

Push-forwards for Witt groups of schemes

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Abstract. Using suitable closed symmetric monoidal structures on derived categories of schemes, as well as adjunctions of the type $(L f^*, R f_*)$ and $(R f_*, f^!)$ (i.e. Grothendieck duality theory), we define push-forwards for coherent Witt groups along proper morphisms between separated noetherian schemes. We also establish fundamental theorems for these push-forwards (e.g. base change and projection formula) and provide some computations.

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Introduction

Push-forwards, also known as transfers or norm maps, exist for many cohomology theories over schemes, e.g. for K -theory, (higher) Chow groups and algebraic cobordism. They are undoubtedly a useful tool for understanding and computing those cohomology theories. The present article is about the construction of such push-forward maps for the coherent Witt groups of schemes defined in the seminal work of Balmer¹. A reader familiar with cohomology theories might think that constructing a push-forward is probably straightforward. He (or she) should be warned: Witt groups are not an oriented cohomology theory. In particular, push-forwards are, in some sense, only conditionally defined. For example, when X and Y are connected noetherian schemes of finite Krull dimension, smooth over a field, the Witt groups depend on a line bundle L used to define the duality and the push-forward takes the form (see Theorem 6.7)

$$\mathbb{W}^{i+\dim X}(X, \omega_X \otimes f^* L) \rightarrow \mathbb{W}^{i+\dim Y}(Y, \omega_Y \otimes L)$$

¹The modern definition of Witt groups using triangulated categories with dualities [2] can be applied either to the derived category of complexes of locally free sheaves to obtain “locally free” Witt groups or to the derived category of complexes with coherent cohomology to obtain “coherent” Witt groups. As with K -theory, it is the latter that is naturally covariant along proper morphisms, as we prove in this article. All schemes considered are over $\mathbb{Z}[1/2]$ so that the derived categories involved are $\mathbb{Z}[1/2]$ -linear and the machinery of triangular Witt groups applies.

where ω is the canonical bundle (the highest nontrivial exterior power of the cotangent bundle) and L is an arbitrary line bundle on Y . In particular, if we pick a line bundle K over X , there is no push-forward starting from $W^i(X, K)$ if K is not isomorphic to $\omega_X \otimes f^*L$ for some L (up to a square $M^{\otimes 2}$, as $W^i(Y, K) \cong W^i(Y, K \otimes M^{\otimes 2})$ so only the class of K in $\text{Pic}(X)/2$ really matters). This fundamental difference with oriented cohomology theories, where the push-forward is always possible, significantly changes classical computations, as one sees *e.g.* in [18], [39], [4]. The groups $W^{\dim X - i}(X, \omega_X)$ can be considered as a homology theory analogous to a non oriented complex homology theory in topology, but the construction of the push-forward here relies on triangulated monoidal methods.

Besides this article and its precursors on regular schemes [10] and [11], there are already several articles available on the construction of push-forwards in special cases. In [16], Gille defined push-forwards along finite morphisms in the affine case. His approach is quite elementary in the sense that he uses direct computations involving explicit injective resolutions etc. It is useful to get a hand on concrete forms. In [31] and [32], Nenashev adapts the oriented cohomology techniques of Panin and Smirnov to the non-oriented case of locally free Witt groups. He thus obtains push-forwards along projective morphisms between smooth quasi-projective varieties over fields. Still another approach using stable \mathbb{A}^1 -representability of Witt groups can be found in [24]. We understand that there is also some unpublished work of C. Walter on this subject. Our approach is different, and it applies to a much larger class of situations: it uses derived functors and Grothendieck duality, so the dualities that appear are canonical and do not depend on choices as the other constructions mentioned above. If necessary, choices can be made in order to compare our constructions with others in the special cases where the latter are defined. Fundamental properties such as base change are proved in a simple and conceptual way, and we furthermore obtain the full generality of singular schemes. An example of how those properties can be used for very concrete computations can be found in the computation of Balmer and the first author of the Witt group of Grassmann varieties [4].

Let us now explain why we use triangulated *monoidal* methods, even though there is no mention of a tensor product in the definition of Witt groups of triangulated categories. In fact, the proof of many results amounts to verifying that a certain number of diagrams of morphisms of functors such as (2) below are commutative. It might be possible to check this by hand in every concrete situation; however, it would be extremely painful: try it for example in the simple case of a regular closed immersion. Hence, some kind of systematic method is needed. Our solution to this problem is the use of a convenient setting involving a tensor product, an adjoint internal Hom, functors of the type f^* , f_* and $f^!$ and the adjunction relationships between them, that is some variant of the so-called Grothendieck six functors formalism in an arbitrary triangulated category. In this setting, we have shown in [12] that all the necessary diagrams commute, whereas this article exploits the existence of this structure on

various concrete triangulated categories. Here is a brief sketch of what is involved: Witt groups are defined for triangulated categories \mathcal{C} equipped with a duality, *i.e.* with a contravariant endofunctor D on \mathcal{C} together with a bidual isomorphism of functors $\text{Id} \rightarrow D^2 = D \circ D$ satisfying $D(\varpi_A) \circ \varpi_{DA} = \text{Id}_{DA}$ for all objects A of \mathcal{C} . A morphism between Witt groups is naturally induced by an exact functor $F : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ (both triangulated categories with dualities resp. (D_1, ϖ_1) and (D_2, ϖ_2)) equipped with an isomorphism of exact functors $\phi : FD_1 \rightarrow D_2F$ which explains how F “commutes” with the dualities. The fact that the isomorphism ϕ is not the identity requires the analysis of its interactions with the other morphisms of functors involved. It is the central problem to solve when proving the main theorems. To start with, this morphism ϕ should make the diagram

$$\begin{array}{ccc}
 F & \xrightarrow{F\varpi_1} & FD_1D_1 \\
 \varpi_2F \downarrow & & \downarrow \phi D_1 \\
 D_2D_2F & \xrightarrow{D_2\phi} & D_2FD_1
 \end{array} \tag{1}$$

commutative. In [12], we discuss such morphisms of functors and diagrams in the setting of closed symmetric monoidal categories. More precisely, let \mathcal{C}_1 and \mathcal{C}_2 be closed symmetric monoidal categories, with tensor product denoted by \otimes and internal Hom denoted by $[-, -]$. Given an object K , the functor $D_K := [-, K]$ together with the canonical natural transformation $\varpi_K : \text{Id} \rightarrow D_K^2$ defines a weak duality functor. Starting with an exact functor $f_* : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ which has a left adjoint f^* (which is monoidal) and a right adjoint $f^!$, there is a natural transformation

$$\zeta : f_*D_{f^!K} \rightarrow D_Kf_*$$

such that the diagram (1), which becomes

$$\begin{array}{ccc}
 f_* & \xrightarrow{f_*\varpi_{f^!K}} & f_*D_{f^!K}D_{f^!K} \\
 \varpi_Kf_* \downarrow & & \downarrow \zeta D_{f^!K} \\
 D_KD_Kf_* & \xrightarrow{D_{f^!K}\zeta} & D_{f^!K}f_*D_K
 \end{array} \tag{2}$$

commutes, as shown in [12]. Therefore, provided $\varpi_K, \varpi_{f^!K}$ and ζ are isomorphisms, f_* induces a morphism of Witt groups

$$W(\mathcal{C}_1, D_{f^!K}, \varpi_{f^!K}) \rightarrow W(\mathcal{C}_2, D_K, \varpi_K)$$

The present article is a description of how to apply this abstract closed monoidal setting to well-chosen derived categories of schemes, with the derived functors Lf^* , Rf_* and its right adjoint $f^!$ constructed by Grothendieck duality theory.

The main result of this article is the definition of a push-forward along a proper morphism $f : X \rightarrow Y$ of separated noetherian schemes. In its most general form (Theorem 4.4), this push-forward is a morphism

$$\mathbf{W}^i(X, f^!K) \xrightarrow{f_*} \mathbf{W}^i(Y, K)$$

where K is a dualizing complex on Y . This push-forward is induced by the derived functor Rf_* and a suitable morphism of functors $\zeta_K : Rf_*D_{f^!K} \rightarrow D_KRf_*$. This means that a form on a complex A for the duality $D_{f^!K}$ is sent to a form on the complex Rf_*A for the duality D_K . We further prove that this push-forward respects composition (Theorem 5.3).

Similarly, for morphisms of finite tor-dimension f we define a pull-back (Theorem 4.1), that is a morphism

$$\mathbf{W}^i(Y, K) \rightarrow \mathbf{W}^i(X, Lf^*K)$$

respecting composition (Theorem 5.2).

We also prove a flat base change theorem (5.5) relating the push-forward and the pull-back in cartesian diagrams and a projection formula in the case of regular schemes (Theorem 5.7). Some explicit computations of transfers are provided in the last section.

We assume that schemes are separated and noetherian for the following technical reasons: quasi-compact and separated are necessary to have an equivalence between the derived category of quasi-coherent sheaves $D(\text{Qcoh}(X))$ and the subcategory $D_{\text{qc}}(X)$ of complexes with quasi-coherent homology in the derived category of all sheaves. Noetherian is used to ensure that the injectives in $\text{Qcoh}(X)$ remain injective in the category of all \mathcal{O}_X -modules. Working without those assumptions would probably require significant improvements in the theory of Grothendieck duality; this is beyond the scope of this article which only intends to apply this theory to Witt groups.

Two main cases are discussed. The easier case is when all schemes considered are regular. Then their derived category $D_{\text{b,c}}$ of complexes with coherent and bounded homology is preserved under the derived tensor product \otimes^L and under RHom , the derived internal Hom. This endows $D_{\text{b,c}}$ with a natural structure of symmetric monoidal category. The dualizing complexes (see Definition 2.1) are line bundles or shifted line bundles. The coherent Witt groups are thus defined using the duality $\text{RHom}(-, L)$ for some line bundle L . Furthermore, the derived pull-back Lf^* for any morphism, the derived push-forward Rf_* and its right adjoint $f^!$ for proper morphisms also preserve $D_{\text{b,c}}$. Hence the abstract formalism of [12] applies on the nose, and we therefore obtain push-forwards and their classical properties of composition, base change and projection.

