Bergman space zero sets, modular forms, von Neumann algebras and ordered groups

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Abstract. A_α^2 will denote the weighted L^2 Bergman space. Given a subset S of the open unit disc we define $\Omega(S)$ to be the infimum of $\{s \mid \exists f \in A_{s-2}^2, \ f \neq 0, \ \text{having } S \text{ as its zero set}\}$. By classical results on Hardy space there are sets S for which $\Omega(S) = 1$. Using von Neumann dimension techniques and cusp forms we give examples of S where $1 < \Omega(S) < \infty$. By using a left order on certain Fuchsian groups we are able to calculate $\Omega(S)$ exactly if $\Omega(S)$ is the orbit of a Fuchsian group. This technique also allows us to derive in a new way well known results on zeros of cusp forms and indeed calculate the whole algebra of modular forms for $\mathrm{PSL}_2(\mathbb{Z})$.

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1. Introduction

The weighted Bergman (Hilbert) spaces A_α^2 for $\alpha > -1$ are the spaces of holomorphic functions in the unit disc $\mathbb{D}$ which are square integrable with respect to the measure $(1 - r^2)^\alpha r \, dr \, d\theta$ in $\mathbb{D}$. Obviously, $A_\alpha^2 \subseteq A_\beta^2$ for $\alpha \leq \beta$. The Hardy space $H^2(\mathbb{D})$ (see [36]) is contained in all the Bergman spaces. Given a complex valued function f on a set X, we let Z_f be its zero set

$$Z_f = \{x \in X \mid f(x) = 0\}.$$

We are interested in Z_f for $f \neq 0$ in Bergman space. The case of the Hardy space is completely understood (see [11], apparently[1] by Szegö in about 1915): An (obviously countable) subset $\{z_n \mid n = 1, 2, \ldots\}$ of $\mathbb{D}$ is Z_f for some $f \in H^2 \setminus \{0\}$ iff

$$\sum_n (1 - |z_n|) < \infty.$$

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[1]The editor believes that the author should rather refer to Blaschke, who wrote in 1915 a paper quoted in [11], rather than to Szegö.

Thus, any such set is Z_f for some $f \in A_\alpha^2$ for all $\alpha > -1$, but many other sets may be zero sets and a similar characterisation for Bergman space seems out of reach.

So for a subset $S \subset \mathbb{D}$, we let

$$X_S = \{s \mid \exists f \in A_{s-2}^2, \ f \neq 0, \ S \subseteq Z_f\}.$$

Then set

$$\Omega(S) = \begin{cases} \infty & \text{if } X_S = \emptyset, \\ \inf(X_S) & \text{otherwise.} \end{cases}$$

(The shift in values, $s = \alpha + 2$, is because for Bergman space the reference measure is Lebesgue measure in the disc, i.e. $\alpha = 0$, whereas for cusp forms and von Neumann algebras the reference measure is hyperbolic area, i.e. $s = 2$.)

In [15], $\Omega(S)$ is determined in terms of a notion of "density" of points in S which might be difficult to calculate. In this paper we will show that some progress can be made when S is the orbit of a Fuchsian group Γ acting on $\mathbb{D}$, effectively calculating the "density" of these orbits.

One can ask whether $\Omega(S) = \inf\{s \mid \exists f \in A_{s-2}^2, \ f \neq 0, \ S = Z_f\}$. Every subset of the zero set of an ordinary (unweighted) Bergman space function is the zero set of another function ([17]) but it is unknown whether this is true for weighted Bergman spaces. However the result of [15] mentioned above makes it clear that the only value of s for which it is unknown is precisely $\inf\{s \mid \exists f \in A_{s-2}^2, \ f \neq 0, \ S = Z_f\}$ so this is the same as $\Omega(S)$.

Our techniques make particular use of von Neumann algebras as inspired by Atiyah in [3], both for the existence and non-existence questions. Fuchsian groups act unitarily (projectively) on the Bergman spaces in such a way that they generate what is known as a II_1 factor M (which depends on s and Γ). But these techniques do not so far allow us to get information on Z_f itself so for results on $\Omega(S)$ we use modular forms whose zero sets are explicitly known.

Another key idea was proposed by Curt McMullen – that is, to exploit the Bergman reproducing kernel vectors ε_z satisfying $\langle \varepsilon_z, f \rangle = f(z)$ for f in a Bergman space. Applying Γ (hence, M) to an ε_z gives an M-module which has a von Neumann dimension which may be compared to the von Neumann dimension of the Bergman space itself. Standard von Neumann results then lead to an upper bound for $\Omega(\Gamma(z))$. To obtain a lower bound involves showing the existence of a "trace vector" for M and for this we introduce what appears to be a new technique. Every Fuchsian group Γ contains a left-orderable subgroup Ψ of finite index [16] (Γ is either a free product of cyclic groups or has a surface group of finite index). To each orbit of a left ordered Γ we produce a trace vector for $vN(\Gamma)$ which acts on A_α^2. This produces lower bounds on von Neumann dimension.

By covolume(Γ), in this paper, we will mean the hyperbolic area (constant curvature equals -1) of the quotient space $\mathbb{H}/\Gamma$. The simplest version of our main results is as follows.

Theorem 1.1. *Let Γ be a Fuchsian group[2] and $\Gamma(z)$ be an orbit in $\mathbb{H}$ containing no fixed points for any element of Γ. Then there is a non-zero function in A_α^2 vanishing on $\Gamma(z)$ iff*

$$\alpha > \frac{4\pi}{\text{covolume}(\Gamma)} - 1.$$

This establishes that

$$\Omega(\Gamma(z)) = \frac{4\pi}{\text{covolume}(\Gamma)} + 1$$

for such groups.

The value $(4\pi/\text{covolume}(\Gamma)) + 1$ is obviously special and there is a good reason for this. It is the value for which the von Neumann dimension $\dim_{vN(\Gamma)}(A_\alpha^2)$ is equal to 1 meaning there is a "cyclic and separating trace vector" for $vN(\Gamma)$ in the Hilbert space, and hence an anti-isomorphism between $vN(\Gamma)$ and its commutant on A_α^2. Now in [27], Radulescu has shown that the commutant $vN(\Gamma)'$ is always generated in some sense by cusp forms which thus give a model for $vN(\Gamma)'$. In addition, Voiculescu in [34] has shown that, at least for groups like $\text{PSL}_2(\mathbb{Z})$, $vN(\Gamma)$ has a random matrix model. Thus there exists a random matrix model for cusp forms. This is a theorem, but it is of little use unless one can lay one's hands on an explicit and manageable cyclic and separating trace vector with which to implement the anti-isomorphism with the commutant. If one did have such a vector one might be able to prove some of the numerically well established relations between random matrices and modular forms; see [23]. Indeed this was the motivation for the research that led to the results of this paper.

Interestingly though, our main theorem shows that when $\dim_{vN(\Gamma)}(A_\alpha^2) = 1$, a trace vector, although it exists, can never be obtained by the left order method of this paper, whereas for all other values of α (for which there is a trace vector) the left order method works, starting with a cusp form!

We would like to end the introduction by making quite clear what issue this paper brings to light.

Fix a Fuchsian group $\Gamma < \text{PSL}_2(\mathbb{R})$ and let $\gamma \mapsto u_\gamma$ be its unitary projective action on A_{s-2}^2. An element $\xi \in A_{s-2}^2$ is called *wandering* for Γ iff

$$\langle u_\gamma \xi, \xi \rangle = 0 \quad \text{for all } \gamma \in \Gamma, \gamma \neq \text{id},$$

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[2]In this paper, a Fuchsian group is a discrete subgroup of $\text{PSL}_2(\mathbb{R})$ with finite covolume; see Section 4 (editor's note).

and *tracelike* if, up to a multiplicative constant,

$$\sum_{\gamma \in \Gamma} \frac{|\xi(z)|^2}{|cz + d|^{2s}} = \text{Im}(z)^{-s}.$$

Problem 1. It follows from the von Neumann dimension that if

$$s = \frac{4\pi}{\text{covolume}(\Gamma)} + 1,$$

there is a non-zero function in A^2_{s-2} that is both wandering and tracelike. Find such a function.

A first step might be to find a direct (non-von Neumann algebraic) proof of the equivalence of the wandering and tracelike conditions when $s = (4\pi/\text{covolume}(\Gamma)) + 1$. For other values of s they are mutually exclusive.

2. Background in von Neumann algebras

We begin the paper by giving an account of von Neumann dimension, and self-contained calculations of some von Neumann dimensions, which are slightly different from the calculations of [4, 13, 27, 28], and require no knowledge of the discrete series for $\text{PSL}_2(\mathbb{R})$.

A von Neumann algebra M is a $*$-closed unital algebra of bounded operators on a (complex) Hilbert space $\mathcal{H}$ which is closed under the topology of pointwise convergence on $\mathcal{H}$. The *commutant* M' of M is the algebra of all bounded operators that commute with M. It is also a von Neumann algebra and has the same centre as M.

A vector $\xi \in \mathcal{H}$ is called *cyclic* for M if $M\xi$ is dense in $\mathcal{H}$ and *separating* for M if $x \mapsto x\xi$ is injective on M. ξ is cyclic for M iff it is separating for M'. The vector ξ is said to be a *trace vector* for M if $\langle ab\xi, \xi \rangle = \langle ba\xi, \xi \rangle$ for all $a, b \in M$.

A von Neumann algebra is a factor if it is central, i.e. if its center is $\mathbb{C}$ id. The most obvious factor is the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators. A factor M is called finite if it possesses a trace functional tr: $M \to \mathbb{C}$ with the properties

(1) $\text{tr}(ab) = \text{tr}(ba)$ for all $a, b \in M$;

(2) $\text{tr}(1) = 1$.

The functional is completely determined by these properties. It is positive definite, which means that $\text{tr}(a^*a) > 0$ for $a \neq 0$ so one may form the Hilbert space $L^2(M)$ which is the completion of M with respect to the pre-Hilbert space inner product $\langle a, b \rangle = \text{tr}(b^*a)$. An easy example is the $n \times n$-matrices acting on a Hilbert space of dimension mn with some "multiplicity" m. We will see more interesting examples very soon. An infinite dimensional finite factor is called a II_1 *factor*.

