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W^*-representations of subfactors and restrictions on the Jones index

Sorin Popa

Abstract. A W^**-representation* of a II_1 subfactor $N \subset M$ with finite Jones index, $[M:N] < \infty$, is a non-degenerate commuting square embedding of $N \subset M$ into an inclusion of atomic von Neumann algebras $\oplus_{i\in I}\mathcal{B}(\mathcal{K}_i) = \mathcal{N} \subset^{\mathcal{E}} \mathcal{M} = \oplus_{j\in J}\mathcal{B}(\mathcal{H}_j)$. We undertake here a systematic study of this notion, first introduced by the author in 1992, giving examples and considering invariants such as the (bipartite) *inclusion graph* $\Lambda_{\mathcal{N}\subset\mathcal{M}}$, the *coupling vector* $(\dim({}_M\mathcal{H}_j))_j$ and the *RC-algebra* (relative commutant) $M' \cap \mathcal{N}$, for which we establish some basic properties. We then prove that if $N \subset M$ admits a W^*-representation $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$, with the expectation $\mathcal{E}$ preserving a semifinite trace on $\mathcal{M}$, such that there exists a norm one projection of $\mathcal{M}$ onto M commuting with $\mathcal{E}$, a property of $N \subset M$ that we call *weak injectivity/amenability*, then $[M:N]$ equals the square norm of the inclusion graph $\Lambda_{\mathcal{N}\subset\mathcal{M}}$.

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To Vaughan Jones, in memoriam

1. Introduction

A tracial von Neumann algebra (M, τ) is naturally represented as the algebra of left multiplication operators on the Hilbert space L^2M, obtained by completing M in the norm $\|\cdot\|_2$ given by the trace τ. This is called the *standard representation* of M. In fact, M acts on L^2M on the right as well, giving a representation of M^{op}, the opposite of the algebra M. The left-right multiplication algebras M, M^{op} commute, one being the centralizer of the other, $M^{\mathrm{op}} = M'$. Any other representation $M \subset \mathcal{B}(\mathcal{H})$ of M as a von Neumann algebra, or *left Hilbert M-module* ${}_M\mathcal{H}$, is of the form

$$\mathcal{H} \simeq \oplus_k L^2(M)p_k,$$

for some projections $\{p_k\}_{k\in K} \subset \mathcal{P}(M)$, with the action of M by left multiplication.

If M is a factor, then

$$\dim({}_M\mathcal{H}) \stackrel{\text{def}}{=} \sum_k \tau(p_k) \in [0,\infty]$$

characterizes the isomorphism class of ${}_M\mathcal{H}$. The M-module ${}_M\mathcal{H}$ can be alternatively described as the (left) M-module $e_{00}L^2(M^\infty, \mathrm{Tr})p$, where M^∞ is the II_∞ factor $M\overline{\otimes}\mathcal{B}(\ell^2K)$, Tr denotes its normal semifinite faithful (n.s.f.) trace $\tau \otimes \mathrm{Tr}_{\mathcal{B}(\ell^2K)}$, $e_{00} \in \mathcal{B}(\ell^2K)$ is a rank one projection, and $p \in M^\infty$ is any projection satisfying

$$\mathrm{Tr}(p) = \dim({}_M\mathcal{H}).$$

Thus, the commutant M' of M in $\mathcal{B}(\mathcal{H})$ is a II_1 factor iff $\dim({}_M\mathcal{H}) < \infty$, and if this is the case then $M' \simeq (M^t)^{\mathrm{op}}$, where $t = \dim({}_M\mathcal{H}) = \mathrm{Tr}(p)$.

This summarizes Murray–von Neumann famous theory of continuous dimension for type II factors. The number $\dim({}_M\mathcal{H}) \in [0,\infty]$ is called the *dimension* of the M-module ${}_M\mathcal{H}$. When viewed as the amplifying number t, measuring the ratio between the size of M and $M' = (M^t)^{\mathrm{op}}$ in $\mathcal{B}(\mathcal{H})$, it is called the *coupling constant*.

We consider in this paper the analogue for a subfactor $N \subset M$ of finite Jones index, $[M:N] < \infty$, of the representations of a single II_1 factor. This concept was introduced in [40, Section 2], but we undertake here a systematic study of this notion and its applications. Thus, a *W^*-representation* of $N \subset M$ is a non-degenerate embedding of $N \subset M$ into an inclusion of atomic von Neumann algebras with expectation,

$$\oplus_{i\in I}\mathcal{B}(\mathcal{K}_i) = \mathcal{N} \subset^{\mathcal{E}} \mathcal{M} = \oplus_{j\in J}\mathcal{B}(\mathcal{H}_j).$$

This means M is embedded as a von Neumann algebra into $\mathcal{M}$ such that: $N \subset \mathcal{N}$, with the restriction of $\mathcal{E}$ to M equal to E_N, the τ-preserving expectation of M onto N (*commuting square* condition); the span of $M\mathcal{N}$ equals $\mathcal{M}$ (*non-degeneracy* condition).

These conditions imply that any orthonormal basis of M over N is an orthonormal basis for $\mathcal{E}$, so the index $\mathrm{Ind}(\mathcal{E})$ of this expectation equals $[M:N]$ and the *inclusion* (bipartite) *graph* $\Lambda = \Lambda_{\mathcal{N}\subset\mathcal{M}} = (b_{ij})_{i\in I, j\in J}$ of $\mathcal{N} \subset \mathcal{M}$ satisfies

$$\|\Lambda\|^2 \leq [M:N],$$

where b_{ij} gives the multiplicity of $\mathcal{B}(\mathcal{K}_i)$ in $\mathcal{B}(\mathcal{H}_j)$ and Λ is viewed as an $I \times J$ matrix (or element in $\mathcal{B}(\ell^2J, \ell^2I)$).

The role of the Murray–von Neumann dimension, or coupling constant, is played here by the *dimension/coupling vectors*

$$\vec{d}_M = \big(\dim({}_M\mathcal{H}_j)\big)_j, \quad \vec{d}_N = \big(\dim({}_N\mathcal{K}_i)\big)_i.$$

For these two vectors to have all entries finite it is sufficient that one of the entries is finite, and if this is the case then

$$\Lambda^t \Lambda(\vec{d}_M) = [M : N]\vec{d}_M, \quad \vec{d}_N = \Lambda(\vec{d}_M).$$

Thus, while for a single factor all representations are stably equivalent, W^*-representations of subfactors appear, from the outset, as a far more complex notion.

Indeed, a subfactor $N \subset M$ admits a large variety of W^*-representations, with a central role played by the *standard representation*, $\mathcal{N}^{\mathrm{st}} \subset^{\mathcal{E}^{\mathrm{st}}} \mathcal{M}^{\mathrm{st}}$, whose inclusion graph $\Lambda_{\mathcal{N}^{\mathrm{st}} \subset \mathcal{M}^{\mathrm{st}}}$ coincides with the transpose of the standard graph $\Gamma_{N \subset M}$ of $N \subset M$, with the coupling vectors given by the standard weights of $\Gamma_{N \subset M}$.

This leads right away to natural concepts of injectivity and amenability, by analogy with the single II_1 factor case. Thus, an inclusion of II_1 factors $N \subset M$ is *injective* if there exists an expectation of $(\mathcal{N}^{\mathrm{st}} \subset^{\mathcal{E}^{\mathrm{st}}} \mathcal{M}^{\mathrm{st}})$ onto $(N \subset M)$, i.e. a norm one projection $\Phi\colon \mathcal{M}^{\mathrm{st}} \to M$ that is $\mathcal{E}^{\mathrm{st}}$-invariant. It is *amenable* if there exists a $(N \subset M)$-*hypertrace* on $(\mathcal{N}^{\mathrm{st}} \subset^{\mathcal{E}^{\mathrm{st}}} \mathcal{M}^{\mathrm{st}})$, i.e. a state φ on $\mathcal{M}^{\mathrm{st}}$ that is $\mathcal{E}^{\mathrm{st}}$-invariant and has M in its centralizer. These two concepts are easily seen to be equivalent and they imply N, M are amenable/injective as single factors, thus being isomorphic to the hyperfinite II_1 factor R by the Connes theorem ([5]). They also imply that $N \subset M$ is the range of a norm-one projection in *any* of its W^*-representations $(N \subset M) \subset (\mathcal{N} \subset^{\mathcal{E}} \mathcal{M})$, once some natural compatibility of higher relative commutants is satisfied (*smoothness*).

Amenability of $N \subset M$ was introduced in [39, 40, 44] and shown there to be equivalent to the condition $N, M \simeq R$ and $\Gamma_{N \subset M}$ amenable, i.e.

$$\|\Gamma_{N \subset M}\|^2 = [M : N].$$

It was also shown equivalent to the fact that $N \subset M$ can be exhausted by higher relative commutants of a "$(N \subset M)$-compatible tunnel" of subfactors, $M \supset N \supset P_1 \supset P_2 \supset \cdots$, obtained by iterative choices of the downward basic construction for induction/reduction by projections $p \in P_n' \cap M_{k(n)}$, at each step n, thus being completely classified by its standard invariant, $\mathcal{G}_{N \subset M}$. We revisit these results in Section 4 of the paper (see Theorem 4.5).

The complexity of W^*-representation theory for a subfactor naturally leads to several weaker amenability properties as well. Thus, a subfactor of finite Jones index $N \subset M$ is *weakly injective* (resp. *weakly amenable*) if it admits a W^*-representation

$$(N \subset M) \subset (\mathcal{N} \subset^{\mathcal{E}} \mathcal{M})$$

with an $\mathcal{E}$-invariant normal semifinite faithful (*n.s.f.*) trace, such that $(N \subset M)$ is the range of a norm-one projection from $(\mathcal{N} \subset^{\mathcal{E}} \mathcal{M})$ (resp. if $(\mathcal{N} \subset^{\mathcal{E}} \mathcal{M})$ has a

$(N \subset M)$-hypertrace). One should note that the standard representation does satisfy this "traciality" property (more on this below), so amenability/injectivity does imply weak amenablity/injectivity.

Again, these two notions are equivalent and they imply $N, M \simeq R$ (by [5]). They are also hereditary, i.e. if $(Q \subset P) \subset (N \subset M)$ is a non-degenerate commuting square of subfactors and $N \subset M$ is weakly amenable, then so is $Q \subset P$. A main result in this paper is the following:

1.1 Theorem. *If an extremal (e.g. irreducible) inclusion of* II_1 *factors with finite index* $N \subset M$ *is weakly amenable, then* $[M : N]$ *is the square norm of a bipartite graph. More precisely, if* $(N \subset M) \subset (\mathcal{N} \subset^{\mathcal{E}} \mathcal{M})$ *is a tracial* W^**-representation so that* $N \subset M$ *is the range of a norm-one projection, then* $[M : N] = \|\Lambda_{\mathcal{N}\subset\mathcal{M}}\|^2$.

The above result indicates that W^*-representations may help detect (and explain!) restrictions on the set $\mathcal{C}(M)$ of indices of irreducible subfactors of a given II_1 factor M, especially for $M = R$

Recall in this respect Jones fundamental result in [20], showing that the index of any subfactor lies in the spectrum

$$\{4\cos^2(\pi/n) \mid n \geq 3\} \cup [4, \infty).$$

One of his proofs of this result amounts to showing that if $[M : N] < 4$ then it must equal the square norm of the standard graph $\Gamma_{N\subset M}$, and using the fact that the set $\mathbb{E}^2$ of square norms of bipartite graphs has only the values $4\cos^2(\pi/n)$ when < 4 (see [21]).

The set $\mathbb{E}^2$ contains the half line $[2 + \sqrt{5}, \infty)$, but $\mathbb{E}^2 \cap [1, 2 + \sqrt{5}]$ is a closed countable set, consisting of an increasing sequence of accumulation points converging to $2 + \sqrt{5}$, the first of which being $4 = \lim_n 4\cos^2(\pi/n)$ (cf. [8]; see also [12]). Yet it is known that any number > 4 can occur as index of an irreducible subfactor ([38, 41]), and that

$$\mathcal{C}(L\mathbb{F}_\infty) = \{4\cos^2(\pi/n) \mid n \geq 3\} \cup [4, \infty)$$

(see [51]). On the other hand, if M is constructed out of a free ergodic probability measure preserving action of a non-elementary hyperbolic group, then $\mathcal{C}(M) = \{1, 2, 3, \dots\}$ (see [53]). So $\mathcal{C}(M)$ appears to depend in very subtle ways on the nature of the factor M.

But the most important question along these lines, of calculating $\mathcal{C}(R)$, remained open. Our work in this paper attempts to provide some tools for approaching the "restrictions" part of this problem, more specifically for showing that $\mathcal{C}(R) \subset \mathbb{E}^2$.

By [14], if an irreducible subfactor $N \subset M$ satisfies

$$4 < [M : N] \leq 2 + \sqrt{5},$$

then its standard graph $\Gamma_{N\subset M}$ equals A_∞. Equivalently, the higher relative commutants $N' \cap M_n$ in the Jones tower $N \subset M \subset_{e_0} M_1 \subset_{e_1} M_2 \subset \cdots$ are generated by the Jones projections $e_0, e_1, \ldots$. So in order to show $\mathcal{C}(R) \subset \mathbb{E}^2$, it is sufficient to prove that any A_∞-subfactor of R has index equal to the square norm of a bipartite graph. Our belief is that in fact any hyperfinite A_∞-subfactor is weakly amenable. If true, then Theorem 1.1 above would imply that $\mathcal{C}(R) \subset \mathbb{E}^2$.

In order to make this speculation more specific, we need to fix some terminology and explain ways of producing W^*-representations.

Thus, a W^*-representation $\mathcal{N} \subset \mathcal{M}$ is *irreducible* if $\mathcal{Z}(\mathcal{M}) \cap \mathcal{Z}(\mathcal{N}) = \mathbb{C}$. This is equivalent to $\Lambda_{\mathcal{N}\subset\mathcal{M}}$ being connected as a graph (irreducible as a matrix).

The W^*-representation is *tracial* if it admits a n.s.f. trace Tr that is $\mathcal{E}$-invariant. If $(N \subset M) \subset (\mathcal{N} \subset^{\mathcal{E}} \mathcal{M})$ is tracial and has finite couplings, with the n.s.f. trace Tr given by $\vec{d}_M$ being $\mathcal{E}$-preserving, then the W^*-representation is *canonically tracial*.

The simplest example of a W^*-representation occurs from a *graphage* of $N \subset M$, i.e. a non-degenerate commuting square $(Q \subset P) \subset (N \subset M)$, with Q, P finite dimensional, by taking the basic construction inclusion

$$(N \subset M) \subset (\mathcal{N} = \langle N, e_P^M \rangle \subset \langle M, e_P^M \rangle = \mathcal{M})$$

(cf. [40, §2.1]; see Lemma 3.2.2 below). It is easy to see that such W^*-representations have finite couplings and are canonically tracial.

Another isomorphism invariant for a W^*-representation $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$, besides the inclusion graph $\Lambda_{\mathcal{N}\subset\mathcal{M}}$ and the coupling vectors, is the isomorphism class of the *relative commutant* (*RC*) algebra $M' \cap \mathcal{N}$. If $\mathcal{N} \subset \mathcal{M}$ has finite couplings, then $M' \cap \mathcal{N}$ identifies naturally to a von Neumann subalgebra of $(M^t)^{\mathrm{op}}$, so it is finite.

A W^*-representation $(N \subset M) \subset (\mathcal{N} \subset^{\mathcal{E}} \mathcal{M})$ is *exact*, if $M \vee (M' \cap \mathcal{N}) = \mathcal{M}$. Such a representation is irreducible if and only if its RC-algebra $M' \cap \mathcal{N}$ is a factor.

We are particularly interested in irreducible exact W^*-representation with finite couplings. They are all of the form

$$\mathcal{N}_P := N \vee P^{\mathrm{op}} \subset^{\mathcal{E}_P} M \vee P^{\mathrm{op}} =: \mathcal{M}_P,$$

where $P \subset M$ is an irreducible subfactor and M acts here by left multiplication on $L^2(M_\infty)$, P acts on the right, and $\mathcal{E}_P$ is the unique expectation extending $E_N \otimes \mathrm{id}_{P^{\mathrm{op}}}$, M_∞ denoting the enveloping algebra of the Jones tower for $N \subset M$.

The standard W^*-representation $(N \subset M) \subset (\mathcal{N}^{\mathrm{st}} \subset^{\mathcal{E}^{\mathrm{st}}} \mathcal{M}^{\mathrm{st}})$, corresponds to the case $P = M$ of this construction, i.e. to $(N \subset M) \subset (N \vee M^{\mathrm{op}} \subset M \vee M^{\mathrm{op}})$.

Taking the direct sum $\oplus_P (\mathcal{N}_P \subset^{\mathcal{E}_P} \mathcal{M}_P)$, over all isomorphism classes of irreducible $P \subset M$, one obtains the universal exact W^*-representation with finite couplings

$$(N \subset M) \subset (\mathcal{N}^{\mathrm{u,fc}} \subset^{\mathcal{E}^{\mathrm{u,fc}}} \mathcal{M}^{\mathrm{u,fc}}).$$

We say that $N \subset M$ is *ufc-amenable* (resp. *ufc-injective*) if this representation has a $(N \subset M)$-hypertrace (resp. norm-one projection). Our conjecture is that any hyperfinite A_∞-subfactor $N \subset M$ is ufc-amenable, and thus

$$[M : N] = \|\Lambda_{\mathcal{N}^{\mathrm{u,fc}} \subset \mathcal{M}^{\mathrm{u,fc}}}\|^2.$$

The paper is organized as follows. Section 2 recalls some basic facts in Jones theory of subfactors, and its generalization for arbitrary W^*-inclusions. In Section 3 we develop the concept of W^*-representation of subfactors, give formal definitions, prove general results and provide examples. In Section 4 we recall from [40,44] the definition of amenability and injectivity for a subfactor $N \subset M$, as well as a result establishing the equivalence of amenability/injectivity with a series of other properties of $N \subset M$ (see Theorem 4.5). In Section 5 we introduce the concepts of weak-amenability and weak-injectivity and prove Theorem 1.1 (see Theorem 5.4). We also define here ufc-amenability and ufc-injectivity of a subfactor $N \subset M$ (see Definition 5.7) and state a result establishing the equivalence of these two properties with several other structural properties of $N \subset M$ (Theorem 5.8), which we will prove in a follow up to this paper. Section 6 contains many comments and open problems.

We mention that this paper is an outgrowth of our unpublished 1997 note entitled *Biduals associated to subfactors, hypertraces and restrictions for the index*.

2. Preliminaries

We recall in this section some basic facts about Jones' index theory for inclusions of II_1 factors and, more generally, for inclusions of von Neumann algebras. We typically use the notations M, N, P, Q, B for tracial von Neumann algebras, with the generic notation τ for the corresponding (faithful normal) trace state on it, sometimes with an index specifying the algebra on which it is defined (e.g. τ_M for the trace on M). The generic notation for arbitrary von Neumann algebras will be $\mathcal{M}, \mathcal{N}, \mathcal{P}, \mathcal{Q}$, etc.

For basics on II_1 factors and tracial von Neumann algebras we refer to [2] and for general von Neumann algebras to [59].

2.1. Jones index for subfactors and the basic construction. If $B \subset M$ is a von Neumann subalgebra of the tracial von Neumann algebra M, then $E_B = E_B^M$ denotes the unique trace preserving conditional expectation of M onto B. It extends to a projection e_B of L^2M onto L^2B, which is selfadjoint and positive on L^2M, when viewed as the space of square summable operators affiliated to M.

Identifying M with its standard representation $M \hookrightarrow \mathcal{B}(L^2M)$ (as left multiplication operators on L^2M), one has:

(i) $e_B x e_B = E_B(x) e_B$ for all $x \in M$;

(ii) $B = \{x \in M \mid [x, e_B] = 0\}$; $B \ni b \mapsto b e_B$ is an onto isomorphism;

(iii) $\vee_{u \in \mathcal{U}(B)} u e_B u^* = 1$.

The von Neumann algebra $\langle M, e_B \rangle$ (or $\langle M, B \rangle$) generated in $\mathcal{B}(L^2 M)$ by M and e_B is called the *extension of M by B*. It coincides with

$$(B^{\mathrm{op}})' = (J_M B J_M)' \cap \mathcal{B}(L^2 M)$$

(the commutant of the operators of right multiplication by elements in B). The span of $\{x e_B y \mid x, y \in M\}$ is a wo-dense *-subalgebra of $\langle M, e_B \rangle$ whose wo-closure contains M. Thus, this wo-closure is equal to $\langle M, e_B \rangle$. This construction of the new inclusion $M \subset \langle M, e_B \rangle$ from an initial inclusion $B \subset M$, is called *Jones basic construction*.

Given an inclusion of II_1 factors $N \subset M$, the *Jones index of N in M*, denoted $[M : N]$, is defined as the Murray–von Neumann coupling constant of N when acting on the Hilbert space $L^2 M$ by left multiplication operators (as a subalgebra of M), or equivalently the Hilbert-dimension of $L^2 M$ as a (left) N-Hilbert module,

$$[M : N] = \dim_N L^2 M.$$

Thus, $[M : N] < \infty$ iff $N' \cap \mathcal{B}(L^2 M)$ is a II_1 factor, while $[M : N] = \infty$ iff N' is of type II_∞. If the index is finite, then $M \subset M_1 := \langle M, e_N \rangle$ is an inclusion of II_1 factors with index $[M_1 : M] = [M : N]$ and the trace state on M_1 satisfies

$$\tau_{M_1}(x e_N y) = \lambda \tau_M(xy), \quad \forall x, y \in M,$$

where $\lambda = [M : N]^{-1}$.

The index of subfactors is multiplicative, in the sense that if $P \subset N \subset M$ are subfactors, then $[M : P] = [M : N][N : P]$.

Letting $(M_{-1} \subset M_0) = (N \subset M)$ and $e_0 = e_N$, this allows constructing iteratively a whole *tower* of inclusions of II_1 factors, $M_{-1} \subset M_0 \subset_{e_0} M_1 \subset_{e_1} \subset M_2 \subset \cdots$, with each M_{i+1}, $i \geq 0$, being generated by M_i and a projection e_i of trace $\lambda = [M : N]^{-1}$, having index $[M_{i+1} : M_i] = [M : N]$ and satisfying the properties:

(a) $e_i x e_i = E^{M_i}_{M_{i-1}}(x) e_i$ for all $x \in M_i$;

(b) $\{e_i\}' \cap M_i = M_{i-1}$;

(c) $\tau(x e_i) = \lambda \tau(x)$ for all $x \in M_i$.

