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A relative Sierpiński theorem: Erratum to "Nonhyperbolic Coxeter groups with Menger boundary"

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Abstract. The purpose of this erratum is to correct the proof of Proposition 2.3 of [HHS]. A classical theorem of Sierpiński states that every subspace of dimension at most one in the 2-dimensional disc D^2 can be topologically embedded in the Sierpiński carpet. The proof of Proposition 2.3 of [HHS] implicitly provides a relative version of Sierpiński's theorem. Unfortunately the proof of Proposition 2.3 given in [HHS] is incorrect.

We provide two brief proofs that each fill this gap. One is a self-contained argument suited specifically for the needs of [HHS], and in the other we explicitly prove a relative embedding theorem that produces embeddings in the Sierpiński carpet with certain prescribed boundary values.

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1. Introduction

The *k-pointed star* E_k is the graph obtained as the cone of a discrete set of k points $p_1, \dots, p_k$. Let $e_1, \dots, e_k$ be the edges of E_k, which we think of as embeddings of $[0,1]$ into E_k parametrized such that $e_i(0) = e_j(0)$ for all $i, j \in \{1, \dots, k\}$. Proposition 2.3 of [HHS] states the following:

Proposition 1.1 ([HHS], Prop. 2.3). *Let $P_1, \dots, P_k \in \mathcal{P}$ be distinct peripheral circles in the Sierpiński carpet $\mathcal{S}$, and fix points $p_i \in P_i$. There is a topological embedding $h\colon E_k \hookrightarrow \mathcal{S}$ such that $h \circ e_i(1) = p_i$ for each $i \in \{1, \dots, k\}$. Furthermore the image of E_k intersects the union of all peripheral circles precisely in the given points $p_1, \dots, p_k$.*

The result claimed in this proposition is correct, but the proof given in [HHS] contains an error. The purpose of this erratum is to explain the nature of this error and how to correct it. As explained in [HHS, Prop. 2.3], in order to prove Proposition 1.1, it suffices to prove the following.

Proposition 1.2. *Let Q be a compact surface of genus zero with boundary circles $P_1, \dots, P_k$, and fix points $p_i \in P_i$. Let T be a countable subset of the interior of Q. Then there exists a topological embedding $h \colon E_k \hookrightarrow Q - T$ such that $h \circ e_i(1) = p_i$ for each i and such that the image of E_k intersects the boundary ∂Q precisely in the given points $p_1, \dots, p_k$.*

The proof of Proposition 1.2 given in [HHS] involves applying the Baire Category Theorem to the space of embeddings $\mathcal{E}$ of E_k into Q. Unfortunately, this proof is incorrect since the space $\mathcal{E}$ is not complete. Thus one may not apply the Baire Category Theorem in this context.

2. Correcting Proposition 1.2

This section contains two different proofs of Proposition 1.2. The first proof uses a simple cardinality argument. This proof applies only to star graphs, but is completely elementary and self-contained. The second proof is much more general; it extends a proof of Sierpiński's embedding theorem for 1-dimensional planar sets to give a relative embedding theorem. The extension of Sierpiński's result from the original setting to the relative case takes inspiration from Hatcher's exposition of Kirby's torus trick in [Hat].

In the first proof of Proposition 1.2, we focus on just the special case of $k = 4$. This case is all that is required for the main results of [HHS]. The general case follows by essentially the same reasoning.

Proof of Proposition 1.2. Let Q be a compact surface of genus zero with boundary components P_1, P_2, P_3, P_4, and let T be a countable set in the interior of Q. Choose an embedding f of Q in the Euclidean plane such that $f(Q) \subset D^2$, the image $f(P_4)$ is the boundary circle S^1, and the peripheral circles P_i (for $i = 1, 2, 3$) are mapped onto the circles with radius $\frac{1}{4}$ and centers at $(-\frac{1}{2}, 0)$ and $(0, \pm\frac{1}{2})$. We also choose f so that for $i = 1, 2, 3$, the point $f(p_i)$ is the point on the circle $f(P_i)$ closest to $(0,0)$ and so that $f(p_4) = (1, 0)$. Such an embedding is illustrated in Figure 1. For the sake of simplicity, we identify Q with its image $f(Q)$ in the plane.

