groups

$$\pi_{-*} : C_q(Y, \mathcal{F}_p) \to C_q(X, \mathcal{F}_p)$$

that commutes with the boundary map. We denote by $\tilde{Y}$ the union of all faces of Y on which $d\pi_-$ is injective, i.e., $\tilde{Y}$ is the union of faces of Y on which the restriction of π_- is a bijection. We denote by τ the inverse map of $\pi_{-|\tilde{Y}}$.

We start by constructing a section s of the map $\pi_{-*} : H_{p,q}(Y) \to H_{p,q}(X)$. Given a (p,q)-cell σ in X, choose a facet F_σ of X containing the support of σ. Then σ induces via $\tau|_{F_\sigma}$ a (p,q)-cell $\tau_*(\sigma)$ in Y. Note that a different choice (if any) of F_σ gives rise to a different (p,q)-chain, differing from $\tau_*(\sigma)$ by a framing divisible by u_-; this will not be important in what follows. By linearity, we obtain a linear map

$$\tau_* : C_q(X, \mathcal{F}_p) \to C_q(Y, \mathcal{F}_p).$$

If σ is a (p,q)-cycle in X, then by construction $\partial\tau_*(\sigma)$ has support contained in $\pi_-^{-1}(Y_- \cup Y_+^p)$ and has a framing divisible by u_-. Hence $\partial\tau_*(\sigma)$ is the boundary in Y of a vertical (p,q)-chain σ_v, and we define

$$s(\sigma) = \tau_*(\sigma) - \sigma_v.$$

The map s is a section of the map π_*, in particular it is injective. In the rest of the proof we identify $H_{p,q}(X)$ and its image by s in $H_{p,q}(Y)$.

Next, the same construction than the construction of the tropical Gysin map in Section 5.1 provides a linear map

$$\kappa : H_{p-1,q-1}(Y_- \cap Y_+^p) \to H_{p,q}(Y).$$

With a proof analogous to the proof in Lemma 5.4 that the map $\iota_+ \times \kappa$ is an isomorphism, we obtain that the linear map $s \times \kappa : H_{p,q}(X) \times H_{p-1,q-1}(Y_- \cap Y_+^p) \to H_{p,q}(Y)$ is also an isomorphism. □

Applying Lemma 5.10 in the particular case when Y_+ is empty, we recover the result by Shaw that tropical homology groups are invariant under tropical modifications.

Corollary 5.11 (Shaw, [Sha3, Theorem 4.13]). *Let $Y \subset \mathbb{T}\tilde{\Delta}^i_{n+1}$ be a tropical modification of $X \subset E_-$. Then for any pair (p,q), the linear map*

$$\pi_{-*} : H_{p,q}(Y) \longrightarrow H_{p,q}(X)$$

is an isomorphism.

Any tropical linear space of dimension m in $\mathbb{T}P^n$ is obtained from $\mathbb{T}P^m$ by a finite sequence of tropical modifications along linear tropical divisors, hence they have the same tropical Hodge diamond. There are many ways to compute tropical homology groups of $\mathbb{T}P^m$ (see for example [BIMS, Example 7.27] and [IKMZ, Corollary 2]), with which we obtain the following well-known statement.

Corollary 5.12. *Let L be a tropical linear space of dimension m in $\mathbb{T}P^n$. Then one has*

$$h_{p,p}(L) = 1 \quad \forall p = 0, 1, \ldots, n, \qquad \text{and} \qquad h_{p,q}(L) = 0 \text{ otherwise.}$$

5.3. Back to tropical surfaces. Now we specialise results from Sections 5.1 and 5.2 to the case of floor composed tropical surfaces. Throughout the whole section, we consider the family of tropical curves $(C_{k,d})_{d\geq 1}$ in $\mathbb{T}P^{k+1}$ we constructed in Theorem 4.3, and the tropical plane L_k which contains them. We denote by $\Sigma_{k,d}$ the $\mathbb{T}P^1$-bundle over $C_{k,d}$ in $\mathbb{T}\tilde{\Delta}^1_{k+2}$, and by $L_{k,d,d-1}$ the birational tropical modification of L_k along $C_{k,d} - C_{k,d-1}$.