In particular, the *λ-sequence of Jones projections* $\{e_i\}_{i \geq 0}$ with the trace τ satisfy the conditions:

(a′) $e_i e_{i \pm 1} e_i = \lambda e_i$;

(b′) $[e_i, e_j] = 0$ for all $j > i + 1$;

(c′) $\tau(xe_{i+1}) = \lambda\tau(x)$ for all $x \in \mathrm{Alg}(\{e_0, e_1, \dots, e_i\})$.

Jones' celebrated theorem shows that the conditions (a′)–(c′) imply the restrictions on the index

$$[M : N] = \lambda^{-1} \in \{4\cos^2(\pi/n) \mid n \geq 3\} \cup [4, \infty),$$

with this latter set called the *Jones spectrum*. The key ingredient in his proof of these restrictions on the index is the fact that axioms (a′)–(c′) above imply that $A_n = C^*(1, e_0, \dots, e_n)$ is finite dimensional, with $e_0 \vee \cdots \vee e_n$ a central projection in it, with the trace of its complement equal to

$$P_{n+1}(\lambda) = P_n(\lambda) - \lambda P_{n-1}(\lambda)$$

whenever $P_n(\lambda), P_{n-1}(\lambda) > 0$, where the polynomials $P_n(t)$, $n \geq -1$, are defined recursively by the formulas

$$P_{-1} = 1, \quad P_0 = 1, \quad P_{n+1}(t) = P_n(t) - tP_{n-1}(t), \; n \geq 0.$$

2.2. Extremal subfactors. It is shown in [20] that if $N \subset M$ is a subfactor of finite index and p is a projection in $N' \cap M$, then

$$[pMp : Np] = \tau_M(p)\tau_{N'}(p)[M : N]$$

(*Jones local index formula*). Thus, if $p_1, \dots, p_n \in N' \cap M$ is a partition of 1 with projections, then

$$[M : N] = \sum_i \frac{[p_i M p_i : N p_i]}{\tau_M(p_i)}.$$

Note that, in particular, this implies $\dim(N' \cap M) \leq [M : N] < \infty$, and more generally

$$\dim(M_i' \cap M_j) \leq [M_j : M_i] = [M : N]^{j-i} < \infty.$$

Following [40, §1.2.5], an inclusion of II_1 factors $N \subset M$ is called *extremal* if $\tau_M(p) = \tau_{N'}(p)$ for all $p \in \mathcal{P}(N' \cap M)$, or equivalently

$$[pMp : Np] = [M : N]\tau(p)^2, \quad \forall p \in \mathcal{P}(N' \cap M) \text{ non-zero.}$$

Recall from [31, Corollary 4.5] that this condition is equivalent to $E_{N' \cap M}(e) = \lambda 1$ for any Jones projection $e \in M$ (i.e. a projection whose expectation on N is equal to $\lambda 1$). By [45] or the appendix in [44], this is also equivalent to the fact that the norm closure of the convex hull of $\{ueu^* \mid u \in \mathcal{U}(N)\}$ contains $\lambda 1$.

Irreducible subfactors are automatically extremal. As pointed out in [31], if $N \subset M$ is extremal and $4 < [M : N] < 3 + 2\sqrt{2}$, then $N \subset M$ is irreducible. So for subfactors with small index > 4, extremality is same as irreducibility.

2.3. The standard invariant and graph of a subfactor. The higher relative commutants in the Jones tower $\{M_i' \cap M_j\}_{j \geq i \geq -1}$ form a lattice of finite dimensional von Neumann algebras, with a trace τ inherited from $\cup_i M_i$ with inclusions

$$M_i' \cap M_j \subset M_l' \cap M_k,$$

whenever $-1 \leq l \leq i \leq j \leq k$. Moreover, the Jones projection e_j lies in $M_{j-1}' \cap M_{j+1}$ and implements the τ-preserving expectation of $M_i' \cap M_j$ onto $M_i' \cap M_{j-1}$ for all $j > i$. Also, any two such expectations commute,

$$E_{M_i' \cap M_j} E_{M_k' \cap M_l} = E_{M_k' \cap M_l} E_{M_i' \cap M_j} = E_{M_k' \cap M_j}.$$

In other words $\{M_i' \cap M_j\}_{j \geq i \geq -1}$ is a lattice of *commuting square* inclusions.

The lattice of higher relative commutants, with its trace and Jones projections, is clearly an isomorphism invariant for $N \subset M$. It is called the *standard invariant of* $N \subset M$ and denoted $\mathcal{G}_{N \subset M}$. See [41] for more on this object and of a way to axiomatize it as an abstract object, called *standard λ-lattice*. It is pointed out in [41] that, due to the *duality* result in [31], which shows that there is a natural identification between the tower $(M_{i+1} \subset M_{i+2} \subset \cdots)$ and the $[M : N]$-amplification of $(M_{i-1} \subset M_i \subset \cdots)$, all information is in fact contained in the first two rows of the lattice, i.e.

$$\mathcal{G}_{N \subset M} = \big(\{M_i' \cap M_j\}_{j \geq i = 0,-1}, \{e_i\}_{i \geq 0}, \tau\big).$$

The first row of consecutive inclusions

$$\mathbb{C} = M_0' \cap M_0 \subset M_0' \cap M_1 \subset \cdots$$

in $\mathcal{G}_{N \subset M}$ is determined by a pointed bi-partite connected graph, called the *standard* (or *principal*) *graph* of $N \subset M$ and denoted $\Gamma_{N \subset M} = (a_{kl})_{k \in K, l \in L}$, with the *even vertices* being indexed by a set K containing the "initial" vertex $*$, the *odd vertices* indexed by a set L, with $a_{kl} \geq 0$ denoting the number of edges between k and l (see e.g. [40, Section 1.3.5]).

To explain this in detail, let

$$K_0 = \{*\}, \quad L_i = \{l \in L \mid \exists a_{kl} \neq 0, k \in K_{i-1}\},$$
$$K_i = \{k \in K \mid \exists a_{kl} \neq 0, l \in L_i\}, \quad i \geq 1,$$

and note that $K = \cup_{i \geq 0} K_i$, $L = \cup_{i \geq 1} L_i$ (because $\Gamma_{N \subset M}$ is connected). The set of irreducible components of $M' \cap M_{2i}$, $i \geq 0$ (resp. $M' \cap M_{2i-1}$, $i \geq 1$) is identified with K_i (resp. L_i), with the embeddings $K_i \subset K_{i+1}$, $i \geq 0$ (resp. $L_i \subset L_{i+1}$, $i \geq 1$) being implemented by the 1-to-1 map from $\mathcal{Z}(M' \cap M_{2i})$ into $\mathcal{Z}(M' \cap M_{2i+2})$ (resp. $\mathcal{Z}(M' \cap M_{2i-1})$ into $\mathcal{Z}(M' \cap M_{2i+1})$):

$$\mathcal{Z}(M' \cap M_{2i}) \ni z \mapsto \text{unique } z' \in \mathcal{Z}(M' \cap M_{2i+2}) \text{ with } z e_{2i+1} = z' e_{2i+1},$$

(similarly for $L_i \subset L_{i+1}$). Each index $k \in K$ (resp. $l \in L$) can also be viewed as labeling the irreducible subfactor $Mp \subset pM_{2i}p$ (resp. $Mq \subset qM_{2i-1}q$) with p a minimal projection in the kth direct summand of $M' \cap M_{2i}$ (resp. lth summand of $M' \cap M_{2i-1}$) for any $i \geq 0$ with $K_i \ni k$ (resp. any $i \geq 1$ with $L_i \ni l$).

With these conventions, the bipartite graph (diagram) for the embedding

$$M' \cap M_{2i} \subset M' \cap M_{2i+1}, \quad i \geq 0$$

(resp. $M' \cap M_{2i-1} \subset M' \cap M_{2i}$, $i \geq 1$) is given by

$$\Gamma_{|K_i} = (a_{kl})_{k \in K_i, l \in L_{i+1}}$$

(resp. $\Gamma^t_{|L_i} = (b_{lk})_{l \in L_i, k \in K_i}$, where $b_{lk} = a_{kl}$).

Let us now assume $N \subset M$ is extremal. In this case, the trace τ on the finite dimensional algebras $M_i' \cap M_j$ is uniquely determined by the *standard vectors* $\vec{v} = (v_k)_{k \in K}$, $\vec{u} = (u_l)_{l \in L}$, given by square roots of indices of irreducible inclusions appearing in the Jones tower, as follows.

First note that Jones local index formula combined with the duality imply that if $k \in K$ (resp. $l \in L$) then the index $[pM_{2i}p : Mp]$ (resp. $[pM_{2i-1}p : Mp]$) is the same for any i with the property that $k \in K_i$ (resp. $l \in L_i$) and any p minimal projection in the kth (resp. lth) summand of $M' \cap M_{2i}$ (resp. of $M' \cap M_{2i-1}$). One defines

$$v_k = [pM_{2i}p : Mp]^{1/2} \quad \text{and} \quad u_l = [pM_{2i-1}p : Mp]^{1/2}.$$

Thus, $v_* = 1$ and if $N' \cap M = \mathbb{C}$, then the single point set $L_1 = \{l_1\}$ satisfies

$$u_{l_1} = [M : N]^{1/2}.$$

Also, $\vec{u} = \Gamma^t(\vec{v})$, $\Gamma\Gamma^t(\vec{v}) = \lambda^{-1}\vec{v}$, where $\Gamma = \Gamma_{N \subset M} = (a_{kl})_{k,l}$ is now viewed as a $K \times L$ matrix.

The trace of a minimal projection p in the kth summand of $M' \cap M_{2i}$ (for i large enough so that $k \in K_i$) is then given by $\tau(p) = \lambda^i v_k$. Similarly, if q is a minimal projection in the lth summand of $M' \cap M_{2i+1}$ then $\tau(q) = \lambda^{i+\frac{1}{2}} u_k$.

The norm of the standard graph of the subfactor coincides with the growth rate of the higher relative commutants

$$\|\Gamma_{N \subset M}\| = \lim_{n \to \infty} (\dim(M' \cap M_n))^{1/n}$$

and satisfies the estimate $\|\Gamma_{N \subset M}\|^2 \leq [M : N]$ (see [40, §1.3.5]).

2.4. Alternative definitions of the index. Let us recall two alternative ways of defining the index, from [31].

Given a von Neumann subalgebra B of a tracial von Neumann algebra (M, τ), there exists a family of elements $\{\xi_i\}_i \subset L^2M$ such that

$$E_B(\xi_j^* \xi_i) = \delta_{ij} p_i \in \mathcal{P}(B) \quad \text{and} \quad x = \sum_i \xi_i E_B(\xi_i^* x) \text{ for all } x \in M,$$

where the convergence is in L^2M. Equivalently, $\{\xi_i\}_i \subset L^2M$ are so that $\xi_i e_B$ are partial isometries with $\sum_i \xi_i e_B \xi_i^* = 1$.

Such $\{\xi_i\}_i \subset L^2M$ is called an *orthonormal basis* (*o.b.*) of M over B, and it is unique in an appropriate sense (cf. [31, Proposition 1.3]). The finite partial sums in $\sum_i \xi_i \xi_i^*$ lie in L^1M_+ and they form an increasing net with "limit"

$$Z = \sum_i \xi_i \xi_i^* = Z_0 p_0 + (\infty)(1 - p_0),$$

where $p_0 \in \mathcal{P}(\mathcal{Z}(M))$ and Z_0 is a positive densely defined operator affiliated with $\mathcal{Z}(M)$, with $Z = Z(B \subset M)$ independent of the choice of the o.b. Thus, if M is a II_1 factor, then

$$Z = \sum_i \xi_i \xi_i^* = \alpha 1$$

with $\alpha \in [1, \infty]$ only depending on the isomorphism class of $B \subset M$.

In case $N \subset M$ is an inclusion of II_1 factors, then by [31, §1.3.3] one has

$$[M : N] = Z(N \subset M),$$

or in other words

$$[M : N] = \sum_i m_i m_i^*$$

for any o.b. $\{m_i\}_i$ of M over N. Thus, if $[M : N] < \infty$, then any o.b. is made up of "bounded elements" $m_i \in M$, with $\|m_i\| \leq [M : N]^{1/2}$, and one can in fact choose them all but possibly one so that $E_N(m_i^* m_i) = 1$.

Another characterization of the Jones index is given in [31, Theorem 2.2], where it is shown that the quantity

$$\lambda(E_N) := \sup\{c \geq 0 \mid E_N(x) \geq cx, x \in M_+\},$$

which measures the "flattening" of the expected N-value of positive elements in M, satisfies $\lambda(E_N) = [M : N]^{-1}$. This characterization is key to calculating the Connes–Størmer *relative entropy* $H(M|N)$ (see [7]) of a subfactor $N \subset M$ ([31, Theorem 4.6]), which in particular provides the characterization

$$\text{“}N \subset M \text{ extremal iff } H(M|N) = \ln[M : N]\text{”.}$$

2.5. W^*-inclusions with finite index. The above two formulas for the Jones index of II_1 subfactors, which only depend on the expectation from the ambient algebra onto its subalgebra, give the possibility of defining the index for arbitrary inclusions of von Neumann algebras (or a W^*-*inclusion*) with conditional expectation, $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$, in two alternative ways.

The easiest to define is as follows. Denote

$$\lambda(\mathcal{E}) \stackrel{\text{def}}{=} \sup\{c \geq 0 \mid \mathcal{E}(x) \geq cx, \ \forall x \in \mathcal{M}_+\}$$

and define $\mathrm{Ind}(\mathcal{E}) = \lambda(\mathcal{E})^{-1}$. We call the latter the *index* of $\mathcal{E}$ (also referred to as the *probabilistic index*, or the *Pimsner–Popa index*, of the expectation $\mathcal{E}$).

Note that if $\mathrm{Ind}(\mathcal{E}) < \infty$, then $\mathcal{E}$ is automatically normal and faithful (see e.g. [45, Lemma 1.1]). As we have seen in Section 2.4, if $N \subset M$ are II_1 factors with $\mathcal{E} = E_N$ the trace preserving expectation, then $\mathrm{Ind}(E_N) = [M : N]$.

This definition has many advantages for various limiting arguments (see below). But it is useful to view it in combination with the definition of the index based on o.b. of $\mathcal{M}$ over $\mathcal{N}$, with respect to the expectation $\mathcal{E}$. Since this is closely related to the notion of basic construction for arbitrary W^*inclusions $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$, we recall from [58] that if $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is the standard representation of the von Neumann algebra $\mathcal{M}$, then there exists a "canonical" projection $e = e_{\mathcal{N}} \in \mathcal{B}(\mathcal{H})$ such that

(i) $exe = \mathcal{E}(x)e$ for all $x \in \mathcal{M}$;

(ii) $\mathcal{N} = \{e\}' \cap \mathcal{M}$; $x \in \mathcal{N}$, $xe = 0$ iff $x = 0$;

(iii) $\vee\{ueu^* \mid u \in \mathcal{U}(\mathcal{M})\} = 1$;

with e unique with these properties, up to spatial isomorphism. One denotes by $\langle \mathcal{M}, e\rangle$ the von Neumann algebra generated by $\mathcal{M}$ and e, which has $\mathrm{sp}\{xey \mid x, y \in \mathcal{M}\}$ as a wo-dense *-subalgebra. This is the *basic construction* for arbitrary W^*-inclusions with expectation. We still call e the *Jones projection* implementing $\mathcal{E}$. We see by this definition that $e\langle \mathcal{M}, e\rangle e = \mathcal{N}e \simeq \mathcal{N}$, so a "corner" of central support 1 of $\langle \mathcal{M}, e\rangle$ is equal to $\mathcal{N}$.

There does exist in this generality an analogue of o.b. of $\mathcal{M}$ over $\mathcal{N}$ with respect to $\mathcal{E}$. For our purposes, it is sufficient to discuss this under two types of assumptions: when either $\mathrm{Ind}(\mathcal{E}) < \infty$; or when $\mathcal{M}$ is finitely generated as a right $\mathcal{N}$ module. In both cases it is immediate to show that there exists a family $\{m_j\}_j \subset \mathcal{M}$ such that

$$\mathcal{E}(m_i^* m_j) = \delta_{ij} p_j \in \mathcal{P}(\mathcal{N}) \quad \text{and} \quad \sum_j m_j e m_j^* = 1,$$

and which we call an *orthonormal basis* (o.b.) of $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$.

One has

$$\mathrm{Ind}(\mathcal{E}) \le \Big\| \sum_j m_j m_j^* \Big\| \le (\mathrm{Ind}(\mathcal{E}))^2$$

(cf. [42, §1.1.6]). Thus, $\mathrm{Ind}(\mathcal{E}) < \infty$ iff $Z(\mathcal{E}) := \sum_j m_j m_j^*$ is bounded, in which case this element, which lies in $\mathcal{Z}(\mathcal{M})$ and is ≥ 1, does not depend on the o.b.

The o.b. properties imply that each $X \in \langle \mathcal{M}, e\rangle$ can be written as

$$X = \sum_{i,j} m_i y_{ij} e m_j^*$$

for some unique $y_{ij} \in p_i \mathcal{N} p_j$, i.e. $\langle \mathcal{M}, e\rangle$ is an "amplification" of $\mathcal{N}$, with $\mathcal{Z}(\mathcal{N})$ naturally identifying with $\mathcal{Z}(\langle \mathcal{M}, e\rangle)$, via the map $\mathcal{Z}(\mathcal{N}) \ni z \mapsto z'$, where z' is the unique element in $\mathcal{Z}(\langle \mathcal{M}, e\rangle)$ such that $z'e = ze$.

If either $\mathcal{N}, \mathcal{M}$ are properly infinite, or if they are type II_1, or if they are both type $\mathrm{I}_{\mathrm{fin}}$ but with each type I_n summand of $\mathcal{N}$ having multiplicity $\le n$ in each homogeneous type I_m summand of $\mathcal{M}$, then one actually has

$$\mathrm{Ind}(\mathcal{E}) = \Big\| \sum_j m_j m_j^* \Big\|,$$

for any o.b. $\{m_j\}_j$ of $\mathcal{M}$ relative to $\mathcal{N}$ (see [42, Theorem 1.1.6]).

Thus, under this dimension condition, one has $\mathrm{Ind}(\mathcal{E}) = \mathrm{Ind}(\mathcal{E} \otimes \mathrm{id})$ for the inclusion $(\mathcal{N} \subset \mathcal{M}) \overline{\otimes} \mathcal{B}(\mathcal{H})$. Note that this stability of the index implies the "stability" of the inequality $\mathcal{E}(x) \ge \lambda x$ for all $x \in \mathcal{M}_+$, where $\lambda = \lambda(\mathcal{E})$, in the sense that more than being positive, the map $(\mathcal{E} - \lambda \mathrm{id}_{\mathcal{M}})\colon \mathcal{M} \to \mathcal{M}$ is completely positive.

If $Z = \sum_i m_i m_i^*$ is bounded, then $\mathcal{E}_1\colon \langle \mathcal{M}, e\rangle \to \mathcal{M}$ defined by $\mathcal{E}_1(xey) = xyZ^{-1}$, defines a normal conditional expectation with $\mathrm{Ind}(\mathcal{E}_1) < \infty$. More precisely, by [42, Lemma 1.2.1] one has

$$\mathrm{Ind}(\mathcal{E}_1) = \|\mathcal{E}(Z)\|; \quad \{m_i e Z^{1/2}\}_i \text{ is an o.b. for } \mathcal{E}_1;$$
$$\Big\| \sum_j m_j e Z e m_j^* \Big\| = \|\mathcal{E}(Z)\|.$$

An important feature of the probabilistic index $\mathrm{Ind}(\mathcal{E})$ of an expectation $\mathcal{E}$ is that it behaves well to "limit operations". For instance, if $(\mathcal{N}_n \subset \mathcal{M}_n)$ is a sequence of W^*-inclusions that are embedded into $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ such that $\mathcal{E}(\mathcal{M}_n) = \mathcal{N}_n$ for all n, and $\mathcal{M}_n \nearrow \mathcal{M}$, $\mathcal{N}_n \nearrow \mathcal{N}$, then

$$\lim_n \mathrm{Ind}(\mathcal{E}_n) = \mathrm{Ind}(\mathcal{E}),$$

and if $f_n \in \mathcal{P}(\mathcal{N})$, $f_n \nearrow 1$, then

$$\lim_n \mathrm{Ind}(\mathcal{E}(f_n \cdot f_n)) = \mathrm{Ind}(\mathcal{E})$$

(see [31,32]). Also, if $\mathcal{E}_i : \mathcal{M} \to \mathcal{N}$ is a net of expectations with decreasing (finite) index, equivalently $\lambda(\mathcal{E}_i) \nearrow \lambda_0 > 0$, then any Banach-limit $\mathcal{E}$ of the $\mathcal{E}_i$ will be an expectation of $\mathcal{M}$ onto $\mathcal{N}$ that satisfies $\mathcal{E}(x) \geq \lambda_0 x$ for all $x \in \mathcal{M}_+$ (so it is automatically normal and faithful). Thus, if one defines the *minimal index* $\mathrm{Ind}_{\min}(\mathcal{N} \subset \mathcal{M}) \in (0, \infty]$ of an arbitrary inclusion of von Neumann algebras as the infimum of $\mathrm{Ind}(\mathcal{E})$, over all normal expectations $\mathcal{E}$ of $\mathcal{M}$ onto $\mathcal{N}$ (with the convention that it is equal to ∞ if there exists none), then

$$\mathrm{Ind}_{\min}(\mathcal{N} \subset \mathcal{M}) < \infty$$

implies that there exists an expectation $\mathcal{E}_0$ with

$$\mathrm{Ind}(\mathcal{E}_0) = \mathrm{Ind}_{\min}(\mathcal{N} \subset \mathcal{M}).$$

In other words, the minimal index "is attained", if finite. Note that if $\mathcal{E}_i$ are λ_i-Markov, then $\mathcal{E}_0$ follows $\lambda_0 = \lim_i \lambda_i$ Markov. It is easy to see that if $Q \subset P$ are finite dimensional as above, with connected inclusion bipartite graph $\Lambda = \Lambda_{Q \subset P}$, then there exists a unique expectation $E \colon P \to Q$ with $\mathrm{Ind}(E) = \mathrm{Ind}_{\min}(Q \subset P)$, and it is exactly the expectation preserving the trace on P implemented by the Perron–Frobenius eigenvector $\vec{t}$ of $\Lambda^t \Lambda$, thus being $\lambda = \|\Lambda\|^{-2}$ Markov (see e.g. [40, Section 1.1.7]). In particular, $\|\Lambda\|^2 \leq \mathrm{Ind}(E)$ for any expectation $E \colon P \to Q$.