Definition 2.1. Let Γ be a (countable) discrete group. The *von Neumann algebra of* Γ, which we will write $vN(\Gamma)$, is the von Neumann algebra on $\ell^2(\Gamma)$ generated by the left regular representation $\gamma \mapsto \lambda_\gamma$, where $\lambda_\gamma(f)(\gamma') = f(\gamma^{-1}\gamma')$.

More generally, if $\omega\colon \Gamma \times \Gamma \to \mathbb{T}$ is a unit circle valued 2-cocycle, $vN_\omega(\Gamma)$ is generated on $\ell^2(\Gamma)$ by the unitaries λ_γ^ω, where[3] $\lambda_\gamma^\omega(f)(\gamma') = \omega(\gamma, \gamma') f(\gamma')$ (so group multiplication is "twisted" by a cocycle).

It is well known (see [9,33]) that $vN(\Gamma)$ is a II_1 factor iff Γ is icc, i.e. all non-trivial conjugacy classes of Γ are infinite. $vN_\omega(\Gamma)$ is a II_1 factor if Γ is icc. The trace on $vN_\omega(\Gamma)$ is given by

$$\mathrm{tr}(\lambda_\gamma^\omega) = \begin{cases} 1 & \text{if } \gamma = \mathrm{id}, \\ 0 & \text{otherwise.} \end{cases}$$

3. The von Neumann dimension

Let M be a finite factor. We will assign a positive real number, or ∞, which we will call $\dim_M(\mathcal{H})$ to any (separable) Hilbert space on which M acts. It will completely characterise the Hilbert space as a (Hilbert space) M-module up to unitary equivalence. In the finite dimensional case it will be m/n, where m is the multiplicity above, thus measuring in some sense the number of copies of the M-module M inside $\mathcal{H}$.

A type II_∞ factor is the closure of the algebra of all finitely supported matrices with entries in a fixed II_1 factor M acting on the direct sum of infinitely many copies of the Hilbert spaces on which M acts. A II_∞ factor has a "trace" given by adding up the traces of the diagonal matrix entries. It is not defined everywhere but one may talk of "trace class" operators in a II_∞ factor just as one does for $\mathcal{B}(\mathcal{H})$ (see [29]). If M is a II_1 factor on $\mathcal{H}$ we can "amplify" it to act diagonally on $\oplus_{n=1}^\infty \mathcal{H}$. Its commutant is then a II_∞ factor. All II_∞ factors arise in this way.

We will now assume basic facts about type-II_∞ factors, traces on them and comparison of projections in a factor; see [9,33].

Proposition 3.1. *If $\mathcal{H}$ is any Hilbert space on which M acts then there is an M-linear isometry*

$$u\colon \mathcal{H} \to \oplus_{n=1}^\infty L^2(M).$$

Proof. The factor M acts diagonally on the direct sum $\mathcal{H} \oplus (\oplus_{n=1}^\infty L^2(M))$. The commutant M' contains the two projections $p = 1 \oplus 0$ and $q = 0 \oplus 1$. Since the commutant is a II_∞ factor and q is certainly infinite, we obtain a partial isometry

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[3]Should be $\lambda_\gamma^\omega(f)(\gamma') = \omega(\gamma'^{-1}, \gamma) f(\gamma^{-1}\gamma')$ (editor's note).

$u \in M'$ such that $u^*u = p$ and $uu^* \leq q$. Identify $\mathcal{H}$ with the image of p, we have our u. ∎

Note that if v is any other M-linear isometry as above then vv^* is equivalent in M' to uu^*. Note also that, on $\oplus_{n=1}^{\infty} L^2(M)$, the commutant M' admits a canonically normalised trace tr_{L^2} such that the trace of any projection onto one of the $L^2(M)$'s is equal to 1.

Definition 3.2. With notation as above

$$\dim_M(\mathcal{H}) = \mathrm{tr}_{L^2}(uu^*).$$

Notes 3.3. (1) Observe that if M is the scalars $\mathbb{C}$ then this definition gives exactly the usual definition of the dimension $\dim \mathcal{H}$ of a separable Hilbert space. If M is the $n \times n$ matrices, we obtain $(\dim \mathcal{H})/n^2$.

(2) With this philosophy one may canonically normalise the trace on M' by defining

$$\mathrm{Tr}_{M'}(a) = \mathrm{tr}_{L^2}(uau^*).$$

It is not hard to show that $\mathrm{Tr}_{M'}$ is $\dim_M(\mathcal{H})$ times the normalised trace on M'. Further, if $a: \mathcal{H} \to \mathcal{K}$ and $b: \mathcal{K} \to \mathcal{H}$ is a bounded linear map between Hilbert spaces over M, then

$$\mathrm{Tr}_{M'}(ab) = \mathrm{Tr}_{M'}(ba).$$

(3) Our definition is not the same as that of Murray and von Neumann in Chapter X of [24] where it measures the relative mobility of M and M' as follows. Take any non-zero $\xi \in \mathcal{H}$ and consider the two closed subspaces $\overline{M\xi}$ and $\overline{M'\xi}$ of $\mathcal{H}$ with orthogonal projections p and q respectively. Clearly, $p \in M'$ and $q \in M$, so we may form the ratio $\mathrm{tr}_M(q)/\mathrm{tr}_{M'}(p)$. This was shown in [24] to be independent of ξ. With this fact one may easily show it is equal to our $\dim_M(\mathcal{H})$. This ratio became known as the "coupling constant" but calling it the von Neumann dimension is more revealing. One reason it is a little obscure in [24] is that the authors defined a theory for all types (I, II_1, II_∞, III) of factors, each one requiring its own treatment.

The next theorem collects some elementary properties of $\dim_M \mathcal{H}$.

Theorem 3.4. *With the notation above.*

(i) $\dim_M(\mathcal{H}) < \infty$ *iff* M' *is a* II_1 *factor.*

(ii) $\dim_M(\mathcal{H}) = \dim_M(\mathcal{K})$ *iff* M *on* $\mathcal{H}$ *and* M *on* $\mathcal{K}$ *are unitarily equivalent.*

(iii) $\dim_M(\oplus_i \mathcal{H}_i) = \sum_i \dim_M \mathcal{H}_i$.

(iv) $\dim_M(L^2(M)q) = \mathrm{tr}_M(q)$ *for any projection* $q \in M$.

(v)   *If p is a projection in M, $\dim_{pMp}(p\mathcal{H}) = \operatorname{tr}_M(p)^{-1} \dim_M(\mathcal{H})$.*

For the next two properties we suppose M' is finite, hence a II_1 factor with trace $\operatorname{tr}_{M'}$ (and $\operatorname{tr}_{M'}(1) = 1$).

(vi)   *If p is a projection in M', $\dim_{Mp}(p\mathcal{H}) = \operatorname{tr}_{M'}(p) \dim_M \mathcal{H}$.*

(vii) *$(\dim_M \mathcal{H})(\dim_{M'} \mathcal{H}) = 1$.*

(viii) *There is a cyclic vector for M iff $\dim_M \mathcal{H} \leq 1$.*

(ix)   *There is a separating vector, indeed a trace vector, for M iff $\dim_M \mathcal{H} \geq 1$.*

(x)   *If $p\xi = \xi$ for $\xi \in \mathcal{H}$ and p a projection in M, then $\dim_M(\overline{M\xi}) \leq \operatorname{tr}_M(p)$.*

Proof. These are all standard results due to Murray and von Neumann [24]. For proofs based on our definition, see [13] or [21]. Property (x) is easiest proved using the Murray–von Neumann definition. Clearly, one can reduce to the case $\mathcal{H} = \overline{M\xi}$, and then

$$M'\xi = M'p\xi = pM'\xi \subseteq p\mathcal{H},$$

so if q is projection onto $\overline{M'\xi}$, then $\operatorname{tr}_M(q) \leq \operatorname{tr}_M(p)$. ∎

Proposition 3.5. *Let Γ be an icc discrete group and $\gamma \mapsto v_\gamma$ be a projective unitary group representation on $\mathcal{H}$ with 2-cocycle ω. Suppose there is a projection q on $\mathcal{H}$ such that*

$$v_\gamma q v_\gamma^{-1} \perp q \quad \forall \gamma \in \Gamma, \ \gamma \neq \mathrm{id}, \quad and \quad \sum_{\gamma \in \Gamma} v_\gamma q v_\gamma^{-1} = 1.$$

Then there is a Γ-linear unitary $U: \mathcal{H} \to \ell^2(\Gamma) \otimes q\mathcal{H}$ with $Uv_\gamma U^{-1} = \lambda_\gamma^\omega \otimes \mathrm{id}$ for $\gamma \in \Gamma$.

Proof. Choose an orthonormal basis $\{\eta_i \mid i = 1, 2, 3, \ldots\}$ of $q\mathcal{H}$. Then by the two conditions of the proposition $\{v_\gamma \eta_i \mid \gamma \in \Gamma, \ i = 1, 2, 3, \ldots\}$ is an orthonormal basis for $\mathcal{H}$. Defining U by $U(v_\gamma \eta_i) = \varepsilon_\gamma \otimes \eta_i$ gives the desired unitary, where ε_γ is the characteristic function of $\{\gamma\}$ in $\ell^2(\Gamma)$. ∎

Corollary 3.6. *Suppose Γ, v, q, ω and U are as in Proposition 3.5. Then the action of Γ on $\mathcal{H}$ makes it into a $vN_\omega(\Gamma)$-module and if p is a projection on $\mathcal{H}$ commuting with v_γ for all γ, then*

$$\dim_{vN_\omega(\Gamma)} \mathcal{H} = \operatorname{Tr}_{B(\mathcal{H})}(pqp) = \operatorname{Tr}_{B(\mathcal{H})}(qpq),$$

where $\operatorname{Tr}_{B(\mathcal{H})}$ is the usual trace (sum of the diagonal elements for a positive operator [29]) on $B(\mathcal{H})$.