Lemma 5.13. *For any integer $k \geq 1$ and $d \geq 2$, the tropical Hodge diamond of $L^o_{k,d,d-1} = L_{k,d,d-1} \setminus E_+$ is the following*

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ k\cdot g(C_{1,d-1}) & & k\cdot\big[d\cdot(d-1)+g(C_{1,d-1})\big]+(k-1)\cdot(2d-3) & & 0 \\ & k-1 & & 0 & \\ & & 0 & & \end{array}$$

Furthermore, both natural maps $H_{2,0}(\Sigma^{oo}_{k,d-1}) \to H_{2,0}(L^o_{k,d,d-1})$ and $H_{1,1}(\Sigma^{oo}_{k,d-1}) \to H_{1,1}(L^o_{k,d,d-1})$ are injective.

Proof. The case $p = 0$ is clear since $H_{0,q}(L^o_{k,d,d-1}) = H_q(L^o_{k,d,d-1};\mathbb{R})$ and that $L^o_{k,d,d-1}$ is contractible. The non-vanishing tropical Hodge numbers of a segment in $\mathbb{R}^n$ are precisely $h_{0,0} = h_{1,0} = 1$. Hence $L_{k,d,d-1}$ has the following tropical Hodge diamond by Lemma 5.10 and Theorem 4.3

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & k\cdot\big[1+d\cdot(d-1)\big]-2(k-1)\cdot(d-1) & & 0 \\ & k-1 & & 0 & \\ & & 1 & & \end{array}$$

The first Chern class of $\Sigma_{k,d}$ is non-null by Example 5.5. We consider the decomposition of $L_{k,d,d-1}$ into the union of $L^o_{k,d,d-1}$ and of a connected and simply connected neighbourhood of $C_{k,d-1}$ in $L_{k,d,d-1}$. The Mayer–Vietoris sequence together with Lemma 5.3 and Example 5.8 give that $H_{1,2}(L^o_{k,d,d-1}) = 0$, and the following long exact sequences

$$(6)\quad 0 \longrightarrow H_{2,2}(L^o_{k,d,d-1}) \longrightarrow H_{2,2}(L_{k,d,d-1}) \longrightarrow H_{1,1}(C_{k,d-1}) \longrightarrow \\ \longrightarrow H_{2,1}(L^o_{k,d,d-1}) \longrightarrow H_{2,1}(L_{k,d,d-1}) \longrightarrow \\ \longrightarrow H_{1,0}(C_{k,d-1}) \longrightarrow H_{2,0}(L^o_{k,d,d-1}) \longrightarrow 0$$

and

$$(7)\quad 0 \longrightarrow H_{0,1}(C_{k,d-1}) \longrightarrow H_{1,1}(C_{k,d-1}) \times H_{1,1}(L^o_{k,d,d-1}) \longrightarrow \\ \longrightarrow H_{1,1}(L_{k,d,d-1}) \longrightarrow H_{1,0}(C_{k,d-1}) \longrightarrow \\ \longrightarrow H_{1,0}(C_{k,d-1}) \times H_{1,0}(L^o_{k,d,d-1}) \longrightarrow 0$$

The map $H_{2,2}(L_{k,d,d-1}) \to H_{1,1}(C_{k,d-1})$ is an isomorphism, so we obtain $H_{2,2}(L^o_{k,d,d-1}) = 0$ from (6). Next, the map $H_{2,1}(L_{k,d,d-1}) \to H_{1,0}(C_{k,d-1})$ is the zero map, since the support of the image of any cycle is contained in disconnecting edges of $C_{k,d-1}$. Hence we obtain statement concerning $H_{2,1}(L^o_{k,d,d-1})$ and $H_{2,0}(L^o_{k,d,d-1})$ from (6).