The general case, when schemes are not assumed to be regular, is more complicated. Indeed, in this case \otimes^L or RHom do not necessarily preserve bounded

homology and so there is no nice closed symmetric monoidal category structure on the category $D_{b,c}$ as the following affine example illustrates. Choose a field k and set $X = \text{Spec}(k[\epsilon]/\epsilon^2)$. Then consider the complex with k concentrated in degree zero. A projective resolution of k is given by

$$\cdots \xrightarrow{\cdot\epsilon} k[\epsilon]/\epsilon^2 \xrightarrow{\cdot\epsilon} k[\epsilon]/\epsilon^2 \xrightarrow{\cdot\epsilon} k[\epsilon]/\epsilon^2 \longrightarrow k \longrightarrow 0.$$

Thus $k \otimes^L k$ is the complex

$$\cdots \xrightarrow{0} k \xrightarrow{0} k \xrightarrow{0} k \longrightarrow 0$$

which has unbounded homology. On the other hand, the unbounded derived category D_{qc} of complexes with quasi-coherent cohomology admits a closed symmetric monoidal structure; this is not completely obvious, see Theorem 1.2. But this category is not suitable to define Witt groups, because there is no obvious (strong) duality on it and, anyway, as Eilenberg swindle type of arguments show for K -theory, unbounded categories are not the good framework to define cohomology theories. Still, the closed symmetric monoidal structure on D_{qc} is useful to prove systematically the commutativity of diagrams such as (1). That is, we can use the framework of [12] to prove this commutativity in the large closed symmetric monoidal category D_{qc} and then notice that all functors used in the definition of the duality ($\text{RHom}(-, K)$ for some suitable K) and the push-forward ($\text{R}f_*$) actually restrict to $D_{b,c}$ under mild additional assumptions. Thus, the commutativity of the diagrams involved is proved in large categories by general closed symmetric monoidal methods, but the diagrams actually often live in a smaller category whose Witt groups are interesting.

A technical point arising is the construction of the functors involved in the symmetric monoidal structure as well as $\text{L}f^*$, $\text{R}f_*$ and $f^!$ on the unbounded derived category D_{qc} . This relies on the work of Spaltenstein [35], the articles of Neeman [29] and [30], and on the very useful notes of Lipman [28], which are a reference on Grothendieck duality and contain very detailed explanations of all constructions.

The article is organized as follows. In Section 1, we recall the closed symmetric monoidal structures of the different categories we use. In Section 2, we use these structures to define triangulated categories with dualities and related Witt groups. Section 3 contains results on the derived functors $\text{L}f^*$ and $\text{R}f_*$ and on Grothendieck duality, *i.e.* the construction of the right adjoint $f^!$ of $\text{R}f_*$. Section 4 contains the main result of the paper, namely Theorem 4.4. We explain there explains how to use [12] to obtain the definition of push-forwards for the coherent Witt groups of schemes. It also contains a definition of the finite tor-dimension (*e.g.* flat) pull-back (Theorem 4.1). Section 5 explains the behavior of the push-forward and the pull-back under composition, and proves a base change formula relating them. Section 6 explains possible reformulations of the push-forward in different contexts and Section 7 studies in detail

the push-forward in the case of finite field extensions, regular embeddings and projective bundles, which is useful for computations and also allows a comparison with the transfer maps of other authors when they are defined. Everything except some specific computations in the last section works both for Grothendieck–Witt groups GW and Witt groups W . For simplicity, we stated all results for W only.

The present article is a generalization of the main results of the unpublished preprints [10] and [11] on regular schemes. To keep this article short, some applications established in [11] (dévissage/localization, Witt motives and partial results about their decomposition for cellular varieties) are not included here. Most important, all the abstract theorems about triangulated symmetric monoidal functors and adjunctions between them which are crucial for proving the theorems of this article are proven in the long article [12].

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1. Closed symmetric monoidal categories

Let Sch denote the category of separated noetherian schemes and $\mathcal{R}eg$ its full subcategory of regular schemes. For any scheme X , let $K(X)$ (resp. $D(X)$) denote the homotopy (resp. derived) category of homological complexes of \mathcal{O}_X -modules (without any restriction). We then add subscripts $+$ for bounded above, *i.e.* bounded where the differentials go, $-$ for bounded below, b for bounded, *i.e.* below and above, qc for quasi-coherent and c for coherent to characterise the derived categories of complexes whose *homology* is as the subscript. For example $D_{b,c}(X)$ is the derived category of complexes of \mathcal{O}_X -modules with coherent and bounded homology, and $D_{qc}(X)$ is the derived category of complexes with quasi-coherent homology. Note that we work with homological notation to be compatible with the literature on Witt groups, but it is easy to switch to cohomological notation by moving bounding subscripts to superscripts and exchanging $+$ and $-$ *i.e.* $D_+ = D^-$.

For any scheme X , the usual tensor product \otimes and internal Hom of complexes together with the obvious structure morphisms coming from the corresponding ones for sheaves turn $K(X)$ into a suspended closed symmetric monoidal category in the sense of Section 3 in [12]. This is completely classical and is detailed in [12], Appendix, where a discussion on sign choices can be found. In particular, we have a functor $T: K(X) \rightarrow K(X)$ given by $(TA)_n = T(A_{n-1})$.

Theorem 1.1. *Let X be a scheme.*

(1) *The tensor product on $K(X)$ admits a left derived functor*

$$\otimes^{\mathbb{L}}: D(X) \times D(X) \rightarrow D(X)$$

together with unit, associativity and symmetry morphisms.

(2) *It restricts to*

$$D_{\text{qc}}(X) \times D_{\text{qc}}(X) \rightarrow D_{\text{qc}}(X).$$

(3) *When $X \in \mathcal{R}eg$, it furthermore restricts to*

$$D_{\text{b,c}}(X) \times D_{\text{b,c}}(X) \rightarrow D_{\text{b,c}}(X).$$

(4) *The internal Hom on $K(X)$ has a right derived functor RHom*

$$D(X)^{\circ} \times D(X) \rightarrow D(X)$$

which is a right adjoint to the derived tensor product in the usual special sense (natural in the three variables).

(5) *When $X \in \mathcal{S}ch$, RHom restricts to*

$$D_{\text{b,c}}(X)^{\circ} \times D_{\text{b,c}}(X) \rightarrow D_{\text{c}}(X)$$

as the usual RHom (computed by replacing the second variable by a quasi-isomorphic complex of injectives in $\text{Qcoh}(X)$).

(6) *When $X \in \mathcal{R}eg$, this last restricted RHom arrives in $D_{\text{b,c}}(X)$.*

Proof. See Theorem A in [35] or [28], 2.5.7, for the existence of the derived tensor product. It is based on the existence of a q-flat (also called K-flat) resolution for any complex C , i.e. the existence of a quasi-isomorphism $Q_C \rightarrow C$ where Q_C is a complex such that $(- \otimes Q_C)$ preserves quasi-isomorphisms. These resolutions can even be constructed functorially (see [28], 2.5.5). The derived tensor product can then be constructed by taking q-flat resolutions of both variables. The former case is used to define the unit morphism and the latter case to define the associativity and symmetry morphisms directly from the ones of $K(X)$ (see [35], Theorem A, or [28], 2.5.9). See [28], 2.5.8, for the fact that $\otimes^{\mathbb{L}}$ restricts to D_{qc} . In the regular case, by Point (3) of Proposition A.4, we can replace any complex in $D_{\text{b,c}}(X)$ by a bounded complex of locally free sheaves, in which case the derived tensor product obviously maps to $D_{\text{b,c}}(X)$.

Similarly, the derived internal Hom is constructed using q-injective (also called K-injective) resolutions: see Section 1 in [35] for the definition of a q-injective complex

and [35], Theorem A, or [28], 2.4.5, for the existence of RHom . Adjointness is also stated in [35], Theorem A (see also [28], 2.6.1, for more details).

We now consider $\mathrm{RHom}(A, B)$ with $A, B \in D_{b,c}(X)$ for $X \in \mathcal{S}ch$. By Corollary A.6, the right derived functor RHom here is computed as the one in [23], Proposition II.3.3. This proves Point (5). In the regular case, we can compute RHom by a locally free resolution of the first variable and then, up to isomorphism, also replace the second variable by a complex of locally free sheaves. As explained above, both these complexes can be chosen to be bounded, and since $\mathcal{H}om(A, B)$ is coherent when A and B are [19], 5.3.5, this proves Point (6). \square

Now the subtle point is that $\mathrm{RHom}(M, N)$ is not necessarily an object in $D_{\mathrm{qc}}(X)$ when M and N are. To fix this, we use the *quasicoherator* $Q: \mathrm{Mod}(X) \rightarrow \mathrm{Qcoh}(X)$ as introduced in Lemma 3.2, p. 187 in [8], which is right adjoint to the inclusion $\mathrm{Qcoh}(X) \subset \mathrm{Mod}(X)$. On an affine space $X = \mathrm{Spec}(A)$, it takes a sheaf of \mathcal{O}_X -modules to the quasi-coherent sheaf associated to the A -module of its global sections by the tilde construction. Its right derived functor is denoted by $\mathrm{R}Q$, as considered in [1], Remark 0.4, [28], Exercises 4.2.3, or B.16 in [36]. It is a right adjoint to the inclusion $D_{\mathrm{qc}}(X) \subseteq D(X)$, and $A \cong \mathrm{R}Q(A)$ when $A \in D_{-, \mathrm{qc}}(X)$ (in particular for $A \in D_{b,c}(X)$) by [8], Exposé II, Proposition 3.5.2. An alternative construction of $\mathrm{R}Q$ can be obtained from Theorem 4.1 in [29].

Theorem 1.2. *Let X be a scheme.*

- (1) *The derived tensor product \otimes^{L} together with the obvious morphisms turns $D(X)$ into a symmetric monoidal category, closed by the RHom , and suspended in the sense of Section 3 in [12].*
- (2) *If $X \in \mathcal{S}ch$, the functor*

$$\mathrm{R}Q \circ \mathrm{RHom}: D_{\mathrm{qc}}(X)^o \times D_{\mathrm{qc}}(X) \rightarrow D_{\mathrm{qc}}(X)$$

is a right adjoint (in the usual special way, see [26], (v), p. 97) to the restricted tensor product \otimes^{L} on D_{qc} . This turns $D_{\mathrm{qc}}(X)$ into a suspended closed symmetric monoidal category.