Proof. The commutant M' of $vN_\omega(\Gamma)$ on $\ell^2(\Gamma) \otimes q\mathcal{H}$ is the tensor product of $vN_\omega(\Gamma)'$ and $B(q\mathcal{H})$ and the correctly normalised trace on it is the tensor product of the trace on $vN_\omega(\Gamma)'$ (on $\ell^2(\Gamma)$) and the usual trace on $B(q\mathcal{H})$. Thus, since $\varepsilon_{\mathrm{id}}$ is a trace vector for $vN_\omega(\Gamma)'$, for $x \geq 0$ in M, we have

$$\mathrm{Tr}_{M'}(x) = \sum_i \langle x(\varepsilon_{\mathrm{id}} \otimes \eta_i), \varepsilon_{\mathrm{id}} \otimes \eta_i \rangle = \mathrm{Tr}_{B(\ell^2(\Gamma) \otimes q\mathcal{H})}(exe),$$

where e is orthogonal projection onto $\varepsilon_{\mathrm{id}} \otimes q\mathcal{H}$.

Now Up is a $vN_\omega(\Gamma)$-linear isometry from $\mathcal{H}$ to $\ell^2(\Gamma) \otimes q\mathcal{H}$ so that, by the definition of von Neumann dimension,

$$\dim_{vN_\omega(\Gamma)} p\mathcal{H} = \mathrm{Tr}_{B(\ell^2(\Gamma) \otimes p\mathcal{H})}(eUpU^*e).$$

However, $U^*eU = q$, so that

$$\dim_{vN_\omega(\Gamma)} p\mathcal{H} = \mathrm{Tr}_{B(\mathcal{H})}(qpq). \qquad \blacksquare$$

A commonly encountered situation in which the hypotheses of Proposition 3.5 are satisfied is when Γ acts as deck transformations for a covering space $\pi\colon \mathcal{M} \to \mathcal{N}$ between manifolds. Then if Γ preserves a smooth measure and D is a fundamental domain, Proposition 3.5 applies to the Hilbert space $\mathcal{H} = L^2(\mathcal{M})$ together with the projection q onto $L^2(D)$. This is the setup for Atiyah's covering space L^2 index theorem [3]. (See also [7] and [8, Section 1.5].) We will use it in a slightly modified form where the natural measure is not preserved.

Remark 3.7. A rather different use of von Neumann dimension occurs in [18]. Given a subfactor N of a II_1 factor M the Hilbert space $L^2(M)$ is a left N-module and one defines $[M : N] = \dim_N(L^2(M))$. Although the von Neumann dimension itself takes on all positive real values, it turns out that $[M : N]$ must be, if finite, in the set $\{4\cos^2 \pi/n : n = 3, 4, 5, \ldots\} \cup [4, \infty)$. One recognises the squares of the numbers in the usual generators of the Hecke groups (see [10]).

The context of this paper originated in 1982 in an attempt to find a relation between the Hecke groups and subfactors. That is still a long way off as is the attempt to exploit the rich structure of modular forms for a Fuchsian group to produce "exotic" subfactors like those of [2, 14] (see also [19, 20, 22] for more examples and details).

4. Fuchsian groups and L^2 holomorphic functions on $\mathbb{H}$

A *Fuchsian group* Γ is by definition a discrete finite covolume subgroup of $\mathrm{PSL}_2(\mathbb{R})$. (Finite covolume is not always assumed in the literature.) If Σ is a compact Riemann

surface of genus ≥ 2, its universal covering space is the upper half plane $\mathbb{H}$ (as a complex manifold). $\mathrm{PSL}_2(\mathbb{R})$ is the group of complex automorphisms of $\mathbb{H}$ so $\pi_1 \Sigma$ is a cocompact Fuchsian group. It is also icc. The unit disc $\mathbb{D}$ is holomorphically the same as $\mathbb{H}$ under the Cayley transform $C \colon \mathbb{H} \to \mathbb{D}$:

$$C(z) = \frac{z - i}{z + i}, \quad C^{-1}(w) = \frac{w + 1}{i(w - 1)}.$$

And the action

$$g(z) = \frac{az + b}{cz + d} \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $\mathrm{PSL}_2(\mathbb{R})$ becomes, after conjugation by C, $w \mapsto C(g(C^{-1}(w)))$. The action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}$ preserves the measure $\mu_0 = (dx\,dy)/y^2$, which is the measure from a hyperbolic metric of constant curvature -1. On $\mathbb{D}$ the measure becomes

$$\nu_0 = 4 \frac{dx\,dy}{(1 - |w|^2)^2}.$$

Proposition 4.1. *For g as above,*

$$\mathrm{Im}(g(z)) = \frac{\mathrm{Im}(z)}{|cz + d|^2}.$$

All Fuchsian groups are icc (see [1]). If Γ is a Fuchsian group it has a fundamental domain which means that $L^2(\mathbb{H}, d\mu_0)$ satisfies the hypotheses of Proposition 3.5, so that Γ generates a II_1 factor with II_∞ commutant on $L^2(\mathbb{H}, d\mu_0)$.

For each real $s > 1$, we define the measure $\mu_s = y^{s-2}\,dx\,dy$ on $\mathbb{H}$. The measure μ_s is not invariant under $\mathrm{PSL}_2(\mathbb{R})$, but for any L^1 function F, we have

$$\int_{\mathbb{H}} F(z)\,\mathrm{Im}(z)^s \frac{dx\,dy}{y^2} = \int_{\mathbb{H}} F(g(z))\,\mathrm{Im}(g(z))^s \frac{dx\,dy}{y^2}$$

$$= \int_{\mathbb{H}} F(g(z)) \frac{y^s}{|cz + d|^{2s}} \frac{dx\,dy}{y^2}$$

so that, choosing a branch of $(cz + d)^s$ for each g,

$$\left(\check{\pi}_s(g^{-1})f\right)(z) = \frac{1}{(cz + d)^s} f(g(z))$$

defines a unitary operator on $L^2(\mathbb{H}, d\mu_s)$, preserving holomorphic functions.

Remark 4.2. We will choose the following branch of log to define $(cz + d)^s$:

$$\log(cz + d) = \int_\kappa \frac{c}{cz + d}\,dz + \frac{i\pi}{2},$$

where κ is the straight line from i to z. Exercise: show that $\check{\pi}_s(g^{-1}) = \check{\pi}_s(g)^{-1}$.

(The reason for using s rather than $\alpha = s - 2$ is that the measure $(1 - r^2)^{-2} r \, dr \, d\theta$ is hyperbolic measure, invariant under the usual action of $\mathrm{PSL}_2(\mathbb{R})$ on the disc, which is more natural when it comes to Fuchsian groups than the usual Lebesgue measure for Bergman space.)

Now if we consider the function $j : \mathrm{SL}_2(\mathbb{R}) \times \mathbb{H} \to \mathbb{C}$ defined by $j(g, z) = cz + d$, it is easy to check the cocycle condition

$$j(gh, z) = j(g, h(z)) j(h, z),$$

so that if s is equal to a *positive integer* p, the map $g \mapsto \check{\pi}_p(g)$ defines a unitary representation of $\mathrm{SL}_2(\mathbb{R})$ which preserves holomorphic functions.

If p is even, $\check{\pi}(-\mathrm{id})$ is the identity so that $\check{\pi}$ passes to $\mathrm{PSL}_2(\mathbb{R})$. If p is odd, $\check{\pi}(-\mathrm{id}) = -\mathrm{id}$ so $\check{\pi}$ is only a projective representation.

Remark 4.3. For an arbitrary real positive s, $\gamma \mapsto \check{\pi}(\gamma)$ is a projective unitary representation. To see this just take the sth power of the cocycle relation for j above to obtain that $j(gh, z)^s$ and $j(g, h(z))^s j(h, z)^s$ differ by a complex number of absolute value equal to one.

The projective representation $\check{\pi}$ cannot, for non-integral s, be lifted to an honest representation of $\mathrm{PSL}_2(\mathbb{R})$ since then it would be a discrete series representation which it is not; see [5]. But when restricted to Γ the relevant cohomology obstruction may vanish (this is the case for $\mathrm{PSL}_2(\mathbb{Z})$) so one may still get an honest representation of Γ. That there are Fuchsian groups for which the relevant cohomology obstruction does not vanish will be treated in Appendix A.

If s is not an integer the cocycle condition for j does not imply a cocycle condition for j^{-p} so one only obtains a projective representation for $\check{\pi}$. It can be considered a unitary representation of the universal cover of $\mathrm{PSL}_2(\mathbb{R})$ via Bargmann [5].

Proposition 4.4. *If $f \in L^2(\mathbb{D}, dv_s)$ then $f \mapsto \check{f}$, where*

$$\check{f}(z) = \left(\frac{2}{z + i} \right)^s f \left(\frac{z - i}{z + i} \right)$$

defines a unitary from $L^2(\mathbb{D}, dv_s)$ to $L^2(\mathbb{H}, d\mu_s)$, which intertwines the two projective representations of $\mathrm{PSL}_2(\mathbb{R})$.

Proof. This can be proved by extending the action on functions from $\mathrm{SL}(2, \mathbb{R})$ to $\mathrm{SL}(2, \mathbb{C})$ and conjugating by the Cayley transform. Unitarity can be checked directly. ∎

Definition 4.5. Let P_s be orthogonal projection from $L^2(\mathbb{H}, d\mu_s)$ onto the closed subspace spanned by functions which are holomorphic. This subspace is the *weighted*

Bergman space A_α^2 with $\alpha = s - 2$. We will use the notation indifferently for functions on $\mathbb{D}$ or $\mathbb{H}$. The projective representation π_s of $\mathrm{PSL}_2(\mathbb{R})$ is defined to be the restriction of $\check{\pi}$ to A_{s-2}^2.

Remark 4.6. These Hilbert spaces of analytic functions are "reproducing kernel" Hilbert spaces. The parameter in the literature is usually $\alpha = s - 2$. This means that for each $z \in \mathbb{H}$, there is a $\varepsilon_z \in A_{s-2}^2$ such that

$$\langle \varepsilon_z, f \rangle = f(z).$$

This follows from the continuity of point evaluation.

As noted, if s is an even positive integer we get an honest unitary representation of $\mathrm{PSL}_2(\mathbb{R})$, but not for s odd.