Another feature of the probabilistic index when considered for an expectation of a C*-inclusion $C \subset^E B$, is that it allows taking the bidual W^*-inclusion $C^{**} \subset^{E^{**}} B^{**}$, which satisfies

$$\mathrm{Ind}(E^{**}) = \mathrm{Ind}(E) < \infty$$

(see [40, Section 2.4] and Section 3.3 of this paper). The [31]-index is also key in relating the relative Dixmier property for W* and C*-inclusions with the finiteness of the index ([44, 45]), as well as in the calculation of the relative entropy for finite dimensional W^*-inclusions $Q \subset P$ (see [31, Section 6]) and for inductive limits of such inclusions (see [32]).

Finally, let us mention that if the W^*-inclusion $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ is *irreducible*, i.e. if one has a trivial relative commutant, $\mathcal{N}' \cap \mathcal{M} = \mathbb{C}$, then $\mathcal{E}$ is the unique expectation of $\mathcal{M}$ onto $\mathcal{N}$, so one can use the Jones notation for the index $[\mathcal{M} : \mathcal{N}]$, even when the factors $\mathcal{N}, \mathcal{M}$ are infinite (non-tracial). For arbitrary W^*-inclusions of infinite factors with finite index expectation $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ (see [25]), one usually considers $\mathcal{E}$ to be the expectation of minimal index ([15]), determined uniquely by the fact that the (scalar) values $\mathcal{E}(q)$ of $\mathcal{E}$ on minimal projections $q \in \mathcal{N}' \cap \mathcal{M}$ are proportional to $[q\mathcal{M}q : \mathcal{N}q]^{1/2}$ (see [15]). As pointed out in [25], in this case one can take an o.b. of just one element $\{m\} \subset \mathcal{M}$, which thus satisfies

$$X = m\mathcal{E}(m^* X), \quad \forall X \in \mathcal{M}$$

and

$$\mathcal{E}(m^*m) = 1, \quad mm^* = \lambda^{-1}1.$$

2.6. λ-Markov inclusion, the associated tower and enveloping algebra. Following [20, §3.3.1], [40, §1.1.4], and [42, §1.2], a W^*-inclusion with expectation $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ that has finite index and satisfies

$$\sum_j m_j m_j^* = \lambda^{-1}1 \in \mathbb{C}1$$

for some (thus any) o.b. is called a *λ-Markov inclusion*.

If $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ is λ-Markov and $\mathcal{M} \subset^{e} \langle \mathcal{M}, e\rangle$ is its basic construction, then $\mathcal{E}_1 : \mathcal{M}_1 \to \mathcal{M}$ defined by $\mathcal{E}_1(xey) = \lambda xy$, $x, y \in \mathcal{M}$, is a conditional expectation of $\mathcal{M}_1 = \langle \mathcal{M}, e\rangle$ onto $\mathcal{M}_0 = \mathcal{M}$ and $\mathcal{M}_0 \subset^{\mathcal{E}_1} \mathcal{M}_1$ is again λ-Markov, with o.b. $\{\lambda^{-1/2} m_i e\}_i$.

Thus, like in the II_1 factor case, one can iterate this construction and obtain a whole *tower of λ-Markov inclusions* $\mathcal{M}_{n-1} \subset^{\mathcal{E}_n}_{e_{n-1}} \mathcal{M}_n$, $n \geq 1$, where we have put $e_0 = e$. Moreover, the composition of expectations $\mathcal{E}_1 \circ \cdots \circ \mathcal{E}_n \circ \cdots$ implements a trace τ on $\mathrm{Alg}(\{e_n\}_{n\geq 0})$ with $\tau(e_n) = \lambda$ for all n, and the sequence of projections $e_0, e_1, \ldots$ satisfies properties (a′)–(c′) of Section 2.1 with respect to this trace and λ. Thus, λ^{-1} lies in the Jones spectrum and $(\{e_n\}_{n\geq 0}, \tau)$ gives a λ-sequence of projections.

Moreover, the inductive limit of the Jones tower $\mathcal{M}_n$ associated with a λ-Markov W^*-inclusion $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ gives rise to a canonical *enveloping* von Neumann algebra, denoted $\mathcal{M}_\infty$.

2.6.1 Lemma. *Let $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M} \subset^{\mathcal{E}_1}_{e_0} \mathcal{M}_1 \subset \cdots$ be a λ-Markov tower of W^*-inclusions. Let ϕ be a normal faithful state on $\mathcal{N}$ and still denote by ϕ the state on $\cup_n \mathcal{M}_n$ which on $\mathcal{M}_n$ is defined by $\phi(X) = \phi \circ \mathcal{E} \circ \mathcal{E}_1 \circ \cdots \circ \mathcal{E}_n(X)$. Denote $(\mathcal{M}_\infty, \mathcal{H}_\phi)$ the GNS completion of $(\cup_n \mathcal{M}_n, \phi)$. Then, we have:*

(i) *The spatial isomorphism class of $(\mathcal{M}_\infty, \mathcal{H}_\phi)$ does not depend on ϕ.*

(ii) *The tower $\mathcal{M}_n$ is naturally embedded into $\mathcal{M}_\infty$ and there exist unique ϕ-preserving conditional expectations $\widetilde{\mathcal{E}}_n : \mathcal{M}_\infty \to \mathcal{M}_{n-1}$ satisfying $\widetilde{\mathcal{E}}_{n|\mathcal{M}_m} = \mathcal{E}_n \circ \cdots \circ \mathcal{E}_m$ for all $m \geq n \geq 0$, where $\mathcal{E}_0 = \mathcal{E}$.*

Moreover, if Tr *is an n.s.f. (normal semifinite faithful) trace on $\mathcal{M}$ that is $\mathcal{E}$-invariant, then* $\mathrm{Tr} := \mathrm{Tr} \circ \widetilde{\mathcal{E}}_0$ *defines an n.s.f. trace on $\mathcal{M}_\infty$ and all $\widetilde{\mathcal{E}}_n$ leave invariant this trace. Also, if $\lambda \neq 1$, then we have:*

(a) *If $\mathcal{M}$ has no finite direct summand, then $\mathcal{M}_\infty$ is of type* II_∞.

(b) *If* $\mathrm{Tr}(1) < \infty$, *then $\mathcal{M}_\infty$ is* II_1. *If in addition $\mathcal{N} \subset \mathcal{M}$ are finite dimensional with irreducible inclusion graph, then $\mathcal{M}_\infty$ is a type* II_1 *factor.*

Proof. Parts (i), (ii) are standard applications of Takesaki's classic results in [58]. The fact that Tr n.s.f. implies $\mathrm{Tr} \circ \mathcal{E} \circ \mathcal{E}_1 \circ \cdots \circ \mathcal{N}_n$ is n.s.f. on $\mathcal{M}_n$ follows easily from the fact that $\mathrm{Ind}(\mathcal{N} \subset \mathcal{M}_n) = \lambda^{-n-1} < \infty$. The last part is trivial and we leave it as an exercise. ■

Note that if $\mathcal{M}$ is a factor, then any $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ with finite index is Markov. In particular, if $N \subset M$ are II_1 factors with $[M : N] < \infty$ and $\lambda = [M : N]^{-1}$, then $N \subset M$ is λ-Markov.

If $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ is a W^*-inclusion with $\mathrm{Ind}(\mathcal{E}) = \lambda^{-1} < \infty$, then any other normal expectation of $\mathcal{M}$ onto $\mathcal{N}$ is of the form $\mathcal{E}_A = \mathcal{E}(A^{1/2} \cdot A^{1/2})$ for some $A \in (\mathcal{N}' \cap \mathcal{M})_+$ with $\mathcal{E}(A) = 1$. Thus, such A satisfies

$$1 = \mathcal{E}(A) \geq \lambda A,$$

so $A \leq \lambda^{-1} 1$. This same argument applied to $\mathcal{E}_A$ shows that if $\lambda_A = \lambda(\mathcal{E}_A) > 0$, then $A \geq \lambda_A 1$. Note that $\{m_j\}_j$ is o.b. for $\mathcal{E}$ iff $\{m_j A^{-1/2}\}_j$ is o.b. for $\mathcal{E}_A$.

A W^*-inclusion $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ may have finite index without being Markov, yet for some A the expectation $\mathcal{E}_A$ becomes Markov. For this, one needs $A \in (\mathcal{N}' \cap \mathcal{M})_+$ to satisfy

$$\sum_j m_j A^{-1} m_j^* \in \mathbb{C}1.$$

For instance, by [20, Theorem 3.2], if $\oplus_{i \in I} \mathbb{M}_{n_i}(\mathbb{C}) = Q \subset^E P = \oplus_{j \in J} \mathbb{M}_{m_j}(\mathbb{C})$ is a finite dimensional W^*-inclusion with E preserving a normal faithful trace state τ on P, given by the weight vector $\vec{t} = (t_j)_j$, then E is λ-Markov iff $\vec{t}$ is a Perron–Frobenius eigenvector for $\Lambda^t \Lambda$, corresponding to

$$\lambda^{-1} = \|\Lambda^t \Lambda\| = \|\Lambda\|^2,$$

where $\Lambda = \Lambda_{Q \subset P}$ is the inclusion bipartite graph of $Q \subset P$, viewed as an $I \times J$ matrix, i.e. $\Lambda^t \Lambda \vec{t} = \lambda^{-1} \vec{t}$.

2.7. Atomic W^*-inclusions. Let $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ be an inclusion of atomic von Neumann algebras with $\mathrm{Ind}(\mathcal{E}) = \lambda^{-1} < \infty$. Thus, both $\mathcal{N}, \mathcal{M}$ are direct sums of type I factors,

$$\mathcal{N} = \oplus_{i \in I} \mathcal{B}(\mathcal{K}_i), \quad \mathcal{M} = \oplus_{j \in J} \mathcal{B}(\mathcal{H}_j).$$

It is trivial to see that if b_{ij} denotes the multiplicity of $\mathcal{B}(\mathcal{K}_i)$ in $\mathcal{B}(\mathcal{H}_j)$ (i.e. $b_{ij}^2 = \dim(\mathcal{B}(\mathcal{K}_i)' \cap \mathcal{B}(\mathcal{H}_j))$, then $b_{ij} \leq \lambda^{-1}$ (if $b_{ij} \leq \dim(\mathcal{K}_i)$, then one actually has $b_{ij} \leq \lambda^{-1/2}$, see [31, Section 6] or [40, p. 200]). The finiteness of the index also implies that for each $j \in J$ (resp. $i \in I$) the number of $i \in I$ (resp. $j \in J$) with $b_{ij} \neq 0$ is finite.

Thus, such an inclusion $\mathcal{N} \subset \mathcal{M}$ is described by a bipartite graph (diagram) $\Lambda_{\mathcal{N}\subset\mathcal{M}} = (b_{ij})_{i\in I, j\in J}$, where the number b_{ij} of edges between the vertices i and j is equal to the multiplicity of $\mathcal{B}(\mathcal{K}_i)$ in $\mathcal{B}(\mathcal{H}_j)$. We alternatively view Λ as an $I \times J$ matrix with integer non-negative entries b_{ij}. One trivially has that

$$\mathcal{Z}(\mathcal{N}) \cap \mathcal{Z}(\mathcal{M}) = \mathbb{C}1$$

iff Λ is connected (equivalently, Λ is irreducible as a matrix).

We are particularly interested in the case $\mathcal{N}, \mathcal{M}$ are of type I_∞ and $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ is λ-Markov, where $\lambda^{-1} = \mathrm{Ind}(\mathcal{E})$. Assume this is the case and let $\mathcal{M} \subset_e^{\mathcal{E}_1} \mathcal{M}_1 = \langle \mathcal{M}, e \rangle$ be the associated basic construction, with e denoting the Jones projection. Then $\mathcal{M}_1$ is an amplification of $\mathcal{N}$, so it is again an atomic von Neumann algebra and by [20, §3.3.2] there is a natural isomorphism

$$\mathcal{Z}(\mathcal{N}) \ni z \mapsto z' \in \mathcal{Z}(\mathcal{M}_1),$$

where z' is the unique element in $\mathcal{Z}(\mathcal{M}_1)$ with $z'e = ze$. Moreover, if one identifies the set labeling the direct summands of $\mathcal{M}_1$ with I, via this identification, then the $J \times I$ bipartite graph $\Lambda_{\mathcal{M}\subset\mathcal{M}_1}$ is given by

$$(\Lambda_{\mathcal{N}\subset\mathcal{M}})^t = (b'_{ji})_{j\in J, i\in I},$$

where $b'_{ji} = b_{ij}$.

An important case is when the atomic W^*-inclusion $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ is both Markov and $\mathcal{E}$ leaves invariant some normal semifinite faithful (n.s.f.) trace Tr on $\mathcal{M}$. A W^*-inclusion $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ with the property that $\mathcal{M}$ admits a $\mathcal{E}$-invariant n.s.f. trace, is called *tracial*. If $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ is tracial and λ-Markov, then an $\mathcal{E}$-invariant n.s.f. trace Tr on $\mathcal{M}$ is called a *λ-Markov trace*.

The proof of Theorem 3.3.2 in [20] can be easily adapted to general atomic inclusions to completely characterize when a tracial $\mathcal{E}$ is Markov:

2.7.1 Lemma. *Let $\oplus_{i\in I}\mathcal{B}(\mathcal{K}_i) = \mathcal{N} \subset^{\mathcal{E}} \mathcal{M} = \oplus_{j\in J}\mathcal{B}(\mathcal{H}_j)$ be an atomic W^*-inclusion of finite index with inclusion graph $\Lambda = \Lambda_{\mathcal{N}\subset\mathcal{M}} = (b_{ij})_{i\in I, j\in J}$. Then we have:*

1° $\|\Lambda\|^2 \le \mathrm{Ind}(\mathcal{E})$.

2° *Assume $\mathcal{E}$ preserves a n.s.f. trace* Tr *on $\mathcal{M}$, with $\vec{t} = (t_j)_j$ its weight vector, i.e. t_j is the trace of a minimal projection in $\mathcal{B}(\mathcal{H}_j)$. Denote $\lambda = \mathrm{Ind}(\mathcal{E})^{-1}$. Then*

$$\widetilde{\mathrm{Tr}}(xey) = \lambda\,\mathrm{Tr}(xy), \quad x, y \in \mathcal{M},$$

defines a n.s.f. trace on $\mathcal{M}_1$ and the following conditions are equivalent:

(i) *$\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ is λ-Markov;*

(ii) $\widetilde{\mathrm{Tr}}_{|\mathcal{M}} = \mathrm{Tr}$;

(iii) $\Lambda^t \Lambda(\vec{t}) = \lambda^{-1}\vec{t}$.

3° *If the equivalent conditions* (i)–(iii) *in* 2° *above are satisfied and* Λ *is finite, then*

$$\lambda^{-1} = \|\Lambda\|^2.$$

Proof. 1° Let $f_i \in \mathcal{P}(\mathcal{N})$ be an increasing net of finite rank projections such that $f_i \nearrow 1$ and denote Λ_i the inclusion graph of the finite dimensional inclusions $f_i \mathcal{N} f_i \subset f_i \mathcal{M} f_i$. We clearly have

$$\lim_i \|\Lambda_i\| = \|\Lambda\| \quad \text{and} \quad \mathrm{Ind}\, \mathcal{E}(f_i \cdot f_i) \leq \mathrm{Ind}\, \mathcal{E}, \quad \forall i.$$

We already pointed out in Section 2.5 that in the finite dimensional case, one has $\mathrm{Ind}\, \mathcal{E}(f_i \cdot f_i) \geq \|\Lambda_i\|^2$. Thus,

$$\|\Lambda\|^2 = \lim_i \|\Lambda_i\|^2 \leq \limsup_i \mathrm{Ind}\, \mathcal{E}(f_i \cdot f_i) \leq \mathrm{Ind}\, \mathcal{E}.$$

2° The fact that $\widetilde{\mathrm{Tr}}$ is a trace follows from the fact that

$$(x_1 e y_1)(x_2 e y_2) = x_1 \mathcal{E}(y_1 x_2) e y_2, \quad (x_2 e y_2)(x_1 e y_1) = x_2 \mathcal{E}(y_2 x_1) e y_1,$$

so applying $\widetilde{\mathrm{Tr}}$ gives

$$\begin{aligned} \widetilde{\mathrm{Tr}}\big((x_1 e y_1)(x_2 e y_2)\big) &= \lambda\, \mathrm{Tr}\big(x_1 \mathcal{E}(y_1 x_2) y_2\big) = \lambda\, \mathrm{Tr}\big(\mathcal{E}(y_1 x_2) y_2 x_1\big) \\ &= \lambda\, \mathrm{Tr}\big(\mathcal{E}(y_1 x_2)\mathcal{E}(y_2 x_1)\big) = \lambda\, \mathrm{Tr}\big(x_2 \mathcal{E}(y_2 x_1) y_1\big) \\ &= \widetilde{\mathrm{Tr}}\big((x_2 e y_2)(x_1 e y_1)\big). \end{aligned}$$

(i) $\Rightarrow$ (ii). Let $\{m_k\}_k \subset \mathcal{M}$ be an o.b. with respect to $\mathcal{E}$. If $x \in \mathcal{M}$ is finite rank, then by applying $\widetilde{\mathrm{Tr}}$ to $x = x \sum_k m_k e m_k^*$, we get

$$\widetilde{\mathrm{Tr}}(x) = \sum_k \widetilde{\mathrm{Tr}}(x m_k e m_k^*) = \lambda\, \mathrm{Tr}\Big(x \sum_k m_k m_k^*\Big) = \lambda^{-1}\lambda\, \mathrm{Tr}(x) = \mathrm{Tr}(x).$$

(ii) $\Rightarrow$ (iii). The equality $\widetilde{\mathrm{Tr}}(x) = \mathrm{Tr}(x)$ for all $x \in \mathcal{M}$, implies

$$\widetilde{\mathrm{Tr}}(ex) = \lambda\, \mathrm{Tr}(x) = \lambda \widetilde{\mathrm{Tr}}(x),$$

and thus $\mathcal{E}_1$ is the $\widetilde{\mathrm{Tr}}$-preserving expectation of $\mathcal{M}_1$ onto $\mathcal{M}$.

Denote $\vec{s} = \Lambda(\vec{t})$ and note that s_i gives the trace Tr (thus also trace $\widetilde{\mathrm{Tr}}$) of any minimal projection f_i in $\mathcal{B}(\mathcal{K}_i)$. Then $f_i e$ is a minimal projection in the ith summand of $\mathcal{M}_1$ and from the above it has trace $\widetilde{\mathrm{Tr}}(f_i e) = \lambda s_i$. One thus gets $\Lambda\Lambda^t(\lambda\vec{s}) = \vec{s}$ and $\Lambda^t(\lambda\vec{s}) = \vec{t}$. Thus,

$$\Lambda^t \Lambda(\vec{t}) = \Lambda^t(\vec{s}) = \lambda^{-1}\vec{t}.$$

(iii) $\Rightarrow$ (i). The fact that $\vec{s} = \Lambda(\vec{t})$ together with (iii) shows that the trace Tr_1 on $\mathcal{M}_1$ given by the vector $\lambda\vec{s}$ has the property that restricted to $\mathcal{M}, \mathcal{N}$ coincides with Tr and that $\mathrm{Tr}_1(ex) = \lambda\,\mathrm{Tr}(x)$, for all finite rank $x \in \mathcal{M}$. This in turn implies that the unique Tr_1-preserving expectation $\mathcal{E}_1'$ of $\mathcal{M}_1$ onto $\mathcal{M}$ satisfies $\mathcal{E}_1'(e) = \lambda 1$, and thus coincides with $\mathcal{E}_1$.

3° Since $\Lambda^t\Lambda(\vec{t}) = \lambda^{-1}\vec{t}$ and $\vec{t}$ has positive entries, if Λ is finite then one necessarily has $\lambda^{-1} = \|\Lambda^t\Lambda\|$, by the Perron–Frobenius theorem (see, e.g., [11]). ■

2.7.2 Definition. An atomic properly infinite W^*-inclusion $(\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}, \mathrm{Tr})$ with an $\mathcal{E}$-invariant semifinite trace Tr that satisfies the equivalent conditions (i)–(iii) in Lemma 2.7.1.2° above is called a *tracial λ-Markov* atomic inclusion.

The above lemma shows that such an object is completely determined by its inclusion bipartite graph $\Lambda = \Lambda_{\mathcal{N}\subset\mathcal{M}}$ and the trace vector $\vec{t}$. Indeed, by Lemma 2.7.1.2°, given any pair $(\Lambda, \vec{t})$ where $\Lambda = (b_{ij})_{i\in I, J\in J}$ is a bipartite graph and $\vec{t} = (t_j)_{j\in J}$ has positive entries and satisfies $\Lambda^t\Lambda(\vec{t}) = \lambda^{-1}\vec{t}$, there exists a unique inclusion of type I_∞ atomic von Neumann algebras with a n.s.f. trace Tr and Tr-preserving expectation $\mathcal{E}$ such that $(\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}, \mathrm{Tr})$ is a tracial λ-Markov with $\Lambda_{\mathcal{N}\subset\mathcal{M}} = \Lambda$ and Tr given by $\vec{t}$.

We call such a pair $(\Lambda, \vec{t})$ a *Markov weighted bipartite graph*. If in addition we fix an even vertex $j_0 \in J$ and renormalize $\vec{t}$ so that $t_{j_0} = 1$, then $(\Lambda, j_0, \vec{t})$ is called a *pointed* Markov weighted graph. When specifying the scalar λ with $\Lambda^t\Lambda(\vec{t}) = \lambda^{-1}\vec{t}$, we call it a (pointed) λ-Markov weighted graph.

We will show in Proposition 2.7.5 below that such an object $(\Lambda, j_0, \vec{t})$ is also equivalent to a sequence of inclusions of finite dimensional C*-algebras $\{A_n\}_{n\geq 0}$ with a trace state τ and a representation of the Jones λ-projections $\{e_n\}_{n\geq 1}$, satisfying certain properties. It turns out that this set of characterizing properties (axioms) is surprisingly minimal (cf. [46, Remark 1.4.3, Theorem 1.5]).

2.7.3 Proposition. *Let $\mathbb{C} = A_0 \subset A_1 \subset A_2 \subset \cdots$ be a sequence of inclusions of finite dimensional C^*-algebras, with a faithful trace τ on $\cup_n A_n$ and a representation of the Jones λ-sequence of projections $\{e_n\}_{n\geq 1} \subset \cup_n A_n$, such that:*

(i) $e_n \in A_{n-1}' \cap A_{n+1}$ *for all* $n \geq 1$*;*

(ii) $e_n x e_n = E_{A_{n-1}}(x) e_n$ *for all* $x \in A_n$, $n \geq 1$.