We wish to find an embedding of E_4 in $Q - T \subseteq D^2$ that maps $e_i(1)$ to p_i for each $i \in \{1, 2, 3, 4\}$. Consider the uncountable family of embeddings $g_t \colon E_4 \to Q$, for $t \in [-\frac{1}{8}, \frac{1}{8}]$, defined as follows and shown in Figure 1. For each $i \in \{1, 2, 3, 4\}$, let $g_t(e_i)$ be the straight line segment from the point

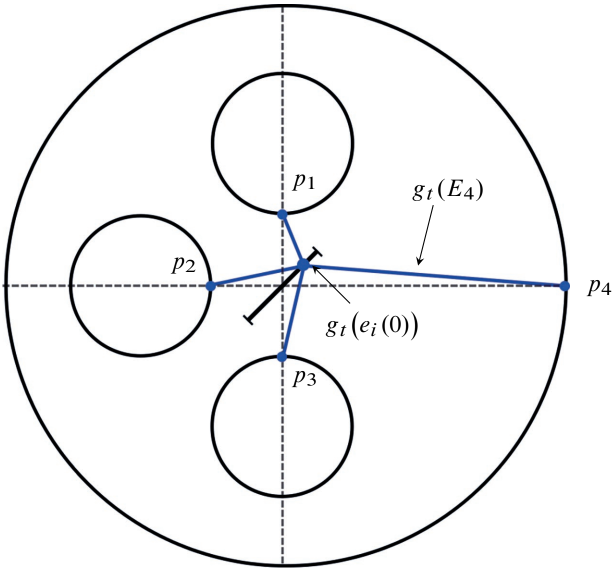

FIGURE 1
Finding an embedding of E_4 in $Q - T$

$g_t(e_i(0)) = (t, -t)$ to the point $g_t(e_i(1)) = p_i$. Since T is countable and the sets $g_t(E_4)$ are pairwise disjoint except at the endpoints p_i, there are at most countably many values of t such that the image of g_t intersects T. Choose t_0 such that $g_{t_0}(E_k)$ is disjoint from T. Then g_{t_0} is the desired embedding $E_4 \to Q - T$. □

The proof above depends on the existence of a 1-parameter family of pairwise disjoint embeddings of E_k in Q. As such the proof does not appear to easily generalize to embeddings of other planar graphs. Below we discuss a different, more elaborate proof of Proposition 1.2 that holds for embeddings of a much broader family of 1-dimensional spaces.

In the rest of this erratum, the *dimension* of a normal topological space X is its covering dimension, i.e., the supremum of all integers n such that every finite open cover of X admits a finite open refinement of order at most n. (See Engelking [Eng] for details.)

Proposition 2.1. *Let A be any topological space of dimension at most $n-1$ that embeds in the closed disc D^n. Let T be a countable set in the interior of D^n. Then for any embedding $f: A \to D^n$ there exists an embedding $g: A \to D^n - T$ such that $f^{-1}(S^{n-1}) = g^{-1}(S^{n-1})$ and f and g are equal on the preimage of S^{n-1}.*

Proof. Let $f\colon A \to D^n$ be an embedding of a space of dimension at most $n-1$ in the closed disc of radius 1. Since $f(A)$ does not contain an n-dimensional disc, the image of A has empty interior in D^n. Let C be the intersection of $f(A)$ with the interior of D^n, and let T be any countable set in the interior of D^n. We identify the interior of D^n with $\mathbb{R}^n$ via a radial reparametrization.

In his text on dimension theory, Engelking gives a proof of Sierpiński's embedding theorem that depends on the following key result (see [Eng, Thm. 1.8.9]). For each subset C of $\mathbb{R}^n$ with empty interior and each countable subset T of $\mathbb{R}^n$, there exists an embedding $h\colon C \to \mathbb{R}^n$ such that $h(C)$ is disjoint from T and the map h is bounded in the sense that the Euclidean distance $d\big(h(c), c\big)$ is less than a fixed constant for all $c \in C$.

Thus, we get an embedding of C into the interior of D^n that misses the countable set T. As in [Hat], due to the boundedness condition, this embedding extends via the identity on ∂D^n to an embedding $\overline{h}\colon f(A) \to D^n$. The desired embedding $g\colon A \to D^n - T$ is given by the composition $g = \overline{h} \circ f$. □

The argument above applies to an embedding in a disc relative to its boundary. Now we show how to modify an embedding of A in a compact manifold to avoid a countable set T, while not moving the part of A that lies in the boundary of the manifold. The following result gives an alternative proof of Proposition 1.2.

Theorem 2.2. *If A is a space of dimension at most $n-1$ with an embedding $f\colon A \to M$ in a compact n-manifold M with boundary, and T is a countable set in the interior of M, then there exists an embedding $g\colon A \to M - T$ with $g = f$ on $A \cap \partial M$.*

Proof. Cover the compact manifold M with a finite collection of closed n-discs $\{D_i\}$ whose interiors cover the interior of M. Then apply the result of Proposition 2.1 to each of the discs D_i to push A off of the part of T contained in the interior of that disc by a move that equals the identity outside of that disc. Since the cover is finite, the composition of this sequence of moves gives an embedding with the desired properties. □

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