Let Γ be a Fuchsian group with fundamental domain F. We have seen that $\check{\pi}_s$ restricted to Γ defines a projective unitary representation of the II_1 factor $M = vN_\omega(\Gamma)$, where ω is the 2-cocycle with values in the circle which comes from the chosen branch of the logarithm of $cz + d$ on $\mathbb{H}$. To calculate the von Neumann dimension $\dim_M(A_{s-2}^2)$ we will use an orthonormal basis of A_{s-2}^2. We will work in $\mathbb{D}$ where it is obvious that the powers of z are orthogonal so all we need to do is normalise them. The result is very well known (see [13, 28]), but we include the calculation for the convenience of the reader.

Proposition 4.7. (1) *Let*

$$e_n(w) = \sqrt{\frac{s-1}{4\pi}} \sqrt{\frac{s(s+1)\cdots(s+n-1)}{n!}} w^n \quad \text{for } w \in \mathbb{D}.$$

Then e_n is an orthonormal basis for A_{s-2}^2.

(2) *Let*

$$f_n(z) = \sqrt{\frac{s-1}{4\pi}} \sqrt{\frac{s(s+1)\cdots(s+n-1)}{n!}} \left(\frac{2}{z+i}\right)^s \left(\frac{z-i}{z+i}\right)^n \quad \text{for } z \in \mathbb{H}.$$

Then f_n is an orthonormal basis for A_{s-2}^2.

Proof. It is trivial that $\langle e_n, e_m \rangle = 0$ for $n \neq m$, so writing $w = u + iv$, we only need to calculate

$$\|w^n\|^2 = \int_{\mathbb{D}} |w|^{2n} (1 - |w|^2)^{s-2} 4 \, du \, dv = 4 \int_0^{2\pi} \int_0^1 r^{2n} (1 - r^2)^{s-2} r \, dr \, d\theta.$$

Putting $t = r^2$, we get

$$4\pi \int_0^1 t^n (1 - t)^{s-2} \, dt = 4\pi \beta(n + 1, s - 1) = 4\pi \frac{\Gamma(n+1)\Gamma(s-1)}{\Gamma(n+s)}.$$

Expanding the Γ functions we get the result for e_n and the result for f_n follows from Proposition 4.4 ∎

Theorem 4.8. *With notation as above*

$$\dim_M(A^2_{s-2}) = \frac{s-1}{4\pi}\text{covolume}(\Gamma).$$

Proof. We will do the calculation in the $\mathbb{D}$ model. By Corollary 3.6, we have to calculate

$$\sum_{n=0}^{\infty}\int_F |e_n(w)|^2\left(1-|w|^2\right)^{s-2}4\,du\,dv$$

$$= \frac{s-1}{4\pi}\sum_{n=0}^{\infty}\int_F \frac{s(s+1)\cdots(s+n-1)}{n!}r^{2n}4\,du\,dv.$$

Everything in sight is positive so one can commute summing and integration. We have

$$(1-r^2)^{-s} = \sum_{n=0}^{\infty}\frac{s(s+1)\cdots(s+n-1)}{n!}r^{2n},$$

which gives

$$\dim_M(A^2_{s-2}) = \frac{s-1}{4\pi}\int_F \frac{du\,dv}{v^2},$$

so we get

$$\dim_M(A^2_{s-2}) = \frac{s-1}{4\pi}\text{covolume}(\Gamma),$$

as required. ∎

Notes 4.9. Special cases.

(1) $\Gamma = \text{PSL}_2(\mathbb{Z})$. Here the covolume (= hyperbolic area of fundamental domain) is, by Gauss–Bonnet or direct integration over F, equal to $\pi/3$. So, for $s > 1$, we have

$$\dim_{vN(\text{PSL}_2(\mathbb{Z}))} A^2_{s-2} = \frac{s-1}{12}.$$

Since Γ is in this case the free product of two cyclic groups the projective representation actually lifts to an honest one so we are dealing with $vN(\text{PSL}_2(\mathbb{Z}))$.

(2) If Σ is a compact Riemann surface of genus $g > 1$ with hyperbolic metric, its area is $4\pi(g-1)$, so

$$\dim_{vN(\pi_1(\Sigma))} A^2_{s-2} = (s-1)(g-1).$$

In this case the projective representation does not necessarily lift to an honest one as we will show in Appendix A. However, if s is an odd integer the existence of spin structures shows that the lifting does exist.

Why might these von Neumann dimension formulae actually lead to non-trivial results? The fact that equality of traces in a factor implies equivalence of projections is an ergodic theoretic result ultimately relying on patching together lots of little projections. There are some instances of results using it which are non-trivial. Let us discuss the author's favourite (due to Kaplansky). In fact, it does not even use factoriality!

Theorem 4.10. *Let Γ be a discrete group and $\mathbb{F}$ a field of characteristic zero. Let $\mathbb{F}\Gamma$ be the group algebra. Then $ab = 1 \iff ba = 1$ in $\mathbb{F}\Gamma$.*

Proof. Since the relations $ab = 1$ and $ba = 1$ only involve finitely many scalars we may embed $\mathbb{F}$ in $\mathbb{C}$ and work in $\mathbb{C}\Gamma$ which embeds into $vN(\Gamma)$.

So the result follows from $ab = 1 \iff ba = 1$ in a finite von Neumann algebra M with trace tr. Let M act on some $\mathfrak{H}$.

Suppose $ab = 1$. Then for any $\xi \in \mathfrak{H}$, $ba(b\xi) = b\xi$ so since ba is bounded it suffices to show that the range of b is dense. But if $b = u|b|$ is the polar decomposition of b then u is a partial isometry from the orthogonal complement of the kernel of b to the closure of the image of b. But $u^*u = 1$ since $\ker(b) = 0$ (since $ab = 1$). So, $\mathrm{tr}(uu^*) = 1$ is one which means $uu^* = 1$ so the image of b is dense. ∎

The conclusion of the theorem remains an open problem if one drops the condition that the characteristic of the field be zero. Thus the use of von Neumann algebra in this context can have considerable content and it could be that the results of this paper are quite difficult to obtain by any other means. Here is a sample (it will be one direction of Theorem 1.1 below).

Proposition 4.11. *Let Γ be a Fuchsian group. Then if $s > 1 + (4\pi/\mathrm{covolume}(\Gamma))$ and $z \in \mathbb{H}$, there is a non-zero function in A_{s-2}^2 vanishing on the orbit $\Gamma(z)$.*

Proof. The group Γ is icc by [1]. Let ε_z be the reproducing kernel vector for z, so that $\langle \varepsilon_z, \xi \rangle = \xi(z)$ for all $\xi \in A_{s-2}^2$. Then the von Neumann dimension of the closure of $vN_\omega(\Gamma)\varepsilon_z$ is at most 1 by (viii) of Theorem 3.4. But by Theorem 4.8, the von Neumann dimension of A_{s-2}^2 is greater than 1. So there is a $\xi \in A_{s-2}^2$ orthogonal to $\pi_s(\gamma)\varepsilon_z$ for every γ. Thus, ξ vanishes on $\Gamma(z)$. ∎

By a relatively subtle argument with cusp forms it is possible to prove this result without the use of von Neumann algebras for $\mathrm{PSL}_2(\mathbb{Z})$ (see the discussion after Definition 7.2), but a proof in full generality might be very complicated.

5. Wandering vectors and trace vectors

For convenience, we introduce the following definition which appears to be well accepted.

Definition 5.1. If π is a (projective unitary) representation of a group Γ on a Hilbert space $\mathcal{H}$, then

(1) A (non-zero) vector $\xi \in \mathcal{H}$ is called a *wandering vector* for π if

$$\langle \xi, \pi(\gamma)(\xi) \rangle = 0 \quad \text{for all } \gamma \neq 1 \text{ in } \Gamma.$$

(2) A subspace $V \subseteq \mathcal{H}$ is called a *wandering subspace* if

$$\pi(\gamma)(V) \perp V \quad \text{for all } \gamma \neq 1 \text{ in } \Gamma.$$

Note that any non-zero element of a wandering subspace is a wandering vector and orthogonal vectors in a wandering subspace produce wandering vectors with orthogonal orbits.

Definition 5.2. If M is a von Neumann algebra on $\mathcal{H}$, a non-zero vector $\eta \in \mathcal{H}$ is called a *trace vector* for M if $\langle x\eta, \eta \rangle$ is a multiple of the trace of x for every $x \in M$.

There is a simple relationship between the two concepts:

Proposition 5.3. *If π is a (projective unitary) representation of a group Γ on a Hilbert space $\mathcal{H}$ and ξ is a wandering vector for π, then ξ is a trace vector for the von Neumann algebra M generated by $\pi(\Gamma)$. Moreover, on the closure of the subspace $[\pi(\Gamma)\xi]$, M is isomorphic to the twisted group von Neumann algebra $vN_\omega(\Gamma)$ (where ω is the 2-cocycle of the projective representation), acting on $L^2(M)$.*

6. Proof of the main theorem

We will use the following easy result on Bergman space functions ("popping zeros").

Lemma 6.1. *Let $f \in A^2_{s-2}$ be a non-zero function with a zero of order k at w, i.e. $f^{(j)}(w) = 0$ for $j = 0, 1, 2, \ldots, k-1$, but $f^{(k)}(w) \neq 0$. Then the functions $(z-w)^{-j} f(z)$ for $j = 1, 2, \ldots, k$ are in A^2_{s-2}.*

Proof. By the transitivity of the action of $\mathrm{PSL}_2(\mathbb{R})$ ($= \mathrm{SU}(1,1)$), we may assume $z = 0$. Write

$$f = \sum_{n=k}^{\infty} c_n e_n,$$

where

$$e_n(z) = \sqrt{\frac{s-1}{4\pi}} \sqrt{\frac{s(s+1)\cdots(s+n-1)}{n!}} z^n$$

are the orthonormal basis constructed in Proposition 4.7. We know that c_n is square summable. The limit of the sequence

$$a_n = \sqrt{\frac{s(s+1)\cdots(s+n+k-1)}{s(s+1)\cdots(s+n-1)}\frac{n!}{(n+k)!}}$$

is 1, so a_n is bounded. The holomorphic function $g(z) = z^{-k}f(z)$ has Taylor series

$$\sum_{n=0}^{\infty} c_{n+k} z^{-k} e_{n+k}(z) = \sum_{n=0}^{\infty} a_n c_{n+k} e_n(z).$$

Thus, $g(z) \in A^2_{s-2}$ and so is $z^j g(z)$ for $1 \le j < k$. ∎

Now let Γ be an *orderable* Fuchsian group and let $\gamma \mapsto \pi_s(\gamma)$ be the (projective) unitary representation on A^2_{s-2} that we have been considering.