Then we have:

(a) *For each $n \geq 1$ and $z \in \mathcal{Z}(A_{n-1})$, there exists a unique $z' \in \mathcal{Z}(A_{n+1})$ such that $z e_n = z' e_n$, and the resulting map $z \mapsto z'$ implements an embedding*

$$\mathcal{Z}(A_{n-1}) \hookrightarrow \mathcal{Z}(A_{n+1}).$$

(b) *If J_n (resp. I_n) labels the set of simple summands in A_{2n} (resp. A_{2n+1}) and we identify J_n (resp. I_n) with a subset of J_{n+1} (resp. I_{n+1}) and let $J = \cup_n J_n$, $I = \cup_n I_n$, then there exists a unique pointed $J \times I$ bipartite graph $(\Gamma, \{j_0\})$, where $\{j_0\} = J_0 \subset J$ such that*

$$\Lambda_{A_{2n} \subset A_{2n+1}} =_{K_n} \Gamma \quad \text{and} \quad \Lambda_{A_{2n+1} \subset A_{2n+2}} =_{L_n} \Gamma^t, \quad n \geq 0.$$

(c) *There exist unique vectors $\vec{t} = (t_j)_{j \in J}$, $\vec{s} = (s_i)_{i \in I}$ such that*

$$t_{j_0} = 1, \quad \Gamma^t(\vec{t}) = \vec{s}, \quad \Gamma\Gamma^t(\vec{t}) = \lambda^{-1}\vec{t},$$

and $\lambda^n t_j$ gives the trace of a minimal projection in the jth summand of A_{2n}, while $\lambda^n s_i$ gives the trace of a minimal projection in the ith summand of A_{2n+1}.

Proof. Let $B_{0n} = A_n$ for $n \geq 0$, and define

$$B_{11} = \mathbb{C} = B_{12}, \quad B_{1n} = \mathrm{Alg}\{1, e_k \mid 2 \leq k \leq n-1\} \quad \text{for } n \geq 3.$$

Then $(B_{ij})_{j \geq i; i=0,1}$ is clearly a generalized λ sequence of commuting squares, in the sense of [46, Definition 1.3]. Thus, by [46, Theorem 1.5], it follows that

$$\mathrm{Ind}(E^{A_n}_{A_{n-1}}) \leq \lambda^{-1} \quad \text{for all } n \geq 1,$$

and that there exists a Jones λ-tower of factors $M_{-1} \subset M_0 \subset M_1 \subset_{e_1} M_2 \subset \cdots$, with $A_n \subset M_n$ satisfying $E_{A_n} E_{M_{n-1}} = E_{A_{n-1}}$, $n \geq 1$ (the commuting square relation, see Section 2.8 below).

But then the proof of the existence of a unique graph and weight vector in [37, Proposition 2.1, Corollary 2.2] for a sequence of inclusions of finite dimensional C*-algebras $\mathbb{C} = A_0 \subset A_1 \subset A_2 \subset \cdots$ with Jones λ-projections works exactly the same in this more general case. ∎

2.7.4 Definition. Following [46, §1.4.3], a sequence of inclusions of finite dimensional C*-algebras with a faithful trace and a representation of the Jones λ-sequence of projections $(\{A_n\}_{n \geq 0}, \{e_n\}_{n \geq 1}, \tau)$ satisfying Proposition 2.7.3 (i),(ii) above is called a *λ-sequence of inclusions*, with $(\Gamma, j_0, \vec{t})$ in (c) above being its associated pointed weighted graph.

2.7.5 Proposition. *Let $(\Lambda, j_0, \vec{t})$ be a pointed bipartite graph with a λ-Markov weight vector. Let $(\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}, \mathrm{Tr})$ be the associated tracial λ-Markov atomic W^*-inclusion and $\mathcal{N} \subset \mathcal{M} \subset^{\mathcal{E}_1}_{e_0} \mathcal{M}_1 \subset^{\mathcal{E}_2}_{e_1} \mathcal{M}_2 \cdots$ its Jones tower, with the semifinite trace Tr on $\cup_n \mathcal{M}_n$ that is invariant to all $\mathcal{E}_n$. Let $p = p_{j_0}$ be a minimal projection in $\mathcal{B}(\mathcal{H}_{j_0})$. Then the sequence of inclusions*

$$(A_0 \subset A_1 \subset_{e_1'} A_2 \subset \cdots) \simeq (p\mathcal{M}p \subset p\mathcal{M}_1 p \subset_{e_1 p} p\mathcal{M}_2 p \subset \cdots)$$

with the trace state $\tau = \mathrm{Tr}(p \cdot p)$ and Jones projections $e_i' = e_i p$, is a λ-sequence of inclusions, with its graph Γ given by Λ^t.

Proof. This is trivial by the properties of tracial λ-Markov inclusion and its Jones tower. ■

2.8. Commuting square embeddings. If $\mathcal{Q} \subset^{\mathcal{F}} \mathcal{P}$, $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ are W^*-inclusions with expectation then a *commuting square* (*c.sq.*) *embedding* (or simply an *embedding*) of $\mathcal{Q} \subset^{\mathcal{F}} \mathcal{P}$ into $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ is a von Neumann algebra inclusion $\mathcal{P} \subset \mathcal{M}$ so that $\mathcal{Q} \subset \mathcal{N}$, $\mathcal{E}(\mathcal{P}) = \mathcal{Q}$, $\mathcal{E}_{|\mathcal{P}} = \mathcal{F}$. Note that this trivially implies $\operatorname{Ind}(\mathcal{F}) \leq \operatorname{Ind}(\mathcal{E})$ and that if these indices are finite, then any o.b. $\{\xi_i\}_i \subset \mathcal{P}$ for $\mathcal{F}$ is an orthonormal system for $\mathcal{E}$, which can be completed to an o.b. $\{\eta_j\}_j \subset \mathcal{M}$ for $\mathcal{E}$, thus

$$\sum_i \xi_i \xi_i^* \leq \sum_j \eta_j \eta_j^*.$$

This property was first considered in [33, p. 29 and §1.2.2] (cf. also [34]) to study the "interaction" between subalgebras of a tracial von Neumann algebra M, calculate relative commutants and normalizers of subalgebras. If $P, Q \subset M$ are von Neumann subalgebras, and E_P, E_Q denote as usual the trace preserving expectations onto them, then the commuting square relation amounts to

$$E_P E_Q = E_Q E_P = E_{P \cap Q}.$$

This is equivalent to $E_P(Q) \subset P$ and also to $e_P e_Q$ being a projection in $\mathcal{B}(L^2 M)$. In case M is the von Neumann algebra $L\Gamma$ of a discrete group Γ, then any two subgroups $G, H \subset \Gamma$ give rise to subalgebras $P = LG$, $Q = LH$ satisfying this relation. So one can view the c.sq. relation for subalgebras of a II_1 factor as a natural "lattice-like" condition.

Commuting square embeddings are the natural "morphisms" between W^*-inclusions with expectation. All such morphisms considered here will be taken between inclusions of "same index", a property that we call "non-degeneracy" (following [40, §1.1.5]). More precisely, the commuting square embedding of $\mathcal{Q} \subset^{\mathcal{F}} \mathcal{P}$ into $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ is *non-degenerate* if one has that $\operatorname{sp} \mathcal{P}\mathcal{N}$ is weakly dense in $\mathcal{M}$ (note that in all cases of interest for us $\mathcal{P}$ is finitely generated right $\mathcal{Q}$-module, where the condition becomes $\operatorname{sp} \mathcal{P}\mathcal{N} = \mathcal{M}$).

Note that if $\operatorname{Ind}(\mathcal{E}) < \infty$, or if all algebras involved are tracial and the expectations involved are trace preserving (with respect to the trace on the largest algebra $\mathcal{M}$), then this is equivalent to saying that any o.b. for $\mathcal{F}$ is an o.b. for $\mathcal{E}$. So one necessarily has $\operatorname{Ind}(\mathcal{E}) = \operatorname{Ind}(\mathcal{F})$. Moreover, if $\mathcal{M} \subset \mathcal{M}_1 = \langle \mathcal{M}, e_{\mathcal{N}} \rangle$ is the basic construction for $\mathcal{E}$, then $\operatorname{sp}\{x e_{\mathcal{N}} y \mid x, y \in \mathcal{P}\}$ is a *-subalgebra of $\mathcal{M}_1$ and its weak closure contains $\mathcal{P}$ and identifies naturally with the basic construction $\mathcal{Q} \subset^{\mathcal{F}} \mathcal{P} \subset \mathcal{P}_1 := \langle \mathcal{P}, e_{\mathcal{Q}} \rangle$.

If the above c.sq. is non-degenerate, it follows trivially that $\mathcal{Q} \subset^{\mathcal{F}} \mathcal{P}$ is λ-Markov iff $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ is λ-Markov, with $\lambda = \lambda(\mathcal{E}) = \lambda(\mathcal{F})$. Also, if one has a c.sq. embedding

$(\mathcal{Q} \subset^{\mathcal{F}} \mathcal{P}) \subset (\mathcal{N} \subset^{\mathcal{E}} \mathcal{M})$ and both $\mathcal{Q} \subset \mathcal{P}$ and $\mathcal{N} \subset \mathcal{M}$ are λ-Markov (same λ) then the commuting square follows non-degenerate.

If these conditions are satisfied, then we say that $(\mathcal{Q} \subset^{\mathcal{F}} \mathcal{P}) \subset (\mathcal{N} \subset^{\mathcal{E}} \mathcal{M})$ is a *λ-Markov commuting square*. By the above observations about basic construction for non-degenerate commuting squares, such c. sq. gives rise to a whole tower of λ-Markov c.sq. embeddings

$$\begin{array}{ccccccc} \mathcal{N} & \subset^{\mathcal{E}} & \mathcal{M} & \subset^{\mathcal{E}_1}_{e_0} & \mathcal{M}_1 & \subset^{\mathcal{E}_2}_{e_1} & \cdots \\ \cup & & \cup & & \cup & & \\ \mathcal{Q} & \subset^{\mathcal{F}} & \mathcal{P} & \subset^{\mathcal{F}_1}_{e_0} & \mathcal{P}_1 & \subset^{\mathcal{F}_2}_{e_1} & \cdots \end{array}$$

where e_i implements both $\mathcal{E}_i$ and $\mathcal{F}_i$,

$$\mathcal{E}_{i+1}(e_i) = \lambda, \quad \mathcal{M}_{i+1} = \operatorname{sp} \mathcal{M}_i e_i \mathcal{M}_i, \quad \mathcal{P}_{i+1} = \operatorname{sp} \mathcal{P}_i e_i \mathcal{P}_i.$$

Finally, let us recall from [40, Proposition 1.1.6] that if M is tracial and $N, P \subset M$ are von Neumann subalgebras such that

$$E_P E_N = E_N E_P = E_Q,$$

where $Q = N \cap P$ (but no finite index assumption on any of the expectations) then the resulting commuting square embedding $(Q \subset P) \subset (N \subset M)$ is non-degenerate iff the embedding $(Q \subset N) \subset (P \subset M)$ is non-degenerate.

From a remark above, if such a commuting square is non-degenerate then one can take its basic construction both "horizontally"

$$\big(Q \subset P \subset \langle P, Q \rangle\big) \subset \big(N \subset M \subset \langle M, N \rangle\big)$$

and "vertically"

$$\big(Q \subset N \subset \langle N, Q \rangle\big) \subset \big(P \subset M \subset \langle M, P \rangle\big),$$

with the canonical trace $\mathrm{Tr}_{|\langle M,N \rangle}$ restricted to $\langle P, Q \rangle$ equal to $\mathrm{Tr}_{|\langle P,Q \rangle}$, resp. $\mathrm{Tr}_{|\langle M,P \rangle}$ restricted to $\langle N, Q \rangle$ equal to $\mathrm{Tr}_{|\langle N,Q \rangle}$. Moreover, one has a canonical Tr-preserving expectation $E^{\langle M,N \rangle}_{\langle P,Q \rangle}$ (resp. $E^{\langle M,P \rangle}_{\langle N,Q \rangle}$) between the corresponding extension algebras and its restriction to M is equal to E^M_P (resp. E^M_N). In addition, the resulting embeddings

$$(N \subset M) \subset \big(\langle N, Q \rangle \subset \langle M, P \rangle\big) \quad \text{and} \quad (P \subset M) \subset \big(\langle P, Q \rangle \subset \langle M, N \rangle\big)$$

are non-degenerate. Also, if the two rows (resp. columns) of the initial commuting square are Markov, then so is the row (resp. column) basic construction inclusion.

2.9. Subfactors of R from Markov commuting squares. An important class of Markov commuting squares are the ones with all the algebras involved finite dimensional and all expectations trace preserving. We explain here how such an object produces an extremal hyperfinite subfactor as an inductive limit of the associated Jones tower (cf. [12, 63]; this construction was found independently in early 1984 by Pimsner–Popa).

Let us consider a finite dimensional c.sq. embedding $(P_{00} \subset P_{01}) \subset (P_{10} \subset P_{11})$, with a (faithful) trace state τ on the largest algebra P_{11} and all expectations involved being τ-preserving. The λ-Markov condition means that the c.sq. is non-degenerate and the row inclusions are λ-Markov. In addition we assume this c.sq. is so that both row inclusion graphs $\Lambda_{P_{00}\subset P_{01}}$ and $\Lambda_{P_{10}\subset P_{11}}$ are irreducible. We call such an object a *Markov cell*.

As shown in Section 2.8 and Lemma 2.6.1 (b), such an object $(P_{00} \subset P_{01}) \subset (P_{10} \subset P_{11})$ gives rise to a tower of λ-Markov c.sq., with row enveloping II_1 factors $P_{1\infty}$, $P_{0\infty}$

$$\begin{array}{ccccccc} P_{10} & \subset & P_{11} & \subset & P_{12} & \cdots \nearrow & P_{1,\infty} \\ \cup & & \cup & & \cup & & \cup \\ P_{00} & \subset & P_{01} & \subset & P_{02} & \cdots \nearrow & P_{0,\infty} \end{array}$$

Moreover, due to the non-degeneracy at each step, the commuting square

$$\begin{array}{ccc} P_{10} & \subset & P_{1\infty} \\ \cup & & \cup \\ P_{00} & \subset & P_{0\infty} \end{array}$$

follows non-degenerate. This implies

$$\mathrm{Ind}(E^{P_{10}}_{P_{00}}) = \mathrm{Ind}(E^{P_{1\infty}}_{P_{0\infty}}) = [P_{1\infty} : P_{0\infty}].$$

In addition, since the algebras involved are II_1 factors, the inclusion $P_{0\infty} \subset P_{1\infty}$ is λ_{01}-Markov, where $\lambda_{01} = [P_{1\infty} : P_{0\infty}]^{-1}$. So $P_{00} \subset P_{10}$ follows λ_{01}-Markov as well, and so do all the vertical inclusions $P_{0n} \subset P_{1n}$.

Thus,

$$[P_{1\infty} : P_{0\infty}] = \lambda_{01}^{-1} = \|\Lambda_{P_{0n}\subset P_{1n}}\|^2, \quad \forall n \geq 0.$$

Moreover, by the formulas for relative entropy in [32], it follows that

$$H(P_{1\infty}|P_{0\infty}) = \lim_n H(P_{1n}|P_{0n}) = -\ln \lambda_{01},$$

and hence $H(P_{1\infty}|P_{0\infty}) = \ln[P_{1\infty} : P_{0\infty}]$. By [31, Corollary 4.5], this implies that $P_{0\infty} \subset P_{1\infty}$ is extremal.

In conclusion, a Markov cell

$$\begin{array}{ccc} P_{10} & \subset & P_{11} \\ \cup & & \cup \\ P_{00} & \subset & P_{01} \end{array}$$

which, by definition, is only assumed Markov "horizontally", follows Markov "vertically" as well, with the inclusion graphs satisfying

$$\Lambda_{P_{10}\subset P_{11}} \circ \Lambda_{P_{00}\subset P_{10}} = \Lambda_{P_{01}\subset P_{11}} \circ \Lambda_{P_{00}\subset P_{01}},$$

and

$$\|\Lambda_{P_{10}\subset P_{11}}\| = \|\Lambda_{P_{00}\subset P_{01}}\|, \quad \|\Lambda_{P_{00}\subset P_{10}}\| = \|\Lambda_{P_{01}\subset P_{11}}\|.$$

Moreover, the resulting inclusion of enveloping II_1 factors $P_{0\infty} \subset P_{1\infty}$ gives an extremal hyperfinite subfactor of index $\alpha = \|\Lambda_{P_{00}\subset P_{10}}\|^2$. As we have seen in Section 2.2, if $4 < \alpha < 3 + 2\sqrt{2}$, the subfactor follows irreducible.

Given a finite bipartite graph Λ, *the commuting square problem for* Λ consists in constructing a Markov cell with the vertical inclusion graph $\Lambda_{P_{00}\subset P_{10}}$ equal to Λ.

2.10. The sets $\mathcal{C}(R)$ and $\mathbb{E}^2$. Following [12], we denote by $\mathbb{E}$ the set of norms of bipartite graphs (finite or infinite) and let $\mathbb{E}^2 := \{\alpha^2 \mid \alpha \in \mathbb{E}\}$. Equivalently, $\mathbb{E}^2$ is the set of square norms of matrices with non-negative integer entries. The set $\mathbb{E}$ was first described in [8, 16]. A detailed account can be found in [12, Section A]. We recall some properties, using the notations therein.

First of all, notice that $\mathbb{E}^2$ is a closed set consisting of an increasing sequence of accumulation points, followed by the half line $[2 + \sqrt{5}, \infty]$.

One has $\mathbb{E}^2 \cap (0, 4] = \{4\cos^2(\pi/n) \mid n \geq 3\} \cup \{4\}$, the only bipartite graphs of square norm < 4 being the Coxeter graphs A_n, D_n, E_6, E_7, E_8. The bipartite graphs of square norm 4 are $A_\infty, D_\infty, A_{-\infty,\infty}, D_n^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)} = E_9$.

There is a gap right after the first accumulation point 4, with $\|E_{10}\|^2 \approx 4.0265\ldots$ being the first (smallest) element in $\mathbb{E}^2 \cap (4, \infty)$. Then one has an increasing sequence of accumulation points c_n^2, $n \geq 3$, converging to $c_\infty^2 = 2 + \sqrt{5}$. Each c_n^2 for $n \geq 3$ is an accumulation point both from below and from above. The set of $\mathbb{E}_0^2$ of square norms of finite bipartite graphs is a dense subset of $\mathbb{E}^2$.

Following [20, 21], given a II_1 factor M one denotes $\mathcal{C}(M)$ the set of indices of irreducible subfactors of M. We also let $\mathcal{E}(M)$ be the set of indices of extremal subfactors of M. One obviously has

$$\mathcal{C}(M^t) = \mathcal{C}(M), \quad \mathcal{E}(M^t) = \mathcal{E}(M), \quad \forall t > 0.$$

Also, one has

$$\mathcal{C}(M_1) \cdot \mathcal{C}(M_2) \subset \mathcal{C}(M_1 \overline{\otimes} M_2)$$

and similarly for $\mathcal{E}$. Since $R\overline{\otimes}R \simeq R$ this implies $\mathcal{C}(R)$, $\mathcal{E}(R)$ are multiplicative semigroups inside the Jones spectrum $\{4\cos^2(\pi/n) \mid n \geq 3\} \cup [4, \infty)$, that contains the integers.

By Jones' theorem, $\{4\cos^2(\pi/n) \mid n \geq 3\} \subset \mathcal{C}(R)$ and, as we have seen in Section 2.9 above, several other $\alpha \in \mathbb{E}_0^2$ have been shown to be contained in $\mathcal{C}(R)$ as well. Solving the commuting square problem for all finite bipartite graphs would show that in fact $\mathbb{E}_0^2 \subset \mathcal{E}(R)$.

3. W^*-representations of subfactors

In this section we'll define the analogue for a finite index subfactor $N \subset M$ of the notion of Hilbert-module (or W^*-representation) of a single II_1 factor M. Roughly speaking, this will be an inclusion of "multi-Hilbert modules" $\oplus_i ({}_N\mathcal{K}_i) \hookrightarrow \oplus_j ({}_M\mathcal{H}_j)$, a structure that is rigorously described as a non-degenerate embedding of $N \subset M$ into the atomic W^*-inclusion $\oplus_i \mathcal{B}(\mathcal{K}_i) \subset^{\mathcal{E}} \oplus_j \mathcal{B}(\mathcal{H}_j)$.

While in the case of a single factor M a (left) Hilbert M-module ${}_M\mathcal{H}$ (or W^*-representation $M \hookrightarrow \mathcal{B}(\mathcal{H})$) comes with the Murray–von Neumann dimension $\dim({}_M\mathcal{H})$ (or coupling constant $c_{M,M'}$ of $M \hookrightarrow \mathcal{B}(\mathcal{H})$), the role of this for an inclusion of II_1 factors will be played by the dimension/coupling vector $(\dim({}_M\mathcal{H}_j))_j$.

3.1. Some basic definitions. A non-degenerate (normal) embedding of a finite index extremal subfactor $N \subset M$ into an atomic W^*-inclusion $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ is called a W^*-*representation* of $N \subset M$. Thus, $\mathcal{N}$, $\mathcal{M}$ are direct sums of type I_∞ factors, i.e.

$$\oplus_i \mathcal{B}(\mathcal{K}_i) = \mathcal{N} \subset^{\mathcal{E}} \mathcal{M} = \oplus_j \mathcal{B}(\mathcal{H}_j).$$

Note that by Section 2.8, $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ follows automatically λ-Markov for $\lambda = [M : N]^{-1}$. Thus, by Lemma 2.7.1, its *inclusion* (bipartite) *graph* $\Lambda = \Lambda_{\mathcal{N}\subset\mathcal{M}}$ satisfies

$$\|\Lambda\| \leq [M : N]^{1/2}.$$

The representation $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ is *irreducible* if

$$\mathcal{M}' \cap \mathcal{N} = \mathcal{Z}(\mathcal{M}) \cap \mathcal{Z}(\mathcal{N}) = \mathbb{C},$$

or equivalently if $\Lambda_{\mathcal{N}\subset\mathcal{M}}$ is connected as a graph (irreducible as a matrix).

Two representations $\mathcal{N}_l \subset^{\mathcal{E}_l} \mathcal{M}_l$, $l = 0, 1$, are *equivalent* (or *isomorphic*) if there exists an isomorphism $\theta\colon \mathcal{M}_0 \simeq \mathcal{M}_1$, with $\theta(\mathcal{N}_0) = \mathcal{N}_1$, $\theta \circ \mathcal{E}_0 = \mathcal{E}_1 \circ \theta$, that intertwines the corresponding embeddings of $N \subset M$. The two representations are *stably equivalent* (or *stably isomorphic*) if there exist projections $p_l \in M' \cap \mathcal{N}_l$ such that

$$p_l \mathcal{N}_l p_l \subset^{\mathcal{E}_l(p_l \cdot p_l)} p_l \mathcal{M}_l p_l, \quad l = 0, 1,$$

are equivalent.