- (3) *If $X \in \mathcal{R}eg$, the usual RHom is a right adjoint (in the usual special way) to the restricted tensor product \otimes^{L} on $D_{b,c}$. This turns $D_{b,c}(X)$ into a suspended closed symmetric monoidal category.*

Proof. The closed symmetric monoidal structure on $D(X)$ easily follows from Theorem 1.1. The fact that it is suspended follows, as explained in [12], Section 3, from the suspended bifunctor structure of RHom . The symmetric monoidal structure on $D_{\mathrm{qc}}(X)$ simply follows from the fact that \otimes^{L} restricts to it. The fact that it is closed

is a formal consequence of the fact that $D(X)$ is closed and that RQ is a right adjoint to the (monoidal) inclusion $\iota: D_{qc}(X) \subseteq D(X)$:

$$\begin{aligned} \text{Hom}_{qc}(A \otimes^L B, C) &= \text{Hom}(\iota(A \otimes^L B), \iota C) \simeq \text{Hom}(\iota A \otimes^L \iota B, \iota C) \\ &\simeq \text{Hom}(\iota A, \text{RHom}(\iota B, \iota C)) \simeq \text{Hom}_{qc}(A, RQ \text{RHom}(\iota A, \iota B)). \end{aligned}$$

(The closedness – that is the existence of the right adjoint to the derived tensor product – can also be deduced from Brown representability, in the spirit of the examples following [29], Theorem 4.1.) Point (3) follows from the same considerations, using Theorem 1.1 (3) and (6). \square

Notation 1.3. To shorten the notation, let $[-, -]$ denote the functor RHom , right adjoint to the tensor product on the derived category D and let $[-, -]'$ denote the functor $RQ \circ \text{RHom}$, right adjoint to the tensor product on the derived category D_{qc} .

Since the derived quasi-coherator is the identity on $D_{-,qc}$ (see above), if $[A, B] \in D_{-,qc}$ then $[A, B] \cong [A, B]'$.

We finish this section by pointing out a comment of Neeman: exploiting the fact that for $X \in Sch$, there are enough flat objects in D_{qc} and his representability result more extensively gives an alternative approach for constructing a closed symmetric monoidal structure on $D_{qc}(X)$ directly without passing through $D(X)$.

2. Witt groups

From now on, we assume that all schemes are defined over $\mathbb{Z}[1/2]$.

To define a Witt group, we need a strong duality on a triangulated category. Using the previous framework of triangulated closed symmetric monoidal categories, we recall how $[-, K]$ and $[-, K]'$ define dualities. The purpose of this section is to compare the restrictions of these dualities to the subcategory $D_{b,c}$ and to discuss when these dualities are strong on it. For any object K , let \sharp_K (resp. \sharp'_K) denote the contravariant exact functor $[-, K]$ (resp. $[-, K]'$).

Following [12], Section 3.2, applied to the closed symmetric monoidal structure on $D(X)$ with X an arbitrary scheme, we may define the *bidual morphism*

$$\varpi_K: \text{Id} \rightarrow \sharp_K \sharp'_K$$

as a morphism of triangulated endofunctors of $D(X)$. From Corollary 3.2 in [12], we obtain that $(D(X), \sharp_K, \varpi_K)$ is a triangulated category with weak duality (in the sense of [12], Definition 2.1.1, so ϖ_K is not necessarily an isomorphism). Similarly, when $X \in Sch$, we obtain a triangulated category with weak duality $(D_{qc}(X), \sharp'_K, \varpi'_K)$.

Definition 2.1. Let K be an object of $D_{\text{qc}}(X)$. It is a *dualizing complex* (or it is *dualizing*) if

- the functor $[-, K]$ preserves $D_{\text{b,c}}(X)$ and
- the bidual morphism ϖ_K is an isomorphism on $D_{\text{b,c}}(X)$.

If furthermore it has finite injective dimension, *i.e.* it is quasi-isomorphic to a finite complex of injectives, we say it is an *injectively bounded dualizing complex*.

In the terminology of Definition 2.1.1 in [12], the second condition says that \sharp_K is a strong duality on the subcategory $D_{\text{b,c}}(X)$.

Note that for any $X \in \text{Sch}$, a dualizing complex K is automatically in $D_{\text{b,c}}(X)$ since the natural morphism $K \rightarrow [\mathcal{O}_X, K]$ coming from the monoidal structure is an isomorphism and \mathcal{O}_X is coherent. In particular, our definition is exactly the “modern” (Definition 3.1, [30]) by Lemma 3.5 of *loc. cit.* Also note that our injectively bounded dualizing complexes are the “old” dualizing complexes of [23], V. §2.

Proposition 2.2. *Let $X \in \text{Sch}$ and $K \in D_{\text{qc}}(X)$ be a dualizing complex. Then the functors \sharp_K and \sharp'_K coincide and the bidual morphisms ϖ_K and ϖ'_K are equal on the subcategory $D_{\text{b,c}}(X)$.*

Proof. Since $[A, K] \in D_{\text{b,c}}(X)$ for any $A \in D_{\text{b,c}}(X)$, we have $[A, K]' \cong [A, K]$ by the remark after Notation 1.3 which proves that $\sharp'_K \cong \sharp_K$. The bidual morphisms are then equal by the large commutative diagram considered in the proof of Theorem 4.1.2 in [12], in which the f^* should be replaced by the inclusion $D_{\text{qc}}(X) \subset D(X)$, which is monoidal by definition of the tensor product on $D_{\text{qc}}(X)$. \square

Example 2.3. (1) A dualizing complex tensored by a shifted line bundle is still a dualizing complex. In fact, this is the only freedom: by Lemma 3.9 in [30] (see also [23], Theorem V.3.1, for the injectively bounded case), a dualizing complex is unique up to tensoring by shifted line bundles (the shift can be different on different connected component of X).

(2) On a Gorenstein scheme X (*e.g.* regular), \mathcal{O}_X itself is dualizing, so by the previous point, the only dualizing complexes are the shifted line bundles.

Note that on a regular scheme, the category $D_{\text{b,c}}(X)$ itself is closed symmetric monoidal. It follows that dualizing complexes are dualizing objects in the sense of Definition 3.2.2 in [12] in the category $D_{\text{b,c}}(X)$, for $X \in \text{Sch}$.

Theorem 2.4. *Let $X \in \text{Sch}$ and K be dualizing. Then $(D_{\text{b,c}}(X), \sharp_K, \varpi_K)$ is a triangulated category with strong duality in the sense of Definition 2.1.1 in [12]. Let it be denoted by \mathcal{C}_K and its Witt groups by $W^i(X, K)$, $i \in \mathbb{Z}$ ([12], Definition 2.1.5).*

Proof. The functor $\sharp_K = \sharp'_K$ is a contravariant endofunctor of $D_{b,c}(X)$ and $\varpi_K = \varpi'_K$ is an isomorphism on this category by definition of dualizing complexes by Proposition 2.2. The necessary commutative diagrams that \sharp'_K and ϖ'_K must satisfy simply follow from the fact that they are already satisfied in $D_{qc}(X)$ since $(D_{qc}(X), \sharp'_K, \varpi'_K)$ is a triangulated category with weak duality. \square

We may thus think of the triangulated category with duality $(D_{b,c}(X), \sharp_K, \varpi_K)$ as being restricted from $(D(X), \sharp_K, \varpi_K)$ or from $(D_{qc}(X), \sharp'_K, \varpi'_K)$, both structures coinciding on $D_{b,c}(X)$.

Remark 2.5. In [2], all dualities considered are strict, *i.e.* they strictly commute with the suspension, but this assumption is only there for simplicity. Instead, in Definition 2.1.1 of [12], we only assume commutativity up to a natural isomorphism, and all theorems in [2] are still true in this more general situation.

Remark 2.6. Recall (see e.g. [40], Definition 10.5.1) that for a left exact functor f between exact categories, the right derived functor really is a couple (Rf, s) with $s: qf \rightarrow (Rf)q$ and q the morphism from the homotopy category to the derived category. It is only the couple (Rf, s) which is unique up to unique isomorphism and therefore deserves being called *the* right derived functor, despite the standard abbreviated notation Rf . Consequently, the various derived functors, for example $R\mathrm{Hom}(-, K)$ (used to define the duality) and Rf_* (used below to define the push-forward) together with all the morphisms of functors defining the symmetric monoidal structure can be considered as abstract exact functors and morphisms of exact functors. With them, it is possible to define coherent Witt groups and push-forwards by the methods discussed in this article, since these methods only involve the abstract triangulated categories and functors, *i.e.* the framework of [12]. But as such, there is no uniqueness of all these constructions. It is only if we keep as extra data all the structural morphisms of the derived functors (the s part of the couples), and thus the relationship between the closed symmetric monoidal structure on $K(X)$ and the one on $D(X)$, that the whole derived construction becomes unique up to unique isomorphism, thus in particular the induced dualities, pull-backs and push-forwards.

3. The functors Lf^* , Rf_* and $f^!$

We now introduce the functors Lf^* , Rf_* and $f^!$ associated to a morphism of schemes f and explain how they behave with respect to the monoidal structures. The first two functors are derived functors, whereas the third one is right adjoint to Rf_* at the level of derived categories, but is not the derived functor of some underlying functor on the category of \mathcal{O}_X -modules. The construction of $f^!$ is the heart of Grothendieck duality theory, for which we refer the reader to [23], [37], [29], [13] or [28].

Proposition 3.1. *Let $f : X \rightarrow Y$ be a morphism of schemes.*

- (1) *The functor f^* admits a left derived functor $Lf^* : D(Y) \rightarrow D(X)$ which restricts to $D_{qc}(Y) \rightarrow D_{qc}(X)$.*
- (2) *If f is of finite tor-dimension (see e.g. Examples (2.7.6), [28]) or if $X, Y \in \mathcal{R}eg$, then Lf^* restricts to $D_{b,c}(Y) \rightarrow D_{b,c}(X)$.*

Proof. For existence, see [35], Theorem A (iii) or Proposition 6.7, or Example 2.7.3 in [28]. For the fact that it restricts to D_{qc} , see [28], 3.9.1. It restricts to $D_{b,c}$ in the finite tor-dimension case because Lf^* is then bounded and it respects the coherence of the cohomology by [23], Proposition II.4.4, bearing in mind Proposition A.7. The case $X, Y \in \mathcal{R}eg$ follows from Point (3) of Proposition A.4 and Proposition A.7. \square

Proposition 3.2. *The usual isomorphism $f^*(A \otimes B) \rightarrow f^*A \otimes f^*B$ induces an isomorphism of triangulated bifunctors (in the sense of [12], Definition 1.4.14)*

$$\alpha : Lf^*(- \otimes -) \rightarrow Lf^*(-) \otimes Lf^*(-)$$

which turns Lf^ into a suspended symmetric monoidal functor in the sense of [12], Section 4.*

Proof. See [35], Proposition 6.8. The morphism α is defined as the corresponding one on $K(X)$ after having replaced both variables by q-flat resolutions. It is already an isomorphism on $K(X)$. The commutative diagrams required (compatibility with the associativity, unit and symmetry of the monoidal structures) then easily follow from the corresponding ones on $K(X)$, using Proposition A.3, Points (1) and (3). \square

By Proposition 4.1.1 in [12] applied to the symmetric monoidal structure and Lf^* on $D(X)$, there is a natural morphism

$$\beta : Lf^*[-, -] \rightarrow [Lf^*(-), Lf^*(-)].$$

We also obtain a morphism

$$\beta' : Lf^*[-, -]' \rightarrow [Lf^*(-), Lf^*(-)]'.$$

using $D_{qc}(X)$ instead of $D(X)$.

