

Approximations of delocalized eta invariants by their finite analogues

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Abstract. For a given self-adjoint first order elliptic differential operator on a closed smooth manifold, we prove a list of results on when the delocalized eta invariant associated to a regular covering space can be approximated by the delocalized eta invariants associated to finite-sheeted covering spaces. One of our main results is the following. Suppose M is a closed smooth spin manifold and \tilde{M} is a Γ -regular covering space of M . Let $\langle \alpha \rangle$ be the conjugacy class of a non-identity element $\alpha \in \Gamma$. Suppose $\{\Gamma_i\}$ is a sequence of finite-index normal subgroups of Γ that distinguishes $\langle \alpha \rangle$. Let π_{Γ_i} be the quotient map from Γ to Γ/Γ_i and $\langle \pi_{\Gamma_i}(\alpha) \rangle$ the conjugacy class of $\pi_{\Gamma_i}(\alpha)$ in Γ/Γ_i . If the scalar curvature on M is everywhere bounded below by a sufficiently large positive number, then the delocalized eta invariant for the Dirac operator of \tilde{M} at the conjugacy class $\langle \alpha \rangle$ is equal to the limit of the delocalized eta invariants for the Dirac operators of M_{Γ_i} at the conjugacy class $\langle \pi_{\Gamma_i}(\alpha) \rangle$, where $M_{\Gamma_i} = \tilde{M}/\Gamma_i$ is the finite-sheeted covering space of M determined by Γ_i . In another main result of the paper, we prove that the limit of the delocalized eta invariants for the Dirac operators of M_{Γ_i} at the conjugacy class $\langle \pi_{\Gamma_i}(\alpha) \rangle$ converges, under the assumption that the rational maximal Baum–Connes conjecture holds for Γ .

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In memory of Vaughan Jones

1. Introduction

The delocalized eta invariant for self-adjoint elliptic operators was first introduced by Lott [22] as a natural extension of the classical eta invariant of Atiyah–Patodi–Singer [1–3]. It is a fundamental invariant in the studies of higher index theory on manifolds with boundary, positive scalar curvature metrics on spin manifolds and rigidity problems in topology. More precisely, the delocalized eta invariant can be

used to detect different connected components of the space of positive scalar curvature metrics on a given closed spin manifold [9, 21]. Furthermore, it can be used to give an estimate of the connected components of the moduli space of positive scalar curvature metrics on a given closed spin manifold [34]. Here the moduli space is obtained by taking the quotient of the space of positive scalar curvature metrics under the action of self-diffeomorphisms of the underlying manifold. As for applications to topology, the delocalized eta invariant can be applied to estimate the size of the structure group of a given closed topological manifold [31]. The delocalized eta invariant is also closely related to the Baum–Connes conjecture. The second and third authors showed that if the Baum–Connes conjecture holds for a given group Γ , then¹ the delocalized eta invariant associated to any regular Γ -covering space is an algebraic number [36]. In particular, if a delocalized eta invariant is transcendental, then it would lead to a counterexample to the Baum–Connes conjecture. We refer the reader to [35] for a more detailed survey of the delocalized eta invariant and its higher analogues.

The delocalized eta invariant, despite being defined in terms of an explicit integral formula, is difficult to compute in general, due to its non-local nature. The main purpose of this article is to study when the delocalized eta invariant associated to the universal covering of a space can be approximated by the delocalized eta invariants associated to finite-sheeted coverings, where the latter are easier to compute.

Let us first recall the definition of delocalized eta invariants. Let M be a closed manifold and D a self-adjoint elliptic differential operator on M . Suppose Γ is a discrete group and \tilde{M} is a Γ -regular covering space of M . Denote by \tilde{D} the lift of D from M to \tilde{M} . For any non-identity element $\alpha \in \Gamma$, the delocalized eta invariant $\eta_{\langle\alpha\rangle}(\tilde{D})$ of \tilde{D} at the conjugacy class $\langle\alpha\rangle$ is defined to be

$$(1.1) \quad \eta_{\langle\alpha\rangle}(\tilde{D}) := \frac{2}{\sqrt{\pi}} \int_0^\infty \sum_{\gamma \in \langle\alpha\rangle} \int_{\mathcal{F}} \text{tr}(K_t(x, \gamma x)) \, dx \, dt,$$

where $K_t(x, y)$ is the Schwartz kernel of the operator $\tilde{D}e^{-t^2\tilde{D}^2}$ and \mathcal{F} is a fundamental domain of M_Γ under the action of Γ .

We point out that it is still open question whether the convergence of the integral in line (1.1) holds in general. A list of cases where the convergence is known to hold is given right after Definition 3.1. In particular, if Γ is finite, then the integral in line (1.1) always converges.

Now we consider finite-sheeted cover of M given by finite-index normal subgroups of Γ .

¹There is also an extra technical assumption that the conjugacy class $\langle\alpha\rangle$ used in the definition of the delocalized eta invariant is required to have polynomial growth.

Definition 1.1. Suppose that $\{\Gamma_i\}$ is a sequence of finite-index normal subgroups of Γ . For any non-trivial conjugacy class $\langle \alpha \rangle$ of Γ , we say that $\{\Gamma_i\}$ distinguishes $\langle \alpha \rangle$ if, for any finite set F in Γ , there exists $k \in \mathbb{N}_+$ such that

$$\forall \beta \in F, \beta \notin \langle \alpha \rangle \implies \pi_{\Gamma_i}(\beta) \notin \langle \pi_{\Gamma_i}(\alpha) \rangle$$

for all $i \geq k$.

Let $M_{\Gamma_i} = \tilde{M}/\Gamma_i$ be the associated finite-sheeted covering space of M and D_{Γ_i} the lift of D from M to M_{Γ_i} . The delocalized eta invariant $\eta_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(D_{\Gamma_i})$ of D_{Γ_i} is defined similarly to line (1.1), where π_{Γ_i} is the canonical quotient map from Γ to Γ/Γ_i . Suppose $\{\Gamma_i\}$ distinguishes the conjugacy class $\langle \alpha \rangle$ of a non-identity element $\alpha \in \Gamma$, which is necessary for the following discussions. We prove a list of results that answer positively either one or both of the following questions.

- (1) Does $\lim_{i \rightarrow \infty} \eta_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(D_{\Gamma_i})$ exist?
- (2) If $\lim_{i \rightarrow \infty} \eta_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(D_{\Gamma_i})$ exists, is the limit equal to $\eta_{\langle \alpha \rangle}(\tilde{D})$?

For simplicity, we assume that M is a closed spin manifold equipped with a Riemannian metric of positive scalar curvature throughout the paper. Positive scalar curvature implies \tilde{D} has a spectral gap. In fact, the majority of results² in this paper can be proved in the same way under the assumption that \tilde{D} has a spectral gap or a sufficiently large spectral gap.

Here is one of the main results of our paper.

Theorem 1.2. *With the above notation, assume that \tilde{D} is invertible and $\{\Gamma_i\}$ distinguishes the conjugacy class $\langle \alpha \rangle$ of a non-identity element $\alpha \in \Gamma$. If the maximal Baum–Connes assembly map for Γ is rationally an isomorphism, then the limit*

$$\lim_{i \rightarrow \infty} \eta_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(D_{\Gamma_i})$$

stabilizes, that is,

$$\exists k > 0 \text{ such that } \eta_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(D_{\Gamma_i}) = \eta_{\langle \pi_{\Gamma_k}(\alpha) \rangle}(D_{\Gamma_k}) \text{ for all } i \geq k.$$

Here we say $\{\Gamma_i\}$ distinguishes the conjugacy class $\langle \alpha \rangle$ if for any finite set F in Γ , there exists $k \in \mathbb{N}_+$ such that

$$\forall \beta \in F, \beta \notin \langle \alpha \rangle \implies \pi_{\Gamma_i}(\beta) \notin \langle \pi_{\Gamma_i}(\alpha) \rangle$$

for all $i \geq k$.

By a theorem of Higson and Kasparov [18, Theorem 1.1], the maximal Baum–Connes assembly map is an isomorphism for all a-T-menable groups. We have the following immediate corollary.

²Such as Theorem 1.2, Theorem 1.4, Theorem 5.8 and Proposition 6.3.

Corollary 1.3. *With the above notation, assume that \tilde{D} is invertible and $\{\Gamma_i\}$ distinguishes the conjugacy class $\langle \alpha \rangle$ of a non-identity element $\alpha \in \Gamma$. If Γ is a - T -menable, then the limit*

$$\lim_{i \rightarrow \infty} \eta_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(D_{\Gamma_i})$$

stabilizes.

Note that Theorem 1.2 and its corollary above only addresses the first question, that is, only the convergence of $\lim_{i \rightarrow \infty} \eta_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(D_{\Gamma_i})$. On the other hand, if in addition there exists a smooth dense subalgebra³ \mathcal{A} of the reduced group C^* -algebra $C_r^*(\Gamma)$ of Γ such that $\mathbb{C}\Gamma \subset \mathcal{A}$ and the trace map⁴ $\text{tr}_{\langle \alpha \rangle}: \mathbb{C}\Gamma \rightarrow \mathbb{C}$ extends continuously to a trace map $\text{tr}_{\langle \alpha \rangle}: \mathcal{A} \rightarrow \mathbb{C}$, then we have

$$\lim_{i \rightarrow \infty} \eta_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(D_{\Gamma_i}) = \eta_{\langle \alpha \rangle}(\tilde{D}).$$

See the discussion at the end of Section 4 for more details.

Here is another main result of our paper.

Theorem 1.4. *With the above notation, suppose $\{\Gamma_i\}$ distinguishes the conjugacy class $\langle \alpha \rangle$ of a non-identity element $\alpha \in \Gamma$. If the spectral gap of \tilde{D} at zero is sufficiently large, then we have*

$$\lim_{i \rightarrow \infty} \eta_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(D_{\Gamma_i}) = \eta_{\langle \alpha \rangle}(\tilde{D}).$$

Here “sufficiently larger spectral gap” means that the spectral gap of \tilde{D} at zero is greater than σ_Γ , where σ_Γ is the constant given in Definition 5.1. In particular, if the group Γ has subexponential growth, then it follows from Definition 5.1 that $\sigma_\Gamma = 0$. In this case, if \tilde{D} has a spectral gap, then it is automatically sufficiently large, hence the following immediate corollary.

Corollary 1.5. *With the above notation, suppose $\{\Gamma_i\}$ distinguishes the conjugacy class $\langle \alpha \rangle$ of a non-identity element $\alpha \in \Gamma$. If Γ has subexponential growth and \tilde{D} has a spectral gap at zero, then we have*

$$\lim_{i \rightarrow \infty} \eta_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(D_{\Gamma_i}) = \eta_{\langle \alpha \rangle}(\tilde{D}).$$

There are other variants of Theorem 1.4 above. We refer the reader to Theorem 5.8 and Proposition 6.3 for details.

³A smooth dense subalgebra of $C_r^*(\Gamma)$ is a dense subalgebra of $C_r^*(\Gamma)$ that is closed under holomorphic functional calculus.

⁴The trace map $\text{tr}_{\langle \alpha \rangle}$ is given by the formula: $\sum_{\beta \in \Gamma} a_\beta \beta \mapsto \sum_{\beta \in \langle \alpha \rangle} a_\beta$.

The paper is organized as follows. In Section 2, we review some basic facts about conjugacy separable groups and certain geometric C^* -algebras. In Section 3, we review some basics of delocalized eta invariants. In Section 4, we prove one of main results, Theorem 1.2, and discuss some of its consequences. In Section 5 and 6, we prove Theorem 1.4 and its variants.

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2. Preliminaries

In this section, we review some basic facts about conjugacy separable groups and certain geometric C^* -algebras.

2.1. Conjugacy separable groups. We will prove our main approximation results for a particular class of groups, called conjugacy separable groups. In this subsection, we review some basic properties of conjugacy separable groups. In the following, all groups are assumed to be finitely generated, unless otherwise specified.

Definition 2.1. Let Γ be a finitely generated discrete group. We say that $\gamma \in \Gamma$ is conjugacy distinguished if for any $\beta \in \Gamma$ that is not conjugate to γ , there exists a finite-index normal subgroup Γ' of Γ such that the image of β in Γ/Γ' is not conjugate to γ .

If every element in Γ is conjugacy distinguished, then we say that Γ is conjugacy separable. In other words, we have the following definition of conjugacy separability.

Definition 2.2. A finitely generated group Γ is conjugacy separable if for any $\gamma_1, \gamma_2 \in \Gamma$ that are not conjugate, there exists a finite-index normal subgroup Γ' of Γ such that the image of γ_1 and γ_2 in Γ/Γ' are not conjugate.

For any normal subgroup Γ' of Γ , we denote by $\pi_{\Gamma'}$ the quotient map from Γ to Γ/Γ' .

Definition 2.3. Suppose that $\{\Gamma_i\}$ is a sequence of finite-index normal subgroups of Γ . For any non-trivial conjugacy class $\langle \alpha \rangle$ of Γ , we say that $\{\Gamma_i\}$ distinguishes $\langle \alpha \rangle$, if for any finite set F in Γ there exists $k \in \mathbb{N}_+$ such that

$$\forall \beta \in F, \beta \notin \langle \alpha \rangle \implies \pi_{\Gamma_i}(\beta) \notin \langle \pi_{\Gamma_i}(\alpha) \rangle$$

for all $i \geq k$.

If $\alpha \in \Gamma$ is conjugacy distinguished, then such sequence always exists. More generally, let \mathfrak{N} be the net of all normal subgroups of Γ with finite indices. If $\alpha \in G$ is

conjugacy distinguished in the sense of Definition 2.1, then \mathfrak{N} distinguishes $\langle \alpha \rangle$, that is, for any finite set $F \subset \Gamma$, there exists a finite index normal subgroup Γ_F of Γ such that

$$\forall \beta \in F, \beta \notin \langle \gamma \rangle \implies \pi_{\Gamma'}(\beta) \notin \langle \pi_{\Gamma'}(\gamma) \rangle$$

for all $\Gamma' \in \mathfrak{N}$ with $\Gamma' \supseteq \Gamma_F$.

Let $\mathbb{C}\Gamma$ be the group algebra of Γ and $\ell^1(\Gamma)$ be the ℓ^1 -completion of $\mathbb{C}\Gamma$. For any normal subgroup Γ' of Γ , the quotient map $\pi_{\Gamma'}: \Gamma \rightarrow \Gamma/\Gamma'$ naturally induces an algebra homomorphism $\pi_{\Gamma'}: \mathbb{C}\Gamma \rightarrow \mathbb{C}(\Gamma/\Gamma')$, which extends to a Banach algebra homomorphism $\pi_{\Gamma'}: \ell^1(\Gamma) \rightarrow \ell^1(\Gamma/\Gamma')$.

For any conjugacy class $\langle \gamma \rangle$ of Γ , let $\text{tr}_{\langle \gamma \rangle}: \mathbb{C}\Gamma \rightarrow \mathbb{C}$ be the trace map defined by the formula:

$$\sum_{\beta \in \Gamma} a_\beta \beta \mapsto \sum_{\beta \in \langle \gamma \rangle} a_\beta.$$

The following lemma is obvious.