Stable isomorphism of W^*-representations involves "reducing" the commuting square embedding $(N \subset M) \subset (\mathcal{N} \subset^{\mathcal{E}} \mathcal{M})$ by a projection in $M' \cap \mathcal{N}$. This latter algebra is of course an isomorphism invariant for the representation.

We call $M' \cap \mathcal{N}$ the *RC-algebra* (relative commutant W^*-algebra) of the W^*-representation. The interesting case is when $M' \cap \mathcal{N}$ is a factor. If this is the case, then we say that $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ is *RC-factorial*. Note that the Murray–von Neumann *type* of the RC-algebra/factor is an isomorphism invariant of the representation. More on this in Section 3.6.

Since $N \subset M$ is λ-Markov for $\lambda = [M : N]^{-1}$ and W^*-representations are non-degenerate embeddings, a representation $(N \subset M) \subset (\mathcal{N} \subset^{\mathcal{E}} \mathcal{M})$ is a λ-Markov commuting square embedding, so by Section 2.8 it gives rise to a tower of representations

$$(N \subset M \subset_{e_0} M_1 \subset_{e_1} \cdots) \subset (\mathcal{N} \subset^{\mathcal{E}} \mathcal{M} \subset^{\mathcal{E}_1}_{e_0} \mathcal{M}_1 \subset_{e_1} \cdots).$$

Since $M_{-1} = N \subset M = M_0$ are II_1 factors, by using the *downward basic construction* in [40, §1.2.3], one can choose (up to conjugacy by a unitary in $N = M_{-1}$) a projection $e_{-1} \in M_0$ with $E_N(e_{-1}) = \lambda 1$ and define $M_{-2} := \{e_{-1}\}' \cap M_{-1}$. Then M_{-2} follows a II_1 subfactor of index λ^{-1} with the property that $M_{-2} \subset M_{-1} \subset_{e_{-1}} M_0$ is a basic construction. One can make such choices of Jones projections recursively, thus obtaining a *tunnel* of factors $\cdots \subset_{e_{-2}} M_{-1} \subset_{e_{-1}} M_0$.

Moreover, by [42, §§1.2.6–1.2.9], if for each $i \leq -1$ we define $\mathcal{M}_{i-1} = \{e_i\}' \cap \mathcal{M}_i$ and $\mathcal{E}_i : \mathcal{M}_i \to \mathcal{M}_{i-1}$ by

$$\mathcal{E}_i(X) = \lambda \sum_j m^i_j X m^{i\,*}_j,$$

where $\{m^i_j\}_j$ is an o.b. of $\{e_n\}''_{n \geq i}$ over $\{e_n\}''_{n \geq i+1}$ (in $\mathcal{M}_\infty$), then

$$(M_{i-1} \subset M_i \subset_{e_{i-1}} M_{i+1}) \subset (\mathcal{M}_{-n-1} \subset \mathcal{M}_{-n} \subset_{e_{-n}} \mathcal{M}_{-n+1}), \quad n \in \mathbb{Z},$$

are all representations. We call such a double sequence of representations a *tower-tunnel* of representations.

3.2. Two classes of examples. This concept of W^*-representation of a subfactor was introduced in [40, §2], where one also notices the following class of examples (see [40, Proposition 2.1]).

3.2.1 Example. Let $N \subset M$ be an extremal inclusion of II_1 factors with finite Jones index. Let $(Q \subset P) \subset (N \subset M)$ be a non-degenerate commuting square with $Q \subset P$ finite dimensional. We will call such $Q \subset P$ a *graphage* of $N \subset M$. Since $N \subset M$ is $\lambda = [M : N]^{-1}$ Markov and the commuting square is non-degenerate, $Q \subset P$

follows λ-Markov as well. By the remark at the end of Section 2.8, if one takes the basic construction of this commuting square vertically and one denotes

$$\mathcal{N} = \langle N, Q\rangle \subset^{\mathcal{E}} \langle M, P\rangle = \mathcal{M},$$

where $\mathcal{E} = E^{\langle M,P\rangle}_{\langle N,Q\rangle}$, then $\mathcal{E}_{|M} = E^M_N$ and $(N \subset M) \subset (\mathcal{N} \subset^{\mathcal{E}} \mathcal{M})$ is a non-degenerate W^*-embedding. Since $\mathcal{N} \subset \mathcal{M}$ is an amplification of $Q \subset P$, it follows that $\mathcal{N}, \mathcal{M}$ are atomic, so $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ is a representation of $N \subset M$. Moreover, since $\mathcal{E} = E^{\langle M,P\rangle}_{\langle N,Q\rangle}$ preserves the canonical trace $\mathrm{Tr}_{\langle M,P\rangle}$, we have that $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ is both λ-Markov and tracial. Also, since $\mathcal{N} \subset \mathcal{M}$ is an amplification of $Q \subset P$, the bipartite graph $\Lambda_{\mathcal{N}\subset\mathcal{M}}$ identifies naturally with $\Lambda_{Q\subset P}$, so the representation is irreducible iff $\Lambda_{Q\subset P}$ is irreducible (equivalently, $\mathcal{Z}(P) \cap \mathcal{Z}(Q) = \mathbb{C}$). Also, we have

$$[M : N] = \|\Lambda_{\mathcal{N}\subset\mathcal{M}}\|^2 = \|\Lambda_{Q\subset P}\|^2.$$

Another class of examples of W^*-representations of a given subfactor $N \subset M$ comes from the following trivial observation.

3.2.2 Lemma. *If $(N \subset M) \subset (\tilde{N} \subset^{\tilde{E}} \tilde{M})$ is a non-degenerate commuting square embedding of the extremal inclusion of II_1 factors $N \subset M$ into another inclusion of factors with expectation, then any representation*

$$(\tilde{N} \subset^{\tilde{E}} \tilde{M}) \subset (\mathcal{N} \subset^{\mathcal{E}} \mathcal{M})$$

(that is, a non-degenerate embedding of $\tilde{N} \subset^{\tilde{E}} \tilde{M}$ into an atomic W^*-inclusion with expectation $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$) gives a W^*-representation $(N \subset M) \subset (\mathcal{N} \subset^{\mathcal{E}} \mathcal{M})$, by composing the embeddings.

3.3. Tracial representations. A representation $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ of a subfactor $N \subset M$ is *tracial* if the atomic W^*-inclusion $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ is tracial, i.e. there exists an n.s.f. trace Tr on $\mathcal{M}$ such that $\mathrm{Tr} \circ \mathcal{E} = \mathrm{Tr}$. As seen in Example 3.2.1 above, a representation arising from a graphage $(Q \subset P) \subset (N \subset M)$ does have this property. The existence of a graphage is a rather strong structural property of $N \subset M$, which in particular implies

$$[M : N] = \|\Lambda_{Q\subset P}\|^2 \in \mathbb{E}^2.$$

So a II_1 subfactor $N \subset M$ with A_∞-graph and index in the set $(4, 2 + \sqrt{5}) \setminus \mathbb{E}^2$ (which by [38] exists for any α lying in this set) does not have any graphage. A tracial representation can be viewed as a "dim graphage" of $N \subset M$.

Note that the tower/tunnel of reps associated with a tracial W^*-representation $(N \subset M) \subset (\mathcal{N} \subset^{\mathcal{E}} \mathcal{M})$ are all tracial.

3.3.1 Proposition. *Let $(N \subset M) \subset (\mathcal{N} \subset^{\mathcal{E}} \mathcal{M})$ be an irreducible representation with finite inclusion graph $\Lambda = \Lambda_{\mathcal{N}\subset\mathcal{M}}$.*

1° *If* Tr *is a n.s.f. $\lambda = [M : N]^{-1}$ Markov trace on $\mathcal{N} \subset \mathcal{M}$, then it is necessarily $\mathcal{E}$-invariant.*

2° *$\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ is tracial if and only if $\|\Lambda\|^2 = [M : N]$.*

Proof. By Lemma 2.7.1, any n.s.f. trace Tr on $\mathcal{M}$ for which the Tr-preserving expectation $\mathcal{E}'$ is λ'-Markov, for some $\lambda' > 0$, forces $\lambda' = \|\Lambda\|^{-2}$ and Tr be given by a weight vector proportional to the (unique) Perron–Frobenius eigenvector of $\Lambda^t\Lambda$ corresponding to eigenvalue $\|\Lambda\|^2$. Thus, condition 1° implies $\|\Lambda\|^2 = [M : N]$. In particular, $\operatorname{Ind} \mathcal{E} = \operatorname{Ind}(\mathcal{E}')$. Since $\mathcal{E}'$ is the unique expectation with index $\|\Lambda\|^2$, this implies $\mathcal{E} = \mathcal{E}'$.

This also proves $\Leftarrow$ in 2°, while the opposite implication follows from Lemma 2.6.1. ∎

3.4. The coupling vector of a representation. Given an $(N \subset M)$-representation

$$\oplus_{i\in I}\mathcal{B}(\mathcal{K}_i) = \mathcal{N} \subset^{\mathcal{E}} \mathcal{M} = \oplus_{j\in J}\mathcal{B}(\mathcal{H}_j),$$

we denote

$$\vec{d}_M(\mathcal{N} \subset \mathcal{M}) = (d_M(j))_{j\in J} \quad (\text{resp. } \vec{d}_N(\mathcal{N} \subset \mathcal{M}) = (d_N(i))_{i\in I}),$$

the vectors with entries $d_M(j) = \dim({}_M\mathcal{H}_j) \in (0, \infty]$ (resp. $d_N(i) = \dim({}_N\mathcal{K}_i) \in (0, \infty]$), and call them the *coupling vectors* (or *dimension vectors*) of the representation. If $d_M(j) < \infty$, $d_N(i) < \infty$ for all i, j, then we say that the representation has *finite couplings* (or *finite dimension vectors*).

We will next prove that if this is the case, then $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ is automatically tracial whenever $\Lambda_{\mathcal{N}\subset\mathcal{M}}$ is finite, with the dimension vector $\vec{d}_M$ giving the weights of the $\mathcal{E}$-invariant n.s.f. trace Tr. Another important class of tracial representations with finite coupling vector giving the weights of Tr will be discussed in Section 3.8.

Recall first some well known facts about the dimension of Hilbert modules over a II_1 factor and the way it relates to Jones index (see e.g. [20]).

3.4.1 Lemma. *Let $N \subset M$ be an inclusion of II_1 factors and $\mathcal{H}$, $\mathcal{H}'$ some (left) Hilbert M-modules. Then we have:*

(a) $\dim({}_M(\mathcal{H} \oplus \mathcal{H}')) = \dim({}_M\mathcal{H}) + \dim({}_M\mathcal{H}')$.

(b) *When viewing $\mathcal{H}$ as an N-module, one has* $\dim({}_N\mathcal{H}) = [M : N]\dim({}_M\mathcal{H})$.

(c) *If p is a projection in M, then* $\dim({}_{pMp}\,p(\mathcal{H})) = \tau(p)^{-1}\dim({}_M\mathcal{H})$.

3.4.2 Proposition. *Let $\oplus_l \mathcal{B}(\mathcal{K}_i) = \mathcal{N} \subset^{\mathcal{E}} \mathcal{M} = \oplus_j \mathcal{B}(\mathcal{H}_j)$ be an irreducible representation of $N \subset M$ with inclusion graph $\Lambda = \Lambda_{\mathcal{N}\subset\mathcal{M}}$ and dimension vectors*

$$\vec{d}_M = (d_M(j))_{j\in J}, \quad \vec{d}_N = (d_N(i))_{i\in I}.$$

If $d_M(j_0) < \infty$, or $d_N(i_0) < \infty$, for some $j_0 \in J$, or $i_0 \in I$, then the representation has finite couplings and we have:

1° $\vec{d}_N = \Lambda(\vec{d}_M)$, $\Lambda^t(\vec{d}_N) = [M : N]\vec{d}_M$, $\Lambda^t\Lambda(\vec{d}_M) = [M : N]\vec{d}_M$.

2° *The n.s.f. trace* Tr *on $\mathcal{M}$ given by the weight vector $\vec{d}_M$ is a $\lambda = [M : N]^{-1}$ Markov n.s.f. trace for $\mathcal{N} \subset \mathcal{M}$.*

3° *If Λ is finite, then the n.s.f. trace* Tr *on $\mathcal{M}$ given by the weights $\vec{d}_M$ is $\mathcal{E}$-invariant.*

Proof. Note that proving the "entry by entry" equalities $\Lambda(\vec{d}_M) = \vec{d}_N$ and $\Lambda^t(\vec{d}_N) = [M : N]\vec{d}_M$ with the entries being in $(0, \infty]$ (so a priori not all finite) implies both the fact that "$d_M(j_0) < \infty$ for some $j_0 \in J$, or $d_N(i_0) < \infty$ for some $i_0 \in I$, implies all entries of both $\vec{d}_M, \vec{d}_N$ are finite", and the fact that $\Lambda^t\Lambda(\vec{d}_M) = [M : N]\vec{d}_M$.

Let $\Lambda = (b_{ij})_{i\in I, j\in J}$ as usual. Since $\mathcal{H}_j = \oplus_i \mathcal{K}_i^{\oplus b_{ij}}$, by Lemma 3.4.1 (a),(b), we have

$$[M : N]\dim({}_M\mathcal{H}_j) = \dim({}_N\mathcal{H}_j) = \sum_i b_{ij}\dim({}_N\mathcal{K}_i),$$

showing that $\Lambda^t(\vec{d}_N) = [M : N]\vec{d}_M$.

Let $\mathcal{N} \subset \mathcal{M} \subset_e^{\mathcal{E}_1} \mathcal{M}_1 = \oplus_{i\in I}\mathcal{B}(\mathcal{K}_i')$ be the basic construction for $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$, with $N \subset M \subset_e M_1$ represented in it. Note that $(N \subset \mathcal{N})$ is isomorphic to

$$e(M_1 \subset \mathcal{M}_1)e = (eM_1e \subset e\mathcal{M}_1e),$$

and that $e\mathcal{B}(\mathcal{K}_i')e = \mathcal{B}(e(\mathcal{K}_i'))$, so that $N \subset \mathcal{B}(\mathcal{K}_i)$ is the "same as" $eM_1e \subset \mathcal{B}(e(\mathcal{K}_i'))$. By Lemma 3.4.1 (c), since $\tau_{M_1}(e) = [M : N]^{-1}$, it follows that

$$\dim({}_{M_1}\mathcal{K}_i') = \tau_{M_1}(e)\dim({}_{eM_1e}e(\mathcal{K}_i')) = [M : N]^{-1}\dim({}_N\mathcal{K}_i), \tag{3.4.2.1}$$

implying that for the dimension vectors, we have

$$\vec{d}_{M_1} = [M : N]^{-1}\vec{d}_N. \tag{3.4.2.2}$$

By applying the first part of the proof to the representation of $M \subset M_1$ into $\oplus_j \mathcal{B}(\mathcal{H}_j) = \mathcal{M} \subset \mathcal{M}_1 = \oplus_i \mathcal{B}(\mathcal{K}_i')$, with its inclusion bipartite graph/matrix $\Lambda_{\mathcal{M}\subset\mathcal{M}_1}$ identified with $\Lambda^t = ((b_{ij})_{i,j})^t$, it follows that

$$\Lambda(\vec{d}_M) = (\Lambda_{\mathcal{M}\subset\mathcal{M}_1})^t\vec{d}_M = [M_1 : M]\vec{d}_{M_1},$$

so by (3.4.2.2) and by using the fact that $[M_1 : M] = [M : N]$, we get $\vec{d}_N = \Lambda(\vec{d}_M)$, end the rest of the equalities in 1°.

Part 2° is now an immediate consequence of part 1° and of Lemma 2.7.1.2°, while part 3° follows from part 2° and Proposition 3.3.1. ∎

3.4.3 Definition. If a W^*-representation $(N \subset M) \subset (\mathcal{N} \subset^{\mathcal{E}} \mathcal{M})$ has finite couplings and the n.s.f. trace Tr implemented by its coupling vector is $\mathcal{E}$-invariant, then we say that it is *canonically tracial*.

3.5. Smooth representations. A representation $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ of a subfactor $N \subset M$ is called *smooth* if in the associated tower of representations

$$(N \subset M \subset_{e_0} M_1 \subset \cdots) \subset (\mathcal{N} \subset^{\mathcal{E}} \mathcal{M} \subset_{e_0}^{\mathcal{E}_1} \mathcal{M}_1 \subset \cdots)$$

one has $N' \cap M_j \subset \mathcal{N}' \cap \mathcal{M}_j$ (see [40, Section 2.3]). It is trivial to see that this condition implies

$$M_i' \cap M_j \subset \mathcal{M}_i' \cap \mathcal{M}_j, \quad \forall j \geq i \geq -1$$

(where $\mathcal{M}_{-1} = \mathcal{N}$, $\mathcal{M}_0 = \mathcal{M}$, $M_{-1} = N$, $M_0 = M$) and that these relations are equivalent to

$$\mathcal{M}_i' \cap M_j = M_i' \cap M_j, \quad \forall i, j.$$

This compatibility relation between the higher relative commutants of $N \subset M$ and $\mathcal{N} \subset \mathcal{M}$ is natural to impose, a fact that's amply emphasized by results in [40, 44]. More generally, a non-degenerate commuting square embedding of $N \subset M$ into an arbitrary W^*-inclusion with expectation $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ is smooth if $M_i' \cap M_j \subset \mathcal{M}_i' \cap \mathcal{M}_j$ for all $j \geq i \geq -1$, with the same convention of notations as above.

3.6. Exact representations. Following [40, Section 2.4], a representation $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ for a subfactor $N \subset M$ is called *exact*, if $\mathcal{M} = M \vee M' \cap \mathcal{N}$. We then also say that it has the *relative bi-commutant* property.

We briefly recall below some facts about exact representations, referring to [40, Section 2.4] for more details.

Note that an exact representation is irreducible iff $\mathcal{P} = M' \cap \mathcal{N}$ is a factor and that if this is the case then each factorial type I_∞ direct summand $\mathcal{B}(\mathcal{H})$ of $\mathcal{M}$ (resp. $\mathcal{B}(\mathcal{K})$ of $\mathcal{N}$) is an irreducible binormal representation of $M \otimes \mathcal{P}$ (resp. $N \otimes \mathcal{P}$), equivalently an irreducible Hilbert bimodule ${}_M\mathcal{H}_{\mathcal{P}^{\mathrm{op}}}$ (resp. ${}_N\mathcal{K}_{\mathcal{P}^{\mathrm{op}}}$). We will call $\mathcal{P} = M' \cap \mathcal{N}$ the *RC-factor* (or *exacting factor*) of the exact representation.

Let $\mathcal{N} \subset \mathcal{M} \subset_{e_0}^{\mathcal{E}_1} \mathcal{M}_1 \subset \cdots$ be the Jones tower for $\mathcal{N} \subset \mathcal{M}$ and denote by $\mathcal{M}_\infty$ its enveloping von Neumann algebra. We then have $M_n' \cap \mathcal{N} = M' \cap \mathcal{N} = \mathcal{P}$ for all $0 \leq n \leq \infty$, and $\mathcal{M}_n = M_n \vee \mathcal{P}$ for all $-1 \leq n \leq \infty$. In particular, this shows that an exact representation is smooth.

Exact representations arise concretely as follows. For simplicity, let us assume N, M are separable II_1 factors and we only look for separable exact representations of $N \subset M$. Take P to be a subfactor of $M^{\mathrm{op}} \overline{\otimes} \mathcal{B}(\ell^2\mathbb{N})$, like for instance $P = M^{\mathrm{op}}$. Denote by $M \otimes_{\mathrm{bin}} P$ the completion of $M \otimes P$ in the maximal binormal C*-norm, obtained from all binormal representations of $M \otimes P$. It is easy to see that the norm it induces on the subalgebra $N \otimes P$ is equal to its own maximal binormal C*-norm, thus identifying $N \otimes_{\mathrm{bin}} P$ as a C*-subalgebra of $M \otimes_{\mathrm{bin}} P$. Moreover, the map

$$E_N \otimes \mathrm{id}_P : M \otimes P \to N \otimes P$$

extends to a conditional expectation, still denoted $E_N \otimes \mathrm{id}$, of $M \otimes_{\mathrm{bin}} P$ onto $N \otimes_{\mathrm{bin}} P$. Since the inequality $E_N(x) \geq \lambda x$ for all $x \in M_+$, is stable (i.e. difference is completely positive), it follows that one still has

$$E_N \otimes \mathrm{id}_P(X) \geq \lambda X, \quad \forall X \in (M \otimes_{\mathrm{bin}} P)_+.$$

This expectation extends to an expectation from the von Neumann algebras they entail. To see this rigorously, note first that by taking biduals of $N \otimes_{\mathrm{bin}} P \subset^{E_N \otimes \mathrm{id}} M \otimes_{\mathrm{bin}} P$, one gets an inclusion of von Neumann algebras with expectation

$$(N \otimes_{\mathrm{bin}} P)^{**} \subset^{\mathcal{E}} (M \otimes_{\mathrm{bin}} P)^{**},$$

where $\mathcal{E} = (E_N \otimes \mathrm{id})^{**}$ still satisfies $\mathcal{E}(X) \geq \lambda X$ for all $X \geq 0$. Due to weak density of $N \otimes_{\mathrm{bin}} P \subset M \otimes_{\mathrm{bin}} P$ inside it, for which one has the formula

$$X = \sum_j m_j \mathcal{E}(m_j^* X), \quad \forall X \in M \otimes_{\mathrm{bin}} P,$$

one has this formula for all $X \in (M \otimes_{\mathrm{bin}} P)^{**}$.

By [40, Section 2.4] or [42, §1.1.2 (iii)], the inequality $\mathcal{E}(X) \geq \lambda X$ for all $X \geq 0$, ensures that the central support of the atomic parts of $(N \otimes_{\mathrm{bin}} P)^{**}$, $(M \otimes_{\mathrm{bin}} P)^{**}$ coincide, and so do the central supports of the parts where M, P (resp. N, P) are represented normally, and for which each factorial direct summand is separable. Thus, if one denotes $\mathcal{N}_P^u \subset^{\mathcal{E}} \mathcal{M}_P^u$ this atomic inclusion, then each factorial direct summand of $\mathcal{M}_P^u$ (resp. $\mathcal{N}_P^u$) is an irreducible binormal representation of $M \otimes_{\mathrm{bin}} P$ (resp. $N \otimes_{\mathrm{bin}} P$).