Proposition 3.3. *Let $X, Y \in Sch$ and $A, B \in D_{qc}(Y)$. Assuming $[A, B] \in D_{qc}(Y)$ and $[Lf^*A, Lf^*B] \in D_{qc}(X)$, the morphisms $\beta_{A,B}$ and $\beta'_{A,B}$ coincide. In particular, when K and Lf^*K are dualizing, β_K and β'_K coincide.*

Proof. This follows from the commutative diagram

$$\begin{array}{ccc}
 Lf^*[A, B]' & \xrightarrow{\beta'} & [Lf^*A, Lf^*B]' \\
 \downarrow & & \downarrow \\
 Lf^*[A, B] & \xrightarrow{\beta} & [Lf^*A, Lf^*B]
 \end{array}$$

in which the vertical maps become identities under the assumptions. This diagram is formally obtained from the definitions of β and β' out of the closed monoidal structures. □

Proposition 3.4. *When $X, Y \in \text{Sch}$ and $f : X \rightarrow Y$ is of finite tor-dimension or when $X, Y \in \text{Reg}$ and for any $f : X \rightarrow Y$, the natural morphism β is an isomorphism on objects in $D_{b,c}$.*

Proof. This follows from [28], Proposition 4.6.6, for f of finite tor-dimension, the first variable coherent and the second in D_- , so in particular for both in $D_{b,c}$. Note that the ρ of loc. cit. coincides with our β by definition (compare [28], (3.5.4.5), and [12], Proposition 4.1.1). In the regular case, by Point (3) of Proposition A.4, we can assume our objects are bounded complexes of locally free sheaves, in which case the result follows from [28], Proposition 4.6.7. □

Proposition 3.5. *Let $f : X \rightarrow Y$ be a morphism of schemes.*

- (1) *The functor f_* admits a right derived functor $Rf_* : D(X) \rightarrow D(Y)$.*
- (2) *The functor Rf_* restricts to $D_{qc}(X) \rightarrow D_{qc}(Y)$ when f is quasi-compact and separated, in particular if X and Y are in Sch , see Corollary 6.1.10 in [22].*
- (3) *The functor Rf_* restricts to $D_{b,c}(X) \rightarrow D_{b,c}(Y)$ when f is proper and Y is quasi-compact.*

Proof. For existence, see Theorem A (iii) in [35] or Examples 2.7.3 in [28]. For the fact that it restricts to $D_{qc}(-)$ see [28], 3.9.2. In the proper case with Y quasi-compact, it restricts to $D_{b,qc}(-)$ by [28], 3.9.2, and it then preserves coherence of the cohomology by Theorem 3.2.1 in [21]. Note that we use Definition 5.3.1 in [19] to define coherent modules on non necessarily noetherian schemes. □

Proposition 3.6. *For any morphism f of schemes, the functor Rf_* is a right adjoint to Lf^* on $D(-)$ and consequently on all full subcategories to which both functors restrict.*

Proof. See see Theorem A (iii) in [35] or Proposition 3.2.1 in [28]. □

By Proposition 4.2.5, [12] applied to the monoidal structure and the functors on the categories $D(-)$, we obtain the projection morphism

$$\pi: \mathbf{R}f_*(-) \otimes - \rightarrow \mathbf{R}f_*(- \otimes \mathbf{L}f^*(-))$$

Theorem 3.7. *Let $f: X \rightarrow Y$ be quasi-compact and quasi-separated e.g. proper. Then the projection morphism π is an isomorphism on D_{qc} .*

Proof. This is Proposition 3.9.4 in [28]. □

Theorem 3.8. *For any separated morphism $f: X \rightarrow Y$ with X and Y separated and quasi-compact, the functor $\mathbf{R}f_*: D_{\text{qc}}(X) \rightarrow D_{\text{qc}}(Y)$ has a right adjoint $f^!$.*

Proof. See Example 4.2 in [29] and use that $D_{\text{qc}}(-)$ and $D(\text{Qcoh}(-))$ are equivalent for separated quasi-compact schemes by Proposition A.4 (1). □

Proposition 3.9. *Let $f: X \rightarrow Y$ be a proper morphism of separated noetherian schemes and let K be a dualizing complex on Y . Then $f^!K$ is a dualizing complex on X . If K is an injectively bounded dualizing complex i.e. dualizing in the sense of [23], V. §2, then $f^!K$ is injectively bounded too.*

Proof. For the case of injectively bounded complexes, see [23], V, §8, or Corollary 3, [37]. For the general case, we reproduce here a proof of Neeman. Since the question of whether $f^!K$ is dualizing is local on X , we may assume Y is affine and restrict to an affine open set U of X . As f is of finite type, we have a factorization $U \rightarrow \mathbb{A}^n \times Y \rightarrow Y$ for some n where the left arrow is a closed embedding. Taking the closure of U in $\mathbb{P}^n \times Y$, we see that U can be embedded as an open subset of a closed subset of some $Y \times \mathbb{P}^n$. Hence we only have to show that closed immersions, open immersions and projections $Y \times \mathbb{P}^n \rightarrow Y$ respect dualizing complexes. The case of closed immersions is done in [30], see Theorem 3.14, Remark 3.17 and Lemma 3.18; closed immersions are finite. The case of open immersions is Theorem 3.12 in [30]. For projective morphisms $f: \mathbb{P}_Y^n \rightarrow Y$, one uses that $\mathbf{R}\text{Hom}(A, f^!K) \cong \mathbf{R}\text{Hom}(A, f^!\mathcal{O} \otimes f^*K) \cong \mathbf{R}\text{Hom}(A, f^*K) \otimes f^!\mathcal{O}$, using Lemma 5.6 below and that $f^!\mathcal{O}$ is a shifted line bundle by [23], Section VII.4. Then one checks the conditions of Definition 2.1 on objects of the form f^*B and $\mathcal{O}(i)$ which by a theorem of Beilinson [7] generate $D_{\text{b,c}}(\mathbb{P}_Y^n)$ as a thick triangulated category. □

4. Pull-back and push-forward for Witt groups

We can now state the main result of this article: the definition of the push-forward for coherent Witt groups along proper morphisms (Theorem 4.4). This section also

contains a definition of the pull-backs for morphisms of finite tor-dimension (Theorem 4.1).

Let $f : X \rightarrow Y$ be a morphism of schemes. By [12], Theorem 4.1.2, applied to the monoidal categories $D(-)$, $\beta_K : Lf^* \#_K \rightarrow \#_{Lf^*K} Lf^*$ defines a duality preserving functor $\{Lf^*, \beta_K\}$ between triangulated categories with weak dualities, from $(D(Y), \#_K, \varpi_K)$ to $(D(X), \#_{Lf^*K}, \varpi_{Lf^*K})$, for any object K of $D_{qc}(Y)$.

Theorem 4.1. *Let $f : X \rightarrow Y$ be a morphism of schemes such that*

- *the objects K and Lf^*K are dualizing.*
- *Lf^* preserves $D_{b,c}$,*
- *β_K is an isomorphism on $D_{b,c}(Y)$,*

Then $\{Lf^, \beta_K\}$ induces a morphism on Witt groups,*

$$f^* : W^i(Y, K) \rightarrow W^i(X, Lf^*K),$$

*which we call pull-back. This pull-back therefore exists in particular if K and Lf^*K are dualizing and*

- *f is of finite tor-dimension and $X, Y \in Sch$ or*
- *for any f and $X, Y \in Reg$ in which case K dualizing implies Lf^*K dualizing by Example 2.3.*

Proof. This follows from Theorem 4.1.2 and Lemma 2.2.6 (1) in [12]. The theorem of loc. cit. ensures the existence of the appropriate commutative diagrams in $D(X)$. The requirements in the lemma of loc. cit. that the dualities given by K and f^*K restrict as strong dualities to $D_{b,c}$ are satisfied by assumption, and the requirement that β_K is an isomorphism when restricted to $D_{b,c}$ follows from Proposition 3.4. \square

Remark 4.2. Note that we obtain the very same pull-back when starting with the monoidal structure on D_{qc} instead of D . This follows from Proposition 3.3.

Remark 4.3. In Theorem 3.12 of [30] it is proved that if f is an open immersion, then Lf^*K is automatically dualizing if K is dualizing.

Let $X, Y \in Sch$, let $K \in D_{qc}(Y)$ and let $f : X \rightarrow Y$ be a separated morphism. From [12], Theorem 4.2.9, applied to the closed monoidal category $D_{qc}(X)$, we obtain a morphism of functors $\zeta_K : Rf_* \#_{f^!K} \rightarrow \#_K Rf_*$. By loc. cit., the pair $\{Rf_*, \zeta_K\}$ is duality preserving, *i.e.* Diagram (1) commutes.

Theorem 4.4. *Let $X, Y \in \text{Sch}$ and $f : X \rightarrow Y$ be a separated morphism such that Rf_* preserves $D_{b,c}$. Let K and $f^!K$ be dualizing. Then $\{Rf_*, \zeta_K\}$ induces a morphisms of Witt groups*

$$f_* : W^i(X, f^!K) \rightarrow W^i(Y, K)$$

which we call push-forward. This push-forward is therefore defined in particular if f is proper and K is dualizing (see Proposition 3.9).

Proof. This follows from Theorem 4.2.9 and Lemma 2.2.6.(1) in [12]. For the theorem of loc. cit., consider the triangulated closed monoidal category D_{qc} . The fact that ζ_K is an isomorphism follows from Proposition 4.3.3 in [12] using Theorem 3.7. Then, apply the Lemma of loc. cit. to the subcategories $D_{b,c}$, to which the dualities restrict by definition of a dualizing object. Note that when X and Y are regular, the complete proof works using directly $D_{b,c}$ as the triangulated closed monoidal category in Theorem 4.2.9 of [12]. □

5. Properties

We now show that both push-forwards and pull-backs respect composition and that they commute in an appropriate way (“base change”) provided certain standard conditions hold. We also prove a projection formula for regular schemes.