(Recall that a group Γ is *orderable* if it admits a total order $<$ which is invariant under left translation, i.e. $\alpha < \beta \iff \gamma\alpha < \gamma\beta$ for all γ. Free groups are orderable as are fundamental groups of surfaces [31].)

Theorem 6.2. *Suppose $O_1, O_2, \ldots, O_n$ are disjoint orbits in $\mathbb{D}$ of Γ. Let $f \in A^2_{s-2}$ be non-zero, with a zero of order at least v_i on all points of O_i. Then there is a wandering subspace W of dimension $t = \sum_{i=1}^{n} v_i$ for $\pi_s(\Gamma)$, and $\pi_s(\gamma)(f) \in W^\perp$ for all γ.*

Proof. To make the argument clear let us begin with the case of the orbit of a single point z, with f having zeros at $\gamma(z)$ for all $\gamma \in \Gamma$.

Choose a left ordering $<$ of Γ and define the closed subspaces U and V of A^2_{s-2} to be

$$U = \{\xi \mid \xi(\gamma(z)) = 0 \text{ for } \gamma \le \text{id}\}, \quad V = \{\xi \mid \xi(\gamma(z)) = 0 \text{ for } \gamma < \text{id}\}.$$

We will now show that a vector in the orthogonal complement $U^\perp \cap V$ of U in V is a wandering vector for Γ.

Suppose $\xi \in U^\perp \cap V$. Then for $\gamma < \text{id}$ and any other $\lambda \le \text{id}$,

$$\gamma\lambda \le \gamma \,\text{id} = \gamma < \text{id},$$

so

$$\pi_s(\gamma^{-1})\xi(\lambda(z)) = \frac{1}{(cz+d)^s}\xi(\gamma\lambda(z)) = 0 \quad (\text{since } \xi \in V),$$

which means that $\pi_s(\gamma^{-1})\xi \in U$, and thus

$$\langle \pi_s(\gamma^{-1})\xi, \xi \rangle = 0$$

since $\xi \in U^{\perp}$. This also means, by unitarity,

$$\langle \pi_s(\gamma)\xi, \xi \rangle = 0.$$

So ξ is wandering.

Moreover, for all γ, $\pi(\gamma)(f)$ vanishes on the entire orbit so it is in U, hence it is orthogonal to ξ.

So we just need to check that the containment of U in V is strict. For this, divide f enough times by a linear function vanishing at z as in Lemma 6.1.

For the general case we essentially repeat the argument. Fix a "base point" z_i in each O_i. Given f satisfying the hypotheses of the theorem let

$$U = \{\xi \mid \xi^{(j)}(\gamma(z_i)) = 0 \text{ for } \gamma \leq \mathrm{id} \ \forall i \text{ and all } 0 \leq j < v_i\}$$
$$V = \{\xi \mid \xi^{(j)}(\gamma(z_i)) = 0 \text{ for } \gamma < \mathrm{id} \ \forall i \text{ and all } 0 \leq j < v_i\}.$$

Clearly, $U \subseteq V$ and put $W = U^{\perp} \cap V$. We claim W is a wandering subspace for $\pi_s(\Gamma)$.

For suppose $\xi, \eta \in U^{\perp} \cap V$. Then for $\gamma < \mathrm{id}$ and any other $\lambda \leq \mathrm{id}$,

$$\gamma\lambda \leq \gamma \, \mathrm{id} = \gamma < \mathrm{id},$$

so

$$\pi_s(\gamma^{-1})\xi(\lambda(z_i)) = \frac{1}{(cz+d)^s}\xi(\gamma\lambda(z_i)) = 0 \quad \forall i.$$

The factor $1/(cz+d)^s$ does not change the nature of the zeros so $\pi_s(\gamma^{-1})\xi \in U$, and thus

$$\langle \pi_s(\gamma^{-1})\xi, \eta \rangle = 0$$

since $\eta \in U^{\perp}$. Which, since ξ and η are arbitrary in W, also means by unitarity that

$$\langle \pi_s(\gamma)\xi, \eta \rangle = 0 \quad \text{for } all \ \gamma.$$

Thus, W is wandering. Moreover, $\pi_s(\gamma)(f) \in U$, so $\pi_s(\gamma)(f) \perp W \ \forall \gamma \in \Gamma$.

We will now show that the dimension of W is at least $t = \sum_{i=1}^{n} v_i$. Without loss of generality we may assume that, for each i, the order of the zero at z_i of f is exactly v_i. (It suffices to apply Lemma 6.1 at each z_i to the given non-zero f.) For such an f, we claim that the t functions

$$\frac{f}{(z-z_i)^j} \quad \text{for } i = 1, \ldots, n \text{ and } j = 1, \ldots, v_i$$

are in V and are linearly independent modulo U. They are in V by Lemma 6.1 and the fact that the nature of the zeros of f on the rest of the orbits is unchanged by multiplication by powers of $z - z_i$. Suppose $a_{i,j}$ are constants with

$$\sum_{i,j} a_{i,j} \frac{f}{(z - z_i)^j} = g \quad \text{for some } g \in U.$$

Then g has a zero of order at least v_i at z_i, so the meromorphic function g/f is holomorphic at each z_i. This forces all the $a_{i,j}$ to be zero. ∎

We now deduce some consequences of Theorem 6.2. We start with Theorem 1.1, which is the most straightforward. (The condition on the freeness of the action on the orbit in Theorem 1.1 is significant. See Remark 7.3.) For convenience of reading we recall Theorem 1.1.

Theorem 6.3. *Let Γ be a Fuchsian group and $\Gamma(z)$ be an orbit in $\mathbb{H}$ containing no fixed points for any element of Γ. Then there is a non-zero function in A^2_{s-2} vanishing on $\Gamma(z)$ iff*

$$s > 1 + \frac{4\pi}{\text{covolume}(\Gamma)}.$$

Proof. ($\Longrightarrow$) By [16], Γ is either a free product of finitely many cyclic groups or has a surface group of finite index. Either way there is an orderable subgroup $\Psi < \Gamma$ with $n = [\Gamma : \Psi] < \infty$. Let $<$ be a left ordering on Ψ. Suppose by way of contradiction that $f \in A^2_{s-2}$ is non-zero but vanishes on $\Gamma(z)$.

By the fixed point hypothesis, $\Gamma(z)$ consists of n disjoint Ψ-orbits so apply Theorem 6.2 to obtain a wandering subspace of dimension at least n. Choosing an orthonormal basis we obtain n vectors ξ_i so that if M is the II_1 factor $vN_\omega(\Psi)$, then each ξ_i is a trace vector and the M-modules $\overline{M\xi_i}$ are mutually orthogonal, of von Neumann dimension one. Hence,

$$n \leq \dim_M A^2_{s-2} = n \dim_{vN_\omega(\Gamma)} A^2_{s-2},$$

which forces $\dim_{vN_\omega(\Gamma)} A^2_{s-2} \geq 1$. In other words,

$$s \geq 1 + \frac{4\pi}{\text{covolume}(\Gamma)}.$$

To see that s cannot be equal to $1 + (4\pi/\text{covolume}(\Gamma))$, observe that by Theorem 6.2, f itself is orthogonal to $vN_\omega(\Gamma)(V^\perp \cap W)$, which already has von Neumann dimension equal to one, a contradiction.

($\Longleftarrow$) See Proposition 4.11. ∎

Corollary 6.4. *If Γ is an arbitrary Fuchsian group then for all z, $\Omega(\Gamma(z))$ is strictly greater than* 1.

Proof. If $\Gamma_0 \subseteq \Gamma$, then $X_{\Gamma(z)} \subseteq X_{\Gamma_0(z)}$, so $\Omega(\Gamma_0(z)) \leq \Omega(\Gamma(z))$. As before, [16], any Fuchsian group Γ has a left orderable subgroup Γ_0 of finite index and the von Neumann dimension multiplies by $[\Gamma : \Gamma_0]$ on restricting to the Γ_0. So by Theorem 6.3, $s \in \Omega(\Gamma_0(z))$ for s sufficiently close to 1. ∎

The next corollary can be proved by other means, e.g. equidistribution (McMullen).

Corollary 6.5. *For any Fuchsian group Γ and any $z \in \mathbb{D}$,*

$$\sum_{\gamma \in \Gamma} \left(1 - |\gamma(z)|\right) \text{ diverges.}$$

Proof. If the sum converged there would be a Hardy space function vanishing on $\Gamma(z)$ and Hardy space is contained in the Bergman spaces. ∎

In [15], a density $D^+(S)$ called the "upper asymptotic κ-density" is defined for subsets S of the unit disc. It is shown on page 131 of that book that the condition

$$D^+(S) \leq \frac{1 + \alpha}{p}$$

is necessary and the condition

$$D^+(S) < \frac{1 + \alpha}{p}$$

is sufficient for A to be an A_α^p-zero set.

Corollary 6.6. *If Γ and z are as in Theorem 1.1, then*

$$D^+(\Gamma(z)) = \frac{2\pi}{\text{covolume}(\Gamma)}.$$

Proof. Putting $p = 2$ in the condition from [15] above, we get

$$D^+(\Gamma(z)) = \frac{1 + \alpha}{2} = \frac{s - 1}{2} = \frac{2\pi}{\text{covolume}(\Gamma)}$$

from Theorem 1.1. ∎

Remark 6.7. Once we have $D^+(\Gamma(z))$, we know that when $\Gamma(z)$ is an A_α^p-zero set for all p by [15]. Thus, the L^2 methods of this paper solve, thanks to [15], an L^p problem for all p.