Obviously, any irreducible separable exact representation of $N \subset M$ arises this way. Doing this construction for all subfactors $P \subset M^{\mathrm{op}} \overline{\otimes} \mathcal{B}(\ell^2\mathbb{N})$, then choosing one irreducible representation for each isomorphism class of such a rep., then taking direct sum, gives a representation of $N \subset M$ that we denote $\mathcal{N}^u \subset^{\mathcal{E}} \mathcal{M}^u$ and that we call the *universal exact* (or *binormal*) W^*-representation of $N \subset M$.

One should note that if $\mathcal{N} \subset \mathcal{M}$ arises as an irreducible $\mathcal{N}_P \subset \mathcal{M}_P$, for some $P \subset M^{\mathrm{op}} \overline{\otimes} \mathcal{B}(\ell^2\mathbb{N})$, then one has $\mathcal{P} = M' \cap \mathcal{N} \supset P$ but the inclusion can be strict (see Remark 3.7.3 for concrete examples). We call $\mathcal{P}$ the *RC-envelope* of P.

We denote

$$\Lambda^u = \Lambda^u_{N \subset M} = (b_{lk})_{l \in L^u, k \in K^u}$$

the inclusion graph of $\mathcal{N}^u \subset \mathcal{M}^u$, where K^u labels the set of atoms in $\mathcal{Z}(\mathcal{M}^u)$, L^u labels the set of atoms of $\mathcal{Z}(\mathcal{N}^u)$. One also denotes by $\mathcal{H}_k$ (resp $\mathcal{K}_l$) the irreducible binormal $M - P$ (resp. $N - P$) Hilbert bimodule corresponding to the atom $k \in K^u$ (resp. $l \in L^u$).

Note that $\mathcal{Z}(\mathcal{M}^u) \cap \mathcal{Z}(\mathcal{N}^u)$ is atomic, in fact any atom of $\mathcal{Z}(\mathcal{M}^u)$, $\mathcal{Z}(\mathcal{N}^u)$ is majorized by a (unique) atom of this intersection, corresponding to a connected component of Λ^u. Each such direct summand

$$(\mathcal{N} \subset \mathcal{M}) = (\mathcal{N}^u \subset \mathcal{M}^u)s(q_0)$$

gives an irreducible representation of $N \subset M$, i.e. with $\mathcal{Z}(\mathcal{N}) \cap \mathcal{Z}(\mathcal{M}) = \mathbb{C}$.

The construction of an irreducible sub-representation $\mathcal{N} \subset \mathcal{M}$ of $(\mathcal{N}^u \subset^{\mathcal{E}} \mathcal{M}^u)$ can be obtained more directly as follows. Start with a (separable) irreducible Hilbert $(M - P)$-bimodule $\mathcal{H}_0$, corresponding to some central atom $q_0 \in \mathcal{Z}(\mathcal{M}^u)$, labeled by $0 \in K^u$ (note that by Connes' theory of correspondences, this is equivalent to an embedding with trivial relative commutant $P \hookrightarrow (M^{\mathrm{op}})^\alpha$ for some $0 < \alpha \leq \infty = \aleph_0$; see [35]). Set $J_0 = \{0\}$. Then take all irreducible $(N - P)$-Hilbert subbimodules $\mathcal{K}_i$ appearing in ${}_N(\mathcal{H}_0)_P$, indexed by $i \in I_1$, denoting b_{i0} its multiplicity. Then take all irreducible $(M - P)$-Hilbert bimodules $\mathcal{H}_j$ with ${}_N(\mathcal{K}_i)_P \leq {}_N(\mathcal{H}_j)_P$, for some $i \in I_1$, indexed by $j \in J_1$, or equivalently the irreducible direct summands of ${}_M(L^2M \otimes_N \mathcal{H}_0)_P$. Denote b_{ij} the corresponding multiplicity. One continues recursively with ${}_N(\mathcal{K}_i)_P$, $i \in I_n$ being the irreducible $(N - P)$-submodules of

$${}_N(L^2M^{\otimes_N (n-1)} \otimes_N \mathcal{H}_0)_P,$$

and ${}_M(\mathcal{H}_j)_P$, $j \in J_n$ the irreducible $(M - P)$-submodules of

$${}_M(L^2M^{\otimes_N n} \otimes_N \mathcal{H}_0)_P,$$

while denoting b_{ij} the multiplicity of this subbimodule. One has natural identifications

$$I_n \hookrightarrow I_{n+1}, \quad J_n \hookrightarrow J_{n+1},$$

due to 2-periodicity in the Jones tower and the fact that ${}_Q L^2(M_n)_N$ is isomorphic to ${}_Q(L^2M^{\otimes_N (n+1)})_N$ for any $n \geq 0$, where $Q \in \{N, M\}$. If one denotes $J = \cup_n J_n$, $I = \cup_n I_n$, then it is immediate to see that

$$(\mathcal{N} \subset \mathcal{M}) = \big(\oplus_{i \in I} \mathcal{B}(\mathcal{K}_i) \subset \oplus_{j \in J} \mathcal{B}(\mathcal{H}_j)\big),$$

with the inclusion bipartite graph given by $\Lambda = (b_{ij})_{i \in I, j \in J}$ (see [40, Section 2.4] for more on this).

3.7. The standard W^*-representations. (See [40, Section 2.4] and [44, Section 5]). The "concrete construction" of exact representations of $N \subset M$ from irreducible subfactors P of amplifications of M, shows that one has "plenty" of examples of such representations, as many as irreducible subfactors of M^α, $0 < \alpha \leq \infty$, one can have (see however Problems 6.1.6, 6.1.7 and accompanying remarks).

The simplest case of such a construction is when $P = M$ and $\mathcal{H}_0$ is the Hilbert bimodule ${}_M L^2 M_M$, i.e. the standard representation of M. We call the corresponding irreducible direct summand of $\mathcal{N}^u \subset \mathcal{M}^u$ the *standard W^*-representation* of $N \subset M$ and denote it $\mathcal{N}^{\mathrm{st}} \subset^{\mathcal{E}^{\mathrm{st}}} \mathcal{M}^{\mathrm{st}}$. So the RC-factor $\mathcal{P}$ for the standard representation of $N \subset M$ is M^{op} itself.

One quick way to describe this representation is to consider in $\mathcal{B}(L^2(M_\infty))$ the inclusion

$$\mathcal{N}^{\mathrm{st}} = N \vee M^{\mathrm{op}} \subset M \vee M^{\mathrm{op}} = \mathcal{M}^{\mathrm{st}},$$

with the expectation $\mathcal{E}^{\mathrm{st}}$ given by

$$X \mapsto \sum_j m_j e_0 X e_0 m_j^*$$

for $X \in M \vee M^{\mathrm{op}}$, where $\{m_j\}_j$ is here an o.b. of $\{e_n\}''_{n\geq 1}$ over $\{e_n\}''_{n\geq 2}$ and as usual $N \subset M \subset_{e_0} M_1 \subset_{e_1} M_2 \subset \cdots \nearrow M_\infty$ is the Jones tower for $N \subset M$. This coincides withe the unique expectation extending $E_N^M \otimes \mathrm{id}_{M^{\mathrm{op}}}$.

One can show that, up to isomorphism of representations, $\mathcal{N}^{\mathrm{st}} \subset^{\mathcal{E}^{\mathrm{st}}} \mathcal{M}^{\mathrm{st}}$ is the unique irreducible exact representation $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ of $N \subset M$ for which there exists a direct summand $\mathcal{B}(\mathcal{H}_0)$ of $\mathcal{M}$ such that $\dim({}_M\mathcal{H}_0) = 1$ (equivalently, ${}_M\mathcal{H}_0 = {}_M L^2 M$) and such that after cutting by the support projection of $\mathcal{B}(\mathcal{H}_0)$ in $\mathcal{Z}(\mathcal{M})$, one has

$$\mathcal{P} = M' \cap \mathcal{B}(\mathcal{H}_0) \simeq M' \cap \mathcal{B}(L^2 M) = M^{\mathrm{op}}.$$

Recall from [40, Section 2.4] that $\mathcal{N}^{\mathrm{st}} \subset^{\mathcal{E}^{\mathrm{st}}} \mathcal{M}^{\mathrm{st}}$ is tracial, with inclusion graph $\Lambda = \Lambda_{\mathcal{N}^{\mathrm{st}} \subset \mathcal{M}^{\mathrm{st}}}$ naturally identifying with the transpose of the standard graph of $N \subset M$, $\Gamma_{N \subset M} = (a_{kl})_{k \in K, l \in L}$. Namely, $\Lambda = (b_{lk})_{l \in L, k \in K}$, where $b_{lk} = a_{kl}$, the "pointed" set $* \in K$ labeling the irreducible subfactors in the "even levels" of the Jones tower, $M \subset M_{2n}$, and L labeling the irreducible subfactors in the "odd levels" of the Jones tower, $N \subset M_{2n}$. More precisely, one has

$$\oplus_{l \in \mathcal{K}_l} \mathcal{B}(\mathcal{K}_l) = \mathcal{N}^{\mathrm{st}} \subset \mathcal{M}^{\mathrm{st}} = \oplus_{k \in K} \mathcal{B}(\mathcal{H}_k),$$

where $\{\mathcal{H}_k\}_{k \in K}$ (resp. $\{\mathcal{K}_l\}_{l \in L}$) is the list of all irreducible Hilbert $(M - M)$ (resp. $(N - M)$)-bimodules in ${}_M L^2(M_n)_M$ (resp. ${}_N L^2(M_n)_M$), $n \geq 0$. The weight vectors giving the canonical $\mathcal{E}^{\mathrm{st}}$-preserving n.s.f. trace Tr were denoted in [40, §2.4] as $\vec{v} = (v_k)_{k\in K}$ and $\vec{u} = (u_l)_{l\in L}$, with v_k (resp. u_l) the square root of the index of the irreducible subfactor $Mp \subset pM_{2n}p$ (resp. $Nq \subset qM_{2n}q$), where p (resp. q) is a minimal projection in $M' \cap M_{2n}$ (resp. $N' \cap M_{2n}$) labeled by $k \in K_n$ (resp. $l \in L_n$), the significance of $K_n \subset K$, $L_n \subset L$ being as in Section 2.3.

Note that the considerations in [40, Section 2.4] show that $\vec{v}, \vec{u}$ coincide with the dimension vectors $\vec{d}_M, \vec{d}_N$ of the representation $(N \subset M) \subset (\mathcal{N}^{\mathrm{st}} \subset^{\mathcal{E}^{\mathrm{st}}} \mathcal{M}^{\mathrm{st}})$, i.e. $v_k = \dim({}_M\mathcal{H}_k), u_l = \dim({}_N\mathcal{K}_l)$ for all $k \in K, l \in L$. Thus, with the terminology we introduced in Section 3.3, the standard representation has finite couplings and it is canonically tracial, i.e. the n.s.f. trace Tr given by the coupling vector is $\mathcal{E}^{\mathrm{st}}$-invariant.

We also consider the *dual standard W^*-representation* of $N \subset M$, defined again on $L^2(M_\infty)$ by

$$(N \vee N^{\mathrm{op}} = \mathcal{N}^{\mathrm{st}'} \subset^{\mathcal{E}^{\mathrm{st}'}} \mathcal{M}^{\mathrm{st}'} = M \vee N^{\mathrm{op}}),$$

with the expectation $\mathcal{E}^{\mathrm{st}'}$ being the unique expectation extending $E^M_N \otimes \mathrm{id}_{N^{\mathrm{op}}}$. With the same reasoning as above, its inclusion graph $\Lambda' = \Lambda_{\mathcal{N}^{\mathrm{st}'} \subset \mathcal{M}^{\mathrm{st}'}}$ identifies naturally with the dual standard graph $\Gamma'_{N\subset M}$, which in turn coincides with the standard graph $\Gamma_{N^{\mathrm{op}} \subset M^{\mathrm{op}}}$, of the opposite subfactor $(N \subset M)^{\mathrm{op}} = (N^{\mathrm{op}} \subset M^{\mathrm{op}})$ (see [40, Section 2.4]). Also, this representation has finite couplings and it is canonically tracial, with the coupling vectors given by the canonical weights $\vec{u}', \vec{v}'$ of $\Gamma'_{N\subset M}$.

3.7.1 Definition. The atomic W^*-inclusions involved in the standard representation and its dual form a natural commuting square embedding

$$(\mathcal{N}^{\mathrm{st}'} \subset^{\mathcal{E}^{\mathrm{st}'}} \mathcal{M}^{\mathrm{st}'}) \subset (\mathcal{N}^{\mathrm{st}} \subset^{\mathcal{E}^{\mathrm{st}}} \mathcal{M}^{\mathrm{st}})$$

given by

$$\begin{array}{ccc} N \vee M^{\mathrm{op}} & \subset & M \vee M^{\mathrm{op}} \\ \cup & & \cup \\ N \vee N^{\mathrm{op}} & \subset & M \vee N^{\mathrm{op}} \end{array}$$

with the vertical expectations being extensions of $\mathrm{id}_N \otimes E^{M^{\mathrm{op}}}_{N^{\mathrm{op}}}$ (resp. $\mathrm{id}_M \otimes E^{M^{\mathrm{op}}}_{N^{\mathrm{op}}}$), with all expectations involved preserving the canonical n.s.f. trace Tr on $M \vee M^{\mathrm{op}}$, and with vertical inclusion graphs given by

$$\Lambda_{N\vee N^{\mathrm{op}} \subset N \vee M^{\mathrm{op}}} = \Gamma_{N\subset M} \quad (\text{resp. } \Lambda_{M\vee N^{\mathrm{op}} \subset M\vee M^{\mathrm{op}}} = (\Gamma'_{N\subset M})^t).$$

This object is obviously an isomorphism invariant of the subfactor $N \subset M$. We call it the *standard λ-commuting square* (or *standard λ-cell*) of $N \subset M$, and denote it $\mathcal{C}_{N\subset M}$.

Let us note that, as an invariant of $N \subset M$, the standard commuting square $\mathcal{C}_{N\subset M}$ contains the same amount of information as (i.e. it is equivalent to) the standard invariant $\mathcal{G}_{N\subset M}$. Recall in this respect that an isomorphism of standard invariants (viewed as abstract objects as in [41]) means a trace preserving isomorphism between the union of the finite dimensional algebras involved which takes the $_{ij}$ algebras of the lattices one onto the other and the Jones-sequences of projections one onto the other. For standard λ-cells, an isomorphism is the usual notion of Tr-preserving isomorphism of commuting squares.

3.7.2 Theorem. 1° *Let $N \subset M$ be an extremal inclusion of* II_1 *factors. Then $\mathcal{C}_{N\subset M}$ identifies naturally with the tracial Markov commuting square $(\mathcal{A}_{-1}^{-1} \subset \mathcal{A}_0^{-1}) \subset (\mathcal{A}_{-1}^{0} \subset \mathcal{A}_0^{0})$ in* [51, Section 2].

2° *If $(Q \subset P)$ is another extremal inclusions of* II_1 *factors, then $\mathcal{C}_{N\subset M} \simeq \mathcal{C}_{Q\subset P}$ if and only if $\mathcal{G}_{N\subset M} \simeq \mathcal{G}_{Q\subset P}$.*

Proof. 1° The first part is trivial by the definition of $\mathcal{C}_{N\subset M}$ and by the way the commuting square $(\mathcal{A}_{-1}^{-1} \subset \mathcal{A}_0^{-1}) \subset (\mathcal{A}_{-1}^{0} \subset \mathcal{A}_0^{0})$ is constructed in [51].

2° Letting $M_{-1} = N \subset M = M_0 \subset_{e_0} M_1 \subset_{e_1} M_2 \subset \cdots$ be the Jones tower for $N \subset M$, one has a Jones tower for the (Markov) standard λ-cell in Notation 3.8.1

$$\begin{array}{ccccc} M_{-1} \vee M^{\mathrm{op}} & \subset & M_0 \vee M^{\mathrm{op}} & \subset_{e_0} & M_1 \vee M^{\mathrm{op}} \cdots \\ \cup & & \cup & & \cup \\ M_{-1} \vee N^{\mathrm{op}} & \subset & M_0 \vee N^{\mathrm{op}} & \subset_{e_0} & M_1 \vee N^{\mathrm{op}} \cdots \end{array}$$

with the Jones tower $M_{-1} \subset M_0 \subset_{e_0} M_1 \subset_{e_1} \cdots$ represented in the bottom row (and in the top row by composition of embeddings). It is immediate to see that

$$(M_i \vee M^{\mathrm{op}})' \cap (M_j \vee N^{\mathrm{op}}) = M_i' \cap M_j,$$

with the Jones projections appearing in the algebras $M_i' \cap M_j$ same way as they do in the higher relative commutants. Moreover, it is easy to see that the compositions of expectations $\mathcal{E}_i^{\mathrm{st}}, i \geq 0$ for the top row implement on these higher relative commutants the trace state τ they inherit from M_∞. Since all these data comes from $\mathcal{C}_{N\subset M}$ viewed as an abstract object, it follows that $\mathcal{C}_{N\subset M}$ uniquely determines $\mathcal{G}_{N\subset M}$.

Conversely, if $\mathcal{G}_{N\subset M}$ is given, then [51, Lemma 2.1] associates to it in a canonical way a tracial Markov commuting $(\mathcal{A}_{-1}^{-1} \subset \mathcal{A}_0^{-1}) \subset (\mathcal{A}_{-1}^{0} \subset \mathcal{A}_0^{0})$, which by part 1° is the same as $\mathcal{C}_{N\subset M}$. ■

3.7.3 Remark. We mentioned that one can have exact representations $\mathcal{N}_P \subset \mathcal{M}_P$, where P is strictly smaller than the RC-factor/envelope. Indeed, if $N \subset M$ is any extremal inclusion of separable II_1 factors with finite index and Jones tower

$$N \subset M \subset M_1 \subset M_2 \cdots \nearrow M_\infty,$$

then by [49] there exists a decreasing sequence of hyperfinite subfactors $R_n \subset M$ such that $\cap_n R_n = \mathbb{C}1$ and $R_n' \cap M_j = M' \cap M_j$ for all $j \leq \infty$. Thus, given any $P = R_n$, we have

$$(N \subset M) \subset (\mathcal{N}_P \subset \mathcal{M}_P) = (N \vee P^{\mathrm{op}} \subset M \vee P^{\mathrm{op}})$$

is isomorphic to the standard representation

$$(N \subset M) \subset (\mathcal{N}^{\mathrm{st}} \subset \mathcal{M}^{\mathrm{st}}) = (N \vee M^{\mathrm{op}} \subset M \vee M^{\mathrm{op}}).$$

3.8. Exact representations with finite couplings. We discuss in this section the class of exact representations with finite couplings, showing they are canonically tracial. We also prove that, after an appropriate amplification, the RC-factor $\mathcal{P}$ can be identified with the mirror image P^{op} of an irreducible subfactor $P \subset M$ satisfying the bicommutant condition $(P' \cap M_\infty)' \cap M = P$. The standard representation is a particular case of this class of exact representations, corresponding to $P = M$, while the dual standard representation corresponds to the case $P = N$.

Thus, let

$$\oplus_{i \in I} \mathcal{B}(\mathcal{K}_i) = \mathcal{N} \subset^{\mathcal{E}} \mathcal{M} = \oplus_{j \in J} \mathcal{B}(\mathcal{H}_j)$$

be an irreducible exact representation of the subfactor $N \subset M$, with inclusion bipartite graph/matrix

$$\Lambda = \Lambda_{\mathcal{N} \subset \mathcal{M}} = (b_{ij})_{i \in I, j \in J}.$$

As usual, denote $\mathcal{P} = M' \cap \mathcal{N}$. Recall from Proposition 3.4.2 that $\mathcal{N} \subset \mathcal{M}$ has finite dimension vectors $\vec{d}_M, \vec{d}_N$ iff there exists $j_0 \in J$ such that

$$\alpha = d_M(j_0) = \dim({}_M\mathcal{H}_{j_0}) < \infty.$$

Note that if this is the case then $\mathcal{P}$ is a II_1 factor which identifies with an irreducible II_1 subfactor of the α-amplification of M^{op}.

Let $t > 0$. Replacing $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ by

$$\mathcal{N} \otimes \mathbb{M}_n(\mathbb{C}) \subset^{\mathcal{E} \otimes \mathrm{id}} \mathcal{M} \otimes \mathbb{M}_n(\mathbb{C})$$

for $n \geq t$, and then taking p to be a projection in the II_1 factor $\mathcal{P} \otimes \mathbb{M}_n(\mathbb{C})$ of normalized trace $\tau(p) = t/n$, one obtains a representation of $N \subset M$ into

$$p\big(\mathcal{N} \otimes \mathbb{M}_n(\mathbb{C}) \subset \mathcal{M} \otimes \mathbb{M}_n(\mathbb{C})\big)p$$

with exactness factor $p(\mathcal{P} \otimes \mathbb{M}_n(\mathbb{C}))p = \mathcal{P}^t$ (the t-amplification of $\mathcal{P}$) and expectation $\mathcal{E} \otimes \mathrm{id}(p \cdot p)$. It is easy to see that the isomorphism class of this representation does not depend on the choice of n and p.

We call this representation of $N \subset M$ the *t-amplification* of $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ and denote it $\mathcal{N}^t \subset^{\mathcal{E}^t} \mathcal{M}^t$. Its inclusion graph $\Lambda_{\mathcal{N}^t \subset \mathcal{M}^t}$ identifies naturally with $\Lambda_{\mathcal{N} \subset \mathcal{M}}$ and if one denotes

$$\oplus_{i \in I} \mathcal{B}(\mathcal{H}_i^t) = \mathcal{N}^t \subset \mathcal{M}^t = \oplus_{j \in J} \mathcal{B}(\mathcal{H}_j^t),$$

then the entries of its dimension vectors are given by

$$\dim({}_M \mathcal{H}_j^t) = t \dim({}_M \mathcal{H}_j), \quad \dim({}_N \mathcal{K}_i^t) = t \dim({}_N \mathcal{K}_i).$$

The amplified representations $\mathcal{N}^t \subset \mathcal{M}^t$, $t > 0$, are obviously stably isomorphic.

Thus, if $\oplus_{i \in I} \mathcal{B}(\mathcal{K}_i) = \mathcal{N} \subset^{\mathcal{E}} \mathcal{M} = \oplus_{j \in J} \mathcal{B}(\mathcal{H}_j) = \mathcal{M}$ is an exact representation with finite couplings and $j_0 \in J$ then, modulo stable equivalence, we may assume $\dim({}_M \mathcal{H}_{j_0}) = 1$. Thus, there exists a unique irreducible subfactor $P \subset M$ such that after reducing with the central support p_{j_0} of $\mathcal{B}(\mathcal{H}_{j_0}) \simeq \mathcal{B}(L^2 M)$, one has $P^{\text{op}} p_{j_0} = \mathcal{P} p_{j_0}$ and $\mathcal{N} \subset \mathcal{M}$ can be reconstructed from the Hilbert $(M - P)$-bimodule ${}_M(\mathcal{H}_{j_0})_P = {}_M L^2 M_P$, as explained in the last paragraph of Section 3.5, in which case we denote it $\mathcal{N}_P \subset \mathcal{M}_P$.