Theorem 5.1. *For any $f : X \rightarrow Y$ and $g : Y \rightarrow Z$,*

- (1) *there is an isomorphism $Lf^* \circ Lg^* \rightarrow L(g \circ f)^*$ between functors on $D(-)$ which is associative in the usual sense.*
- (2) *There is an isomorphism $R(g \circ f)_* \rightarrow Rg_* \circ Rf_*$ between functors on $D(-)$ which is associative in the usual sense, and respects the adjoint couple $(L(-)^*, R(-)_*)$ in the sense of Definition 5.1.5 of [12].*
- (3) *When the schemes are separated and quasi-compact, and both f and g are separated, there is an isomorphism $f^! \circ g^! \rightarrow (g \circ f)^!$ between functors on $D_{qc}(-)$ which is associative in the usual sense, and which respects the adjoint couple $(R(-)_*, (-)^!)$ in the sense of Definition 5.1.5 of [12].*

Proof. For the functors Lf^* on D , the isomorphism is in Theorem A (iii), [35] or [28], 3.6.4. For a proof that it is associative, see Scholium 3.6.10 in [28]. The other points follow from the first one by Lemma 5.1.6 in [12]. □

Theorem 5.2. *The pull-back respects composition: the diagram*

$$\begin{array}{ccccc}
 W^i(Z, K) & \xrightarrow{g^*} & W^i(Y, Lg^*K) & \xrightarrow{f^*} & W^i(X, Lf^*Lg^*K) \\
 & \searrow & & & \downarrow \wr \\
 & & & & W^i(X, L(gf)^*K)
 \end{array}$$

$(gf)^*$

commutes, under the conditions for the existence of the pull-backs f^* and g^* of Theorem 4.1.

Proof. This follows from [12], Theorem 5.1.3 and Corollary 5.1.4, applied to the structures on $D(-)$. □

Theorem 5.3. *The push-forward respects composition: the diagram*

$$\begin{array}{ccccc}
 W^i(X, f^!g^!K) & \xrightarrow{f_*} & W^i(Y, g^!K) & \xrightarrow{f_*} & W^i(Z, K) \\
 \wr \downarrow & & & \nearrow & \\
 W^i(X, (gf)^!K) & & & &
 \end{array}$$

$(gf)_*$

commutes, under the conditions for the existence of the push-forward of Theorem 4.4.

Proof. This follows from [12], Theorem 5.1.9 and Corollary 5.1.10, applied to the structures on $D_{qc}(-)$. □

We now prove a base change formula. Let us consider a pull-back diagram

$$\begin{array}{ccc}
 V & \xrightarrow{\bar{g}} & Y \\
 \bar{f} \downarrow & & \downarrow f \\
 X & \xrightarrow{g} & Z.
 \end{array}$$

By Section 5.2, [12], we obtain a morphism of functors

$$\varepsilon: Lf^*Rg_* \rightarrow R\bar{g}_*L\bar{f}^*$$

between functors on $D(X)$.

Proposition 5.4. *If all schemes are in Sch and the diagram is tor-independent, e.g. f flat, the morphism ε is an isomorphism on $D_{qc}(X)$.*

Proof. The case where f is flat is Proposition 3.9.5 in [28] (all maps between schemes in Sch are “concentrated” in the sense of loc. cit.). The more general case is Theorem 3.10.3 in [28]. \square

Then, when the schemes are in Sch , still by Section 5.2 of [12], ε induces a morphism

$$\gamma: L\bar{f}^*g^! \rightarrow \bar{g}^!Lf^*$$

between functors on $D_{qc}(X)$. It is an isomorphism on the subcategory $D_{-,qc}$ by Corollary 4.4.3, [28]. In particular, γ_K is an isomorphism when K is dualizing (and thus in $D_{b,c}(Z)$).

Theorem 5.5 (Base change). *Under the assumptions of Proposition 5.4 and the ones for the pull-backs along f and \bar{f} and the push-forwards along g and \bar{g} to exist (Theorems 4.1 and 4.4), the pull-back and push-forward satisfy a base change formula: the diagram*

$$\begin{array}{ccc}
 & W^i(V, \bar{g}^!Lf^*K) & \xrightarrow{\bar{g}^*} W^i(Y, Lf^*K) \\
 & \nearrow \gamma & \uparrow f^* \\
 W^i(V, L\bar{f}^*g^!K) & & \\
 \bar{f}^* \uparrow & & \\
 W^i(X, g^!K) & \xrightarrow{g_*} & W^i(Z, K)
 \end{array}$$

commutes.

Proof. This follows from [12], Theorem 5.2.1 and Corollary 5.2.2, applied to the structures on $D_{qc}(-)$, keeping in mind Remark 4.2. \square

We conclude this section with a projection formula for Witt groups, in the case of regular schemes. For this, we first need to introduce another natural morphism that will anyway be of some use even in the case of non regular schemes.

When $f: X \rightarrow Y$ is a separated morphism in Sch , using the functors Lf^* (monoidal), Rf_* and $f^!$ between the categories $D_{qc}(X)$ and $D_{qc}(Y)$, and the fact that the projection morphism π is an isomorphism by Theorem 3.7, we obtain a morphism of functors

$$\theta: f^!(-) \otimes^L Lf^*(-) \rightarrow f^!(- \otimes^L -)$$

by Proposition 4.3.1 in [12].

Lemma 5.6. *The morphism $\theta: f^!A \otimes^L Lf^*B \rightarrow f^!(A \otimes^L B)$ is an isomorphism when B is a perfect complex.*

Proof. The morphism θ is compatible with open immersions by Diagram 41 of Proposition 5.2.5 in [12], and so we can restrict to the case of bounded complexes of vector bundles, then to vector bundles, then again using open immersions to the trivial bundle \mathcal{O}_Y . In that case, one can show that $\theta_{A, \mathcal{O}_Y}$ coincides with the unit morphism of the monoidal structure $f^!(A) \otimes^L \mathcal{O}_X \rightarrow f^!(A)$, and is therefore an isomorphism. By this coincidence we mean that the left diagram

$$\begin{array}{ccc}
 f^!A \otimes^L Lf^*\mathcal{O}_Y \xrightarrow{\theta} f^!(A \otimes^L \mathcal{O}_Y) & Rf_*B \otimes^L \mathcal{O}_Y \xrightarrow{\pi} Rf_*(B \otimes^L Lf^*\mathcal{O}_Y) \\
 \wr \Big| & \wr \Big| & \wr \Big| & \wr \Big| \\
 f^!A \otimes^L \mathcal{O}_X \xrightarrow{\sim} f^!A & Rf_*B \xrightarrow{\sim} Rf_*(B \otimes^L \mathcal{O}_X)
 \end{array}$$

is commutative, in which the left vertical morphism is the fact that Lf^* is monoidal and in particular respects units, and the bottom and right maps are unit morphisms of the monoidal structures. By following the definition of θ given in Proposition 4.3.1 of [12] the commutativity of the left diagram follows from the one of the right hand side, which is in turn implied, using the definition of π in Proposition 4.2.5 of [12], by the compatibility of the unit and monoidal structure morphisms for Lf^* . \square

For any scheme X in $\mathcal{R}eg$, the derived tensor product preserves $D_{b,c}(X)$ (Theorem 1.1 (3)). This gives two different products on Witt groups by the formalism of [18], using Proposition 4.4.6 and Corollary 4.4.7 in [12] applied to the closed monoidal structure of $D_{b,c}(X)$. We fix one of these products (say, the left one) for the following results. When K and L are shifted line bundles, thus dualizing complexes, the product is a pairing

$$W^i(X, K) \times W^j(X, L) \rightarrow W^{i+j}(X, K \otimes L).$$

Theorem 5.7 (Projection formula). *For any proper morphism $f : X \rightarrow Y$ with $X, Y \in \mathcal{R}eg$, the pull-back and push-forward satisfy a projection formula: If K, L are shifted line bundles on Y , $x \in W^i(X, f^!K)$ and $y \in W^j(Y, L)$, then $f_*(I(x.f^*y)) = f_*(x).y$ in $W^{i+j}(Y, K \otimes L)$ with I the isomorphism from $W^i(X, f^!K \otimes f^*L)$ to $W^i(X, f^!(K \otimes L))$ induced by $\theta_{K,L}$.*

Proof. First note that L being a shifted line bundle explains the absence of derivations in the pull-back and tensor products above. Then, the morphism $\theta_{K,L}$ is an isomorphism by Lemma 5.6, thus the result follows from [12], Theorem 5.5.1 and Corollary 5.5.2, applied to the closed monoidal structure on $D_{b,c}$. \square

6. Reformulations in special cases

In this section, we give other canonical ways of writing the push-forward, under additional assumptions.

Notation 6.1. For an equidimensional morphism $f : X \rightarrow Y$ of relative dimension n , let ω_f denote the object $f^!(\mathcal{O}_Y)[-n] := T^{-n} f^!(\mathcal{O}_Y)$.

This notation is motivated by the fact that in several cases, this object can be identified with a geometric object called a relative dualizing sheaf and usually denoted ω_f : see Sections 7.2 and 7.3 for the examples of regular embeddings and projective spaces.

6.1. Relative dualizing sheaf

Theorem 6.2. *Let $f : X \rightarrow Y$ be a proper morphism in Sch , K a dualizing complex on Y such that $f^!K$ is a dualizing complex and assume that $\theta_{\mathcal{O}_Y, K} : f^!\mathcal{O}_Y \otimes^L Lf^*K \rightarrow f^!K$ is an isomorphism. Then we can rewrite the push-forward of Theorem 4.4 as*

$$f_* : W^{i+d}(X, \omega_f \otimes^L Lf^*K) \rightarrow W^i(Y, K).$$

In particular, the hypotheses and therefore the conclusion hold if K is dualizing and either of the two following conditions hold.

- f is quasi-perfect (see below, e.g. of finite tor-dimension) and $f^!K$ is dualizing.
- Y is a Gorenstein scheme, e.g. regular.

Proof. The reformulation of the push-forward is Definitions 6.1.3 and 6.1.4, [12]. When f is quasi-perfect, Proposition 2.1 in [27] shows that θ is an isomorphism on all $D_{qc}(Y)$. Example 2.2 in loc. cit. shows that if f is of finite tor-dimension, it is quasi-perfect. When Y is Gorenstein, the only dualizing complexes are shifted line bundles, for which θ is an isomorphism by Lemma 5.6. □

Let $g : Y \rightarrow Z$ be another proper morphisms in Sch and M a dualizing complex on Z and let

$$\iota_{f,g} : \omega_f \otimes^L Lf^*(\omega_g \otimes^L Lg^*M) \rightarrow \omega_{gf} \otimes^L L(gf)^*M$$

be the morphism defined in Theorem 6.1.5 of [12].