7. Use of cusp forms

Let us restrict initially to the case $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. A cusp form of weight p is a function $f\colon \mathbb{H} \to \mathbb{C}$, which is holomorphic and satisfies

$$f(\gamma(z)) = (cz + d)^p f(z),$$

which means that $f(z + 1) = f(z)$ so that we may write f as a function of $q = e^{2\pi i z}$. The cusp form condition is then

$$f(z) = \sum_{n=1}^{\infty} a_n q^n.$$

The first thing to observe is that

$$|f(z)| \le (\text{constant})(\mathrm{Im}\, z)^{-p/2}.$$

To see this, note that $|f(z)|\,\mathrm{Im}(z)^{p/2}$ is invariant under the action of $\mathrm{PSL}_2(\mathbb{Z})$. (Follows from modularity of f and Proposition 4.1.) But since $f(z) = qg(z)$ with g having a finite limit as $q \to 0$, then $|f(z)|\,\mathrm{Im}(z)^{p/2}$ is bounded on a fundamental domain, hence everywhere.

The first cusp form is the modular discriminant $\Delta(z)$ of weight 12 which as a function of q can be written as

$$q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

It is the 24th power of the Dedekind η function. Cusp forms give a graded algebra under multiplication and can be multiplied by modular forms (same invariance as cusp forms but do not vanish at ∞) to give other cusp forms. See [32].

Now let Γ be an arbitrary Fuchsian group and, following some authors ([35]), we say a cusp form of weight p is a holomorphic function $f\colon \mathbb{H} \to \mathbb{C}$ such that

$$f(\gamma(z)) = (cz + d)^p f(z) \quad \text{and} \quad |f(z)| \le (\text{constant})(\mathrm{Im}\, z)^{-p/2}.$$

Proposition 7.1. *If f is a cusp form of (integer) weight p, let*

$$M_f\colon L^2(\mathbb{H}, y^{s-2}\, dx\, dy) \to L^2(\mathbb{H}, y^{s+p-2}\, dx\, dy)$$

be the operator of multiplication by f. Then M_f is a bounded linear operator intertwining the actions of $\check{\pi}_{s+p}(\gamma)$ and $\check{\pi}_s(\gamma)$, and preserving the subspace of holomorphic functions. Also,

$$M_f^*(\xi)(z) = \mathrm{Im}(z)^p\, \overline{f(z)}\, \xi(z).$$

Proof. Boundedness: For $\xi \in L^2(\mathbb{H}, y^{s-2}\, dx\, dy)$, using the above bound on $|f(z)|$,

$$\|M_f\xi\|^2 = \int_{\mathbb{H}} |f(z)|^2 |\xi(z)|^2 y^{s+p-2}\, dx\, dy$$

$$\leq (\text{constant}) \int_{\mathbb{H}} |\xi(z)|^2 y^{s-2}\, dx\, dy.$$

Also,

$$\check{\pi}_{s+p}(\gamma^{-1})(M_f\xi)(z) = \frac{1}{(cz+d)^{s+p}} f(\gamma(z))\xi(\gamma(z))$$

$$= \frac{(cz+d)^p}{(cz+d)^{s+p}} f(z)\xi(\gamma(z)) = M_f(\check{\pi}_s(\gamma^{-1})\xi)(z).$$

And finally,

$$\langle M_f\xi, \eta \rangle = \int_{\mathbb{H}} f(z)\xi(z)\overline{\eta(z)} y^{s+p-2}\, dx\, dy$$

$$= \int_{\mathbb{H}} \xi(z)\overline{\overline{f(z)}\eta(z)} \operatorname{Im}(z)^p y^{s-2}\, dx\, dy,$$

which is the formula given in the statement of the proposition for M_f^*. ∎

Definition 7.2. If f is a cusp form of weight p, we call T_f the operator from A_α^2 to $A_{\alpha+p}^2$ given by

$$T_f = PM_f,$$

where P is orthogonal projection from $\xi \in L^2(\mathbb{H}, y^{\alpha+p}\, dx\, dy)$ onto Bergman space.

We saw above that for a Fuchsian group Γ there is, for s large enough and any z, simply because of von Neumann dimension, a function $f \in A_{s-2}^2$ vanishing on $\Gamma(z)$. But the von Neumann dimension is a blunt tool and is of no help whatsoever in finding such functions. Cusp forms give us *explicit* functions in Bergman spaces vanishing on orbits under Γ. Indeed if f is a cusp form of weight p vanishing at $z \in \mathbb{H}$, and $\xi \in A_{s-2}^2$ then by Proposition 7.1, $T_f\xi$ is in $\mathcal{H}_{s+p}$ and vanishes on $\Gamma(z)$. For $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ this shows that there are elements of A_{s-2}^2 vanishing at $\Gamma(e^{i\pi/3})$ provided $s > 17$. This is because the Eisenstein series G_2 is a modular form of weight 4 vanishing at $e^{i\pi/3}$ so that ΔG_2 is a cusp form of weight 16 vanishing at $e^{i\pi/3}$. Elements of A_{s-2}^2 for $s > 1$ may be multiplied by $G_2\Delta$ to give the required Bergman space functions.

L. Rolen and I. Wagner have improved this method considerably ([30]) to get explicit elements of A_{s-2}^2 vanishing on $\mathrm{PSL}_2(\mathbb{Z})(z)$ for any $s > 13$: Begin with the modular function $j(z)$ and choose any $w \in \mathbb{H}$. Then $j(z) - w$ is a holomorphic function that vanishes exactly on the $\mathrm{PSL}_2(\mathbb{Z})$ orbit of a z_0 with $j(z_0) = w$. Now multiply by $\Delta(z)$

to obtain a modular form vanishing on the same set. Then choose a branch of $\eta(z)^r$ for r real, small and positive. Then the product $f(z) = (j(z) - w)\Delta\eta(z)^r$ satisfies

$$|f(\gamma(z)|\,\mathrm{Im}(z)^{6+r/4} = |f(z)|.$$

Since $(j(z) - w)\Delta(z)$ has a limit as $q \to 0$ and $|\eta(z)|$ tends to zero as $\mathrm{Im}\,z$ grows, $|f(\gamma(z)|\,\mathrm{Im}(z)^{6+r/4}$ is bounded on a fundamental domain, and hence

$$|f(z)| \leq (\text{constant})\,\mathrm{Im}(z)^{-(6+r/4)}.$$

Thus, as before, multiplication by f defines a bounded operator from A_{s-2} to $A_{s+10+r/2}$. Choosing s close to 1 and r close to zero we get an *explicit* element of A_{t-2}^2 whose zero set is *exactly* the orbit $\mathrm{PSL}_2(\mathbb{Z})(z_0)$ for any $t > 13$.

Remark 7.3. Here is an example showing that the freeness of the action on the orbit of z is essential. Let G_2 be the Eisenstein series modular form for $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ of (smallest) weight 4. Then $G_2(e^{\pi i/3}) = 0$, so G_2 vanishes on the Γ orbit of $e^{\pi i/3}$. Using the same trick as above, multiply G_2 by some branch of $\eta(z)^r$ for r real, small and positive. The resulting holomorphic function f will satisfy

$$|f(z)| \leq (\text{constant})\,\mathrm{Im}(z)^{-(2+r/4)}$$

and so defines by multiplication a bounded map from A_α to $A_{\alpha+4+r/2}$. So if s is slightly bigger than 1 we obtain elements of $A_{3+\epsilon}$ vanishing exactly on the Γ orbit of $e^{\pi i/3}$ for all $\epsilon > 0$.

Remark 7.4. For a cusp forms f of weight p the operator T_f is M-linear where $M = vN_\omega(\Gamma)$ so if we let M acting diagonally on the direct sum $\oplus_{n=0}^\infty A_{s-2+np}$ of Bergman spaces, the T_f define operators in the commutant which is a II_∞ factor. But we can also think of T_f as a map between Bergman spaces intertwining the action of M.

Proposition 7.5. *The closure of $T_f A_{s-2}^2$ is an M-module of von Neumann dimension equal to that of A_{s-2}^2.*

Proof. This is trivial since multiplication by a non-zero holomorphic function is injective so the polar decomposition of T_f gives a unitary equivalence. ∎

Here is a simple consequence of von Neumann dimension in the spirit of Proposition .

Corollary 7.6. *For any $s > 1$ and every cusp form f of weight p, there is a $\xi \in A_{s+p-2}^2$ which is orthogonal to $f\eta$ for all $\eta \in A_{s-2}^2$.*

C. McMullen pointed out that this result is trivial if f has zeros since then the Bergman reproducing kernel vector ε_z is automatically orthogonal to $f\eta$ for all $\eta \in A_{s-2}^2$. For a cusp form with no zeros, like Δ we have not seen a constructive proof.

8. Fixed points

It is possible to improve on Theorem 1.1 by a closer analysis of the orbits of an orderable subgroup of finite index. We are guided by Example 7.3. If Γ is a Fuchsian group, the stabiliser of any point $z \in \mathbb{H}$ is finite and cyclic. Denote by stab_i the stabiliser of a point in an orbit O_i (defined up to conjugacy).

Theorem 8.1. *If Γ is any Fuchsian group and $O_1, O_2, \ldots, O_n$ are disjoint orbits in $\mathbb{D}$ of Γ. Then there is a non-zero function in A_{s-2}^2 with a zero of order at least v_i on all points of O_i iff*

$$s > 1 + \frac{4\pi}{\mathrm{covolume}(\Gamma)} \sum_i \frac{v_i}{|\mathrm{stab}_i|}.$$

Proof. ($\Longrightarrow$) Choose as in Theorem 1.1 an orderable subgroup $\Psi < \Gamma$ with $n = [\Gamma : \Psi] < \infty$. The action of the stabiliser of a point in $\mathbb{H}$ on Γ/Ψ is free since if $\gamma(\mu\Psi) = \mu\Psi$, then $\gamma\mu = \mu\psi$ for some $\psi \in \Psi$, so γ is conjugate to an element of Ψ, but the stabiliser is of finite order and Ψ is torsion free. The action of Γ on O_i is the same as the action on Γ/stab_i. But the spaces $(\mathrm{stab}_i\backslash\Gamma)/\Psi$ and $\mathrm{stab}_i\backslash(\Gamma/\Psi)$ are the same so there are $[\Gamma : \Psi]/|\mathrm{stab}_i|$ disjoint orbits of Ψ in O_i. Thinking of A_{s-2}^2 as a representation of $M = vN_\omega(\Psi)$, the orbit O_i thus contributes $v_i[\Gamma : \Psi]/|\mathrm{stab}_i|$ mutually Ψ-orthogonal trace vectors for M by Theorem 6.2. Thus,

$$\dim_M(A_{s-2}^2) = \frac{s-1}{4\pi}\mathrm{covolume}(\Gamma)[\Gamma : \Psi] \geq [\Gamma : \Psi] \sum_i \frac{v_i}{|\mathrm{stab}_i|}.$$

Moreover, as before the function in the statement of the theorem vanishing on the orbit is actually orthogonal to the M-linear span of the trace vectors so the inequality is strict.