Lemma 2.4. *If $\langle \alpha \rangle$ is a non-trivial conjugacy class of Γ and $\{\Gamma_i\}$ is a sequence of finite-index normal subgroups that distinguishes $\langle \alpha \rangle$, then*

$$\lim_{i \rightarrow \infty} \text{tr}_{(\pi_{\Gamma_i}(\alpha))}(\pi_{\Gamma_i}(f)) = \text{tr}_{\langle \alpha \rangle}(f)$$

for all $f \in \ell^1(\Gamma)$. Moreover, if $f \in \mathbb{C}\Gamma$, then the limit on the left-hand side stabilizes, that is,

$$\exists k > 0 \text{ such that } \text{tr}_{(\pi_{\Gamma_i}(\alpha))}(\pi_{\Gamma_i}(f)) = \text{tr}_{\langle \alpha \rangle}(f) \text{ for all } i \geq k.$$

As we will mainly work with integral operators whose associated Schwartz kernels are smooth, let us fix some notation further and restate the above lemma in the context of integral operators. Let M be a closed manifold and \tilde{M} be the universal covering space of M . Denote the fundamental group $\pi_1(M)$ of M by Γ . Suppose T is a Γ -equivariant bounded smooth function on $\tilde{M} \times \tilde{M}$, that is,

$$T(\gamma x, \gamma y) = T(x, y)$$

for all $x, y \in \tilde{M}$ and $\gamma \in \Gamma$. We say that T has finite propagation if there exists a constant $d > 0$ such that

$$\text{dist}(x, y) > d \implies T(x, y) = 0,$$

where $\text{dist}(x, y)$ is the distance between x and y in \tilde{M} . In this case, we define the propagation of T to be the infimum of such d .

Definition 2.5. A Γ -equivariant bounded function T on $\tilde{M} \times \tilde{M}$ is said to be ℓ^1 -summable if

$$\|T\|_{\ell^1} := \sup_{x,y \in \mathcal{F}} \sum_{\gamma \in \Gamma} |T(x, \gamma y)| < \infty,$$

where \mathcal{F} is a fundamental domain of \tilde{M} under the action of Γ . We shall call $\|T\|_{\ell^1}$ the ℓ^1 -norm of T from now on.

Clearly, every T with finite propagation is ℓ^1 -summable.

If a Γ -equivariant bounded smooth function $T \in C^\infty(\tilde{M} \times \tilde{M})$ is ℓ^1 -summable, then it defines a bounded operator on $L^2(\tilde{M})$ by the formula

$$(2.1) \quad f \mapsto \int_{\tilde{M}} T(x, y) f(y) dy$$

for all $f \in L^2(\tilde{M})$. For notational simplicity, we shall still denote this operator by T .

Now suppose that Γ' is a finite-index normal subgroup of Γ . Let $M_{\Gamma'} = \tilde{M}/\Gamma'$ be the quotient space of \tilde{M} by the action of Γ' . In particular, $M_{\Gamma'}$ is a finite-sheeted covering space of M with the deck transformation group being Γ/Γ' . Let $\pi_{\Gamma'}$ be the quotient map from \tilde{M} to $M_{\Gamma'}$. Any Γ -equivariant bounded smooth function $T \in C^\infty(\tilde{M} \times \tilde{M})$ that is ℓ^1 -summable naturally descends to a smooth function $\pi_{\Gamma'}(T)$ on $M_{\Gamma'} \times M_{\Gamma'}$ by the formula

$$\pi_{\Gamma'}(T)(\pi_{\Gamma'}(x), \pi_{\Gamma'}(y)) := \sum_{\gamma \in \Gamma'} T(x, \gamma y)$$

for all $(\pi_{\Gamma'}(x), \pi_{\Gamma'}(y)) \in M_{\Gamma'} \times M_{\Gamma'}$. Clearly, $\pi_{\Gamma'}(T)$ is a Γ/Γ' -equivariant smooth function on $M_{\Gamma'} \times M_{\Gamma'}$ and, similar to the formula in (2.1), defines a bounded operator on $L^2(M_{\Gamma'})$.

For any non-trivial conjugacy class $\langle \alpha \rangle$ of Γ , we define the following trace map:

$$\mathrm{tr}_{\langle \alpha \rangle}(T) = \sum_{\gamma \in \langle \alpha \rangle} \int_{\mathcal{F}} T(x, \gamma x) dx,$$

for all Γ -equivariant ℓ^1 -summable smooth function $T \in C^\infty(\tilde{M} \times \tilde{M})$, where \mathcal{F} is a fundamental domain of \tilde{M} under the action of Γ . More generally, for each finite-index normal subgroup Γ' of Γ , a similar trace map is defined for Γ/Γ' -equivariant smooth functions on $M_{\Gamma'} \times M_{\Gamma'}$.

With the above notation, Lemma 2.4 can be restated as follows.

Lemma 2.6. *Suppose $\langle \alpha \rangle$ is a non-trivial conjugacy class of Γ and $\{\Gamma_i\}$ is a sequence of finite-index normal subgroups that distinguishes $\langle \alpha \rangle$. Let T be a Γ -equivariant ℓ^1 -summable bounded smooth function on $\tilde{M} \times \tilde{M}$. Then we have*

$$\lim_{i \rightarrow \infty} \mathrm{tr}_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(\pi_{\Gamma_i}(T)) = \mathrm{tr}_{\langle \alpha \rangle}(T).$$

Moreover, if T has finite propagation, the limit on the left-hand side stabilizes, that is,

$$\exists k > 0 \text{ such that } \operatorname{tr}_{(\pi_{\Gamma_i}(\alpha))}(\pi_{\Gamma_i}(T)) = \operatorname{tr}_{(\alpha)}(T) \text{ for all } i \geq k.$$

2.2. Geometric C^* -algebras. In this subsection, we review the definitions of some geometric C^* -algebras, cf. [33, 38] for more details.

Let X be a proper metric space, i.e. every closed ball in X is compact. An X -module is a separable Hilbert space equipped with a $*$ -representation of $C_0(X)$. An X -module is called non-degenerated if the $*$ -representation of $C_0(X)$ is non-degenerated. An X -module is called standard if no non-zero function in $C_0(X)$ acts as a compact operator.

In addition, we assume that a discrete group Γ acts on X properly and cocompactly by isometries. Assume H_X is an X -module equipped with a covariant unitary representation of Γ . If we denote by φ and π the representations of $C_0(X)$ and Γ respectively, this means

$$\pi(\gamma)(\varphi(f)v) = \varphi(\gamma^* f)(\pi(\gamma)v),$$

where $f \in C_0(X)$, $\gamma \in \Gamma$, $v \in H_X$ and $\gamma^* f(x) = f(\gamma^{-1}x)$. In this case, we call (H_X, Γ, φ) a covariant system.

Definition 2.7 ([39]). A covariant system (H_X, Γ, φ) is called admissible if

- (1) H_X is a non-degenerate and standard X -module;
- (2) for each $x \in X$, the stabilizer group Γ_x acts regularly in the sense that the action is isomorphic to the action of Γ_x on $l^2(\Gamma_x) \otimes H$ for some infinite-dimensional Hilbert space H . Here Γ_x acts on $l^2(\Gamma_x)$ by translations and acts on H trivially.

We remark that for each locally compact metric space X with a proper, cocompact and isometric action of Γ , an admissible covariant system (H_X, Γ, φ) always exists. In particular, if Γ acts on X freely, then the condition (2) above holds automatically.

Definition 2.8. Let (H_X, Γ, φ) be a covariant system and T a Γ -equivariant bounded linear operator acting on H_X .

- The propagation of T is defined to be

$$\sup\{d(x, y) : (x, y) \in \operatorname{supp}(T)\},$$

where $\operatorname{supp}(T)$ is the complement (in $X \times X$) of points $(x, y) \in X \times X$ for which there exists $f, g \in C_0(X)$ such that $gTf = 0$ and $f(x) \neq 0, g(y) \neq 0$;

- T is said to be locally compact if fT and Tf are compact for all $f \in C_0(X)$.

Definition 2.9. Let X be a locally compact metric space with a proper and cocompact isometric action of Γ . Let (H_X, Γ, φ) be an admissible covariant system. We denote by $\mathbb{C}[X]^\Gamma$ the $*$ -algebra of all Γ -equivariant locally compact bounded operators acting on H_X with finite propagations. We define the equivariant Roe algebra $C^*(X)^\Gamma$ to be the completion of $\mathbb{C}[X]^\Gamma$ under the operator norm.

Indeed, $C^*(X)^\Gamma$ is isomorphic to $C_r^*(\Gamma) \otimes \mathcal{K}$, the C^* -algebraic tensor product of the reduced group C^* -algebra of Γ and the algebra of compact operators.

Definition 2.10. We define the localization algebra $C_L^*(X)^\Gamma$ to be the C^* -algebra generated by all uniformly bounded and uniformly norm-continuous function

$$f: [0, \infty) \rightarrow C^*(X)^\Gamma$$

such that the propagation of $f(t)$ goes to zero as t goes to infinity. Define $C_{L,0}^*(X)^\Gamma$ to be the kernel of the evaluation map

$$\text{ev}: C_L^*(X)^\Gamma \rightarrow C^*(X), \quad \text{ev}(f) = f(0).$$

Now let us also review the construction of higher rho invariants for invertible differential operators. For simplicity, let us focus on the odd-dimensional case. Suppose M is closed manifold of odd dimension. Let M_Γ be the regular covering space of M whose deck transformation group is Γ . Suppose D is a self-adjoint elliptic differential operator on M and \tilde{D} is the lift of D to M_Γ . If \tilde{D} is invertible, then its higher rho invariant is defined as follows.

Definition 2.11. With the same notation as above, the higher rho invariant $\rho(\tilde{D})$ of an invertible operator \tilde{D} is defined to be

$$\rho(\tilde{D}) := [e^{2\pi i(\chi(\tilde{D}/t)+1)/2}] \in K_1(C_{L,0}^*(M_\Gamma)^\Gamma),$$

where χ (called a normalizing function) is a continuous odd function such that

$$\lim_{x \rightarrow \pm\infty} \chi(x) = \pm 1.$$

By definition, the higher rho invariant $\rho(\tilde{D})$ is a uniformly norm-continuous function from $[0, \infty)$ to $C^*(X)^\Gamma$. It is a secondary invariant that serves as an obstruction for the higher index of \tilde{D} to be both trivial and local (i.e. having small propagation) at the same time, cf. [11]. More precisely, for each fixed t , the unitary $e^{2\pi i(\chi(\tilde{D}/t)+1)/2}$ is a representative of the higher index class of \tilde{D} . On one hand, since \tilde{D} is invertible, \tilde{D} has a spectral gap near zero. It follows that $e^{2\pi i(\chi(\tilde{D}/t)+1)/2}$ converges in norm to the trivial unitary 1, as t goes to zero. On the other hand, the propagation of $e^{2\pi i(\chi(\tilde{D}/t)+1)/2}$ goes

to zero (up to operators with small norm)⁵, as t goes to infinity. For an invertible \tilde{D} , we can choose a representative of the higher index of \tilde{D} to be either trivial or local (i.e. having small propagation), but generally not both at the same time. In other words, the higher rho invariant measures the tension between the triviality and locality of the higher index of an invertible operator

The above discussion has an obvious maximal analogue (cf. [15, Lemma 3.4]).

Definition 2.12. For an operator $T \in \mathbb{C}[X]^\Gamma$, its *maximal norm* is

$$\|T\|_{\max} := \sup_{\varphi} \{\|\varphi(T)\| : \varphi: \mathbb{C}[X]^\Gamma \rightarrow \mathcal{B}(H) \text{ is a } *\text{-representation}\}.$$

The maximal equivariant Roe algebra $C_{\max}^*(X)^\Gamma$ is defined to be the completion of $\mathbb{C}[X]^\Gamma$ with respect to $\|\cdot\|_{\max}$. Similarly, we define

- (1) the maximal localization algebra $C_{L,\max}^*(X)^\Gamma$ to be the C^* -algebra generated by all uniformly bounded and uniformly norm-continuous function $f: [0, \infty) \rightarrow C_{\max}^*(X)^\Gamma$ such that the propagation of $f(t)$ goes to zero as t goes to infinity;
- (2) $C_{L,0,\max}^*(X)^\Gamma$ to be the kernel of the evaluation map

$$\text{ev}: C_{L,\max}^*(X)^\Gamma \rightarrow C_{\max}^*(X), \quad \text{ev}(f) = f(0).$$

Now suppose M is a closed spin manifold. Assume that M is endowed with a Riemannian metric g of positive scalar curvature. Let M_Γ be the regular covering space of M whose deck transformation group is Γ . Suppose D is the associated Dirac operator on M and \tilde{D} is the lift of D to M_Γ . In this case, we can define the maximal higher rho invariant of \tilde{D} as follows.

Definition 2.13. The maximal higher rho invariant $\rho_{\max}(\tilde{D})$ of \tilde{D} is defined to be

$$\rho_{\max}(\tilde{D}) := [e^{2\pi i(\chi(\tilde{D}/t)+1)/2}] \in K_1(C_{L,0,\max}^*(M_\Gamma)^\Gamma),$$

Here χ is again a normalizing function, but the functional calculus for defining $\chi(t^{-1}\tilde{D})$ is performed under the maximal norm instead. See, for example, [16, Section 3] for a discussion of such a functional calculus.

⁵To be precise, one needs to use a normalizing function χ whose distributional Fourier transform has compact support, and furthermore approximate the function $e^{2\pi i x}$ by an appropriate polynomial.

3. Delocalized eta invariants and their approximations

In this section, we review the definition of delocalized eta invariants and formulate the main question of this article.

We assume that M is a closed spin manifold equipped with a Riemannian metric of positive scalar curvature throughout the paper. Let Γ be a finitely generated discrete group and \tilde{M} a Γ -regular covering space of M . Suppose D is the associated Dirac operator on M and \tilde{D} is the lift of D to \tilde{M} .

Positive scalar curvature implies \tilde{D} has a spectral gap. In fact, the majority of results⁶ in this paper also hold true under the assumption that \tilde{D} has a spectral gap or a sufficiently large spectral gap. For simplicity, we shall only discuss the case where M is a closed spin manifold equipped with a Riemannian metric of positive scalar curvature.

Definition 3.1 ([22]). For any conjugacy class $\langle \alpha \rangle$ of Γ , Lott’s *delocalized eta invariant* $\eta_{\langle \alpha \rangle}(\tilde{D})$ of \tilde{D} is defined to be

$$(3.1) \quad \eta_{\langle \alpha \rangle}(\tilde{D}) := \frac{2}{\sqrt{\pi}} \int_0^\infty \text{tr}_{\langle \alpha \rangle}(\tilde{D}e^{-t^2\tilde{D}^2}) dt,$$

whenever the integral converges. Here

$$\text{tr}_{\langle \alpha \rangle}(\tilde{D}e^{-t^2\tilde{D}^2}) = \sum_{\gamma \in \langle \alpha \rangle} \int_{\mathcal{F}} \text{tr}(k_t(x, \gamma x)) dx,$$

where $k_t(x, y)$ is the corresponding Schwartz kernel of the operator $\tilde{D}e^{-t^2\tilde{D}^2}$ and \mathcal{F} is a fundamental domain of \tilde{M} under the action of Γ .