Theorem 6.3. *The push-forward of Theorem 6.2 respects composition: the morphism $\iota_{f,g}$ is an isomorphism and if I denotes the isomorphism of Witt groups induced by $\iota_{f,g}$, then the push-forward on Witt groups defined above satisfies that $g_* f_* = (gf)_* I$.*

Proof. This follows from Theorem 6.1.5 and Lemma 2.2.6 (2) in [12]. Note that $\iota_{f,g}$ is an isomorphism because it is a composition of isomorphisms under the assumptions for the reformulated push-forward to exist. □

Theorem 6.4. *In the situation of Theorem 5.5 and under the assumptions of the reformulation of the push-forward above for the morphisms g and \bar{g} , the base change theorem 5.5 becomes*

$$f^* g_* = \bar{g}_* I \bar{f}_*$$

where I is the isomorphism of Witt groups induced by the isomorphism

$$\iota: L\bar{f}^*(\omega_g \otimes^L Lg^*K) \rightarrow \omega_{\bar{g}} \otimes^L L\bar{g}^*L f^*K.$$

Proof. This follows from Theorem 6.1.7 in [12]. Note that $\gamma_{\mathcal{O}_Z}$ is an isomorphism (see before Theorem 5.5). □

In the regular case, the projection formula 5.7 becomes the following.

Theorem 6.5. *Let $f: X \rightarrow Y$ be a proper equidimensional morphism of relative dimension d with $X, Y \in \mathcal{R}eg$. Then the push-forward of Theorem 6.2 and the pull-back of Theorem 4.1 satisfy $f_*I(x.f^*(y)) = f_*(x).y$ in $W^{i+j}(Y, L \otimes K)$ for any $x \in W^{i+d}(X, \omega_f \otimes f^*L)$ and $y \in W^j(Y, K)$.*

Proof. See [12], Theorem 6.1.9 and Corollary 6.1.10. □

6.2. Smooth schemes over a base. We now fix a base scheme $S \in \mathcal{S}ch$ with a dualizing complex K_S and consider the category $SmPr/S$ of schemes in $\mathcal{S}ch$ that are smooth, equidimensional and proper over S . For such a scheme X , let the structural morphism be denoted by $p_X: X \rightarrow S$ and its relative dimension over S by d_X . Note that any separated morphism between schemes in $SmPr/S$ is proper, being the composition of a closed embedding, its graph, and a proper projection.

Notation 6.6. Let $X \in SmPr/S$. We set $\omega_X = p_X^!(K_S)[-d_X]$. Observe that $\omega_X = \omega_{p_X}$ if $K_S = \mathcal{O}_S$.

Theorem 6.7. *Let $f: X \rightarrow Y$ be a separated morphism, $X, Y \in SmPr/S$ and let L be a line bundle on Y . The push-forward can be written*

$$f_*: W^{i+d_X}(X, \omega_X \otimes f^*L) \rightarrow W^{i+d_Y}(Y, \omega_Y \otimes L)$$

Proof. First, let us note that when pulling back or tensoring by a line bundle, there is nothing to derive. This is why no L appear in front of f^* and \otimes . We then use the definitions in [12] (Definitions 6.3.3 and 6.3.4). We need to check that the morphism

$$\omega_X \otimes L \simeq f^!\omega_Y \otimes f^*L \rightarrow f^!(\omega_Y \otimes L)$$

is an isomorphism. This is the case by Lemma 5.6. □

Theorem 6.8. *The push-forward of Theorem 6.7 respects composition.*

Proof. See [12], Theorem 6.3.5 and Corollary 6.3.6. □

Theorem 6.9. *The push-forward of Theorem 6.7 satisfies flat base change.*

Proof. See [12], Theorem 6.3.7 and Corollary 6.3.8. □

7. Examples

Note that $f^!$ is unique up to unique isomorphism whenever it is defined, because it is always defined as a right adjoint to Rf_* . This allows us to use computations of $f^!$ from [23] and other sources in the examples below.

7.1. Finite field extensions. The simplest example of a proper morphism is the case of a finite field extension E/F giving rise to a finite morphism

$$f : X = \text{Spec}(E) \rightarrow \text{Spec}(F) = Y.$$

The tilde construction gives equivalences of categories $\text{Mod}(F) \simeq \text{Qcoh}(Y)$ and $\text{Mod}(E) \simeq \text{Qcoh}(X)$, and the subcategories of finite dimensional vector spaces correspond to coherent sheaves of modules. We thus describe all objects and functors through these equivalences of categories. The only dualizing complex (up to shifts and isomorphisms) on Y is F itself. The functors $f^* = (- \otimes_F E)$ and $f_* = (-)|_F$ are exact, there is nothing to derive. The functor $f^!$ is given by III, §6, of [23] as $\mathcal{H}om_F(E, -)$ (mapping to E -vector spaces) and the unit and counit of the adjunction $(f_*, f^!)$ are respectively given by

$$\begin{aligned} V &\rightarrow \mathcal{H}om_F(E, V|_F), & \mathcal{H}om_F(E, V')|_F &\rightarrow V', \\ a &\mapsto (e \mapsto e.a), & \phi &\mapsto \phi(1), \end{aligned}$$

for an E -vector space V and an F -vector space V' . For fields, the only nonzero Witt group modulo 4 is W^0 which is the classical Witt group of the field. So we are reduced to study push-forward for forms on vector spaces, *i.e.* complexes concentrated in degree zero. Following the construction, it is easy to check that for any E -vector space V , the morphism $\zeta : f_*[V, f^!F] \rightarrow [f_*V, F]$ coincides with the Cartan isomorphism of F -vector spaces

$$\mathcal{H}om_E(V, \mathcal{H}om_F(E, F))|_F \simeq \mathcal{H}om_F(V|_F, F)$$

which sends a morphism $\phi : V \rightarrow \mathcal{H}om_F(E, F)$ to the morphism $(a \mapsto \phi(a)(1))$. Thus, the push-forward $f_* : W^0(E, \mathcal{H}om_F(E, F)) \rightarrow W^0(F)$ is a Scharlau transfer

(see [34], p. 48) with respect to the usual trace $\text{Tr}: E \rightarrow F$. To see this, note that Tr factors as $E \xrightarrow{\cong} \mathcal{H}om_F(E, F) \rightarrow F$ where the isomorphism is given by $e \mapsto (x \mapsto \text{Tr}(e.x))$ and $\mathcal{H}om_F(E, F) \rightarrow F$ is the evaluation at 1.

7.2. Regular embeddings. Let F be a vector bundle of rank $d > 0$ over X with a regular section $s: \mathcal{O}_X \rightarrow F$, i.e. such that the corresponding embedding $f: Z \subset X$ of the zero locus is a closed regular embedding of codimension d . In this case the augmented Koszul resolution is exact, see Proposition 3.1 (IV, §2) in [14], and thus yields a quasi-isomorphism

$$\begin{array}{ccccccc}
 \mathcal{K}_F & = & (0 \longrightarrow \Lambda^d F^\vee \longrightarrow \Lambda^{d-1} F^\vee \longrightarrow \cdots \longrightarrow F^\vee \longrightarrow \mathcal{O}_X \longrightarrow 0) \\
 \text{qis} \downarrow & & & & & & \downarrow \\
 f_* \mathcal{O}_Z & = & (0 \longrightarrow f_* \mathcal{O}_Z \longrightarrow 0)
 \end{array} \tag{3}$$

from the Koszul complex \mathcal{K}_F to $f_* \mathcal{O}_Z$ concentrated in degree 0. Since f is a closed embedding, thus finite, f_* is exact and coincides with Rf_* . Let $\Delta_F = \Lambda^d F$ be the determinant of F . In this situation, we have $f^! A = f^* \Delta_F[-d] \otimes Lf^* A$ for all $A \in D_{b,c}(X)$; this may be extracted from III, §7, [23], see also Proposition 1 in [37], after applying Lemma 5.6 and using that F is dual to the cotangent sheaf. By tensoring the augmented Koszul resolution with Δ_F and using the canonical isomorphisms $\Lambda^i F^\vee \otimes \Delta_F \cong \Lambda^{d-i} F$ and $f_* \mathcal{O}_Z \otimes \Delta_F \cong f_* f^* \Delta_F$, we obtain the trace map $f_* f^! \mathcal{O}_X \rightarrow \mathcal{O}_X$ (counit of the adjunction $(Rf_*, f^!)$) in the derived category as the composition of a usual map of complexes followed by the inverse of a quasi-isomorphism (\mathcal{O}_X is in degree 0 and $f_* f^* \Delta_F$ in degree $-d$):

$$\begin{array}{ccccccc}
 f_* f^! \mathcal{O}_X & = & (0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow f_* f^* \Delta_F \longrightarrow 0) \\
 \downarrow & & & & & & \downarrow \text{id} \\
 & & (0 \longrightarrow F \longrightarrow \cdots \longrightarrow \Delta_F \longrightarrow f_* f^* \Delta_F \longrightarrow 0) \\
 & & \uparrow s & & & & \\
 \mathcal{O}_X & & (0 \longrightarrow \mathcal{O}_X \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow 0)
 \end{array}$$

Now assume Z is Gorenstein. Then \mathcal{O}_Z is dualizing, and the isomorphism $\mathcal{O}_Z \rightarrow [\mathcal{O}_Z, \mathcal{O}_Z]$ adjoint to $\mathcal{O}_Z \otimes^\mathbb{L} \mathcal{O}_Z \simeq \mathcal{O}_Z$ defines a form on \mathcal{O}_Z , denoted by $\mathbf{1}_Z$. On the other hand, there is a well-known form $\theta_F: \mathcal{K}_F \rightarrow \mathcal{H}om(\mathcal{K}_F, \Delta_F^\vee[d])$ (see §4 in [6]) given by the canonical isomorphism $\Lambda^i F^\vee \simeq (\Lambda^{d-i} F^\vee)^\vee \otimes \Lambda^d F^\vee$ in degree i , with a sign chosen so that when $F = \oplus L_i$ is a direct sum of line bundles, this

form is the tensor product of forms θ_{L_i} on Koszul complexes of length one

$$\begin{array}{ccc}
 \mathcal{K}_{L_i} & = & (0 \longrightarrow L_i^\vee \xrightarrow{s_i^\vee} \mathcal{O}_X \longrightarrow 0) \\
 \theta_{L_i} \downarrow & & \begin{array}{ccc} & -1 \downarrow & \downarrow 1 \\ & L_i^\vee & \mathcal{O}_X \end{array} \\
 \mathcal{H}om(\mathcal{K}_{L_i}, L_i^\vee[1]) & = & (0 \longrightarrow L_i^\vee \xrightarrow{-s_i^\vee} \mathcal{O}_X \longrightarrow 0)
 \end{array} \tag{4}$$

representing elements in $W^1(X, L_i^\vee)$. The following proposition can be considered as a concrete description of the push-forward of $\mathbf{1}_Z$ along f .