The map $H_{1,0}(C_{k,d-1}) \to H_{1,0}(C_{k,d-1}) \times H_{1,0}(L^o_{k,d,d-1})$ is the identity on the first factor, so we obtain from (7) the statements about $h_{1,0}(L^o_{k,d,d-1})$ and $h_{1,1}(L^o_{k,d,d-1})$. Since the map $H_{0,1}(C_{k,d-1}) \to H_{1,1}(C_{k,d-1})$ is the zero map, we obtain the injectivity of the map $H_{1,1}(\Sigma^{oo}_{k,d}) \to H_{1,1}(L_{k,d,d-1})$ from (7). $\square$

For simplicity, we denote by $(X_{k,d})_{d\geq 1}$ rather than $(X_{2,k,d})_{d\geq 1}$ the family of floor composed tropical surfaces constructed in the proof of Theorem 4.5 out of the family $(C^k_d)_{d\geq 1}$ of tropical curves contained in the tropical plane L_k. Since the tropical surface $X_{k,1}$ is a tropical plane, it has the following tropical Hodge diamond by Corollary 5.12:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Proposition 5.14. *For any integers $k \geq 1$ and $d \geq 2$, the tropical surface $X_{k,d}$ has the following tropical Hodge diamond:*

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ h_{2,0}(X_{k,d-1}) + k\cdot g(C_{1,d-1}) & & h_{1,1}(X_{k,d}) & & h_{0,2}(X_{k,d-1}) + k\cdot g(C_{1,d-1}) \\ & h_{2,1}(X_{k,d-1}) + k - 1 & & 0 & \\ & & 1 & & \end{array}$$

where

$$h_{1,1}(X_{k,d}) = h_{1,1}(X_{k,d-1}) + k \cdot \left[d \cdot (d-1) + 2g(C_{1,d-1}) - 1\right] - 2(k-1)\cdot(d-2).$$

Furthermore for any $d \geq 1$, the natural map $H_{2,1}(X_{k,d}) \to H_{1,0}(C_{k,d})$ is the zero map.

Proof. Since $H_{2,1}(X_{k,1}) = 0$, the map $H_{2,1}(X_{k,1}) \to H_{1,0}(C_{k,1})$ is clearly the zero map. We do not compute the tropical Hodge numbers with $q = 0$ here, since they correspond to Betti numbers and have already been computed in Theorem 3.11. We denote by $X^o_{k,d}$ the tropical surface $X_{k,d}$ from which we remove the copy of the curve $C_{k,d}$ located on the boundary. Let $d \geq 2$, and suppose that the proposition is true for $d-1$. Since the map $H_{2,1}(X_{k,d-1}) \to H_{1,0}(C_{k,d-1})$ is the zero map, by the same computation performed in the proof of Lemma 5.13 we obtain that $X^o_{k,d-1}$ has the following tropical Hodge diamond:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ h_{2,0}(X_{k,d-1}) + g(C_{k,d-1}) & & h_{1,1}(X_{k,d-1}) - 1 + g(C_{k,d-1}) & & h_{0,2}(X_{k,d-1}) \\ & h_{2,1}(X_{k,d-1}) & & 0 & \\ & & 0 & & \end{array}$$

We consider the same decomposition of $X_{k,d}$ as in the proof of Proposition 3.11. By the Mayer–Vietoris Theorem together with Lemmas 5.3 and 5.13 and Example 5.8, we obtain that $h_{1,0}(X_{k,d}) = 0$, and the following long exact sequences

$$(8) \quad 0 \longrightarrow H_{2,2}(X_{k,d}) \longrightarrow H_{1,1}(C_{k,d-1}) \longrightarrow H_{2,1}(L^o_{k,d,d-1}) \times H_{2,1}(X^o_{k,d-1}) \longrightarrow \\ \longrightarrow H_{2,1}(X_{k,d}) \longrightarrow H_{1,0}(C_{k,d-1}) \longrightarrow H_{2,0}(L^o_{k,d,d-1}) \times H_{2,0}(X^o_{k,d-1}) \longrightarrow \\ \longrightarrow H_{2,0}(X_{k,d}) \longrightarrow 0$$