It is known that the integral formula (3.1) for $\eta_{\langle \alpha \rangle}(\tilde{D})$ converges if \tilde{D} is invertible and any one of the following conditions is satisfied.

- (1) The scalar curvature of M is sufficiently large (see [12, Definition 3.2] for the precise definition of “sufficiently large”).
- (2) There exists a smooth dense subalgebra of $C_r^*(\Gamma)$ onto which the trace map $\text{tr}_{\langle \alpha \rangle}$ extends continuously (cf. [22, Section 4]). For example, when Γ is a Gromov’s hyperbolic group, Puschnigg’s smooth dense subalgebra [25] is such a subalgebra which admits a continuous extension of the trace map $\text{tr}_{\langle \alpha \rangle}$ for all conjugacy classes $\langle h \rangle$.
- (3) $\langle \alpha \rangle$ has subexponential growth (cf. [12, Corollary 3.4]).

⁶Such as Theorem 1.2, Theorem 1.4, Theorem 5.8 and Proposition 6.3.

In general, it is still an open question when the integral in (3.1) converges for invertible operators.

Now suppose that Γ' is a finite-index normal subgroup of Γ . As before, let $M_{\Gamma'} = \tilde{M}/\Gamma'$ be the associated finite-sheeted covering space of M . Similarly, let $D_{\Gamma'}$ be the lift of D to $M_{\Gamma'}$, and define the delocalized eta invariant $\eta_{\langle\pi_{\Gamma'}(\alpha)\rangle}(D_{\Gamma'})$ of $D_{\Gamma'}$ to be

$$(3.2) \quad \eta_{\langle\pi_{\Gamma'}(\alpha)\rangle}(D_{\Gamma'}) := \frac{2}{\sqrt{\pi}} \int_0^\infty \text{tr}_{\langle\pi_{\Gamma'}(\alpha)\rangle}(D_{\Gamma'} e^{-t^2 D_{\Gamma'}^2}) dt,$$

where $\alpha \in \Gamma$ and $\langle\pi_{\Gamma'}(\alpha)\rangle$ is conjugacy class of $\pi_{\Gamma'}(\alpha)$ in Γ/Γ' . As $M_{\Gamma'}$ is compact, it is not difficult to verify that the integral in (3.2) always converges absolutely.

The above discussion naturally leads to the following questions.

Question 3.2. Given a non-identity element $\alpha \in \Gamma$, suppose $\{\Gamma_i\}$ is a sequence of finite-index normal subgroups that distinguishes the conjugacy class $\langle\alpha\rangle$.

(I) When does $\lim_{i \rightarrow \infty} \eta_{\langle\pi_{\Gamma_i}(\alpha)\rangle}(D_{\Gamma_i})$ exist?

(II) If $\eta_{\langle\alpha\rangle}(\tilde{D})$ is well-defined and $\lim_{i \rightarrow \infty} \eta_{\langle\pi_{\Gamma_i}(\alpha)\rangle}(D_{\Gamma_i})$ exists, when do we have

$$(3.3) \quad \lim_{i \rightarrow \infty} \eta_{\langle\pi_{\Gamma_i}(\alpha)\rangle}(D_{\Gamma_i}) = \eta_{\langle\alpha\rangle}(\tilde{D})?$$

4. Maximal higher rho invariants and their functoriality

In this section, we use the functoriality of higher rho invariants to give some sufficient conditions under which the answer to part (I) of Question 3.2 is positive.

Before we get into the technical details, here is a special case which showcases the main results of this section.

Proposition 4.1. *With the same notation as in Question 3.2, if Γ is a -T-menable and $\{\Gamma_i\}$ is a sequence of finite-index normal subgroups that distinguishes the conjugacy class $\langle\alpha\rangle$ for a non-identity element $\alpha \in \Gamma$, then the limit*

$$\lim_{i \rightarrow \infty} \eta_{\langle\pi_{\Gamma_i}(\alpha)\rangle}(D_{\Gamma_i})$$

stabilizes, that is,

$$\exists k > 0 \text{ such that } \eta_{\langle\pi_{\Gamma_i}(\alpha)\rangle}(D_{\Gamma_i}) = \eta_{\langle\pi_{\Gamma_k}(\alpha)\rangle}(D_{\Gamma_k}) \text{ for all } i \geq k.$$

In particular, $\lim_{i \rightarrow \infty} \eta_{\langle\pi_{\Gamma_i}(\alpha)\rangle}(D_{\Gamma_i})$ exists.

Proof. This is a consequence of Theorem 4.3 below and a theorem of Higson and Kasparov [18, Theorem 1.1]. ■

Given a finitely presented discrete group Γ , let $\underline{E}\Gamma$ be the universal Γ -space for proper Γ -actions. The Baum–Connes conjecture [4] can be stated as follows.

Conjecture 4.2 (Baum–Connes conjecture). *The following map*

$$\text{ev}_*: K_i(C_L^*(\underline{E}\Gamma)^\Gamma) \rightarrow K_i(C^*(\underline{E}\Gamma)^\Gamma)$$

is an isomorphism. Here

$$K_i(C_L^*(\underline{E}\Gamma)^\Gamma) := \varinjlim_Y K_i(C_L^*(Y)^\Gamma),$$

$$K_i(C^*(\underline{E}\Gamma)^\Gamma) := \varinjlim_Y K_i(C^*(Y)^\Gamma),$$

where the limit is taken over all Γ cocompact spaces Y .

Although this was not how the Baum–Connes conjecture was originally stated, the above formulation is equivalent to the original Baum–Connes conjecture, after one makes the following natural identifications:

$$K_i(C_L^*(\underline{E}\Gamma)^\Gamma) \cong K_i^\Gamma(\underline{E}\Gamma) \text{ and } K_i(C^*(\underline{E}\Gamma)^\Gamma) \cong K_i(C_r^*(\Gamma)).$$

Under this notation, we usually write the map

$$\text{ev}_*: K_i(C_L^*(\underline{E}\Gamma)^\Gamma) \rightarrow K_i(C^*(\underline{E}\Gamma)^\Gamma)$$

as follows:

$$\mu: K_i^\Gamma(\underline{E}\Gamma) \rightarrow K_i(C_r^*(\Gamma))$$

and call it the Baum–Connes assembly map. Similarly, there is a maximal version of the Baum–Connes assembly map:

$$\mu_{\max}: K_i^\Gamma(\underline{E}\Gamma) \rightarrow K_i(C_{\max}^*(\Gamma)).$$

The maximal Baum–Connes assembly map μ_{\max} is not an isomorphism in general. For example, μ_{\max} fails to be surjective for non-finite property (T) groups.

Before we discuss the functoriality of higher rho invariants, let us recall the functoriality of higher indices. More precisely, let D be a Dirac-type operator on a closed n -dimensional manifold X . Consider the following commutative diagram

$$\begin{array}{ccc} & & B\Gamma_1 \\ & \nearrow f_1 & \downarrow B\varphi \\ X & & B\Gamma_2 \\ & \searrow f_2 & \end{array}$$

where f_1, f_2 are continuous maps and $B\varphi$ is a continuous map from $B\Gamma_1$ to $B\Gamma_2$ induced by a group homomorphism $\varphi: \Gamma_1 \rightarrow \Gamma_2$. Let X_{Γ_1} (resp. X_{Γ_2}) be the Γ_1 (resp. Γ_2) regular covering space of X induced by the map f_1 (resp. f_2), and $D_{X_{\Gamma_1}}$ (resp. $D_{X_{\Gamma_2}}$) be the lift of D to X_{Γ_1} (resp. X_{Γ_2}). We have the following functoriality of the higher indices:

$$\varphi_* (\text{Ind}_{\max}(D_{X_{\Gamma_1}})) = \text{Ind}_{\max}(D_{X_{\Gamma_2}}) \text{ in } K_n(C_{\max}^*(\Gamma_2)),$$

where $C_{\max}^*(\Gamma_i)$ is the maximal group C^* -algebra of Γ_i , the notation Ind_{\max} stands for higher index in the maximal group C^* -algebra, and

$$\varphi_*: K_n(C_{\max}^*(\Gamma_1)) \rightarrow K_n(C_{\max}^*(\Gamma_2))$$

is the morphism naturally induced by φ .

Now let us consider the functoriality of higher rho invariants. Following the same notation from above, in addition, assume X is a closed spin manifold endowed with a Riemannian metric of positive scalar curvature. In this case, the maximal higher rho invariants $\rho_{\max}(D_{X_{\Gamma_1}})$ of $D_{X_{\Gamma_1}}$ and $\rho_{\max}(D_{X_{\Gamma_2}})$ of $D_{X_{\Gamma_2}}$ are defined. Let $E\Gamma_1$ (resp. $E\Gamma_2$) be universal Γ_1 -space (resp. Γ_2 -space) for free Γ_1 -actions (resp. Γ_2 -actions). Denote by Φ the equivariant map $X_{\Gamma_1} \rightarrow X_{\Gamma_2}$ induced by $\varphi: \Gamma_1 \rightarrow \Gamma_2$, which in turn induces a morphism

$$\Phi_*: K_n(C_{L,0,\max}^*(X_{\Gamma_1})^{\Gamma_1}) \rightarrow K_n(C_{L,0,\max}^*(X_{\Gamma_2})^{\Gamma_2}).$$

By [17], the maximal higher rho invariants are functorial:

$$\Phi_* (\rho_{\max}(D_{X_{\Gamma_1}})) = \rho_{\max}(D_{X_{\Gamma_2}})$$

in $K_n(C_{L,0,\max}^*(X_{\Gamma_2})^{\Gamma_2})$.

Now suppose M is an odd-dimensional closed spin manifold endowed with a positive scalar curvature metric and Γ is a finitely generated discrete group. Let \tilde{M} be a Γ -regular covering space of M and \tilde{D} be the Dirac operator lifted from M . For each finite-index normal subgroup Γ' of Γ , let $M_{\Gamma'} = \tilde{M}/\Gamma'$ be the associated finite-sheeted covering space of M . Denote by $D_{\Gamma'}$ the Dirac operator on $M_{\Gamma'}$ lifted from M .

Theorem 4.3. *With the above notation, given a non-identity element $\alpha \in \Gamma$, suppose $\{\Gamma_i\}$ is a sequence of finite-index normal subgroups that distinguishes the conjugacy class $\langle \alpha \rangle$. If the maximal Baum–Connes assembly map for Γ is rationally an isomorphism, then*

$$\lim_{i \rightarrow \infty} \eta_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(D_{\Gamma_i})$$

stabilizes, that is,

$$\exists k > 0 \text{ such that } \eta_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(D_{\Gamma_i}) = \eta_{\langle \pi_{\Gamma_k}(\alpha) \rangle}(D_{\Gamma_k}) \text{ for all } i \geq k.$$

Proof. We have the short exact sequence of C^* -algebras:

$$0 \rightarrow C_{L,0,\max}^*(E\Gamma)^\Gamma \rightarrow C_{L,\max}^*(E\Gamma)^\Gamma \rightarrow C_{\max}^*(E\Gamma)^\Gamma \rightarrow 0,$$

which induces the following long exact sequence in K -theory:

$$(4.1) \quad \begin{array}{ccccc} K_0(C_{L,0,\max}^*(E\Gamma)^\Gamma) \otimes \mathbb{Q} & \longrightarrow & (C_{L,\max}^*(E\Gamma)^\Gamma) \otimes \mathbb{Q} & \xrightarrow{\mu_0} & K_0(C_{\max}^*(\Gamma)) \otimes \mathbb{Q} \\ \uparrow & & & & \downarrow \partial \\ K_1(C_{\max}^*(\Gamma)) \otimes \mathbb{Q} & \xleftarrow{\mu_1} & K_1(C_{L,\max}^*(E\Gamma)^\Gamma) \otimes \mathbb{Q} & \leftarrow & K_1(C_{L,0,\max}^*(E\Gamma)^\Gamma) \otimes \mathbb{Q} \end{array}$$

Note that $K_i(C_{L,\max}^*(E\Gamma)^\Gamma)$ is naturally isomorphic to $K_i^\Gamma(E\Gamma)$. Similarly, we have

$$K_i(C_{L,\max}^*(E\Gamma)^\Gamma) \cong K_i^\Gamma(E\Gamma).$$

The morphism $K_i^\Gamma(E\Gamma) \rightarrow K_i^\Gamma(\underline{E}\Gamma)$ induced by the inclusion from $E\Gamma$ to $\underline{E}\Gamma$ is rationally injective (cf. [5, Section 7]). It follows that if the rational maximal Baum–Connes conjecture holds for Γ , that is, the maximal Baum–Connes assembly map

$$\mu_{\max}: K_i(C_{L,\max}^*(E\Gamma)^\Gamma) \otimes \mathbb{Q} \rightarrow K_i(C_{\max}^*(\Gamma)) \otimes \mathbb{Q}$$

is an isomorphism, then the maps μ_i in the above commutative diagram are injective and the map ∂ is surjective. In particular, for the higher rho invariant $\rho(\tilde{D})$ of \tilde{D} , there exists

$$[p] \in K_0(C_{\max}^*(E\Gamma)^\Gamma) \cong K_0(C_{\max}^*(\Gamma))$$

such that $\partial[p] = \rho(\tilde{D})$ rationally, that is, $\partial[p] = \lambda \cdot \rho(\tilde{D})$ for some $\lambda \in \mathbb{Q}$.

By the surjectivity of the Baum–Connes assembly map

$$\mu_{\max}: K_i(C_{L,\max}^*(E\Gamma)^\Gamma) \otimes \mathbb{Q} \rightarrow K_i(C_{\max}^*(\Gamma)) \otimes \mathbb{Q},$$

we can assume p is an idempotent with finite propagation in $\mathcal{K} \otimes \mathbb{C}\Gamma$. Indeed, let $[q]$ be an element in

$$K_0(C_{L,\max}^*(E\Gamma)^\Gamma) \otimes \mathbb{Q} \cong K_0^\Gamma(E\Gamma) \otimes \mathbb{Q}$$

such that $\mu_{\max}([q]) = [p]$. It follows from the Baum–Douglas model of K -homology [6] that $[q]$ is the K -homology class of a twisted spin^c Dirac operator and $[p]$ is its higher index. More precisely, there is an even-dimensional spin^c Γ -manifold X together with a Γ -equivariant vector bundle E such that rationally $[p]$ equals the Γ -index of the twisted Dirac operator \not{D}_E on X . For the convenience of the reader, we shall review the construction of this index. Recall that a function χ on \mathbb{R} is called a normalizing function if $\chi: \mathbb{R} \rightarrow [-1, 1]$ is an odd continuous function such that $\chi(x) > 0$ when $x > 0$,

and $\chi(x) \rightarrow \pm 1$ as $x \rightarrow \pm\infty$. Let f be a smooth normalizing function whose distributional Fourier transform has compact support. Let us denote $f(\mathcal{D}_E) = \begin{pmatrix} 0 & F_+ \\ F_- & 0 \end{pmatrix}$. Using the formula

$$f(\mathcal{D}) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(s) e^{isD} ds$$

and the fact that e^{isD} has propagation less than or equal to $|s|$, it follows that $f(\mathcal{D})$ has finite propagation, since the Fourier transform \widehat{f} has compact support. Now we define

$$w = \begin{pmatrix} 1 & F_+ \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -F_- & 1 \end{pmatrix} \begin{pmatrix} 1 & F_+ \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Note that

$$w^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -F_+ \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ F_- & 1 \end{pmatrix} \begin{pmatrix} 1 & -F_+ \\ 0 & 1 \end{pmatrix}.$$

The higher index of \mathcal{D} is given as the following formal difference of idempotents:

$$\left[w \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} w^{-1} \right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right].$$

By construction, we have

$$w \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} w^{-1} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{S} \otimes \mathbb{C}\Gamma,$$

where \mathcal{S} is the algebra of trace class operators on a Hilbert space. In particular, we see that the element $[p] \in K_0(C_{\max}^*(\Gamma))$ from above can be (rationally) represented by a formal difference of idempotents with finite propagation.