Proposition 7.1. *Let Z and X be Gorenstein schemes and $f : Z \rightarrow X$ be a closed regular embedding of codimension d defined as the zero locus of a regular section of a vector bundle F of rank d whose determinant $\Lambda^d F$ is denoted by Δ_F . Then the image of the form $\mathbf{1}_Z : \mathcal{O}_Z \xrightarrow{\cong} [\mathcal{O}_Z, \mathcal{O}_Z]$ (adjoint to $\mathcal{O}_Z \otimes^L \mathcal{O}_Z \simeq \mathcal{O}_Z$) under the composition*

$$W^0(Z, \mathcal{O}_Z) \simeq W^d(Z, f^! \Delta_F^\vee) \xrightarrow{f_*} W^d(X, \Delta_F^\vee)$$

is a form ϕ such that the following diagram in $D(X)$ commutes.

$$\begin{array}{ccc}
 \mathcal{K}_F & \xrightarrow[\cong]{qis} & f_* \mathcal{O}_Z \\
 \theta \downarrow & & \downarrow \phi \\
 \mathcal{H}om(\mathcal{K}_F, \Delta_F^\vee[d]) & \xrightarrow[\cong]{qis} & [f_* \mathcal{O}_Z, \Delta_F^\vee[d]]
 \end{array}$$

Proof. Let $\delta_F : \mathcal{K}_F \otimes \mathcal{K}_F \rightarrow \mathcal{K}_F$ in $D(X)$ be the composition

$$\mathcal{K}_F \otimes \mathcal{K}_F \xrightarrow{qis \otimes qis} f_* \mathcal{O}_Z \otimes^L f_* \mathcal{O}_Z \xrightarrow{\lambda} f_*(\mathcal{O}_Z \otimes^L \mathcal{O}_Z) \xrightarrow{\cong} f_* \mathcal{O}_Z \xrightarrow{qis^{-1}} \mathcal{K}_F$$

where λ is the morphism from Proposition 4.2.1 in [12]. Note that δ_F is in fact represented by a morphism of complexes (not just a fraction): one can check that the map from $\mathcal{K}_F \otimes \mathcal{K}_F$ to \mathcal{K}_F in degree i is a sum of the canonical morphisms $\Lambda^k F \otimes \Lambda^{i-k} F \rightarrow \Lambda^i F$ with appropriate signs. We also consider the map $\sigma_F : \mathcal{K}_F \rightarrow \Delta_F^\vee[d]$ given by

$$\begin{array}{ccc}
 \mathcal{K}_F & = & (0 \longrightarrow \Lambda^d F^\vee \longrightarrow \Lambda^{d-1} F^\vee \longrightarrow \cdots \longrightarrow F^\vee \longrightarrow \mathcal{O}_X \longrightarrow 0) \\
 \sigma_F \downarrow & & \parallel \\
 \Delta_F^\vee[d] & = & (0 \longrightarrow \Delta_F^\vee \longrightarrow 0)
 \end{array}$$

The following three lemmas together clearly imply the proposition.

Lemma 7.2. *The morphism of complexes $x_F: \mathcal{K}_F \rightarrow \mathcal{H}om(\mathcal{K}_F, \Delta_F^\vee[d])$ defined as the composition*

$$\mathcal{K}_F \rightarrow \mathcal{H}om(\mathcal{K}_F, \mathcal{K}_F \otimes \mathcal{K}_F) \xrightarrow{(\delta_F)^*} \mathcal{H}om(\mathcal{K}_F, \mathcal{K}_F) \xrightarrow{(\sigma_F)^*} \mathcal{H}om(\mathcal{K}_F, \Delta_F^\vee[d])$$

coincides with the form θ where the first map is the unit of the adjunction of the tensor product and the internal Hom in the homotopy category.

Lemma 7.3. *The form $\phi: f_*\mathcal{O}_Z \rightarrow [f_*\mathcal{O}_Z, \Delta_F^\vee[d]]$ coincides with the composition*

$$\begin{array}{ccc} f_*\mathcal{O}_Z & \longrightarrow & [f_*\mathcal{O}_Z, f_*\mathcal{O}_Z \otimes^L f_*\mathcal{O}_Z] \xrightarrow{\lambda} [f_*\mathcal{O}_Z, f_*(\mathcal{O}_Z \otimes^L \mathcal{O}_Z)] \\ & & \downarrow \wr \\ [f_*\mathcal{O}_Z, \Delta_F^\vee[d]] & \longleftarrow & [f_*\mathcal{O}_Z, f_*f^!\Delta_F^\vee[d]] \xleftarrow{\simeq} [f_*\mathcal{O}_Z, f_*\mathcal{O}_Z] \end{array}$$

where the first map is the unit of the adjunction of the tensor product and the internal Hom in $D(X)$, the penultimate one is the identification $\mathcal{O}_Z \simeq f^!\Delta_F^\vee[d]$ and the last one is induced by the counit of the adjunction $(f_, f^!)$, i.e. the trace map described above.*

Lemma 7.4. *The composition of Lemma 7.2 coincides with the one of Lemma 7.3 when $f_*\mathcal{O}_Z$ is identified with \mathcal{K}_F using qis .*

Proof of Lemma 7.2. As we are dealing with honest morphisms of complexes we may first reduce to open subsets on which F is a sum of line bundles L_i (note that two morphisms in $D(X)$ are not necessarily equal if they are equal when restricted to all open sets of an affine covering, see for example [5]). We then reduce to the case of codimension $d = 1$, by multiplicativity of Koszul complexes: let $d = d_1 + d_2$ and let $F = F_1 \oplus F_2$ where F_1 (resp. F_2) is the sum of the first d_1 line bundles (resp. last d_2). Let $f_1: \mathcal{O}_{Z_1} \rightarrow X$ (resp. $f_2: \mathcal{O}_{Z_2} \rightarrow X$) be the corresponding regular subschemes. Then $\Delta_F \simeq \Delta_{F_1} \otimes \Delta_{F_2}$ and $\mathcal{K}_F \simeq \mathcal{K}_{F_1} \otimes \mathcal{K}_{F_2}$. We leave it to the reader to show that the diagram

$$\begin{array}{ccccc} \mathcal{K}_{F_1} \otimes \mathcal{K}_{F_2} & \xrightarrow{\simeq} & \mathcal{K}_F & \xrightarrow{x_F} & \mathcal{H}om(\mathcal{K}_F, \Delta_F^\vee[d]) \\ \downarrow x_{F_1} \otimes x_{F_2} & & & & \downarrow \simeq \\ \mathcal{H}om(\mathcal{K}_{F_1}, \Delta_1^\vee[d_1]) \otimes \mathcal{H}om(\mathcal{K}_{F_2}, \Delta_2^\vee[d_2]) & \xrightarrow{\tau} & \mathcal{H}om(\mathcal{K}_{F_1} \otimes \mathcal{K}_{F_2}, \Delta_1^\vee[d_1] \otimes \Delta_2^\vee[d_2]) & & \end{array}$$

commutes, where τ is the morphism defined as in [12] (Definition 4.4.1), using the monoidal structure on the homotopy category. By definition, θ_F, θ_{F_1} and θ_{F_2} make the same diagram commutative when they replace x_F, x_{F_1} and x_{F_2} . Hence it suffices to show the lemma for one line bundle L and its associated Koszul complex of length one, which can be checked by hand. □

Proof of Lemma 7.3. By definition, the form ϕ is given by the composition

$$f_*\mathcal{O}_Z \xrightarrow{f_*\mathbf{1}_Z} f_*[\mathcal{O}_Z, \mathcal{O}_Z] \xrightarrow{\cong} f_*[\mathcal{O}_Z, f^!\Delta_F^\vee[d]] \xrightarrow{\zeta} [f_*\mathcal{O}_Z, \Delta_F^\vee[d]].$$

One proves using the closed monoidal structure that it coincides with the composition

$$f_*\mathcal{O}_Z \xrightarrow{\cong} f_*f^!\Delta_F^\vee[d] \longrightarrow f_*[\mathcal{O}_Z, f^!\Delta_F^\vee[d]] \xrightarrow{\zeta} [f_*\mathcal{O}_Z, \Delta_F^\vee[d]]$$

where the second map is adjoint to the unit morphism of the monoidal structure. Then, looking back at the definition of ζ and μ in Proposition 4.2.2 and Theorem 4.2.9 of [12], one sees that ϕ is the composition around the lower left corner of the commutative diagram

$$\begin{array}{ccccccc}
 f_*\mathcal{O}_Z & \longrightarrow & [f_*\mathcal{O}_Z, f_*\mathcal{O}_Z \otimes^L f_*\mathcal{O}_Z] & \longrightarrow & [f_*\mathcal{O}_Z, f_*(\mathcal{O}_Z \otimes^L \mathcal{O}_Z)] & \longrightarrow & [f_*\mathcal{O}_Z, f_*\mathcal{O}_Z] \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 f_*f^!\Delta_F^\vee[d] & \longrightarrow & [f_*\mathcal{O}_Z, f_*f^!\Delta_F^\vee[d] \otimes^L f_*\mathcal{O}_Z] & \longrightarrow & [f_*\mathcal{O}_Z, f_*(f^!\Delta_F^\vee[d] \otimes^L \mathcal{O}_Z)] & \longrightarrow & [f_*\mathcal{O}_Z, f_*f^!\Delta_F^\vee[d]] \\
 \downarrow & & \downarrow & & \downarrow & \nearrow & \downarrow \\
 f_*(\mathcal{O}_Z, f^!\Delta_F^\vee[d]) & \longrightarrow & [f_*\mathcal{O}_Z, f_*(\mathcal{O}_Z, f^!\Delta_F^\vee[d]) \otimes^L f_*\mathcal{O}_Z] & \longrightarrow & [f_*\mathcal{O}_Z, f_*(\mathcal{O}_Z, f^!\Delta_F^\vee[d]) \otimes^L \mathcal{O}_Z] & & [f_*\mathcal{O}_Z, \Delta_F^\vee[d]]
 \end{array}$$

which thus proves the lemma (all squares in this diagram are commutative by obvious functorial reasons, and the triangle by adjunction). □

Proof of Lemma 7.4. This follows from the computation of the resolution of $f_*\mathcal{O}_Z$ by \mathcal{K}_F when computing the derived functors $[f_*\mathcal{O}_Z, -]$ and $- \otimes^L f_*\mathcal{O}_Z$. □

This finishes the proof of Proposition 7.1. □

Remark 7.5. If $F = F' \oplus L_1$, with L_1 a line bundle, $s = (s', s_1)$, s' and s_1 transverse, the push-forward of $\mathbf{1}_Z$ is zero: decompose the inclusion $Z \subset X$ as $Z \subset Z(s') \subset X$ where $Z(s')$ is the zero locus of s' . Push-forwards respect composition and the push-forward of $\mathbf{1}_Z$ along $Z \subset Z(s')$ is already zero since it is the form (4) which is the cone of a (degenerate) form $s : L_1^\vee \rightarrow \mathcal{O}_X$.