and

$$(9) \quad 0 \longrightarrow H_{1,2}(X_{k,d}) \longrightarrow H_{0,1}(C_{k,d-1}) \longrightarrow H_{1,1}(L^o_{k,d,d-1}) \times H_{1,1}(X^o_{k,d-1}) \longrightarrow \\ \longrightarrow H_{1,1}(X_{k,d}) \longrightarrow H_{1,0}(C_{k,d-1}) \longrightarrow 0$$

The map $H_{2,2}(X_{k,d}) \to H_{1,1}(C_{k,d})$ is clearly an isomorphism. Furthermore the map $H_{1,0}(C_{k,d-1}) \longrightarrow H_{2,0}(L^o_{k,d,d-1}) \times H_{2,0}(X^o_{k,d-1})$ is injective by Lemma 5.13, hence we obtain from (8) that

$$h_{2,1}(X_{k,d}) = h_{2,1}(L^o_{k,d,d-1}) + h_{2,1}(X^o_{k,d-1}) \\ = h_{2,1}(X_{k,d-1}) + k - 1$$

and

$$\begin{aligned} h_{2,0}(X_{k,d}) &= h_{2,0}(X^o_{k,d-1}) \\ &= h_{2,0}(X_{k,d-1}) + g(C_{k,d-1}). \end{aligned}$$

The map $H_{0,1}(C_{k,d-1}) \longrightarrow H_{1,1}(L^o_{k,d,d-1}) \times H_{1,1}(X^o_{k,d-1})$ is injective by Lemma 5.13, hence we obtain from (9) that $h_{1,2}(X_{k,d}) = 0$ and

$$\begin{aligned} h_{1,1}(X_{k,d}) &= h_{1,1}(L^o_{k,d,d-1}) + h_{1,1}(X^o_{k,d-1}) \\ &= h_{1,1}(X_{k,d-1}) + k \cdot \big[d \cdot (d-1) + g(C_{1,d-1})\big] \\ &\qquad - (k-1) \cdot (2d-3) - 1 + g(C_{k,d-1}) \\ &= h_{1,1}(X_{k,d-1}) + k \cdot \big[d \cdot (d-1) + 2g(C_{1,d-1}) - 1\big] \\ &\qquad - 2(k-1) \cdot (d-2). \end{aligned}$$

With the exact same proof of Lemma 5.13, we obtain that the natural map $H_{2,1}(L_{k,d,d-1}) \to H_{1,0}(C_{k,d})$ is the zero map. Hence the map

$$H_{2,1}(X_{k,d}) = H_{2,1}(L_{k,d,d-1}) \times H_{2,1}(X_{k,d-1}) \longrightarrow H_{1,0}(C_{k,d})$$

is the zero map, since the above Mayer–Vietoris sequence also implies that the map $H_{2,1}(X_{k,d-1}) \to H_{1,0}(C_{k,d})$ is the zero map. □

5.4. Proof of Theorem 1.7. We prove the theorem by choosing $X = X_{k,d}$, and by computing its tropical homology groups recursively on d using Proposition 5.14. The theorem holds for $k = 1$ by [Sha1], and so for all numbers $h_{p,q}(X)$ with $(p,q) \neq (1,1)$. Since we have

$$h_{1,1}(X_{k,1}) = k \cdot h^{\mathbb{C}}_{1,1}(1,2) - (k-1),$$

we obtain

$$\begin{aligned} h_{1,1}(X) &= k \cdot h^{\mathbb{C}}_{1,1}(d,2) - (k-1) \cdot (d-1) \cdot (d-2) - (k-1) \\ &= h^{\mathbb{C}}_{1,1}(d,2) + \frac{(k-1) \cdot (d-1) \cdot (2d^2 - 7d + 9)}{3} \end{aligned}$$

as announced. □

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