Let Ψ_i be the canonical quotient map from \tilde{M} to $M_{\Gamma_i} = \tilde{M}/\Gamma_i$ and

$$(\Psi_i)_*: K_1(C_{L,0,\max}^*(\tilde{M})^\Gamma) \rightarrow K_1(C_{L,0,\max}^*(M_{\Gamma_i})^{\Gamma/\Gamma_i})$$

the corresponding morphism induced by Ψ_i . By [17, Theorem 1.1], we have

$$(\Psi_i)_*(\rho_{\max}(\tilde{D})) = \rho(D_{\Gamma_i}) \quad \text{in } K_1(C_{L,0,\max}^*(M_{\Gamma_i})^{\Gamma/\Gamma_i}).$$

By passing to the universal spaces, we have

$$(\Psi_i)_*(\rho_{\max}(\tilde{D})) = \rho(D_{\Gamma_i}) \quad \text{in } K_1(C_{L,0,\max}^*(E(\Gamma/\Gamma_i))^{\Gamma/\Gamma_i}).$$

Consider the following commutative diagram of long exact sequences⁷:

(4.2)

$$\begin{array}{ccccc}
 K_0(C_{L,\max}^*(E\Gamma)^\Gamma) \otimes \mathbb{Q} & \longrightarrow & K_0(C_{\max}^*(\Gamma)) \otimes \mathbb{Q} & \xrightarrow{\partial} & K_1(C_{L,0,\max}^*(E\Gamma)^\Gamma) \otimes \mathbb{Q} \\
 \downarrow & & \downarrow (\pi_{\Gamma_i})_* & & \downarrow (\Psi_i)_* \\
 K_0(C_{L,\max}^*(E(\Gamma/\Gamma_i))^{\Gamma/\Gamma_i}) \otimes \mathbb{Q} & \longrightarrow & K_0(C_r^*(\Gamma/\Gamma_i)) \otimes \mathbb{Q} & \xrightarrow{\partial} & K_1(C_{L,0}^*(E(\Gamma/\Gamma_i))^{\Gamma/\Gamma_i}) \otimes \mathbb{Q},
 \end{array}$$

where

$$(\pi_{\Gamma_i})_*: K_0(C_{\max}^*(\Gamma)) \rightarrow K_0(C_r^*(\Gamma/\Gamma_i))$$

is the natural morphism induced by the canonical quotient map $\pi_{\Gamma_i}: \Gamma \rightarrow \Gamma/\Gamma_i$. Let us denote $(\pi_{\Gamma_i})_*(p)$ by p_i . It follows from the commutative diagram above that

$$(4.3) \quad \partial(p_i) = \rho(D_{\Gamma_i}).$$

By [36, Lemma 3.9 and Theorem 4.3], for each Γ_i , there exists a determinant map

$$\tau_i: K_1(C_{L,0}^*(E(\Gamma/\Gamma_i))^{\Gamma/\Gamma_i}) \rightarrow \mathbb{C}$$

such that

$$\frac{1}{2} \eta_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(D_{\Gamma_i}) = -\tau_i(\rho(D_{\Gamma_i})) = \text{tr}_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(p_i).$$

Since the idempotent p has finite propagation, it follows from Lemma 2.4 that

$$\lim_{i \rightarrow \infty} \eta_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(D_{\Gamma_i}) = 2 \lim_{i \rightarrow \infty} \text{tr}_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(p_i) = 2 \text{tr}_{\langle \alpha \rangle}(p),$$

and the limit stabilizes. ■

Remark 4.4. (1) In Theorem 4.3 above, instead of the assumption that $\{\Gamma_i\}$ is a sequence of finite-index normal subgroups that distinguishes the conjugacy class $\langle \alpha \rangle$, we assume that $\{\Gamma_i\}$ is a decreasing sequence⁸ of finite-index normal subgroups of Γ . The same proof shows that if the maximal Baum–Connes assembly map for Γ is rationally an isomorphism, then

$$\lim_{i \rightarrow \infty} \eta_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(D_{\Gamma_i})$$

stabilizes. On the other hand, to eventually relate the limit $\lim_{i \rightarrow \infty} \eta_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(D_{\Gamma_i})$ to $\eta_{\langle \alpha \rangle}(\tilde{D})$, if the latter exists, one will likely have to assume the condition that $\{\Gamma_i\}$ distinguishes $\langle \alpha \rangle$.

⁷Since Γ/Γ_i is finite, we have $C_{L,0,\max}^*(E(\Gamma/\Gamma_i))^{\Gamma/\Gamma_i} \cong C_{L,0}^*(E(\Gamma/\Gamma_i))^{\Gamma/\Gamma_i}$.

⁸We say $\{\Gamma_i\}$ is a decreasing sequence of finite-index normal subgroups of Γ if $\Gamma_i \supseteq \Gamma_{i+1}$ for all i .

(2) Note that Theorem 4.3 only answers part (I) of Question 3.2. Part (II) of Question 3.2 is still open, even under the assumption that the maximal Baum–Connes conjecture holds for Γ .

(3) Although Theorem 4.3 assumes that the maximal Baum–Connes conjecture holds for Γ , it is clear from the proof that it suffices to assume $\rho_{\max}(\tilde{D})$ is rationally in the image of the composition of the following maps:

$$K_0^\Gamma(E\Gamma) \rightarrow K_0(C_{\max}^*(\Gamma)) \xrightarrow{\partial} K_1(C_{L,0,\max}^*(E\Gamma)^\Gamma).$$

By a theorem of Higson and Kasparov [18, Theorem 1.1], the maximal Baum–Connes conjecture holds for all a-T-menable groups. Together with Theorem 4.3 above, this proves Proposition 4.1 at the beginning of the section.

As mentioned above, the maximal Baum–Connes assembly map μ_{\max} fails to be an isomorphism in general. For example, μ_{\max} fails to be surjective for non-finite property (T) groups. On the contrary, there is no counterexample to the Baum–Connes conjecture, at the time of writing. In particular, the Baum–Connes conjecture is known to hold for all hyperbolic groups [20, 23], many of which have property (T). For this reason, we shall now investigate Question 3.2, in particular, the convergence of

$$\lim_{i \rightarrow \infty} \eta_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(D_{\Gamma_i})$$

when the group Γ satisfies the Baum–Connes conjecture.

One of the first difficulties we face is that reduced group C^* -algebras are not functorial with respect to group homomorphisms in general. As a result, the functoriality of higher rho invariants is, a priori, lost in the reduced C^* -algebra setting. Note that a key step (cf. equation (4.3)) in the proof of Theorem 4.3 is the existence of a “universal” idempotent $p \in \mathcal{S} \otimes \mathbb{C}\Gamma$ such that

$$\partial(p_i) = \rho(D_{\Gamma_i}),$$

where $p_i = (\pi_{\Gamma_i})_*(p)$. In the maximal setting, the existence of such a universal idempotent follows if the rational maximal Baum–Connes conjecture holds for Γ . In the following, we shall discuss some geometric conditions that are sufficient for deriving an analogue of Theorem 4.3 in the reduced setting. How these geometric conditions are related to the (reduced) Baum–Connes conjecture will be explained in Appendix A.

Recall that M is a closed spin manifold equipped with a Riemannian metric h of positive scalar curvature. Let $\varphi: M \rightarrow B\Gamma$ be the classifying map for the covering $\tilde{M} \rightarrow M$, that is, the pullback of $E\Gamma$ by φ is \tilde{M} . In the following, we denote by \mathfrak{B} the Bott manifold, a simply connected spin manifold of dimension 8 with $\hat{A}(\mathfrak{B}) = 1$. This manifold is not unique, but any choice will work for the following discussion.

Definition 4.5. We say a multiple of (M, φ, h) stably bounds with respect to $B\Gamma$ if there exists a compact spin manifold W and a map $\Phi: W \rightarrow B\Gamma$ such that $\partial W = \bigsqcup_{i=1}^{\ell} M'$ and $\Phi|_{\partial W} = \bigsqcup_{i=1}^{\ell} \varphi'$, where (M', φ', h') is the direct product of (M, φ, h) with finitely many copies of \mathfrak{B} and $\bigsqcup_{i=1}^{\ell} M'$ is the disjoint union of ℓ copies of M' .

Definition 4.6. Let \tilde{h} be the metric on \tilde{M} lifted from h . We say a multiple of (\tilde{M}, \tilde{h}) positively stably bounds with respect to $\underline{E}\Gamma$ if there exists a spin cocompact Γ -manifold⁹ \tilde{V} equipped with a Γ -equivariant positive scalar curvature metric $g_{\tilde{V}}$ such that $\partial\tilde{V} = \bigsqcup_{i=1}^{\ell} \tilde{M}'$ (as Γ -manifolds) and $g_{\tilde{V}}$ has product structure near $\partial\tilde{V}$, where (\tilde{M}', \tilde{h}') is the direct product of (\tilde{M}, \tilde{h}) with finitely many copies of \mathfrak{B} .

The following proposition is an analogue of Theorem 4.3 in the reduced setting, under the assumptions that a multiple of (M, φ, h) stably bounds with respect to $B\Gamma$ and a multiple of (\tilde{M}, \tilde{h}) positively stably bounds with respect to $\underline{E}\Gamma$. For example, if M is a lens space equipped with the metric inherited from the standard round metric on S^n and $\Gamma = \pi_1(M)$, then both of these assumptions are satisfied. In general, the validity of these two assumptions is closely related to the reduced Baum–Connes conjecture and the Stolz conjecture on positive scalar curvature metrics. We refer the reader to Appendix [Appendix A](#) for more details.

Proposition 4.7. *Let M be a closed spin manifold equipped with a Riemannian metric h of positive scalar curvature. Given a non-identity element $\alpha \in \Gamma$, suppose $\{\Gamma_i\}$ is a sequence of finite-index normal subgroups of Γ that distinguishes the conjugacy class $\langle \alpha \rangle$. If a multiple of (M, φ, h) stably bounds with respect to $B\Gamma$ and a multiple of (\tilde{M}, \tilde{h}) positively stably bounds with respect to $\underline{E}\Gamma$, then*

$$\lim_{i \rightarrow \infty} \eta_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(D_{\Gamma_i})$$

stabilizes, that is,

$$\exists k > 0 \text{ such that } \eta_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(D_{\Gamma_i}) = \eta_{\langle \pi_{\Gamma_k}(\alpha) \rangle}(D_{\Gamma_k}) \text{ for all } i \geq k.$$

Proof. For notational simplicity, let us assume (M, φ, h) itself bounds with respect to $B\Gamma$, that is, there exists a compact spin manifold W and a map $\Phi: W \rightarrow B\Gamma$ such that $\partial W = M$ and $\Phi|_{\partial W} = \varphi$. Similarly, let us assume (\tilde{M}, \tilde{h}) itself positively stably bounds with respect to $\underline{E}\Gamma$, that is, there exists a cocompact Γ -spin manifold \tilde{V} with $\partial\tilde{V} = \bigsqcup_{i=1}^{\ell} \tilde{M}'$ (as Γ -manifolds) and \tilde{V} is equipped with a Γ -equivariant positive scalar curvature metric that has product structure near $\partial\tilde{V}$. The general case can be proved in exactly the same way.

⁹Here a Γ -manifold is a Riemannian manifold equipped with a proper isometric action of Γ .

Endow W with a Riemannian metric g which has product structure near $\partial W = M$ and whose restriction on ∂W is the positive scalar curvature metric h . Let \tilde{W} be the covering space of W induced by the map $\Phi: W \rightarrow B\Gamma$ and \tilde{g} be the lift of g from W to \tilde{W} . Due to the positive scalar curvature of \tilde{g} near the boundary of \tilde{W} , the corresponding Dirac operator $D_{\tilde{W}}$ on \tilde{W} with respect to the metric \tilde{g} has a well-defined higher index $\text{Ind}(D_{\tilde{W}}, \tilde{g})$ in $KO_{n+1}(C_r^*(\Gamma; \mathbb{R}))$.

Now for each normal subgroup Γ_i of Γ , let $M_{\Gamma_i} = \tilde{M}/\Gamma_i$, $W_{\Gamma_i} = \tilde{W}/\Gamma_i$ and g_i be the lift of g to W_{Γ_i} . Similarly, the corresponding Dirac operator $D_{W_{\Gamma_i}}$ on W_{Γ_i} with respect to the metric g_i has a well-defined higher index

$$\text{Ind}(D_{W_{\Gamma_i}}, g_i) \quad \text{in } KO_{n+1}(C_r^*(\Gamma/\Gamma_i; \mathbb{R})).$$

Moreover, we have

$$\partial(\text{Ind}(D_{W_{\Gamma_i}}, g_i)) = \rho(D_{M_{\Gamma_i}}) \quad \text{in } KO_n(C_{L,0}^*(E(\Gamma/\Gamma_i); \mathbb{R})^{\Gamma/\Gamma_i}),$$

cf. [24, Theorem 1.14] and [33, Theorem A].

By [36, Lemma 3.9 and Theorem 4.3], for each Γ_i , there exists a determinant map

$$\tau_i: K_1(C_{L,0,\max}^*(E(\Gamma/\Gamma_i))^{\Gamma/\Gamma_i}) \rightarrow \mathbb{C}$$

such that

$$\frac{1}{2}\eta_{(\pi_{\Gamma_i}, \alpha)}(D_{M_{\Gamma_i}}) = -\tau_i(\rho(D_{M_{\Gamma_i}})) = \text{tr}_{(\pi_{\Gamma_i}, \alpha)}(\text{Ind}(D_{W_{\Gamma_i}}, g_i)).$$

Therefore, to prove the proposition, it suffices to show that there exists

$$[p] \in KO_{n+1}(C_{\max}^*(\Gamma; \mathbb{R}))$$

such that $[p]$ is represented by a formal difference of idempotents in $\mathcal{S} \otimes \mathbb{C}\Gamma$ and

$$(\pi_{\Gamma_i})_*([p]) = \text{Ind}(D_{W_{\Gamma_i}}, g_i)$$

for all k , where $(\pi_{\Gamma_i})_*: C_{\max}^*(\Gamma; \mathbb{R}) \rightarrow C_r^*(\Gamma/\Gamma_i; \mathbb{R})$ is the morphism induced by the quotient homomorphism $\pi_{\Gamma_i}: \Gamma \rightarrow \Gamma/\Gamma_i$. The existence of such a “universal” K -theory element with finite propagation can be seen as follows.