On the other hand, an example when this push-forward is nonzero can be extracted from [4]. Let k be a field and let $\text{Gr}_k(2, 4)$ be the Grassmannian of 2-planes in \mathbb{A}_k^4 . A nonzero map from \mathbb{A}_k^4 to \mathbb{A}_k induces a section of the dual W^\vee of the universal subbundle $W \subset \mathbb{A}_k^4$ of rank 2. Its zero locus is a copy of \mathbb{P}^2 regularly embedded in $\text{Gr}(2, 4)$. The push-forward of the unit form of \mathbb{P}^2 to $\text{Gr}(2, 4)$ is nonzero by [4] where it is proved that it is an element of a basis of the total Witt group of $\text{Gr}(2, 4)$ as a $W(k)$ -module.

7.3. Projective spaces. Let $Y \in \mathit{Sch}$ be a Gorenstein scheme, let \mathcal{E} be a vector bundle of rank $r + 1$ on Y and let us examine when the unit form on $X = \mathbb{P}_Y(\mathcal{E})$ can be pushed forward to Y along $f: X \rightarrow Y$. Since f is smooth (thus flat), we can use Section 6.1. In the case of a smooth morphism f , the object ω_f of 6.1 is a line bundle, and it is the maximal exterior power of the relative cotangent bundle (see Chapter VII §4 in [23]). Here, since f is a projective bundle, it is given by $\omega_f = f^*(\Delta_{\mathcal{E}})^\vee \otimes \mathcal{O}(-r - 1)$ (see e.g. [15], Appendix B.5.8). If $r + 1$ is even, we can push-forward the unit form $\mathbf{1}_X: \mathcal{O}_X \simeq [\mathcal{O}_X, \mathcal{O}_X]$ from $W^0(X, \mathcal{O}_X)$ by using the composition

$$W^0(X, \mathcal{O}_X) \simeq W^0(X, \mathcal{O}(-r - 1)) = W^0(X, \omega_f \otimes f^*(\Delta_{\mathcal{E}})) \rightarrow W^{-r}(Y, \Delta_{\mathcal{E}})$$

where the first isomorphism is given by tensoring with the canonical form $\phi_r = [\mathcal{O}(-(r + 1)/2) \xrightarrow{\sim} \mathcal{O}((r + 1)/2) \otimes \mathcal{O}(-r - 1)] \cong \mathcal{H}om_{\mathcal{O}}(\mathcal{O}(-(r + 1)/2), \mathcal{O}(-r - 1))$ and the last map is the push-forward in the form of Theorem 6.2. Computing the image of $\mathbf{1}_X$ through this composition means therefore computing the image of ϕ_r by the push-forward. The complex on which $f_*(\phi_r)$ lives is $Rf_*(\mathcal{O}(-(r + 1)/2))$. But this complex is zero by [20], 2.1.15, so $f_*(\phi_r) = 0$. If $r + 1$ is odd, there is no push-forward induced by f with source $W^0(X, \mathcal{O}_X)$ because then there is no line bundle K on Y such that \mathcal{O}_X is equal to $f^*(\Delta_{\mathcal{E}}^\vee) \otimes \mathcal{O}(-r - 1) \otimes f^*(K)$ up to a square in $\mathit{Pic}(Y)$. In other words, pushing forward the unit form of $\mathbb{P}_Y(\mathcal{E})$ to Y is not very interesting: whenever it is possible, we get zero. Of course, there are other forms on $\mathbb{P}^r(\mathcal{E})$ not mapping to 0 under the push-forward, as we will see in the following remark.

Remark 7.6. Let us explain a potential source of confusion. Let $i: \mathit{Spec} k \rightarrow \mathbb{P}_k^r$ be a rational point and L a line bundle on \mathbb{P}_k^r . Since $\mathit{Pic}(\mathit{Spec} k) = 0$, using first an isomorphism $\mathcal{O}_k \simeq \omega_i \otimes i^*(L)$, we can push-forward from $W^0(\mathit{Spec} k, \mathcal{O}_k)$ to $W^r(\mathbb{P}_k^r, L)$ for any L . But for different L , we get *very different* push-forwards. Indeed, for example $W^r(\mathbb{P}_k^r, \mathcal{O}(-r)) = 0$ for odd r (by [39] or [3]) so any push-forward to there is obviously zero, whereas since $\omega_k \simeq \mathcal{O}_k$ the push-forward (written as in Theorem 6.7)

$$W^0(\mathit{Spec} k, \mathcal{O}_k) \simeq W^0(\mathit{Spec} k, \omega_k) \rightarrow W^r(\mathbb{P}_k^r, \omega_{\mathbb{P}_k^r})$$

is certainly nonzero, because we can further compose it by a push-forward back to $W^0(\mathit{Spec} k, \omega_k)$ and since the push-forward respects composition, the composite is the identity. Note that this last case also gives an example of a form on \mathbb{P}_k^r whose push-forward to $\mathit{Spec} k$ is not zero. More generally, this phenomenon of different push-forwards starting from the same group can happen whenever $f^*: \mathit{Pic}(Y) \rightarrow \mathit{Pic}(X)$ is not injective.

A. q-flat and q-injective resolutions

For the convenience of the reader, we include here well-known facts on q-flat or q-injective objects that are repeatedly used in the proofs of this article. Most of them are due to Spaltenstein.

Definition A.1. Let X be a scheme and let A be an object in the homotopy category $K(X)$. We say that A is q-flat (or K-flat) if the triangulated functor $(- \otimes A): K(X) \rightarrow K(X)$ preserves quasi-isomorphisms. We say that A is q-injective (or K-injective) if the triangulated functor $\mathcal{H}om_{\bullet}(-, A): K(X)^o \rightarrow K(X)$ preserves quasi-isomorphisms.

Example A.2. A bounded above complex of flat \mathcal{O}_X -modules is q-flat. A bounded below complex of injectives is q-injective.

A discussion of q-flat and q-injective complexes can be found in Sections 1 and 5 of [35]. See in particular Propositions 1.5 and 5.3.

Proposition A.3. *Let A and B be objects in $K(X)$ or $K(Y)$ and let $f: X \rightarrow Y$ be a morphism of schemes.*

- (1) *If A and B are q-flat, then so is $A \otimes B$.*
- (2) *If A is q-flat and B is q-injective, then $\mathcal{H}om_{\bullet}(A, B)$ is q-injective.*
- (3) *If A is q-flat, then f^*A is q-flat.*

Proof. See [35], Proposition 5.3 and 5.4. □

The following two propositions summarize the equivalences of categories and the properties of injectives that we need. Let $\text{Qcoh}(X)$ denote the abelian category of quasi-coherent sheaves on X and $\text{C}_b(\text{Vect}(X))$ (resp. $\text{C}_b(\text{Coh}(X))$) the category of bounded complexes of locally free (resp. coherent) sheaves on X .

Proposition A.4. *Let $X \in \text{Sch}$.*

- (1) *The natural functor $D(\text{Qcoh}(X)) \rightarrow D_{\text{qc}}(X)$ is an equivalence of categories and thus the same is true for their homologically bounded, bounded below or bounded above subcategories and the subcategories with coherent homology.*
- (2) *The natural functor $D_b(\text{Coh}(X)) \rightarrow D_{b,c}(\text{Qcoh}(X))$ is an equivalence.*
- (3) *If $X \in \mathcal{R}eg$, then the natural functor $D(\text{C}_b(\text{Vect}(X))) \rightarrow D_{b,c}(X)$ is an equivalence of categories.*

Proof. Point (1) is Corollary 5.5 in [9]. In Point (2), fully faithful follows from Theorem 12.1 in [25], second part: For affine schemes, use Example 12.3 in [25]. In general, take a finite affine cover of the noetherian scheme X and then take the direct sum over the coherent sheaves on X obtained by extending the coherent sheaves on the affine subschemes using [22], I.6.9.7. Essentially surjective can be found in [17], Example 2.5.2, which follows from Lemme 2.1.2. c) of [38] and an induction argument. Point (3) can then be proved as follows. Let $C_b(\text{Coh}(X))$ be the category of bounded complexes of coherent sheaves. Decompose the functor $D(C_b(\text{Vect}(X))) \rightarrow D_{b,c}(X)$ as

$$\begin{aligned} D(C_b(\text{Vect}(X))) &\longrightarrow D(C_b(\text{Coh}(X))) \longrightarrow D_b(\text{Coh}(X)) \\ &\longrightarrow D_{b,c}(\text{Qcoh}(X)) \longrightarrow D_{b,c}(X). \end{aligned}$$

All these functors are equivalences of categories: the first one by the fact since X is regular, every coherent sheaf has a finite resolution by locally free sheaves by [33], §7, Point 1, the second one by Lemma 11.7 of [25], the third one by Point (2) and the fourth by Point (1). □

Proposition A.5. *Let $X \in \text{Sch}$.*

- (1) *The category $\text{Qcoh}(X)$ has enough injectives by [36], B.3.*
- (2) *The natural inclusion $\text{Qcoh}(X) \rightarrow \text{Mod}(X)$ preserves injectives by [36], B.4.*
- (3) *Every bounded below complex of quasi-coherent \mathcal{O}_X -modules admits a quasi-isomorphism into a complex of $\text{Qcoh}(X)$ -injectives by (1) and [23], I.4.6.*
- (4) *Every bounded below complex of $\text{Qcoh}(X)$ -injectives is q -injective in both $K(\text{Qcoh}(X))$ and $K(X)$ by (2).*

Corollary A.6. *Let $X \in \text{Sch}$. On objects in $D_{-,qc}(X)$ or $D_-(X)$, the unbounded right derived functors computed by q -injective resolutions (as in [35]) coincide with the more classical bounded below right derived functors computed by using resolutions by bounded below complexes of injectives (as in [23]).*

Proof. By the proposition, any object $A \in D_{-,qc}(X)$ is quasi-isomorphic to a bounded below complex of $\text{Qcoh}(X)$ -injectives which are also $\text{Mod}(X)$ -injective (resp. a bounded below complex of $\text{Mod}(X)$ -injectives) and this complex is q -injective. □

Similarly, for q -flat resolutions, we have a weaker statement, sufficient for our purposes.

Proposition A.7. *Let $X \in \text{Sch}$. On objects in $D_{+,qc}(X)$ or $D_+(X)$, the unbounded left derived functors computed using q -flat resolutions can be computed by using bounded above resolutions consisting of flat \mathcal{O}_X -modules.*

Proof. This follows from the fact that any \mathcal{O}_X -module is a quotient of a flat one (see [28], 2.5.5). \square

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