($\Longleftarrow$) For each i choose $z_i \in O_i$ and let ε_i^j be vectors such that

$$\langle f, \varepsilon_i^j \rangle = f^{(j)}(z_i) \quad \text{for each } 0 \leq j \leq v_{i-1}.$$

If γ_i generates the stabiliser of z_i we can clearly arrange the cocycle ω of the projective representation π_s so that

$$u_i^{|\mathrm{stab}_i|} = 1,$$

u_i being $\pi_s(\gamma_i)$. Moreover, it is clear that $u_i \varepsilon_i^j$ is a multiple of ε_i^j, necessarily by an nth root of unity so that the $u_i \varepsilon_i^j$ are in eigenspaces of the u_i. Hence, they are in the image of projections in $vN_\omega(\Gamma)$ of trace $1/|\mathrm{stab}_i|$. Hence, by Theorem 3.4 (x), we have

$$\dim_{vN_\omega(\Gamma)}\overline{\left(vN_\omega(\Gamma)\varepsilon_i^j\right)} \leq \frac{1}{|\mathrm{stab}_i|}.$$

Since the von Neumann dimension is subadditive, summing over i and j, we get

$$\sum_{i,j} \dim_{vN_\omega(\Gamma)} \overline{\left(vN_\omega(\Gamma)\varepsilon_i^j \right)} \leq \sum_i \frac{v_i}{|\text{stab}_i|},$$

which by hypothesis is less than $\dim_{vN_\omega(\Gamma)} A_{s-2}^2$.

So there is a function $\xi \in A_{s-2}^2$ which is orthogonal to all the $\pi_s(\gamma)\varepsilon_i^j$. This means that ξ vanishes to order at least v_i on each O_i. ∎

We can now extend the calculation in Corollary 6.6 of the density D^+ to all orbits of all Fuchsian groups.

Corollary 8.2. *If Γ, a Fuchsian group, and $z \in \mathbb{D}$ are given with the stabiliser of z having order* stab *then*

$$D^+(\Gamma(z)) = \frac{2\pi}{\text{stab}} \times \text{covolume}(\Gamma).$$

Proof. The proof is as in Corollary 6.6. ∎

Note that the result extends to more than one orbit, and if there were a density calculation for sets with zeros of prescribed order, that density could be calculated for Fuchsian groups.

The following result is surely known to experts.

Corollary 8.3. *Let f be a holomorphic k-differential on a Riemann surface $\mathbb{D}/\Gamma$ of genus g, lifted to give a holomorphic function on $\mathbb{D}$. Then f is square integrable for the measure $(1 - r^2)^{s-2} r\, dr\, d\theta$ for every $s > 2k + 1$, but not for $s = 2k + 1$.*

Proof. With our definition of cusp form, f is a cusp form of weight $2k$ so the multiplication operator M_f is by 7.1 a bounded operator from $A_{-1+\epsilon}^2$ to $A_{-1+2k+\epsilon}^2$ for every $\epsilon > 0$. The constant function 1 is in $A_{-1+\epsilon}^2$ so f itself is in $A_{-1+2k+\epsilon}^2$. On the other hand, the degree of the kth power of the canonical bundle is $2k(g-1)$, so by Riemann–Roch f has $2k(g-1)$ zeros counted with multiplicity. So by Theorem 8.1, since Γ acts freely we must have s strictly greater than

$$1 + 2k(g-1)\frac{4\pi}{\text{covolume}(\Gamma)} = 1 + 2k. \qquad \blacksquare$$

9. Trace vectors for the commutant of Γ

We need an elementary result on Poincaré series, going back to Poincaré [26]. We prove it here because it is usually stated for s a positive integer whereas we need it for

real $s > 1$ (see [6]). If s is a positive even integer the next step after convergence is usually to show that the Poincaré series defines a cusp form. But for real s this will not be the case because of the non-homomorphic nature of the branch of the logarithm.

In the next lemma, F will be a fundamental domain for Γ, a Fuchsian group as usual with the convention established above for the meaning of $cz + d$, s will be a real number bigger than one and a fixed branch of log is used to define $(cz + d)^s$.

Lemma 9.1. *Let* $\xi \in A^2_{s-2}$. *Then the Poincaré series*

$$\sum_{\gamma \in \Gamma} \frac{\xi(\gamma(z))^2}{(cz + d)^{2s}}$$

converges locally uniformly in $\mathbb{H}$ *as does*

$$\sum_{\gamma \in \Gamma} \frac{|\xi(\gamma(z))|^2}{|cz + d|^{2s}},$$

the former to a holomorphic function and the latter to (at least) a continuous one.

Proof. Fix a ball K in F. Putting

$$f_\gamma(z) = \frac{\xi(\gamma(z))^2}{(cz + d)^{2s}},$$

the square of the L^2 norm of ξ is

$$\int_{\mathbb{H}} |\xi(z)|^2 y^s \frac{dx\, dy}{y^2} = \sum_{\gamma \in \Gamma} \int_F |f_\gamma(z)| y^s \frac{dx\, dy}{y^2}.$$

Since f_γ is holomorphic, by the mean value property there is a C such that $|f_\gamma(z)| \leq C \|f_\gamma(z)\|_1$ for all $\gamma \in \Gamma$ and $z \in K$, where by $\| - \|_1$ we mean the 1-norm on the fundamental domain F for the measure $\frac{dx\, dy}{y^{2-s}}$. Since $\sum_\gamma \|f\|_1$ converges, convergence on F of the two functions in the statement of the theorem is guaranteed by the Weierstrass M-test. Locally uniform convergence everywhere follows by varying the fundamental domain. ∎

In [27, 28], Radulescu has given a description of the commutant $M = vN_\omega(\Gamma)'$ on A^2_{s-2}. Given an L^∞ function f on $\mathbb{H}$ that is fixed by the action of Γ (which is the same thing as an L^∞ function on a fundamental domain), one can define the "Toeplitz" operator T_f, which is the composition

$$T_f = PM_f \colon A^2_{s-2} \to A^2_{s-2},$$

where P is the orthogonal projection from $L^2(\mathbb{H})$ onto A^2_{s-2}. In [27], we find the following two theorems.

Theorem 9.2. *The subspace of M spanned by the T_f is dense in the 2-norm*

$$\|x\|_2 = \sqrt{\operatorname{tr}(x^*x)}.$$

Theorem 9.3. *The trace in M of T_f is a multiple of*

$$\int_F f(z) \frac{dx\,dy}{y^2}.$$

Note that by Γ-invariance the integral does not depend on the fundamental domain.

Definition 9.4. An element $\xi \in A^2_{s-2}$ will be called *tracelike* if, for all $z \in \mathbb{H}$,

$$\sum_{\gamma \in \Gamma} \frac{|\xi(z)|^2}{|cz+d|^{2s}} = (\text{constant})\,\operatorname{Im}(z)^{-s}.$$

Corollary 9.5. *A function $\xi \in A^2_{s-2}$ is a trace vector for M iff it is tracelike.*

Proof. By Theorem 9.3 we have, up to constants, for a bounded Γ invariant function on $\mathbb{H}$,

$$\langle T_f \xi, \xi \rangle = \int_F f(z) \frac{dx\,dy}{y^2}.$$

However,

$$
\begin{aligned}
\langle T_f \xi, \xi \rangle &= \int_{\mathbb{H}} f(z) |\xi(z)|^2 y^s \frac{dx\,dy}{y^2} \\
&= \int_F f(z) \sum_{\gamma \in \Gamma} \frac{|\xi(\gamma(z))|^2}{|cz+d|^{2s}} y^s \frac{dx\,dy}{y^2}.
\end{aligned}
$$

The series converges to a continuous function by Lemma 9.1. When we subtract a constant times y^{-s} we get a function orthogonal on F to all bounded measurable functions. The corollary follows by varying the fundamental domain. ∎

Remark 9.6. Cusp forms give us a supply of interesting Toeplitz operators. We have seen in Definition 7.2 that a cusp form f gives a bounded $vN_\omega(\Gamma)$-linear map T_f between Bergman spaces. So if f and g are cusp forms of the same weight p, $T_f^* T_g$ is in M. It is actually the Toeplitz operator for the Γ-invariant bounded function

$$h(z) = \overline{f(z)} g(z) \operatorname{Im}(z)^p.$$

Theorem 9.3 then shows that the trace in M of $T_f^* T_g$ is the integral of h over the fundamental domain with hyperbolic measure, i.e. the well known Petersson inner product [25]. This result was also obtained in [13]. Radulescu also claims in [27] that the Toeplitz operators given by cusp forms are dense in M though it appears he has only proved it for $\text{PSL}_2(\mathbb{Z})$.

This allows us to state a theorem about existence of such functions.