Let Y be the spin Γ -manifold obtained by gluing \tilde{V} and \tilde{W} along their common boundary \tilde{M} . Since the scalar curvature on \tilde{V} is uniformly bounded below by a positive number, it follows from the relative index theorem [10, 32] that

$$\text{Ind}_{\max}(D_Y) = \text{Ind}_{\max}(D_{\tilde{W}}, \tilde{g}) \text{ in } KO_{n+1}(C_{\max}^*(\Gamma; \mathbb{R})).$$

Let $p = \text{Ind}_{\max}(D_Y)$. By the discussion in the proof of Theorem 4.3, the index class $\text{Ind}_{\max}(D_Y)$ can be represented by a formal difference of idempotents in $\mathcal{S} \otimes \mathbb{C}\Gamma$. On the other hand, we have

$$(\pi_{\Gamma_i})_*(\text{Ind}_{\max}(D_{\tilde{W}}, \tilde{g})) = \text{Ind}(D_{W_{\Gamma_i}}, g_i)$$

for all i . To summarize, we have

$$(\pi_{\Gamma_i})_*([p]) = (\pi_{\Gamma_i})_*(\text{Ind}_{\max}(D_{\tilde{W}}, \tilde{g})) = \text{Ind}(D_{W_{\Gamma_i}}, g_i).$$

This finishes the proof. ■

In Theorem 4.3 and Proposition 4.7, we have mainly focused on the part (I) of Question 3.2. In the following, we shall try to answer part (II) of Question 3.2 in some special cases. Note that, a key ingredient of the proofs for Theorem 4.3 and Proposition 4.7 is the existence of a K -theory element¹⁰ $[p_{\max}] \in K_{n+1}(C_{\max}^*(\Gamma) \otimes \mathcal{K})$ that is represented by a formal difference of idempotents in $\mathcal{S} \otimes \mathbb{C}\Gamma$ such that

$$\partial(p_{\max}) = \rho_{\max}(\tilde{D}),$$

where

$$\partial: K_{n+1}(C_{\max}^*(\Gamma)) \rightarrow KO_n(C_{L,0,\max}^*(E\Gamma)^\Gamma)$$

is the usual boundary map in the corresponding K -theory long exact sequence. We shall assume the existence of such a K -theory element p_{\max} throughout the rest of the section.

In addition, suppose there exists a smooth dense subalgebra \mathcal{A} of $C_r^*(\Gamma)$ such that $\mathcal{A} \supset \mathbb{C}\Gamma$ and the trace map $\text{tr}_{(\alpha)}: \mathbb{C}\Gamma \rightarrow \mathbb{C}$ extends to a trace map $\mathcal{A} \rightarrow \mathbb{C}$. In this case, $\text{tr}_{(\alpha)}: \mathcal{A} \rightarrow \mathbb{C}$ induces a trace map

$$\text{tr}_{(\alpha)}: K_0(C_r^*(\Gamma)) \cong K_0(\mathcal{A}) \rightarrow \mathbb{C}$$

and a determinant map (cf. [36])

$$\tau_\alpha: K_1(C_{L,0}^*(E\Gamma)^\Gamma) \rightarrow \mathbb{C}$$

such that the following diagram commutes:

$$\begin{array}{ccc} K_0(C_r^*(\Gamma)) & \xrightarrow{\partial} & K_1(C_{L,0}^*(E\Gamma)^\Gamma) \\ \text{tr}_{(\alpha)} \downarrow & & \downarrow \tau_\alpha \\ \mathbb{C} & \xrightarrow{=} & \mathbb{C}. \end{array}$$

¹⁰In the case of Proposition 4.7, we map KO -theory to K -theory.

Such a smooth dense subalgebra indeed exists if $\langle \alpha \rangle$ has polynomial growth (cf. [13,36]) or Γ is word hyperbolic (cf. [12,25]).

Note that the canonical morphism

$$K_1(C_{L,0,\max}^*(E\Gamma)^\Gamma) \rightarrow K_1(C_{L,0}^*(E\Gamma)^\Gamma)$$

maps $\rho_{\max}(\tilde{D})$ to $\rho(\tilde{D})$. Let p_r be the image of p_{\max} under the canonical morphism

$$K_0(C_{\max}^*(\Gamma)) \rightarrow K_0(C_r^*(\Gamma)).$$

The same argument from the proof of Theorem 4.3 shows that $\partial(p_r) = \rho(\tilde{D})$ and

$$\frac{1}{2}\eta_{\langle \alpha \rangle}(\tilde{D}) = -\tau_\alpha(\rho(\tilde{D})) = \text{tr}_{\langle \alpha \rangle}(p_r).$$

Similarly, for each finite-index normal subgroup $\Gamma_i \subset \Gamma$, let

$$(\pi_{\Gamma_i})_*: K_0(C_{\max}^*(\Gamma)) \rightarrow K_0(C_r^*(\Gamma/\Gamma_i))$$

be the natural morphism induced by the quotient map $\pi_{\Gamma_i}: \Gamma \rightarrow \Gamma/\Gamma_i$. Let us denote $p_i := (\pi_{\Gamma_i})_*(p)$. We have $\partial(p_i) = \rho(D_{\Gamma_i})$ and

$$\frac{1}{2}\eta_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(D_{\Gamma_i}) = -\tau_i(\rho(D_{\Gamma_i})) = \text{tr}_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(p_i),$$

where

$$\tau_i: K_1(C_{L,0}^*(E(\Gamma/\Gamma_i))^{\Gamma/\Gamma_i}) \rightarrow \mathbb{C}$$

is a determinant map induced by the trace map $\text{tr}_{\langle \pi_{\Gamma_i}(\alpha) \rangle}$, cf. [36, Lemma 3.9 and Theorem 4.3]. Since p_{\max} is a formal difference of idempotents in $\mathcal{S} \otimes \mathbb{C}\Gamma$, it follows that the limit $\lim_{i \rightarrow \infty} \text{tr}_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(p_i)$ stabilizes and is equal to $\text{tr}_{\langle \alpha \rangle}(p_r)$. Thus, the limit

$$\lim_{i \rightarrow \infty} \eta_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(D_{\Gamma_i})$$

stabilizes and is equal to $\eta_{\langle \alpha \rangle}(\tilde{D})$.

In particular, as a consequence of the above discussion and Theorem 4.3, we have the following theorem

Theorem 4.8. *If Γ is both a-T-menable and word hyperbolic^u, then*

$$\lim_{i \rightarrow \infty} \eta_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(D_{\Gamma_i}) = \eta_{\langle \alpha \rangle}(\tilde{D}).$$

^uFor example, if Γ is a virtually free group, then it is both a-T-menable and word hyperbolic.

5. Scalar curvature and ℓ^1 -summability

In this section, we show that the answers to both part (I) and part (II) of Question 3.2 are positive when the scalar curvature of the given spin manifold M is bounded below by a sufficiently large positive number.

Throughout this section, assume M is an odd-dimensional closed spin manifold endowed with a positive scalar curvature metric and Γ is a finitely generated discrete group. Let \tilde{M} be a regular Γ -covering space of M and \tilde{D} be the Dirac operator lifted from M . For each finite-index normal subgroup Γ' of Γ , let $M_{\Gamma'} = \tilde{M}/\Gamma'$ be the associated finite-sheeted covering space of M . Denote by $D_{\Gamma'}$ the Dirac operator on $M_{\Gamma'}$ lifted from M .

Let S be a symmetric finite generating set of Γ and ℓ be the associated word length function on Γ . There exist $C > 0$ and $B > 0$ such that

$$(5.1) \quad \#\{\gamma \in \Gamma : \ell(\gamma) \leq n\} \leq Ce^{B \cdot n}$$

for all $n \geq 0$. Let K_Γ be the infimum of all such numbers B .

Furthermore, there exist $\theta_0, \theta_1, c_0, c_1 > 0$ such that

$$(5.2) \quad \theta_0 \cdot \ell(\beta) - c_0 \leq \text{dist}(x, \beta x) \leq \theta_1 \cdot \ell(\beta) + c_1$$

for all $x \in \mathcal{F}$ and $\beta \in \Gamma$, where \mathcal{F} is a fundamental domain of \tilde{M} under the action of Γ . In particular, we may define θ_0 as follows:

$$(5.3) \quad \theta_0 = \liminf_{\ell(\beta) \rightarrow \infty} \left(\inf_{x \in \mathcal{F}} \frac{\text{dist}(x, \beta x)}{\ell(\beta)} \right).$$

Definition 5.1. With the above notation, let us define

$$\sigma_\Gamma := \frac{2K_\Gamma}{\theta_0}.$$

The following theorem answers both part (I) and part (II) of Question 3.2 positively, under the condition that the spectral gap of \tilde{D} at zero is sufficiently large.

Theorem 5.2. *With the same notation as above, suppose $\{\Gamma_i\}$ is a sequence of finite-index normal subgroups that distinguishes the conjugacy class $\langle \alpha \rangle$ of a non-identity element $\alpha \in \Gamma$. If the spectral gap of \tilde{D} at zero is greater than σ_Γ , then*

$$\lim_{i \rightarrow \infty} \eta_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(D_{\Gamma_i}) = \eta_{\langle \alpha \rangle}(\tilde{D}).$$

Proof. It suffices to find a function of t that is a dominating function for all the following functions:

$$\begin{aligned} \mathrm{tr}_{(\alpha)}(\tilde{D}e^{-t^2\tilde{D}^2}) &= \sum_{\gamma \in (\alpha)} \int_{\mathcal{F}} \mathrm{tr}(K_t(x, \gamma x)) dx, \\ \mathrm{tr}_{(\pi_{\Gamma_i}(\alpha))}(D_{\Gamma_i}e^{-t^2D_{\Gamma_i}^2}) &= \sum_{\omega \in (\pi_{\Gamma_i}(\alpha))} \int_{\mathcal{F}} \mathrm{tr}((K_i)_t(x, \omega x)) dx, \end{aligned}$$

and show that $\mathrm{tr}_{(\pi_{\Gamma_i}(\alpha))}(D_{\Gamma_i}e^{-t^2D_{\Gamma_i}^2})$ converges to $\mathrm{tr}_{(\alpha)}(\tilde{D}e^{-t^2\tilde{D}^2})$ as $i \rightarrow \infty$, for each t . Indeed, the theorem then follows by the dominated convergence theorem.

Recall that $K_t(x, y)$ (resp. $(K_i)_t(x, y)$) is the Schwartz kernel of $\tilde{D}e^{-t^2\tilde{D}^2}$ (resp. $D_{\Gamma_i}e^{-t^2D_{\Gamma_i}^2}$). We have the following estimates (cf. [12, Section 3]).

(1) By [12, Lemma 3.8], for any $\mu > 1$ and $r > 0$, there exists a constant $c_{\mu,r} > 0$ such that

$$(5.4) \quad \|K_t(x, y)\| \leq c_{\mu,r} \cdot F_t\left(\frac{\mathrm{dist}(x, y)}{\mu}\right)$$

for $\forall x, y \in \tilde{M}$ with $\mathrm{dist}(x, y) > r$. Here $\|K_t(x, y)\|$ is the operator norm of the matrix $K_t(x, y)$, and the function F_t is defined by

$$F_t(s) := \sup_{n \leq \frac{3}{2} \dim M + 3} \int_{|\xi| > s} \left| \frac{d^n}{d\xi^n} \hat{f}_t(\xi) \right| d\xi,$$

where \hat{f}_t is the Fourier transform of $f_t(x) = xe^{-t^2x^2}$. It follows that for $\mu > 1$ and $r > 0$, there exist $c_{\mu,r} > 0, n_1 > 0$ and $m_1 > 0$ such that

$$(5.5) \quad \|K_t(x, y)\| \leq c_{\mu,r} \frac{(1 + \mathrm{dist}(x, y))^{n_1}}{t^{m_1}} \exp\left(\frac{-\mathrm{dist}(x, y)^2}{4\mu t^2}\right),$$

for all $t > 0$ and for all $x, y \in \tilde{M}$ with $\mathrm{dist}(x, y) > r$.

(2) By [12, Lemma 3.5], there exists $c_2 > 0$ such that

$$(5.6) \quad \sup_{x, y \in M} \|K_t(x, y)\| \leq c_2 \cdot \sup_{k+j \leq \frac{3}{2} \dim M + 3} \|\tilde{D}^k(\tilde{D}e^{-t^2\tilde{D}^2})\tilde{D}^j\|_{\mathrm{op}},$$

for all $x, y \in \tilde{M}$, where $\|\cdot\|_{\mathrm{op}}$ stands for the operator norm. It follows that there exist positive numbers c_2, m_2 and δ such that

$$(5.7) \quad \|K_t(x, y)\| \leq c_2 \frac{1}{t^{m_2}} \exp(-(\sigma_{\Gamma} + \delta)^2 \cdot t^2),$$

for all $t > 0$ and all $x, y \in \tilde{M}$.

In fact, since the manifolds \tilde{M} and M_{Γ_i} have uniformly bounded geometry, the constants $c_{\mu,r}$, n_1 , m_1 , c_2 , m_2 and δ from above can be chosen so that for all $i \geq 1$, we have

$$(5.8) \quad \|(K_i)_t(x, y)\| \leq c_{\mu,r} \frac{(1 + \text{dist}(x, y))^{n_1}}{t^{m_1}} \exp\left(\frac{-\text{dist}(x, y)^2}{4\mu t^2}\right),$$

for all $t > 0$ and for all $x, y \in M_{\Gamma_i}$ with $\text{dist}(x, y) > r$; and

$$(5.9) \quad \|(K_i)_t(x, y)\| \leq c_2 \frac{1}{t^{m_2}} \exp(-(\sigma_\Gamma + \delta)^2 \cdot t^2)$$

for all $t > 0$ and all $x, y \in M_{\Gamma_i}$.

For the rest of the proof, let us fix $r > 0$. Note that we have

$$\begin{aligned} \text{dist}(x, \gamma y) &\leq \text{dist}(x, \gamma x) + \text{dist}(\gamma x, \gamma y) \\ &= \text{dist}(x, \gamma x) + \text{dist}(x, y) \end{aligned}$$

for all $x, y \in \tilde{M}$ and $\gamma \in \Gamma$. Similarly, we have

$$\text{dist}(x, \gamma x) - \text{dist}(x, y) \leq \text{dist}(x, \gamma y).$$

By line (5.2), we have

$$\begin{aligned} \theta_0 \cdot \ell(\gamma) - c_0 - \text{dist}(x, y) &\leq \text{dist}(x, \gamma y) \\ &\leq \theta_1 \cdot \ell(\gamma) + c_1 + \text{dist}(x, y) \end{aligned}$$

for all $x, y \in \tilde{M}$ and $\gamma \in \Gamma$. In particular, there exist $c'_0 > 0$ and $c'_1 > 0$ such that

$$(5.10) \quad \theta_0 \cdot \ell(\gamma) - c'_0 \leq \text{dist}(x, \gamma y) \leq \theta_1 \cdot \ell(\gamma) + c'_1$$

for all $x, y \in \mathcal{F}$ and $\gamma \in \Gamma$, where \mathcal{F} is a precompact fundamental domain of \tilde{M} under the action of Γ . Let us define

$$F := \{\beta \in \Gamma \mid \text{dist}(x, \beta y) \leq r \text{ for some } x, y \in \mathcal{F}\}.$$

Clearly, F is a finite subset of Γ .