3.8.1 Notation. We denote by $\mathcal{N}^{\text{u,fc}} \subset \mathcal{M}^{\text{u,fc}}$ the subrepresentation of $\mathcal{N}^u \subset \mathcal{M}^u$ obtained as the direct sum of irreducible representations that contain a type I_∞ factor summand $\mathcal{B}(\mathcal{H}_j)$ of $\mathcal{M}^u$ with $\dim({}_M \mathcal{H}_j) = 1$. Thus, $\mathcal{N}^{\text{u,fc}} \subset^{\mathcal{E}} \mathcal{M}^{\text{u,fc}}$ is of the form $\oplus_P (\mathcal{N}_P \subset \mathcal{M}_P)$, where the sum is over some irreducible subfactors $P \subset M$. We denote the inclusion graph of $\mathcal{N}^{\text{u,fc}} \subset \mathcal{M}^{\text{u,fc}}$ by $\Lambda^{\text{u,fc}}_{N \subset M}$.

Taking into account that ${}_M(\mathcal{H}_j)_P$, ${}_N(\mathcal{K}_i)_P$ are all sub bimodules of $L^2(M_n)$, $n \geq 0$, whose union is dense in $L^2(M_\infty)$, it follows that ${}_M L^2(M_\infty)_P$ (resp. ${}_N(L^2(M_\infty)_P)$ is the direct sum of irreducible $(M - P)$- (resp. $(N - P)$-) Hilbert bimodules $\{\mathcal{H}_j\}_j$ (respectively $\{\mathcal{K}_i\}_i$), which appear with infinite multiplicity. Thus, when viewed as a direct summand of the universal exact representation $\mathcal{N}^u \subset \mathcal{M}^u$, $\mathcal{N}_P \subset \mathcal{M}_P$ is isomorphic to the W^*-inclusion $N \vee P^{\text{op}} \subset M \vee P^{\text{op}}$, acting on $L^2(M_\infty)$.

In fact, the entire Jones tower for $\mathcal{N}_P = \mathcal{M}_{-1} \subset \mathcal{M}_0 = \mathcal{M}_P$ is represented on $L^2(M_\infty)$, with the consecutive inclusions $\mathcal{M}_n \subset_{e_n} \mathcal{M}_{n+1} = \langle \mathcal{M}_n, e_n \rangle$ given by

$$M_n \vee P^{\text{op}} \subset_{e_n} M_{n+1} \vee P^{\text{op}},$$

and expectation $\mathcal{E}_{n+1} \colon \mathcal{M}_{n+1} \to \mathcal{M}_n$ given by

$$\mathcal{E}_{n+1}(X) = \sum_k m_k^n e_{n+1} X e_{n+1} m_k^{n*}, \quad X \in \mathcal{M}_{n+1},$$

where $\{m_k^n\}_k$ is o.b. for $(\{e_m\}_{m \geq n+2})''$ over $(\{e_m\}_{m \geq n+3})''$, and $N \subset M \subset_{e_0} M_1 \subset_{e_1} \cdots$ being the Jones tower and sequence of projections for $N \subset M$, acting (from the left) on $L^2(M_\infty)$.

Moreover, if one denotes $B = P' \cap M_\infty$, then the $\mathcal{M}_n$'s are all contained in the II_∞ von Neumann algebra

$$\langle M_\infty, e_B \rangle = (J_{M_\infty} B J_{M_\infty})' \cap \mathcal{B}(L^2 M_\infty),$$

with the canonical n.s.f. trace $\mathrm{Tr} = \mathrm{Tr}_{\langle M_\infty, B\rangle}$ (defined as usual by $\mathrm{Tr}(xe_B y) = \tau(xy)$, $x, y \in M_\infty$) being semifinite on each $\mathcal{M}_n$ and preserving $\mathcal{E}_n$, $n \geq 0$. In addition, in $\mathcal{B}(L^2 M_\infty)$ one has $\overline{\cup_n \mathcal{M}_n} = \langle M_\infty, e_B\rangle$ and $\langle M_\infty, e_B\rangle$ naturally identifies this way with the enveloping von Neumann algebra $\mathcal{M}_\infty$ of the Jones tower $\{\mathcal{M}_n\}_n$, i.e. with the GNS completion of $(\cup_n \mathcal{M}_n, \Phi)$, where $\Phi = \phi \circ \mathcal{E} \circ \mathcal{E}_1 \circ \cdots$, with ϕ any n.s.f. weight on $\mathcal{N}$ (e.g. tracial; or a normal faithful state).

To see this, it is sufficient to prove that e_B is contained in $M \vee P^{\mathrm{op}}$. Indeed, because then $ue_B u^*$, $u \in \mathcal{U}(M)$ are in $M \vee P^{\mathrm{op}}$ and $\vee_u u e_B u^* = 1$, showing that Tr is semifinite on $M \vee P^{\mathrm{op}}$. Then recursively, using that $\mathcal{E}_n$ have finite index, Tr follows semifinite on $\mathcal{M}_n$ for all n.

To see that $e_B \in M \vee P^{\mathrm{op}}$, we show that the cyclic projection $[(M \vee P^{\mathrm{op}})'(\hat{1})]$, which belongs to $\mathcal{M}_P = M \vee P^{\mathrm{op}}$, is equal to e_B. Indeed, by cutting with the orthogonal projection e_{M_n}, of $L^2(M_\infty)$ onto $L^2(M_n)$, which is invariant to $\mathcal{M}_P = M \vee P^{\mathrm{op}}$ and commutes with e_B (because one has the commuting square relation $E_{M_n} E_{P' \cap M_\infty} = E_{P' \cap M_n}$), one gets

$$e_{M_n} (M \vee P^{\mathrm{op}})' e_{M_n}(\hat{1}) = (P' \cap M_n)(\hat{1}) \subset B(\hat{1})$$

and the equality follows by letting $n \to \infty$.

Note that by the commuting square relation $e_{M_n} e_B = e_B e_{M_n} = e_{P' \cap M_n}$, by the formula for the canonical trace Tr on $\langle M_\infty, e_B\rangle$, and by the fact that $e_{M_n} \in (M \vee P^{\mathrm{op}})'$, it follows that Tr restricted to $(M \vee P^{\mathrm{op}}) z_n$ (where $z_n \in \mathcal{Z}(\mathcal{M}_P)$ is the support of $\mathcal{M}_P \ni x \mapsto x e_{M_n}$) coincides with $\mathrm{Tr}_{\langle M, e_{P' \cap M_n}\rangle}$ and thus the minimal projections of its direct summands are proportional with the trace τ of the minimal projections in $P' \cap M_n \subset M_n$, which by Section 3.3 are proportional to the entries of $\vec{d}_M$ supported by z_n. This shows that for any finite subset $J_0 \subset J$ the restriction of Tr to $\{p_j\}_{j \in J_0}$, with p_j minimal projection in $\mathcal{B}(\mathcal{H}_j)$, is proportional to $\{\dim({}_M \mathcal{H}_j)\}_{j \in J_0}$. This in turn clearly implies that the weight vector for Tr on $M \vee P^{\mathrm{op}}$ is proportional with the dimension vector $\vec{d}_M$.

We summarize all these fact in the next statement.

3.8.2 Proposition. *Let $N \subset M$ be an extremal subfactor of finite index. Let $P \subset M$ be an irreducible subfactor and denote as above by $\mathcal{N}_P \subset^{\mathcal{E}} \mathcal{M}_P$ the associated exact $(N \subset M)$-representation with finite couplings*

$$\oplus_{i \in I} \mathcal{B}(\mathcal{H}_i) = N \vee P^{\mathrm{op}} \subset^{\mathcal{E}} M \vee P^{\mathrm{op}} = \oplus_{j \in J} \mathcal{B}(\mathcal{H}_j),$$

viewed as acting on $L^2(M_\infty)$, with $P^{\mathrm{op}} = J_{M_\infty} P J_{M_\infty}$. Then we have:

1° *$\mathcal{N}_P \subset^{\mathcal{E}} \mathcal{M}_P$ is canonically tracial. Moreover, the sequence of inclusions with Jones projections $\mathbb{C} = P' \cap M \subset P' \cap M_1 \subset_{e_1} M_2 \subset \cdots$ and the trace state τ inherited from M_∞ forms a λ-sequence of inclusions, in the sense of Definition 2.7.4 and Proposition 2.7.5, with its pointed bipartite graph given by Λ^t and weight vector $\vec{t} = (t_j)_j$ given by $t_j = \dim({}_M\mathcal{H}_j)$.*

2° *The Jones tower $\mathcal{N}_P \subset^{\mathcal{E}} \mathcal{M}_P \subset^{\mathcal{E}_1}_{e_0} \mathcal{M}_1 \subset \cdots$ identifies with the tower $M_n \vee P^{\mathrm{op}} \subset \mathcal{B}(L^2 M_\infty)$ with the enveloping von Neumann algebra $(\mathcal{M}_\infty, \mathrm{Tr})$ identifying with $(\langle M_\infty, B\rangle, \mathrm{Tr}_{\langle M_\infty, B\rangle})$, where $B = P' \cap M_\infty$.*

3° *P^{op} is the RC-factor of $\mathcal{N}_P \subset \mathcal{M}_P$ (i.e. $M' \cap \mathcal{N}_P = P^{\mathrm{op}}$) iff P satisfies the bicommutant condition $(P' \cap M_\infty)' \cap M = P$.*

Proof. We already proved the first part of 1° and part 2°. The second part of 1° is an immediate consequence of Propositions 2.7.3 and 3.4.2.

To prove 3°, recall from Section 3.6 that $M' \cap (N \vee M^{\mathrm{op}}) = M^{\mathrm{op}}$ (because M^{op} is the RC-factor of the standard rep.). Thus,

$$\begin{aligned} M' \cap (N \vee P^{\mathrm{op}}) &= (M' \cap (N \vee M^{\mathrm{op}})) \cap (N \vee P^{\mathrm{op}}) = M^{\mathrm{op}} \cap (N \vee P^{\mathrm{op}}) \\ &= M^{\mathrm{op}} \cap (N \vee P^{\mathrm{op}}) \cap ((P' \cap M_\infty)' \cap M_\infty)^{\mathrm{op}} \\ &= ((P' \cap M_\infty)' \cap M)^{\mathrm{op}} \cap (N \vee P^{\mathrm{op}}). \end{aligned}$$

Thus, if $(P' \cap M_\infty)' \cap M = P$, then the last term is equal to P^{op}, showing that

$$M' \cap (N \vee P^{\mathrm{op}}) = P^{\mathrm{op}}.$$

Conversely, if $M' \cap (N \vee P^{\mathrm{op}}) = \mathcal{P}$, with $P^{\mathrm{op}} \subset \mathcal{P} \subset M^{\mathrm{op}}$ but $P^{\mathrm{op}} \neq \mathcal{P}$, then $\mathcal{P}$ commutes with $(P' \cap M_\infty)^{\mathrm{op}}$, so

$$((P' \cap M_\infty)' \cap M)^{\mathrm{op}} \supset \mathcal{P},$$

showing that $(P' \cap M_\infty)' \cap M$ is strictly larger than P. ∎

3.9. A_∞-subfactors. We say that $N \subset M$ is an *A_∞-subfactor* if its standard graph $\Gamma_{N \subset M}$ is equal to the bipartite graph A_∞. This is equivalent to the "dual" standard graph $\Gamma'_{N \subset M}$ being equal to A_∞, and also to the fact that $[M : N] \geq 4$ and the higher relative commutants in the tower $N \subset M \subset_{e_0} M_1 \subset_{e_1} \cdots$ are generated by the Jones projections, i.e. $M_i' \cap M_j = \{1, e_{i+1}, \ldots, e_{j-1}\}''$ for all $j \geq i + 2$.

The existence of subfactors with A_∞ graph and index equal to 4 of the hyperfinite II_1 factor R was shown by Vaughan Jones in his original paper [20], who gave two alternate constructions: (a) as the inclusion $\{e_n\}''_{n \geq 1} \subset \{e_n\}''_{n \geq 0}$, where e_n is the sequence of Jones projections for the tower of $\mathbb{C} \subset \mathbb{M}_2(\mathbb{C})$; (b) as the inclusion of fixed point algebras of the product action of SU(2) on $\overline{\otimes}_{n\geq 1}(\mathbb{M}_2(\mathbb{C}), \mathrm{tr})_n \subset \overline{\otimes}_{n\geq 0}(\mathbb{M}_2(\mathbb{C}), \mathrm{tr})_n$. By [40, Corollary 5.2.1], there exists in fact a unique index 4 subfactor of R with A_∞ graph, up to conjugacy by an automorphism.

The problem of the existence of A_∞-subfactors with arbitrary index $\lambda^{-1} > 4$, which is closely related to the restriction problem on the Jones index for irreducible subfactors beyond 4, remained open for some time. This was solved in [38], where it was shown that any $\lambda^{-1} > 4$ can occur as the index of an A_∞-subfactor. This was surprising, because initially it was believed that the set of all possible indices of irreducible subfactors has gaps beyond 4 (cf. [21, p. 940]).

The A_∞-subfactors in [38] are canonical, universal objects, obtained through a tracial amalgamated free product construction, which we briefly recall. Let $(\{e_n\}_{n\geq 0}, \tau)$ be a sequence of λ-Jones projections. Such a sequence exists by [20], because of the existence of subfactors of arbitrary index $\lambda^{-1} > 4$ constructed as locally trivial, non-extremal subfactors of R, defined by

$$R_\lambda = \{x + \theta(x) \mid x \in pRp\},$$

where $\theta\colon pRp \simeq (1-p)R(1-p)$, with $\tau(p)\tau(1-p) = \lambda$ (due to the fact that by [27] the fundamental group of R is the entire multiplicative group $(0, \infty)$).

Let Q be a diffuse tracial von Neumann algebra (taken as "initial data"). Denote $A_{i,\infty} = \{e_j \mid j \geq i\}''$ and let $M^\lambda_\infty(Q) = Q\overline{\otimes}A_{1,\infty} *_{A_{1,\infty}} A_{0,\infty}$. Then define $M^\lambda(Q)$ as the smallest von Neumann algebra in $M^\lambda_\infty(Q)$ that contains Q and is stable to the u.c.p. map $x \mapsto \sum_j m_j e_0 x e_0 m_j^*$, where $\{m_j\}_j$ is o.b. of $A_{1,\infty}$ over $A_{2,\infty}$, and let $N^\lambda(Q) = \{e_0\}' \cap M^\lambda(Q)$. Theorem 5.2 of [38] then shows that $N^\lambda(Q) \subset M^\lambda(Q)$ is an inclusion of factors of index λ^{-1} and standard graphs equal to A_∞.

By [51], if $Q = L(\mathbb{F}_\infty)$, then

$$N^\lambda(Q), M^\lambda(Q) \simeq L(\mathbb{F}_\infty),$$

so $L(\mathbb{F}_\infty)$ contains A_∞-subfactors of any index ≥ 4. It is pointed out in [38, §8.1] (cf. [43, Theorem]; see also [46, Theorem 4.5]) that given any extremal subfactor $N \subset M$ of index λ^{-1}, any free ultrafilter ω and any $Q \subset M^\omega$, the subfactor $N^\lambda(Q) \subset M^\lambda(Q)$ can be realized as commuting square embedding into $N^\omega \subset M^\omega$.

The construction in [38] was further refined in [41] to obtain a characterization of all lattices of tracial finite dimensional algebras with λ-Jones projections $\mathcal{G} = (\{A_{ij}\}_{j\geq i\geq -1}, \{e_j\}_{j\geq 0}, \tau)$ that can occur as the standard invariant of a subfactor of index λ^{-1}, i.e. for which there exists an extremal subfactor $N \subset M$ with $\mathcal{G}_{N\subset M} = \mathcal{G}$. The abstract objects $\mathcal{G}$ are called *standard λ-lattices*. Thus, the result in [38] states that the "minimal" lattice, consisting of the algebras generated by the Jones projections,

$$A_{ij} = \mathrm{Alg}\big(\{1, e_k \mid i+1 \leq k \leq j-1\}\big), \quad \forall j \geq i \geq -1,$$

is a standard λ-lattice. This is called the *Temperley–Lieb–Jones* (*TLJ*) *λ-lattice* and denoted $\mathcal{G}^\lambda$. Thus, A_∞-subfactors are the subfactors that have TLJ standard invariant.

But the main interest for us in this paper is the study of A_∞-subfactors of the hyperfinite factor R. By results in [14], any irreducible subfactor $N \subset M$ of index $[M : N] \in (4, (5+\sqrt{13})/2)$ has A_∞ graph. In particular, if $4 < [M : N] < 2+\sqrt{5}$, then $\Gamma_{N\subset M} = A_\infty$. So if $\alpha \in (4, 2+\sqrt{5})$ and $N \subset R$ is an irreducible subfactor of index α of the hyperfinite factor, then $\Gamma_{N\subset R} = A_\infty$. For $\alpha = \|E_{10}\|^2 \approx 4.062\ldots$, which we saw in Section 2.10 is the first number in $\mathbb{E}^2$ that's larger than 4, Ocneanu was able to solve the commuting square problem for E_{10} (see Section 2.9), thus obtaining an example of irreducible hyperfinite subfactor of index α through the method described in Section 2.9 (see [57] for details).

The next two results detail some classes of representations that any A_∞-subfactor has, starting with the standard representation:

3.9.1 Proposition. *Let $N \subset M$ be an A_∞-subfactor and denote $\oplus_{l\in L}\mathcal{B}(\mathcal{K}_l) = \mathcal{N}^{\mathrm{st}} \subset \mathcal{M}^{\mathrm{st}} = \oplus_{k\in K}\mathcal{B}(\mathcal{H}_k)$ its standard representation, with its inclusion graph $\Lambda = \Lambda_{\mathcal{N}^{\mathrm{st}}\subset\mathcal{M}^{\mathrm{st}}}$. Then,*

$$\Lambda^t = A_\infty = \Gamma_{N\subset M} = (a_{kl})_{k\in K, l\in L}.$$

If one denotes $K = \{* = 0, 2, 4, \ldots\}$ the "even" vertices of $\Gamma_{N\subset M}$ and $L = \{1, 3, \ldots\}$ its "odd" vertices, then the entries are given by

$$a_{2n,2n+1} = 1 = a_{2n+2,2n+1}, \quad n \geq 0,$$

with all other $a_{kl} = 0$. The corresponding couplings are given by

$$d_n = \sqrt{P_n(\lambda)/\lambda P_{n-1}(\lambda)} \quad \text{for all } n \geq 0,$$

where $P_{-1}(\lambda) = 1$, $P_0(\lambda) = 1$, $P_{n+1}(\lambda) = P_n(\lambda) - \lambda P_{n-1}(\lambda)$ for all $n \geq 1$.

Proof. The formulas for the entries of the weights d_n, $n \geq 0$, are well known (see e.g. [12]). ∎

Recall from [20] that if $(\{e_n\}_{n\in\mathbb{Z}}, \tau)$ is the two-sided λ-sequence of Jones projections with $\lambda^{-1} > 4$ and we denote $R_n = \{e_i \mid i \leq n-1\}''$ (resp. $P_n = \{e_i \mid i \geq n+1\}''$), $n \in \mathbb{Z}$, then $(R_n \subset_{e_n} R_{n+1})_{n\in\mathbb{Z}}$ (resp. $(P_{n+1} \subset_{e_n} P_n)_{n\in\mathbb{Z}}$) is a Jones tower/tunnel of II_1 factors of index λ^{-1}. Moreover, by [31, Corollary 5.4]; see also [36, Corollary 3.3]), $R_n \subset R_{n+1}$ (resp. $P_{n+1} \subset P_n$) is a locally trivial subfactor with

$$R'_{n-1} \cap R_n = \mathbb{C} f_n + \mathbb{C}(1 - f_n) \quad (\text{resp. } P'_{n+1} \cap P_n = \mathbb{C} f'_n + \mathbb{C}(1 - f'_n)),$$

where f_n, f'_n are projections of trace $t < 1/2$ and $t(1-t) = \lambda$.

3.9.2 Proposition. *Let $N \subset M$ be an A_∞ subfactor and*

$$N \subset M \subset_{e_0} M_1 \subset_{e_1} M_2 \subset_{e_2} \cdots \nearrow M_\infty$$

its Jones tower and enveloping II_1 factor. For each $i \geq 0$, denote

$$\widetilde{M}_i = \{e_n \mid n \geq i+1\}' \cap M_\infty = P_i' \cap M_\infty = (M_i' \cap M_\infty)' \cap M_\infty,$$

with $\widetilde{M}_0 = \widetilde{M}$, $\widetilde{M}_{-1} = \widetilde{N}$. Denote by $\widetilde{E}_n \colon \widetilde{M}_n \to \widetilde{M}_{n-1}$ the map given by

$$x \mapsto \sum_j m_j^n e_n x e_n m_j^{n*}, \quad x \in \widetilde{M}_n,$$

where $\{m_j^n\}_j$ is an o.b. of P_n over P_{n+1}.