Theorem 9.7. *There is a tracelike $\xi \in A_{s-2}^2$ iff*

$$s \leq 1 + \frac{4\pi}{\text{covolume}(\Gamma)}.$$

Moreover, if $s = 1 + (4\pi/\text{covolume}(\Gamma))$, the condition is equivalent *to ξ being a wandering vector for Γ.*

Proof. These are immediate consequences of von Neumann dimension. Theorem 3.4 (ix) proves the first assertion and the equivalence of being a trace vector for M and M' is easy when the von Neumann dimension is one since then the Hilbert space is M-isomorphic to the L^2 closure of M.   ∎

A. Existence of non-trivial central extensions of Fuchsian groups arising from the non-integral values of *s*

We observed in Section 4 that for $s > 1$, the formula

$$\left(\check{\pi}_s(g^{-1})f\right)(z) = \frac{1}{(cz+d)^s} f(g(z))$$

only defines a *projective* unitary representation of $\text{PSL}_2(\mathbb{R})$ on A_{s-2}^2, but that on restriction to a Fuchsian group Γ the representation might be adjusted to become honest. That is the case for instance if Γ is a free product of cyclic groups [16] – simply adjust the unitaries representing the generators so that they have the right order in the unitary group of A_{s-2}^2. It would have simplified the presentation in this paper if we could do the same for all Fuchsian groups, but in this appendix we will show that this is not the case for fundamental groups of surfaces of genus bigger than one.

Theorem A.1. *Let Γ be the Fuchsian group of the fundamental group of a Riemann surface Σ of genus g. The projective representation given by the restriction of $\check{\pi}_s$ (for $s > 1$) is equivalent to an honest representation iff $s \in \frac{1}{g-1}\mathbb{Z}$.*

Proof. Let $\{\gamma_i\}$ be generators for Γ so that the defining relation for Γ is $\prod[\gamma_i, \gamma_{i+1}] = 1$ (see [12]). Then changing liftings $\check{\pi}_s(\gamma)$ of the projective representation of Γ to different unitaries does not affect $\prod[\check{\pi}(\gamma_i), \check{\pi}(\gamma_{i+1})] = 1$ provided the liftings of inverses in Γ are inverse unitaries. By Remark 4.2 this is true for our careful definition of $\check{\pi}$. So the single number

$$\text{obstr}(s) = \prod[\check{\pi}(\gamma_i), \check{\pi}(\gamma_{i+1})]$$

is exactly the obstruction to lifting the restriction of $\check{\pi}$ to an honest unitary representation. Observe also that $s \mapsto \mathrm{obstr}(s)$, when written out as an explicit function of s and z, is a continuous homomorphism from $\mathbb{R}$ to the circle $\mathbb{T}^1$.

Thus, the problem becomes: "what is the kernel of obstr?" (Any even integer s is in the kernel since then $1/(cz + d)^s$ has the cocycle property for all of $\mathrm{PSL}_2(\mathbb{R})$.) This question has nothing to do with Bergman space. We will answer it using line bundles on Σ. We claim that $\ker(\mathrm{obstr}) = \frac{1}{g-1}\mathbb{Z}$. Our proof will use the following construction.

Proposition A.2. *Suppose* $\mathrm{obstr}(s) = 1$ *for some* $s \in \mathbb{R}$, $s > 1$. *Then there is a holomorphic line bundle* $L(s)$ *over* Σ *with the following two properties:*

(1) $L(s + t) = L(s) \otimes L(t)$;

(2) $L(2)$ *is the canonical line bundle* K.

Proof. Since $\mathrm{obstr}(s) = 1$, the projective representation of Γ on A^2_{s-2} may be lifted to an honest representation by defining, for $w = \prod_1^n \alpha_i$, where α_i is one of the generators for each $i = 1, 2, \ldots, n$,

$$\pi(w) = \prod_1^n \check{\pi}(\alpha_i).$$

It follows that

$$\pi(\gamma^{-1})(\xi)(z) = j(\gamma, z)\xi(\gamma(z)),$$

where $j(\gamma, z)$ satisfies the cocycle condition

$$j(\gamma_1\gamma_2, z) = j(\gamma_1, \gamma_2(z))j(\gamma_2, z),$$

and $j(\gamma, z)$ is a product of holomorphic functions of z of the form $\exp(s \log(cz + d))$.

The cocycle condition is exactly what is required to define an action of Γ on the line bundle (over $\mathbb{H}$) $\mathbb{H} \times \mathbb{C}$:

$$\gamma(z, w) = (\gamma(z), j(\gamma, z)w).$$

This action is properly discontinuous so, passing to the quotient, we obtain a line bundle $L(s)$ on Σ, which is holomorphic because j is.

(1) Change of local trivialisations of $L(s)$ are obtained by lifting to $\mathbb{H} \times \mathbb{C}$ and applying elements of Γ, and tensor product of line bundles corresponds to multiplying the cocycle defining the action. Since j is a product of terms $f(s)$ with $f(s + t) = f(s)f(t)$, the same is true for j as a function of s.

(2) When $s = 2$, $1/(cz + d)^s$ is already a cocycle so it is equal to $j(\gamma, z)$. But the canonical line bundle is that of holomorphic one-forms which are locally of the form $f(z)\, dz$ and transform under the action of Γ just as our action on $\mathbb{H} \times \mathbb{C}$ acts on functions.   ∎

We return to the proof of Theorem A.1. The rational number 2 is in ker(obstr), so it suffices to show that

(1) $1/(g - 1) \in \ker(\text{obstr})$;

(2) no rational number $r = 1 + \epsilon$, $0 < \epsilon < 1/(g - 1)$ is in ker(obstr).

Let us begin with (2). Suppose $s = 1 + \frac{m}{n} \in \ker(\text{obstr})$ with $m/n < 1/(g - 1)$. Form the line bundle $L(s)$ of Proposition A.2 over Σ and let its degree be d. Then

$$\otimes^{2n} L(s) \cong \otimes^{n+m} K$$

by Proposition A.2 (2). Equating the degrees of both sides, we get

$$2nd = 2(g - 1)(n + m) \quad \text{or} \quad d = (g - 1)\Big(1 + \frac{m}{n}\Big).$$

But $(g - 1)\frac{m}{n}$ is not an integer, so we have a contradiction.

So we only have to show that $s = 1/(g - 1) \in \ker(\text{obstr})$. Note that another way of phrasing the lifting property for $\check{\pi}_s$ is the following: does there exist a function $\omega \colon \Gamma \to \mathbb{T}$ so that $\gamma \mapsto \omega(\gamma)/(cz + d)^s$ has the cocycle property.

Choose a holomorphic line bundle L over Σ of degree 1. Tensoring L if necessary by a line bundle of degree 0 we may assume that $\otimes^{2g-2} L$ is the canonical line bundle K of holomorphic 1-forms. Now take the universal cover of L to obtain $\tilde{L}$ over $\mathbb{H}$ which may be trivialised so that there is an action of Γ on $\mathbb{H} \times \mathbb{C}$ of the form

$$\gamma(z, w) = \big(\gamma(z), \alpha(\gamma, z)w\big)$$

for some holomorphic cocycle α. The $(2g - 2)$th power of α yields an action on $\mathbb{H} \times \mathbb{C}$ which is equivalent to the action yielding K, i.e. that coming from the cocycle $1/(cz + d)^2$. We conclude there is a non-vanishing holomorphic function $h(z)$ such that

$$\alpha(\gamma, z)^{2g-2} = \frac{1}{(cz + d)^2} h(\gamma(z)) h(z)^{-1}.$$

Since $\mathbb{H}$ is simply connected choose for each γ a branch of $h(\gamma(z))^{1/(2g-2)}$ to obtain

$$\alpha(\gamma, z) = \omega(\gamma) \frac{1}{(cz + d)^s} h(\gamma(z))^{1/(2g-2)} \big(h(z)^{1/(2g-2)}\big)^{-1}$$

for some $2(g - 2)$th roots of unity $\omega(\gamma)$. Since α and $h(\gamma(z))^{1/(2g-2)} (h(z)^{1/(2g-2)})^{-1}$ are (holomorphic) cocycles, so is $\omega(\gamma)/(cz + d)^s$ which means $s \in \ker(\text{obstr})$.   ∎

B. An amusette: Calculation of the algebra of modular forms

We will restrict our attention to $\mathrm{PSL}_2(\mathbb{Z})$ though the method of this section surely applies in great generality. All the results about modular forms in this section are extremely well known and elementary (see [32, 35]), but our derivation of them is somewhat different! The method should be applicable to a Fuchsian group provided there is an analogue of Δ ([10]). Relations in the algebra can be checked using zero sets.

Lemma B.1. *If f is a cusp form of weight p for $\mathrm{PSL}_2(\mathbb{Z})$, then $p \geq 12$.*

Proof. Choose a w which is not a fixed point for $\mathrm{PSL}_2(\mathbb{Z})$. Then $g(z) = j(z) - j(w)$ vanishes on the orbit of w. And fg is a modular form of weight p. Choosing a small $\epsilon > 0$, $|\Delta^\epsilon fg|y^{(p+12\epsilon)/2}$ is invariant under $\mathrm{PSL}_2(\mathbb{Z})$ and bounded on a fundamental domain. Which gives

$$|\Delta^\epsilon fg(z)| \leq (\text{constant})y^{-(p+12\epsilon)/2}.$$

So that for $\xi \in A^2_{s-2}$, we have $\Delta^\epsilon fg\xi \in A^2_{s-2+p+12\epsilon}$ and it vanishes on the orbit of w which contradicts Theorem 1.1 if s is close to 1 and ϵ is small. ∎

Corollary B.2. *There is no non-zero modular form of weight 2.*

Proof. Suppose f were such a modular form. If f has a zero, then multiplying it by a small positive power of Δ times a vector ξ as in Lemma B.1 gives an A^2 function vanishing on an orbit which contradicts Theorem 1.1.

If f vanishes nowhere, one may form Δ/f which is a cusp form of weight less than 12, disallowed by Lemma B.1. ∎

Given the above and the Eisenstein series it is not hard to determine the whole algebra of modular forms. Uniqueness of the modular form of weight 12 is given by dividing by Δ and the maximum modulus theorem, as usual [35]. For weights $p = 4, 6, 8, 10$, just subtract the appropriate multiple of $G_{p/2}$ to obtain a cusp form which must be zero by Lemma B.1.

It is now routine to obtain the whole algebra of modular forms since multiplication by Δ is clearly an injection of modular forms of weight p onto cusp forms of weight $p + 12$ and subtracting the appropriate multiple of the Eisenstein series gives a cusp form. We conclude that the algebra of cusp forms is a graded commutative algebra freely generated by G_2 in degree 4 and G_3 in degree 6 (see [32]).

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