For any given $t > 0$, it follows from line (5.1), (5.5) and (5.10) that the Schwartz kernel K_t is ℓ^1 -summable (cf. Definition 2.5).

Now approximate $f_t(x) = xe^{-t^2x^2}$ by smooth functions $\{\varphi_j\}$ whose Fourier transforms are compactly supported. By applying the estimates in line (5.4) and (5.6) to the Schwartz kernel $K_{\varphi_j(\tilde{D})}$ (resp. $K_{\varphi_j(D_{\Gamma_i})}$) of the operator $\varphi_j(\tilde{D})$ (resp. $\varphi_j(D_{\Gamma_i})$), it is not difficult to see that $K_{\varphi_j(\tilde{D})}$ (resp. $K_{\varphi_j(D_{\Gamma_i})}$) converges to K_t (resp. $(K_i)_t$) in

ℓ^1 -norm (defined in Definition 2.5). Note that $\varphi_j(\tilde{D})$ has finite propagation. Since \tilde{D} locally coincides with D_{Γ_i} , it follows from finite propagation estimates of wave operators that (cf. [17]):

$$K_{\varphi_j(D_{\Gamma_i})}(\pi_{\Gamma_i}(x), \pi_{\Gamma_i}(y)) = \sum_{\beta \in \Gamma_i} K_{\varphi_j(\tilde{D})}(x, \beta y)$$

for all $x, y \in \tilde{M}$. As a consequence of the above discussion, we have

$$(5.11) \quad (K_i)_t(\pi_{\Gamma_i}(x), \pi_{\Gamma_i}(y)) = \sum_{\beta \in \Gamma_i} K_t(x, \beta y).$$

for all $x, y \in \tilde{M}$ and for all $t > 0$. Furthermore, by Lemma 2.6, we have the following convergence:

$$\text{tr}_{(\pi_{\Gamma_i}(\alpha))}(D_{\Gamma_i} e^{-t^2 D_{\Gamma_i}^2}) \rightarrow \text{tr}_{\langle \alpha \rangle}(\tilde{D} e^{-t^2 \tilde{D}^2}) \quad \text{as } j \rightarrow \infty,$$

for each $t > 0$, since $\{\Gamma_i\}$ distinguishes the conjugacy class $\langle \alpha \rangle$.

By line (5.5) and (5.7), there exists a positive number c_3 such that

$$\begin{aligned} \|K_t(x, y)\|^2 &\leq c_3 \frac{(1 + \text{dist}(x, y))^{n_1}}{t^{m_1+m_2}} \exp\left(\frac{-\text{dist}(x, y)^2}{4\mu t^2}\right) e^{-(\sigma_\Gamma + \varepsilon)^2 \cdot t^2} e^{-\varepsilon^2 t^2} \\ &\leq c_3 \frac{(1 + \text{dist}(x, y))^{n_1}}{t^{m_1+m_2}} \exp\left(\frac{-\text{dist}(x, y) \cdot (\sigma_\Gamma + \varepsilon)}{\mu}\right) e^{-\varepsilon^2 t^2}, \end{aligned}$$

for all $x, y \in \tilde{M}$ with $\text{dist}(x, y) > r$, where $\varepsilon = \delta/2$. By choosing $\mu > 1$ sufficiently close to 1, we see that there exist $c_4 > 0$ and $\lambda > 1$ such that

$$\|K_t(x, y)\| \leq c_4 \frac{e^{-\varepsilon^2 t^2}}{t^{(m_1+m_2)/2}} \exp\left(-\frac{\lambda}{2} \cdot \sigma_\Gamma \cdot \text{dist}(x, y)\right)$$

for all $x, y \in \tilde{M}$ with $\text{dist}(x, y) > r$. It follows that there exist $c_5 > 0$ and $m > 0$ such that

$$\begin{aligned} \sum_{\gamma \in \Gamma} \|K_t(x, \gamma y)\| &\leq \sum_{\gamma \in F} c_2 \frac{e^{-(\sigma_\Gamma + \delta)^2 \cdot t^2}}{t^{m_2}} + c_4 \frac{e^{-\varepsilon^2 t^2}}{t^{(m_1+m_2)/2}} \sum_{\gamma \notin F} e^{-\lambda \cdot K_\Gamma \cdot (\ell(\gamma) - c'_0 \tau_0^{-1})} \\ &\leq c_2 \cdot |F| \cdot \frac{e^{-\varepsilon^2 t^2}}{t^{m_2}} + c_4 \frac{e^{-\varepsilon^2 t^2}}{t^{(m_1+m_2)/2}} \sum_{n=0}^{\infty} e^{K_\Gamma \cdot n} e^{-\lambda \cdot K_\Gamma \cdot (n - c'_0 \tau_0^{-1})} \\ &< \frac{c_5}{t^m} e^{-\varepsilon^2 t^2} \end{aligned}$$

for all $t > 0$. In particular, we have

$$\sum_{\gamma \in \langle \alpha \rangle} |\text{tr } K_t(x, \gamma y)| < \frac{c_5}{t^m} e^{-\varepsilon^2 t^2}$$

for all $x, y \in \tilde{M}$ and all $t > 0$. By the same argument, we also have

$$\sum_{\omega \in \langle \pi_{\Gamma_i}(\alpha) \rangle} |\operatorname{tr}(K_i)_t(x', \omega y')| < \frac{c_5}{t^m} e^{-\varepsilon^2 t^2}$$

for all $x', y' \in M_{\Gamma_i}$ and all $t > 0$. Therefore, the functions

$$|\operatorname{tr}_{\langle \alpha \rangle}(\tilde{D} e^{-t^2 \tilde{D}^2})| \quad \text{and} \quad |\operatorname{tr}_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(D_{\Gamma_i} e^{-t^2 D_{\Gamma_i}^2})|$$

are all bounded by the function

$$c_5 \cdot t^{-m} e^{-\varepsilon^2 t^2}.$$

The latter is clearly absolutely integrable on $[1, \infty)$.

We have found above an appropriate dominating function on the interval $[1, \infty)$. Now let us find the dominating function on $(0, 1]$. Since Γ acts on \tilde{M} freely and cocompactly, it follows that there exists $\varepsilon_0 > 0$ such that

$$\operatorname{dist}(x, \gamma x) > \varepsilon_0$$

for all $x \in \tilde{M}$ and all $\gamma \neq e \in \Gamma$. By applying line (5.5), a similar calculation as above shows that there exist $\varepsilon_1 > 0$ and $c_6 > 0$ such that

$$\sum_{\gamma \in \Gamma} \|K_t(x, \gamma x)\| < \frac{c_6}{t^{m_1}} e^{-\varepsilon_1 t^{-2}}$$

for all $x \in \tilde{\mathcal{F}}$ and all $t \leq 1$. The same estimate also holds for $(K_i)_t$. Therefore, on the interval $(0, 1]$, the functions

$$|\operatorname{tr}_{\langle \alpha \rangle}(\tilde{D} e^{-t^2 \tilde{D}^2})| \quad \text{and} \quad |\operatorname{tr}_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(D_{\Gamma_i} e^{-t^2 D_{\Gamma_i}^2})|$$

are all bounded by the function

$$c_6 \cdot t^{-m_1} e^{-\varepsilon_1 t^{-2}}.$$

The latter is absolutely integrable on $(0, 1]$. This finishes the proof. ■

If the group Γ has subexponential growth, then it follows from Definition 5.1 that $\sigma_\Gamma = 0$. In this case, if \tilde{D} has a spectral gap, then it is automatically sufficiently large, hence the following immediate corollary.

Corollary 5.3. *With the above notation, suppose $\{\Gamma_i\}$ distinguishes the conjugacy class $\langle \alpha \rangle$ of a non-identity element $\alpha \in \Gamma$. If Γ has subexponential growth and \tilde{D} has a spectral gap at zero, then we have*

$$\lim_{i \rightarrow \infty} \eta_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(D_{\Gamma_i}) = \eta_{\langle \alpha \rangle}(\tilde{D}).$$

Remark 5.4. Recall that if Γ has subexponential growth, then Γ is amenable and thus a-T-menable [8]. So the convergence of the limit

$$\lim_{i \rightarrow \infty} \eta_{(\pi_{\Gamma_i}(\alpha))}(D_{\Gamma_i})$$

also follows from Proposition 4.1. In fact, Proposition 4.1 implies that the above limit stabilizes. On the other hand, Corollary 5.3 above answers positively both part (I) and part (II) of Question 3.2.

Remark 5.5. In this remark, we shall briefly comment on the condition that the spectral gap of \tilde{D} at zero is greater than σ_Γ in Theorem 5.2. Here is a class of natural examples such that the spectral gap of \tilde{D} at zero is greater than σ_Γ and the higher rho invariant¹² $\rho(\tilde{D})$ is non-zero.

Suppose that N is a closed spin manifold equipped with a positive scalar curvature metric g_N , whose fundamental group $F = \pi_1(N)$ is finite and its higher rho invariant $\rho(\tilde{D}_N)$ is non-trivial. Here \tilde{D}_N is the Dirac operator on the universal covering \tilde{N} of N . For instance, let N to be a lens space, that is, the quotient of the 3-dimensional sphere by a free action of a finite cyclic group. In this case, the classical equivariant Atiyah–Patodi–Singer index theorem implies that the delocalized higher rho invariant of N is non-trivial, cf. [14].

Now let X be an even-dimensional closed spin manifold, whose Dirac operator D_X has non-trivial higher index in $K_0(C_r^*(\Gamma))$, where $\Gamma = \pi_1(X)$. In particular, it follows that D_X defines a non-zero element in the equivariant K -homology $K_0(C_L^*(E\Gamma)^\Gamma)$ of the universal space $E\Gamma$ for free Γ actions. Consider the product space $M = V \times N$ equipped with a metric $g_M = g_X + \varepsilon \cdot g_N$, where g_X is an arbitrary Riemannian metric on X and the metric g_N on N is scaled by a positive number ε . Denote the Dirac operator on the universal covering \tilde{M} of M by \tilde{D}_M . The spectral gap of \tilde{D}_M at zero can always be made sufficiently large, as long as we choose ε to be sufficiently small. To see that $\rho(\tilde{D}_M)$ is non-zero in $K_1(C_{L,0}^*(E(\Gamma \times F))^{\Gamma \times F})$, we apply the product formula for secondary invariants (cf. [33, Claim 2.19] and [40, Corollary 4.15]), which states that the higher rho invariant $\rho(\tilde{D}_M)$ is the product of the K -homology class of D_X and the higher rho invariant $\rho(\tilde{D}_N)$. It follows from the above construction that the higher rho invariant $\rho(\tilde{D}_M)$ is non-zero in $K_1(C_{L,0}^*(E(\Gamma \times F))^{\Gamma \times F})$. In fact, if the Baum–Connes conjecture holds for Γ , then the K -theory group $K_1(C_{L,0}^*(E(\Gamma \times F))^{\Gamma \times F})$ is (at least rationally) generated by the higher rho invariants of the above examples, cf. [37, Theorem 3.7 and Corollary 3.16].

¹²As we have seen in the proofs of Theorem 4.1 and Theorem 4.8, the delocalized eta invariant $\eta_{(\alpha)}(\tilde{D})$ is essentially the pairing between the higher rho invariant $\rho(\tilde{D})$ and the delocalized trace $\text{tr}_{(\alpha)}$, cf. [36, Theorem 4.3].

On the other hand, we also would like to point out that for the operator \tilde{D}_M from the above examples, a straightforward calculation shows the delocalized eta invariant $\eta_{\langle\alpha\rangle}(\tilde{D}_M)$ could be non-zero only in the case when $\alpha = (1, a) \in \Gamma \times F$ with a being a non-identity element of $F = \pi_1(N)$. This essentially reduces the computation of $\eta_{\langle\alpha\rangle}(\tilde{D}_M)$ to the case of finite fundamental groups. Consequently, Question 3.2 has a positive answer for the above examples but for trivial reasons.

By the proof of Theorem 5.2 above, in order to bound the function

$$|\mathrm{tr}_{\langle\alpha\rangle}(\tilde{D}e^{-t^2\tilde{D}^2})|,$$

it suffices to assume the spectral gap of \tilde{D} at zero to be greater than

$$\sigma_{\langle\alpha\rangle} := \frac{2 \cdot K_{\langle\alpha\rangle}}{\theta_0},$$

where θ_0 is the constant from line (5.3) and $K_{\langle\alpha\rangle}$ is non-negative constant such that there exists some constant $C > 0$ satisfying

$$(5.12) \quad \#\{\gamma \in \langle\alpha\rangle : \ell(\gamma) \leq n\} \leq Ce^{K_{\langle\alpha\rangle} \cdot n}$$

for all n . In fact, if we have a uniform control of the spectral gap of D_{Γ_i} at zero and the growth rate of the conjugacy class $\{\langle\pi_{\Gamma_i}(\alpha)\rangle\}$ for all $i \geq 1$, a notion to be made precise in the following, then the same proof above also implies that

$$\lim_{i \rightarrow \infty} \eta_{\langle\pi_{\Gamma_i}(\alpha)\rangle}(D_{\Gamma_i}) = \eta_{\langle\alpha\rangle}(\tilde{D})$$

in this case.

Recall that S a symmetric finite generating set of Γ . For each normal subgroup Γ_i of Γ , the map $\pi_{\Gamma_i}: \Gamma \rightarrow \Gamma/\Gamma_i$ is the canonical quotient map. The set $\pi_{\Gamma_i}(S)$ is a symmetric generating set for Γ/Γ_i , hence induces a word length function ℓ_{Γ_i} on Γ/Γ_i . More explicitly, we have

$$(5.13) \quad \ell_{\Gamma_i}(\omega) := \inf\{\ell(\beta) : \beta \in \pi_{\Gamma_i}^{-1}(\omega)\}$$

for all $\omega \in \Gamma/\Gamma_i$.

Definition 5.6. For a given conjugacy class $\langle\alpha\rangle$ of Γ , we say that $\langle\alpha\rangle$ has uniform exponential growth with respect to a family of normal subgroups $\{\Gamma_i\}$, if there exist $C > 0$ and $A \geq 0$ such that

$$(5.14) \quad \#\{\omega \in \langle\pi_{\Gamma_i}(\alpha)\rangle : \ell_{\Gamma_i}(\omega) \leq n\} \leq Ce^{A \cdot n}.$$

for all $i \geq 1$ and all $n \geq 0$. In this case, we define K_u to be the infimum of all such numbers A .

Definition 5.7. With the above notation, we define

$$\sigma_u := \frac{2 \cdot K_u}{\theta_0},$$

where θ_0 is the constant from line (5.3).

The same argument from the proof of Theorem 5.2 can be used to prove the following.

Theorem 5.8. *With the same notation as in Theorem 5.2, suppose $\{\Gamma_i\}$ is a sequence of finite-index normal subgroups that distinguishes the conjugacy class $\langle \alpha \rangle$ of a non-identity element $\alpha \in \Gamma$. Assume $\langle \alpha \rangle$ has uniform exponential growth with respect to $\{\Gamma_i\}$. If there exists $\varepsilon > 0$ such that the spectral gap of D_{Γ_i} at zero is greater than $\sigma_u + \varepsilon$ for sufficiently large $i \gg 1$, then*

$$\lim_{i \rightarrow \infty} \eta_{(\pi_{\Gamma_i}(\alpha))}(D_{\Gamma_i}) = \eta_{\langle \alpha \rangle}(\tilde{D}).$$

Remark 5.9. Here is a geometric condition on M that guarantees the spectral gap of D_{Γ_i} at zero to be greater than $\sigma_u + \varepsilon$ for all $i \geq 1$. If the scalar curvature of M is strictly bounded below¹³ by $4 \cdot \sigma_u^2$, then it follows from the Lichnerowicz formula that there exists $\varepsilon > 0$ such that the spectral gap of D_{Γ_i} at zero is greater than $\sigma_u + \varepsilon$ for all $i \geq 1$.

6. Separation rates of conjugacy classes

In this section, we introduce a notion of separation rate for how fast a sequence of normal subgroups $\{\Gamma_i\}$ of Γ distinguishes a conjugacy class $\langle \alpha \rangle$ of Γ and use it to answer Question 3.2 in some cases.

Definition 6.1. For each normal subgroup Γ' of Γ , let $\pi_{\Gamma'}: \Gamma \rightarrow \Gamma/\Gamma'$ be the quotient map from Γ to Γ/Γ' . Given a conjugacy class $\langle \alpha \rangle$ of Γ , we define the injective radius of $\pi_{\Gamma'}$ with respect to $\langle \alpha \rangle$ to be

$$(6.1) \quad r(\Gamma') := \max\{n \mid \text{if } \gamma \notin \langle \alpha \rangle \text{ and } \ell(\gamma) \leq n, \text{ then } \pi_{\Gamma'}(\gamma) \notin \langle \pi_{\Gamma'}(\alpha) \rangle\}.$$

Definition 6.2. Suppose that $\{\Gamma_i\}$ is a sequence of finite-index normal subgroups of Γ that distinguishes $\langle \alpha \rangle$. We say that $\{\Gamma_i\}$ distinguishes $\langle \alpha \rangle$ sufficiently fast if there exist $C > 0$ and $R > 0$ such that

$$(6.2) \quad |\langle \pi_{\Gamma_i}(\alpha) \rangle| \leq C e^{R \cdot r(\Gamma_i)}.$$

In this case, we define the separation rate $R_{\langle \alpha \rangle, \{\Gamma_i\}}$ of $\langle \alpha \rangle$ with respect to $\{\Gamma_i\}$ to be the infimum of all such numbers R .

¹³The scalar curvature function $\kappa(x)$ of M satisfies that $\kappa(x) > 4 \cdot \sigma_u^2$ for all $x \in M$.

We have the following proposition.

Proposition 6.3. *Let $\langle \alpha \rangle$ be the conjugacy class of a non-identity element $\alpha \in \Gamma$. Suppose $\{\Gamma_i\}$ is a sequence of finite-index normal subgroups that distinguishes $\langle \alpha \rangle$ sufficiently fast with separation rate $R = R_{\langle \alpha \rangle, \{\Gamma_i\}}$. If $\eta_{\langle \alpha \rangle}(\tilde{D})$ is finite¹⁴ and there exists $\varepsilon > 0$ such that the spectral gap of D_{Γ_i} at zero is greater than¹⁵ $\sigma_R + \varepsilon$ for all sufficiently large $i \gg 1$, where*

$$\sigma_R := \frac{2(K_\Gamma \cdot R)^{1/2}}{\theta_0},$$

then we have

$$\lim_{i \rightarrow \infty} \eta_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(D_{\Gamma_i}) = \eta_{\langle \alpha \rangle}(\tilde{D}).$$

Proof. By assumption, the integral

$$\eta_{\langle \alpha \rangle}(\tilde{D}) := \frac{2}{\sqrt{\pi}} \int_0^\infty \text{tr}_{\langle \alpha \rangle}(\tilde{D} e^{-t^2 \tilde{D}^2}) dt$$

converges. To prove the proposition, it suffices to show that there exists a sequence of positive real numbers $\{s_i\}$ such that $s_i \rightarrow \infty$ as $i \rightarrow \infty$, and

$$(6.3) \quad \lim_{i \rightarrow \infty} \int_0^{s_i} (\text{tr}_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(D_{\Gamma_i} e^{-t^2 D_{\Gamma_i}^2}) - \text{tr}_{\langle \alpha \rangle}(\tilde{D} e^{-t^2 \tilde{D}^2})) dt = 0,$$

$$(6.4) \quad \lim_{i \rightarrow \infty} \int_{s_i}^\infty \text{tr}_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(D_{\Gamma_i} e^{-t^2 D_{\Gamma_i}^2}) dt = 0.$$

Let $K_t(x, y)$ (resp. $(K_i)_t(x, y)$) be the Schwartz kernel of $\tilde{D} e^{-t^2 \tilde{D}^2}$ (resp. $D_{\Gamma_i} e^{-t^2 D_{\Gamma_i}^2}$). Recall that we have (cf. line (5.11))

$$(K_i)_t(\pi_{\Gamma_i}(x), \pi_{\Gamma_i}(y)) = \sum_{\beta \in \Gamma_i} K_t(x, \beta y)$$

for all $x, y \in \tilde{M}$. It follows that

$$|\text{tr}_{\langle \pi_{\Gamma_i}(\alpha) \rangle}(D_{\Gamma_i} e^{-t^2 D_{\Gamma_i}^2}) - \text{tr}_{\langle \alpha \rangle}(\tilde{D} e^{-t^2 \tilde{D}^2})| \leq \sum_{\substack{\gamma \in \pi_{\Gamma_i}^{-1}(\pi_{\Gamma_i}(\alpha)) \\ \text{but } \gamma \notin \langle \alpha \rangle}} \int_{x \in \mathcal{F}} \|K_t(x, \gamma x)\| dx.$$

¹⁴To be precise, $\eta_{\langle \alpha \rangle}(\tilde{D})$ is finite if the integral in line (3.1) converges. In particular, the integral in line (3.1) does *not* necessarily absolutely converge.

¹⁵For example, if the spectral gap of \tilde{D} at zero is greater than $\sigma_R + \varepsilon$ for some positive number ε , then the spectral gap of D_{Γ_i} at zero is greater than $\sigma_R + \varepsilon$ for all $i \geq 1$, cf. Remark 5.9.

By the definition of $r(\Gamma_i)$ in line (6.1), we see that

$$\{\gamma \in \Gamma \mid \gamma \in \pi_{\Gamma_i}^{-1}(\pi_{\Gamma_i}(\alpha)) \text{ but } \gamma \notin \langle \alpha \rangle\} \subseteq \{\gamma \in \Gamma \mid \ell(\gamma) \geq r(\Gamma_i)\}.$$

From line (5.5), we have

$$\sum_{\ell(\gamma) \geq r(\Gamma_i)} \int_{x \in \mathcal{F}} \|K_t(x, \gamma x)\| dx \leq c_{\mu, r} \sum_{m=r(\Gamma_i)}^{\infty} e^{-(\theta_0 \cdot m - c_0)^2 / 4\mu t^2} e^{K_{\Gamma} \cdot m}.$$

Note that

$$\int_0^{s_i} \sum_{m=r(\Gamma_i)}^{\infty} e^{-(\theta_0 \cdot m - c_0)^2 / 4\mu t^2} e^{K_{\Gamma} \cdot m} dt \leq s_i \sum_{m=r(\Gamma_i)}^{\infty} e^{-((\theta_0 \cdot m - c_0)^2 / 4\mu s_i^2) + K_{\Gamma} \cdot m}.$$

The right-hand side goes to zero as $s_i \rightarrow \infty$, as long as there exists $\lambda_1 > 1$ such that

$$(6.5) \quad \frac{(\theta_0 \cdot r(\Gamma_i) - c_0)^2}{4\mu s_i^2} > \lambda_1 \cdot K_{\Gamma} \cdot r(\Gamma_i)$$

for all sufficiently large $i \gg 1$. Since $\{\Gamma_i\}$ distinguishes $\langle \alpha \rangle$, we have that $r(\Gamma_i) \rightarrow \infty$, as $i \rightarrow \infty$. So the condition in line (6.5) is equivalent to

$$s_i^2 < \frac{\theta_0^2 \cdot r(\Gamma_i)}{4\mu \cdot \lambda_1 \cdot K_{\Gamma}}$$

for sufficiently large $i \gg 1$.

On the other hand, by the inequality from line (5.9), there exist $c > 0$ and $\varepsilon > 0$ such that

$$\left| \int_{s_i}^{\infty} \text{tr}_{(\pi_{\Gamma_i}(\alpha))} (D_{\Gamma_i} e^{-t^2 D_{\Gamma_i}^2}) dt \right| \leq c \cdot e^{-(\sigma_R + \varepsilon)^2 \cdot s_i^2} \cdot |\langle \pi_{\Gamma_i}(\alpha) \rangle|$$

for all sufficiently large $i \gg 1$. Note that the right-hand side goes to zero as $s_i \rightarrow \infty$, as long as there exists $\lambda_2 > 1$ such that

$$(6.6) \quad (\sigma_R + \varepsilon)^2 \cdot s_i^2 > \lambda_2 \cdot R \cdot r(\Gamma_i).$$

for all sufficiently large $j \gg 1$. Combining the two inequalities in line (6.5) and (6.6) together, we can choose a sequence of real numbers $\{s_i\}$ that satisfies the limits in both line (6.3) and (6.4), as long as there exists $\lambda_3 > 1$ such that

$$(6.7) \quad \frac{\theta_0^2 \cdot r(\Gamma_i)}{4\mu \cdot \lambda_1 \cdot K_{\Gamma}} > \lambda_3 \frac{R \cdot r(\Gamma_i)}{(\sigma_R + \varepsilon)^2}$$

for all sufficiently large $i \gg 1$. By choosing μ sufficiently close to 1, the inequality in line (6.7) follows from the definition of σ_R . This finishes the proof. \blacksquare

We finish this section with the following calculation of the separation rates of conjugacy classes of $SL_2(\mathbb{Z})$. The group $SL_2(\mathbb{Z})$ is a conjugacy separable group [28]. It has a presentation:

$$\langle x, y \mid x^4 = 1, x^2 = y^3 \rangle,$$

where $x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. In particular, it follows that $SL_2(\mathbb{Z})$ is an amalgamated free product of a cyclic group of order 4 and a cyclic group of order 6. We now show that for any finite order element $\alpha \in SL_2(\mathbb{Z})$, there exists a sequence of finite-index normal subgroups $\{\Gamma_i\}$ of $SL_2(\mathbb{Z})$ that distinguishes $\langle \alpha \rangle$ such that the corresponding separation rate $R_{\langle \alpha \rangle, \{\Gamma_i\}} = 0$.

Since $SL_2(\mathbb{Z})$ is an amalgamated free product of a cyclic group of order 4 and a cyclic group of order 6, every finite order element in $SL_2(\mathbb{Z})$ is conjugate to an element in one of the factors. It follows that every finite order element of $SL_2(\mathbb{Z})$ is conjugate to a power of x and y . Let $\psi: SL_2(\mathbb{Z}) \rightarrow \mathbb{Z}/12\mathbb{Z}$ be the group homomorphism defined by $\psi(x) = 3$ and $\psi(y) = 2$. In particular, we have

$$\begin{aligned} \psi(e) = 0, \quad \psi(x) = 3, \quad \psi(x^2) = \psi(y^3) = 6, \quad \psi(x^3) = 9, \\ \psi(y) = 2, \quad \psi(y^2) = 4, \quad \psi(y^4) = 8, \quad \psi(y^5) = 10. \end{aligned}$$

It follows that any finite order elements γ_1 and γ_2 of $SL_2(\mathbb{Z})$ are conjugate in $SL_2(\mathbb{Z})$ if and only if $\psi(\gamma_1) = \psi(\gamma_2)$.

Now given any finite-index normal subgroup N of $SL_2(\mathbb{Z})$, the group

$$N_1 = N \cap \ker(\psi)$$

is a finite-index normal subgroup of $SL_2(\mathbb{Z})$. By the discussion above, we see that any finite order elements γ_1 and γ_2 of $SL_2(\mathbb{Z})$ are conjugate in $SL_2(\mathbb{Z})$ if and only if they are conjugate in $SL_2(\mathbb{Z})/N_1$. In other words, the set $\{N_1\}$ consisting of a single finite-index normal subgroup distinguishes the conjugacy class $\langle \alpha \rangle$ of any finite order element $\alpha \in SL_2(\mathbb{Z})$. Moreover, the injective radius $r(N_1)$ of $\pi_{N_1}: SL_2(\mathbb{Z}) \rightarrow SL_2(\mathbb{Z})/N_1$ with respect to $\langle \alpha \rangle$ is infinity. It follows that the separation rate $R_{\langle \alpha \rangle, \{N_1\}} = 0$ in this case.

Remark 6.4. Since $SL_2(\mathbb{Z})$ is hyperbolic, Puschnigg’s smooth dense subalgebra \mathcal{A} of $C_r^*(SL_2(\mathbb{Z}))$ admits a continuous extension of the trace map $\text{tr}_{\langle \alpha \rangle}$ for any conjugacy class $\langle \alpha \rangle$ of $SL_2(\mathbb{Z})$ (cf. [25]). In this case, for any element $\alpha \neq e \in \Gamma$, the delocalized eta invariant $\eta_{\langle \alpha \rangle}(\tilde{D})$ is finite¹⁶ (cf. [22, Section 4] and [12, Section 6]). Hence, we can apply Proposition 6.3 to answer positively both part (I) and (II) of Question 3.2 for the group $SL_2(\mathbb{Z})$, when α is a finite order element.

¹⁶For any hyperbolic group and the conjugacy class of any non-identity element, the integral in line (3.1) absolutely converges.

On other hand, since $SL_2(\mathbb{Z})$ is also a-T-menable, we can equally apply Proposition 4.1 to answer positively both part (I) and (II) of Question 3.2 for the group $SL_2(\mathbb{Z})$ (cf. the discussion at the end of Section 4).

Appendix A. Positive scalar curvature and the Stolz conjecture

In this appendix, we shall explain how the geometric conditions given in Definitions 4.5 and 4.6 are related to the reduced Baum–Connes conjecture and the Stolz conjecture on positive scalar curvature metrics.

Definition A.1. Given a topological space Y , let $R_n^{\text{spin}}(Y)$ be the following bordism group of triples (L, f, h) , where L is an n -dimensional compact spin manifold (possibly with boundary), $f: L \rightarrow Y$ is a continuous map, and h is a positive scalar curvature metric on the boundary ∂M . Two triples (L_1, f_1, h_1) and (L_2, f_2, h_2) are bordant if

- (a) there is a bordism (V, F, H) between $(\partial L_1, f_1, h_1)$ and $(\partial L_2, f_2, h_2)$ such that H is a positive scalar curvature metric on V with product structure near ∂L_i and $H|_{\partial L_i} = h_i$, and the restriction of the map $F: V \rightarrow Y$ on ∂L_i is f_i ;
- (b) and the closed spin manifold $L_1 \cup_{\partial L_1} V \cup_{\partial L_2} L_2$ (obtained by gluing L_1 , V , and L_2 along their common boundaries) is the boundary of a spin manifold W with a map $E: W \rightarrow Y$ such that $E|_{L_i} = f_i$ and $E|_V = F$.

The above definition has the following obvious analogue for the case of proper actions.

Definition A.2. Let X be a proper metric space equipped with a proper and cocompact isometric action of a discrete group Γ . We denote by $R_n^{\text{spin}}(X)^\Gamma$ the set of bordism classes of pairs (L, f, h) , where L is an n -dimensional complete spin manifold equipped with a proper and cocompact isometric action of Γ , the map $f: L \rightarrow X$ is a Γ -equivariant continuous map and h is a Γ -invariant positive scalar curvature metric on ∂L . Here the bordism equivalence relation is defined similarly to the non-equivariant case above.

If the action of Γ on X is free and proper, then it follows by definition that

$$R_n^{\text{spin}}(X)^\Gamma \cong R_n^{\text{spin}}(X/\Gamma).$$

Suppose (L, f, h) is an element in $R_n^{\text{spin}}(E\Gamma)^\Gamma \cong R_n^{\text{spin}}(B\Gamma)$, where $B\Gamma = E\Gamma/\Gamma$ is the classifying space for free Γ -actions. Let L_Γ be the Γ -covering space of M induced by the map $f: L \rightarrow B\Gamma$ and D_{L_Γ} be the associated Dirac operator. Due to the positive scalar curvature metric h on ∂L , the Γ -equivariant operator D_{L_Γ} has a well-defined

higher index class $\text{Ind}(D_{L_\Gamma})$ in $K_n(C_r^*(\Gamma))$, cf. [26, Proposition 3.11] and [27]. By the relative higher index theorem [10, 32], we have the following well-defined index map

$$\text{Ind}: R_n^{\text{spin}}(B\Gamma) \rightarrow KO_n(C_r^*(\Gamma; \mathbb{R})), \quad (L, f, h) \mapsto \text{Ind}(D_{L_\Gamma}),$$

where $C_r^*(\Gamma; \mathbb{R})$ is the reduced group C^* -algebra of Γ with real coefficients. Now let \mathfrak{B} be the Bott manifold, a simply connected spin manifold of dimension 8 with $\widehat{A}(\mathfrak{B}) = 1$. This manifold is not unique, but any choice will work for the following discussion. To make the discussion below more transparent, let us choose a \mathfrak{B} that is equipped with a scalar flat curvature metric. The fact that such a choice exists follows for example from the work of Joyce [19, Section 6].

Let (L, f, h) be an element $R_n^{\text{spin}}(B\Gamma)$, that is, L is an n -dimensional spin manifold whose boundary ∂L carries a positive scalar curvature metric h , together with a map $f: L \rightarrow B\Gamma$. Taking direct product with k copies of \mathfrak{B} produces an element (L', f', h') in $R_{n+8k}^{\text{spin}}(B\Gamma)$, where $L' = L \times \mathfrak{B} \times \cdots \times \mathfrak{B}$, $f' = f \circ p$ with the map p being the projection from L' to L , and h' is the product metric of h with the Riemannian metric on \mathfrak{B} . By our choice of \mathfrak{B} above, the Riemannian metric h' also has positive scalar curvature since h does. Define $R_n^{\text{spin}}(B\Gamma)[\mathfrak{B}^{-1}]$ to be the direct limit of the following directed system:

$$R_n^{\text{spin}}(B\Gamma) \xrightarrow{\times \mathfrak{B}} R_{n+8}^{\text{spin}}(B\Gamma) \xrightarrow{\times \mathfrak{B}} R_{n+16}^{\text{spin}}(B\Gamma) \rightarrow \cdots .$$

Since the higher index class $\text{Ind}(D_{L_\Gamma})$ associated to (L, f, h) is invariant under taking direct product with \mathfrak{B} , it follows that the above index map induces the following well-defined index map:

$$\theta: R_n^{\text{spin}}(B\Gamma)[\mathfrak{B}^{-1}] \rightarrow KO_n(C_r^*(\Gamma; \mathbb{R})), \quad (L, f, h) \mapsto \text{Ind}(D_{L_\Gamma}).$$

Conjecture A.3 (Stolz conjecture [29, 30]). *The index map*

$$\theta: R_n^{\text{spin}}(B\Gamma)[\mathfrak{B}^{-1}] \rightarrow KO_n(C_r^*(\Gamma; \mathbb{R}))$$

is an isomorphism.

Similarly, if one works with the universal space $\underline{E}\Gamma$ for proper Γ -actions instead, then the same argument from above also produces a similar index map

$$\Theta: R_n^{\text{spin}}(\underline{E}\Gamma)^\Gamma[\mathfrak{B}^{-1}] \rightarrow KO_n(C_r^*(\Gamma; \mathbb{R})),$$

where $R_n^{\text{spin}}(\underline{E}\Gamma)^\Gamma[\mathfrak{B}^{-1}]$ is the direct limit of the following directed system:

$$R_n^{\text{spin}}(\underline{E}\Gamma)^\Gamma \xrightarrow{\times \mathfrak{B}} R_{n+8}^{\text{spin}}(\underline{E}\Gamma)^\Gamma \xrightarrow{\times \mathfrak{B}} R_{n+16}^{\text{spin}}(\underline{E}\Gamma)^\Gamma \rightarrow \cdots .$$

One has the following analogue of the Stolz conjecture above, which will be called the generalized Stolz conjecture from now on.

Conjecture A.4 (Generalized Stolz conjecture). *The index map*

$$\Theta: R_n^{\text{spin}}(\underline{E}\Gamma)^\Gamma[\mathfrak{B}^{-1}] \rightarrow KO_n(C_r^*(\Gamma; \mathbb{R}))$$

is an isomorphism.

If Γ is torsion-free, then clearly the generalized Stolz conjecture coincides with the original Stolz conjecture. By definition, the surjectivity of the Stolz map θ in Conjecture A.3 implies the surjectivity the generalized Stolz map Θ in Conjecture A.4. On the other hand, the surjectivity of the Stolz map θ follows from the surjectivity of the Baum–Connes assembly map

$$\mu_{\mathbb{R}}: KO_\bullet^\Gamma(\underline{E}\Gamma) \rightarrow KO_\bullet(C_r^*(\Gamma; \mathbb{R}))$$

cf. [37, Corollary 3.15]. At the time of writing of this paper, the injectivity of the Stolz map θ or generalized Stolz map Θ is wild open, and it is not even known in the case where Γ is the trivial group.

Similarly, one could also formulate the maximal version of the (generalized) Stolz conjecture by considering the index maps

$$\begin{aligned} \theta_{\max}: R_n^{\text{spin}}(B\Gamma)[\mathfrak{B}^{-1}] &\rightarrow KO_n(C_{\max}^*(\Gamma; \mathbb{R})), \\ \Theta_{\max}: R_n^{\text{spin}}(\underline{E}\Gamma)^\Gamma[\mathfrak{B}^{-1}] &\rightarrow KO_n(C_{\max}^*(\Gamma; \mathbb{R})), \end{aligned}$$

respectively. Again, the surjectivity of θ_{\max} , hence that of Θ_{\max} , follows from the surjectivity of the maximal Baum–Connes assembly map

$$\mu_{\mathbb{R}}: KO_\bullet^\Gamma(\underline{E}\Gamma) \rightarrow KO_\bullet(C_{\max}^*(\Gamma; \mathbb{R})).$$

A.1. Stable bounding with respect to $B\Gamma$. In this subsection, we shall discuss how the assumption that a multiple of (M, φ, h) stably bounds with respect to $B\Gamma$ (cf. Definition 4.5) is related to the Baum–Connes conjecture and the Stolz conjecture. Here again M is a closed spin manifold equipped with a Riemannian metric h of positive scalar curvature. Let $\varphi: M \rightarrow B\Gamma$ be the classifying map for the covering $\tilde{M} \rightarrow M$, that is, the pullback of $E\Gamma$ by φ is \tilde{M} .

To be more precise, in this subsection, let us assume the Baum–Connes assembly map

$$\mu_{\mathbb{R}}: KO_\bullet^\Gamma(\underline{E}\Gamma) \rightarrow KO_\bullet(C_r^*(\Gamma; \mathbb{R}))$$

is rationally isomorphic¹⁷. There is a long exact sequence for KO -theory of reduced C^* -algebras analogous to commutative diagram (4.1). Now a similar argument as in

¹⁷The rational bijectivity of $\mu_{\mathbb{R}}$ follows from the rational bijectivity of the complex version $\mu: K_\bullet^\Gamma(\underline{E}\Gamma) \rightarrow K_\bullet(C_r^*(\Gamma))$, cf. [7].

the proof of Theorem 4.3 shows the higher rho invariant $\rho(\tilde{D}) = \partial[p]$ (up to rational multiples) for some element $[p] \in KO_{n+1}(C_r^*(\Gamma; \mathbb{R}))$, where $n = \dim M$. Now the rational surjectivity of the Baum–Connes assembly map

$$\mu_{\mathbb{R}}: KO_{\bullet}^{\Gamma}(\underline{E}\Gamma) \rightarrow KO_{\bullet}(C_r^*(\Gamma; \mathbb{R}))$$

implies the rational surjectivity of the Stolz map

$$\theta: R_{\bullet}^{\text{spin}}(B\Gamma)[\mathfrak{B}^{-1}] \rightarrow KO_{\bullet}(C_r^*(\Gamma; \mathbb{R})),$$

cf. [37, Corollary 3.15]. And the rational surjectivity of θ implies that there exists an element $(L, f, h) \in R_{n+1}^{\text{spin}}(B\Gamma)[\mathfrak{B}^{-1}]$ such that $\theta(L, f, h) = [p]$ (up to a rational multiple). Recall that (cf. [24, Theorem 1.14] and [33, Theorem A])

$$\partial(\theta(L, f, g)) = \rho(D_{\partial\tilde{L}}) \quad \text{in } KO_n(C_{L,0}^*(E\Gamma; \mathbb{R})^{\Gamma}),$$

where $\rho(D_{\partial\tilde{L}})$ is the higher rho invariant of $D_{\partial\tilde{L}}$ with respect to the positive scalar curvature metric h . In particular, this implies that

$$\rho(\tilde{D}) = \rho(D_{\partial\tilde{L}}).$$

Hence, as far as $\rho(\tilde{D})$ is concerned, we could work with $(\partial L, f, g)$, which clearly bounds, instead of (M, φ, h) . On the other hand, it is an open question whether the higher rho invariants for M and ∂L remain equal to each other, for corresponding finite-sheeted covering spaces of M and ∂L .

A.2. Positively stable bounding with respect to $\underline{E}\Gamma$. In this subsection, we shall discuss how the assumption that a multiple of (\tilde{M}, \tilde{h}) positively stably bounds with respect to $\underline{E}\Gamma$ (cf. Definition 4.6) is related to the Baum–Connes conjecture and the generalized Stolz conjecture.

Observe that the injectivity of the generalized Stolz map

$$\Theta: R_n^{\text{spin}}(\underline{E}\Gamma)^{\Gamma}[\mathfrak{B}^{-1}] \rightarrow KO_n(C_r^*(\Gamma; \mathbb{R}))$$

has the following immediate geometric consequence. Let (L, f, h) be an element in $R_n^{\text{spin}}(\underline{E}\Gamma)^{\Gamma}$, that is, L is a n -dimensional spin Γ -manifold whose boundary ∂L carries a Γ -invariant positive scalar curvature metric h , together with a Γ -equivariant map $f: L \rightarrow \underline{E}\Gamma$. Suppose the higher index $\text{Ind}(D_L)$ associated to (L, f, h) vanishes, then the injectivity of the generalized Stolz map implies that (L, f, h) is stably Γ -equivariantly cobordant to the empty set. More precisely, if (L', f', h') is the direct product of (L, f, h) with sufficiently many copies of \mathfrak{B} , then (L', f', h') is Γ -equivariantly cobordant to the empty set. In particular, this implies that, if the higher

index of (L, f, h) vanishes, then ∂L positively stably bounds with respect to $\underline{E}\Gamma$, cf. Definition 4.6. More precisely, $\partial L'$ bounds a spin Γ -manifold V such that V admits a Γ -invariant positive scalar curvature metric g_0 , which has product structure near the boundary $\partial V = \partial L'$, and the restriction of g_0 to the boundary is equal to h' .

Now let M be a closed spin manifold equipped with a Riemannian metric h of positive scalar curvature. Let $\varphi: M \rightarrow B\Gamma$ be the classifying map for the covering $\tilde{M} \rightarrow M$ and \tilde{h} the metric on \tilde{M} lifted from h . The above discussion has the following consequence.

Lemma A.5. *With the above notation, suppose a multiple of (M, φ, h) stably bounds with respect to $B\Gamma$. If the Baum–Connes assembly map*

$$\mu_{\mathbb{R}}: KO_{\bullet}^{\Gamma}(\underline{E}\Gamma) \rightarrow KO_{\bullet}(C_r^*(\Gamma; \mathbb{R}))$$

is rationally surjective and the generalized Stolz map

$$\Theta: R_n^{\text{spin}}(\underline{E}\Gamma)^{\Gamma}[\mathfrak{B}^{-1}] \rightarrow KO_n(C_r^*(\Gamma; \mathbb{R}))$$

is rationally injective, then (\tilde{M}, \tilde{h}) positively stably bounds with respect to $\underline{E}\Gamma$.

Proof. For notational simplicity, let us assume (M, φ, h) itself bounds with respect to $B\Gamma$, that is, there exists a compact spin manifold W and a map $\Phi: W \rightarrow B\Gamma$ such that $\partial W = M$ and $\Phi|_{\partial W} = \varphi$.

Endow W with a Riemannian metric g which has product structure near $\partial W = M$ and whose restriction on ∂W is the positive scalar curvature metric h . Let \tilde{W} be the covering space of W induced by the map $\Phi: W \rightarrow B\Gamma$ and \tilde{g} be the lift of g from W to \tilde{W} . Due to the positive scalar curvature of \tilde{g} near the boundary of \tilde{W} , the corresponding Dirac operator $D_{\tilde{W}}$ on \tilde{W} with respect to the metric \tilde{g} has a well-defined higher index $\text{Ind}(D_{\tilde{W}}, \tilde{g})$ in $KO_{n+1}(C_r^*(\Gamma; \mathbb{R}))$.

By the (rational) surjectivity of the Baum–Connes assembly map

$$\mu_{\mathbb{R}}: KO_{\bullet}^{\Gamma}(\underline{E}\Gamma) \rightarrow KO_{\bullet}(C_r^*(\Gamma; \mathbb{R})),$$

there exists a spin Γ -manifold Z (without boundary) such that the higher index $\text{Ind}(D_Z)$ of its Dirac operator D_Z is equal to $-\text{Ind}(D_{\tilde{W}}, \tilde{g})$ (up to a rational multiple). Let Z_1 be the Γ -equivariant connected sum¹⁸ of \tilde{W} with Z . Then Z_1 is a spin Γ -manifold whose boundary is equal to $\partial \tilde{W} = \tilde{M}$. Moreover, the higher index $\text{Ind}(D_{Z_1})$ of the Dirac operator D_{Z_1} is zero. Now it follows from the discussion in this subsection that

$$(\tilde{M}, \tilde{h}) = (\partial Z_1, \tilde{h})$$

positively stably bounds with respect to $\underline{E}\Gamma$. ■

¹⁸The connected sum is performed away from the boundary of \tilde{W} .

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