1° *$\widetilde{M}_n$ are II_1 factors with $\widetilde{M}_n = (M_n' \cap M_\infty)' \cap M_\infty$ and one has:*

(i) *$\widetilde{M}_{n-1} \subset \widetilde{M}_n$ is locally trivial with $\widetilde{M}_{n-1}' \cap \widetilde{M}_n = \mathbb{C} f_n' + \mathbb{C}(1 - f_n')$;*

(ii) *$M_i' \cap \widetilde{M}_j = \widetilde{M}_i' \cap \widetilde{M}_j = \mathrm{Alg}(\{1, e_i, \ldots, e_j; f_i', \ldots, f_j'\})$;*

(iii) *$\widetilde{E}_n$ is a conditional expectation satisfying $\widetilde{E}_n(f_n') = 1 - t$;*

(iv) *$\{\widetilde{M}_n \subset_{e_n}^{\widetilde{E}_{n+1}} \widetilde{M}_{n+1}\}_{n \in \mathbb{Z}}$ is a Jones tower-tunnel;*

(v) *$\widetilde{M}_n = vN(M_n, f_n')$;*

(vi) *$(N \subset^{E_N} M) \subset (\widetilde{N} \subset^{\widetilde{E}} \widetilde{M})$ is a commuting square embedding, i.e. $\widetilde{E}_{|M} = E_N$.*

2° *Let $P = \widetilde{M}$ and consider the Hilbert $M - P$ bimodule ${}_M\mathcal{H}_P = {}_M L^2(\widetilde{M})_{\widetilde{M}}$. The exact irreducible representation $\mathcal{N}_P \subset^{\mathcal{E}} \mathcal{M}_P$ has graph $A_{-\infty,\infty}$ and is not tracial.*

3° *There exists a choice of a tunnel $\cdots \subset_{e_{-2}} M_{-1} \subset_{e_{-1}} M_0$ for $N = M_{-1} \subset M_0 = M$ such that $R_0 = \{e_j \mid j \leq -1\}''$ satisfies $R_0' \cap M = \mathbb{C}$, and more generally $R_0' \cap M_\infty = R_0' \cap R_\infty$. If one denotes by $\mathcal{N}_{R_0} \subset \mathcal{M}_{R_0}$ the exact irreducible representation associated with the irreducible Hilbert $(M - R_0)$-bimodule ${}_M L^2 M_{R_0}$, then $\mathcal{N}_{R_0} \subset \mathcal{M}_{R_0}$ has inclusion graph $\Lambda = A_{-\infty,\infty}$ and has finite couplings. Moreover, if one labels by $\mathbb{Z}$ the consecutive vertices of Λ, with even vertices $J = \{2n \mid n \in \mathbb{Z}\}$, odd vertices $I = \{2n+1 \mid n \in \mathbb{Z}\}$, then the vector $\vec{d}_M$ of M-couplings at even levels is given by $d_{2n} = ((1-t)/t)^n$, $n \in \mathbb{Z}$.*

Proof. 1° Part 1° is essentially [38, Theorem 7.8].

2° The fact that if $P = \widetilde{M}$ the inclusion graph of $\mathcal{N}_P \subset^{\mathcal{E}} \mathcal{M}_P$, is equal to $A_{-\infty,\infty}$ follows immediately from 1°. The corresponding $(N \subset M)$-representation is non-tracial because one has $\widetilde{N} = \widetilde{M}_{-1} \subset \widetilde{M}_0 = \widetilde{M}$ as an intermediate inclusion, i.e. one has

$$(N \subset M) \subset (\widetilde{N} \subset \widetilde{M}) \subset (\mathcal{N}_P \subset^{\mathcal{E}} \mathcal{M}_P),$$

with $\mathcal{E}_{|\widetilde{M}} = \widetilde{E}_0$ and $f_0' \in \widetilde{N}' \cap \widetilde{M}$ satisfying $\widetilde{E}_0(f_0') = 1 - t$, while $\tau(f_0') = t$.

3° The existence of the choice of the tunnel $M = M_0 \supset M_{-1} \supset \cdots$, so that

$$R_0 = \overline{\cup_n M'_{-n} \cap M} = \{e_j \mid j \leq -1\}''$$

has trivial relative commutant in M, and more generally $R'_0 \cap M_\infty = R'_0 \cap R_\infty$, follows from [44, Theorem 4.10 (b)] (see also [50, Lemma 2.7]).

By part 1° of Proposition 3.8.2, the calculation of the M-couplings amounts to the calculation of the trace vector for the λ-sequence of inclusions

$$\mathbb{C} = R'_0 \cap M \subset R'_0 \cap M_1 \subset_{e_1} R'_0 \cap M_2 \subset \cdots,$$

which coincides with $R'_0 \cap R_0 \subset R'_0 \cap R_1 \subset R'_0 \cap R_2 \subset \cdots$, for which this calculation was done in [31, Section 5]. ∎

3.9.3 Remark. Given an A_∞-subfactor $N \subset M$, the irreducible inclusion of II_1 factors

$$M \subset \widetilde{M} = P'_0 \cap M_\infty = (M' \cap M_\infty)' \cap M_\infty$$

in Proposition 3.9.2 is quasi-regular, in the sense of [44, Definition 4.9] or [47, Section 1.4.2]. This follows trivially from the fact that ${}_M L^2(\widetilde{M})_M$ is a submodule of ${}_M L^2(M_\infty)_M$, which is a direct sum of finite index bimodules. Using an argument similar to that found in the proof of Theorem 4.5 of [44], one can in fact show that if $M \subset \widetilde{M} \subset \langle \widetilde{M}, e\rangle \simeq M^\infty$ denotes the basic construction, with its canonical n.s.f. trace Tr, then $M' \cap \langle \widetilde{M}, e\rangle$ is discrete abelian, generated by minimal projections $\{f_k\}_{k\in K}$, with $\mathrm{Tr}(f_k) = v_k$, where $(v_k)_k$ is the standard vector at even levels of the standard graph $\Gamma_{N\subset M}$, and that ${}_M L^2(\widetilde{M})_M = \oplus_{k\in K}({}_M \mathcal{H}_{k\,M})$.

3.10. Untamed representations. By Lemma 3.2.2, any non-degenerate embedding of a given subfactor $N \subset M$ into another subfactor $\widetilde{N} \subset^{\widetilde{E}} \widetilde{M}$ composed with the standard representation of the latter, gives a representation of $N \subset M$. When combined with [46, Theorem 4.10], this gives the following class of representations, which we loosely call *untamed*.

3.10.1 Theorem. *Let $N \subset M$ be an extremal inclusion of separable* II_1 *factors of index* $4 < \lambda^{-1} = [M : N] < \infty$. *Given any standard graph* $(\Gamma, \vec{v})$ *of index* λ^{-1}, *there exists a separable tracial representation* $(N \subset M) \subset (\mathcal{N} \subset^{\mathcal{E}} \mathcal{M})$ *having* λ*-Markov weighted graph* $(\Lambda_{\mathcal{N}\subset\mathcal{M}}, \vec{t})$ *given by* $(\Gamma^t, \vec{v})$.

Proof. Let $Q \subset P$ be an extremal inclusion of II_1 factors of index λ^{-1} and weighted standard graph $(\Gamma, \vec{v})$. Both $\mathcal{G}_{N\subset M}, \mathcal{G}_{Q\subset P}$ contain the TLJ λ-lattice $\mathcal{G}^\lambda$. So by [46, Theorem 4.10], there exists a separable extremal subfactor $\widetilde{N} \subset \widetilde{M}$ with standard invariant $\mathcal{G}^\lambda$ in which both $N \subset M$ and $Q \subset P$ embed as non-degenerate commuting squares. Let $e = e_P^{\widetilde{M}}$ denote the Jones projection for the basic construction

$$P \subset \widetilde{M} \subset_e \langle \widetilde{M}, P\rangle \simeq P^\infty.$$

By applying [40, Proposition 2.1], one thus gets a non-degenerate embedding

$$(\widetilde{N} \subset \widetilde{M}) \subset \big(\langle \widetilde{N}, e\rangle \subset \langle \widetilde{M}, e\rangle\big) \simeq (Q^\infty \subset^{E^\infty} P^\infty),$$

where $(Q^\infty \subset^{E^\infty} P^\infty) = (Q \subset^{E_Q^P} P)\overline{\otimes}\mathcal{B}(\ell^2\mathbb{N})$. Thus, this inclusion further embeds with non-degenerate commuting squares into

$$(\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}) \overset{\text{def}}{=} (\mathcal{Q}^{\text{st}} \subset \mathcal{P}^{\text{st}})\overline{\otimes}\mathcal{B}(\ell^2\mathbb{N}),$$

which is a tracial λ-Markov atomic W^*-inclusion with weighted inclusion graph given by the standard weighted graph of $Q \subset P$, i.e. $(\Gamma, \vec{v})$. Composing the embeddings, we get the representation $(N \subset M) \subset (\mathcal{N} \subset^{\mathcal{E}} \mathcal{M})$ which has all required properties. ∎

3.10.2 Corollary. *Any extremal inclusion of separable* II_1 *factors* $N \subset M$ *of index* $4 < \lambda^{-1} = [M : N] < \infty$ *has a tracial representation with inclusion graph equal to* A_∞ *and* λ*-weights as given in Proposition 3.9.1.*

4. Amenability for graphs, subfactors, and λ-lattices

In this section we recall the definition of amenability for weighted bipartite graphs, standard λ-lattices and subfactors from [39, 40, 44], and revisit some results in [44] about these notions.

4.1 Definition. A Markov weighted graph $(\Lambda, \vec{t}, \lambda)$ is *amenable* if $\|\Lambda\|^2 = \lambda^{-1}$.

The next result establishes a Følner-type characterization of amenability for Markov weighted graphs. Like in [39, Definition 3.1], if $\Lambda = (b_{ij})_{i\in I, j\in J}$ is a bipartite graph and $F \subset J$ is a non-empty set, then we let

$$\Lambda^t\Lambda(F) = \Big\{ j \in J \mid \exists j' \in F \text{ such that } \sum_i b_{ij}b_{ij'} \neq 0 \Big\}$$

and denote $\partial F \overset{\text{def}}{=} \Lambda^t\Lambda(F) \setminus F$.

4.2 Theorem. *Let* $(\Lambda, \vec{t})$ *be a Markov weighted graph, with* $\Lambda = (b_{ij})_{i\in I,\, j\in J}$, $\vec{t} = (t_j)_{j\in J}$ *and* $\Lambda\Lambda^t(\vec{t}) = \lambda^{-1}\vec{t}$. *Then* $(\Lambda, \vec{t})$ *is amenable if and only if it satisfies the following Følner-type condition:*

(4.2.1) *For any* $\varepsilon > 0$, *there exists* $F = F(\varepsilon) \subset J$ *finite such that* $\|\vec{t}_{|\partial F}\|_2 < \varepsilon\|\vec{t}_{|F}\|_2$, *i.e.* $(\sum_{j\in\partial F} t_j^2)^{1/2} < \varepsilon(\sum_{j\in F} t_j^2)^{1/2}$.

Proof. In the case where Λ is the standard graph of a subfactor, this result amounts to $(1') \Leftrightarrow (2)$ of Theorem 5.3 in [44] and (i) $\Leftrightarrow$ (ii) of Theorem 3.5 in [39], whose proof only uses the fact that Λ has non-negative integers as entries, the weight-vector $\vec{t}$ has strictly positive entries, and the fact that $\Lambda^t\Lambda(\vec{t}) = \lambda^{-1}\vec{t}$. We revisit that argument, for convenience.

For $a = (a_j)_j, b = (b_j)_j \in \mathbb{C}^J$, with b finitely supported, we write

$$\langle a, b\rangle = \sum_j a_j \overline{b}_j.$$

If condition (4.2.1) is satisfied, then for any $\varepsilon > 0$, the finite set $F = F(\varepsilon) \subset J$ gives rise to the finitely supported element $t_F = (t_j)_{j\in F} \in \ell^2 J$ and if we denote $F' = \Lambda^t\Lambda(F)$, then we have

$$\begin{aligned}\lambda^{-1}\sum_{j\in F} t_j^2 &= \langle \Lambda^t\Lambda(t_{F'}), t_F\rangle \\ &\leq \|\Lambda\|^2\|t_{F'}\|_2\|t_F\|_2 \leq \|\Lambda\|^2(1+\varepsilon)^{1/2}\|t_F\|_2^2,\end{aligned}$$

where $\Lambda^t\Lambda$ is viewed here as an operator on $\ell^2(J)$ and $\| \ \|_2$ denotes the norm on this Hilbert space. Letting $\varepsilon \to 0$, this shows that $\|\Lambda\|^2 \geq \lambda^{-1}$. Since we also have $\|\Lambda\|^2 \leq \lambda^{-1}$ (see Section 2.7), we get $\|\Lambda\|^2 = \lambda^{-1}$.

Conversely, assume $\|\Lambda\|^2 = \lambda^{-1}$. Let $\Phi = \lambda T^{-1}\Lambda^t\Lambda T$, viewed as a $J \times J$ matrix with non-negative entries, where T is the diagonal matrix with entries $\vec{t} = (t_j)_j$. Note that Φ defines a unital positive linear map from the semifinite von Neumann algebra $P = \ell^\infty J$ into itself. We endow P with the n.s.f. trace Tr which on a finitely supported $a = (a_j)_j \in \ell^\infty J = P$ is given by

$$\mathrm{Tr}(a) = \sum_j a_j t_j^2.$$

The λ-Markovianity condition for $(\Lambda, \vec{t})$ trivially implies $\mathrm{Tr}(\Phi(b)) = \mathrm{Tr}(b)$ for all $b \in P$. By Kadison's inequality, this also implies $\|\Phi(b)\|_{2,\mathrm{Tr}} \leq \|b\|_{2,\mathrm{Tr}}$ for all $b \in L^2(P, \mathrm{Tr})$.

Since $\|\lambda\Lambda^t\Lambda\| = 1$, it follows that for any $\delta > 0$, there exists $F_0 \subset J$ finite such that $T_0 =_{F_0} (\lambda\Lambda^t\Lambda)_{F_0}$ satisfies $1 \geq \|T_0\| \geq 1 - \delta^2/2$. Since T_0 is a symmetric $F_0 \times F_0$ matrix with non-negative entries, by the classic Perron–Frobenius theorem there exists $b_0 \in P_+$ supported by F_0 such that

$$\langle b_0, b_0\rangle = 1 \quad \text{and} \quad T_0 b_0 = \|T_0\| b_0 \geq (1 - \delta^2/2) b_0.$$

Thus, $\lambda\Lambda^t\Lambda(b_0) \geq (1 - \delta^2/2) b_0$.

Denote $b = T^{-1}(b_0) \in P_+$ and notice that $\|b\|_{2,\mathrm{Tr}}^2 = 1$. Also, we have

$$\begin{aligned}\|\Phi(b) - b\|_{2,\mathrm{Tr}}^2 &\le 2 - 2\,\mathrm{Tr}(\Phi(b)b)\\ &= 2 - 2\langle \lambda \Lambda^t \Lambda(b_0), b_0\rangle \le 2 - 2(1 - \delta^2/2) = \delta^2.\end{aligned}$$

Thus, Φ is a Tr-preserving unital c.p. map on P and $b \in L^2(P, \mathrm{Tr})_+$ is a unit vector satisfying

$$\|\Phi(b) - b\|_{2,\mathrm{Tr}} \le \delta, \quad \|\Phi(b)\|_{2,\mathrm{Tr}} \ge 1 - \delta^2/2.$$

By [44, Section A.2], when $\delta < 10^{-4}$ this implies there exists a spectral projection e of b, corresponding to an interval (c, ∞) for some $c > 0$, such that

$$\|\Phi(e) - e\|_{2,\mathrm{Tr}} < \delta^{1/4}\|e\|_{2,\mathrm{Tr}}.$$

The latter inequality implies $\|(1 - e)\Phi(e)\|_{2,\mathrm{Tr}} < \delta^{1/4}\|e\|_{2,\mathrm{Tr}}$. If one denotes by $F \subset J$ the support of $e \in P = \ell^\infty J$, this is easily seen to imply

$$\sum_{j \in \partial F} t_j^2 < \lambda^{-4}\delta^{1/4} \sum_{j \in F} t_j^2.$$

Thus, if one chooses $\delta \le (\lambda^4\varepsilon^2)^4$ at the beginning, then F satisfies (4.2.1) for the given $\varepsilon > 0$. ∎

4.3 Definition. 1° Let $N \subset M$ be an extremal inclusion of II_1 factors with finite index and $\mathcal{N}^{\mathrm{st}} \subset^{\mathcal{E}^{\mathrm{st}}} \mathcal{M}^{\mathrm{st}}$ its standard representation. $N \subset M$ is *injective* if there exists a norm-one projection $\Phi\colon \mathcal{M}^{\mathrm{st}} \to M$ such that $\Phi(\mathcal{N}^{\mathrm{st}}) = N$. It is easy to see that this is equivalent to the condition $\mathcal{E}^{\mathrm{st}} \circ \Phi = \Phi \circ \mathcal{E}^{\mathrm{st}}$. We then also say that Φ is a norm-one projection of $\mathcal{N}^{\mathrm{st}} \subset^{\mathcal{E}^{\mathrm{st}}} \mathcal{M}^{\mathrm{st}}$ onto $N \subset M$.

$N \subset M$ is *amenable* if the standard representation has an $(N \subset M)$-*hypertrace*, i.e. a state φ on $\mathcal{M}^{\mathrm{st}}$ that has M in its centralizer and is $\mathcal{E}^{\mathrm{st}}$-invariant.

2° A standard λ-lattice $\mathcal{G}$ (resp. standard invariant $\mathcal{G}_{N\subset M}$ of an extremal subfactor $N \subset M$) is *amenable* if its standard graph $\Gamma_{\mathcal{G}}$ satisfies the Kesten-type condition $\|\Gamma_{\mathcal{G}}\|^2 = \lambda^{-1}$ (resp. $\|\Gamma_{N\subset M}\|^2 = [M : N]$).

4.4 Proposition. *Let $(N \subset M) \subset (\mathcal{N} \subset^{\mathcal{E}} \mathcal{M})$ be a non-degenerate commuting square embedding of $N \subset M$ into a W^*-inclusion with expectation. There exists a norm one projection $\Phi\colon \mathcal{M} \to M$ such that $\Phi \circ \mathcal{E} = E_N \circ \Phi$ iff there exists a state φ on $\mathcal{M}$ that's $\mathcal{E}$-invariant and has M in its centralizer.*

Proof. If $\Phi\colon \mathcal{M}^{\mathrm{st}} \to M$ is a norm one projection commuting with $\mathcal{E}$ then it is M-bimodular by Tomiyama's theorem and thus for any $x \in M$, $X \in \mathcal{M}$, the state $\varphi = \tau \circ \Phi$

satisfies

$$\begin{aligned}\varphi(xX) &= \tau(\Phi(xX)) = \tau(x\Phi(X)) \\ &= \tau(\Phi(X)x) = \tau(\Phi(Xx) = \varphi(Xx).\end{aligned}$$

Also, $\varphi(X) = \tau(\Phi(X)) = \tau(E_N(\Phi(X)) = \tau(\Phi(\mathcal{E}(X)) = \varphi(\mathcal{E}(X))$.

Conversely, if φ is a state on $\mathcal{M}$ that has M in its centralizer and commutes with $\mathcal{E}$ then one constructs a conditional expectation Φ from $\mathcal{M}$ onto M in the usual way: if $X \in \mathcal{M}$, then $\Phi(X)$ is the unique element in M with the property that $\tau(\Phi(X)x) = \varphi(Xx)$ for all $x \in M$ (see e.g. [40, Proposition 3.2.2]). Since $\varphi = \varphi \circ \mathcal{E}$, it follows that

$$\begin{aligned}\Phi(\mathcal{E}(X)) &= \varphi(\mathcal{E}(X)\cdot) = \varphi(\mathcal{E}(\mathcal{E}(X)\cdot)) = \varphi(\mathcal{E}(X)\mathcal{E}(\cdot) \\ &= \varphi(X\mathcal{E}(\cdot)) = \varphi(XE_N(\cdot)) = E_N(\Phi(X)).\end{aligned}$$ ■

4.5 Theorem. *Let $N \subset M$ be an extremal inclusion of separable II_1 factors with finite index. The following conditions are equivalent:*

(1) *$N \subset M$ is amenable.*

(1′) *$N \subset M$ is injective.*

(2) *Any smooth representation $(N \subset M) \subset (\mathcal{N} \subset^{\mathcal{E}} \mathcal{M})$ has an $(N \subset M)$-hypertrace, i.e. an $\mathcal{E}$-invariant state on $\mathcal{M}$ that has M in its centralizer.*

(2′) *Any smooth representation $(N \subset M) \subset (\mathcal{N} \subset^{\mathcal{E}} \mathcal{M})$ admits a norm one projection of $\mathcal{M}$ onto M that commutes with $\mathcal{E}$.*

(3) *N, M are amenable II_1 factors (equivalently, $N \simeq R \simeq M$) and its standard invariant $\mathcal{G}_{N\subset M}$ is amenable (i.e. $\|\Gamma_{N\subset M}\|^2 = [M : N]$).*

(4) *$N \simeq R \simeq M$ and $\|\Lambda^{u,f}_{N\subset M}\|^2 = [M : N]$, where $\Lambda^{u,f}_{N\subset M}$ denotes the inclusion graph of the universal exact finite representation of $N \subset M$ (arising as direct sum of $\mathcal{N}_P \subset \mathcal{M}_P$, with $\mathcal{M}_P = \oplus_j \mathcal{B}(\mathcal{H}_j)$, $\dim({}_M\mathcal{H}_{j\,P}) < \infty$ for all j).*

(5) *$N \simeq R \simeq M$ and $\|\Lambda\|^2 = [M : N]$ for any connected component Λ of $\Lambda^{u,f}_{N\subset M}$.*

(6) *Given any finite set $F \subset M$ and any $\varepsilon > 0$, there exists a subfactor of finite index $P \subset N$ such that $F \subset_\varepsilon P' \cap M$.*

(7) *There exists a sequence of subfactors with finite index $M \supset N \supset P_1 \supset P_2 \cdots$ such that $P_n' \cap M \nearrow M$.*

(8) *There exists an $(N \subset M)$-compatible tunnel $M \supset N \supset P_1 \supset P_2 \cdots$ (in the sense of* [50, Definition 2.1]*) such that $P_n' \cap M \nearrow M$.*

(9) *$N \subset M$ is isomorphic to a model hyperfinite subfactor $N(\mathcal{G}) \subset M(\mathcal{G})$, obtained as an inductive limit of higher relative commutants (in N and resp. M) of an $(N \subset M)$-compatible tunnel $M \supset N \supset P_1 \supset P_2 \cdots$, whose choice is dictated by the standard invariant $\mathcal{G} = \mathcal{G}_{N\subset M}$.*

Proof. By Theorem 4.2, we have (1) $\Leftrightarrow$ (1′), (2) $\Leftrightarrow$ (2′). One obviously has (2) $\Rightarrow$ (1) while (1) $\Rightarrow$ (2) follows from the implication (2) $\Rightarrow$ (1) of [44, Theorem 7.1]; see page 720 for the proof.

The implications (8) $\Rightarrow$ (7) $\Rightarrow$ (6) are trivial and (6) $\Rightarrow$ (1) is a consequence of (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) in [44, Theorem 7.1]; see page 720 for the proof.

Since $\|\Gamma_{N\subset M}\| = \|\Lambda_{\mathcal{N}^{\mathrm{st}}\subset\mathcal{M}^{\mathrm{st}}}\|$, the implication (1) $\Rightarrow$ (3) is a consequence of the general Theorem 5.3 hereafter (note that it is also a consequence of [40, Theorem 4.4.1] combined with [32, Theorem 3.1 or 3.3]).

(3) $\Rightarrow$ (8). By the implication (2) $\Rightarrow$ (3) in [44, Theorem 7.1] (see top of page 719 for the proof), one first gets that for any $F \subset M$ finite and any $\varepsilon > 0$, there exists an $(N \subset M)$-compatible subfactor $P \subset N$ and a finite dimensional subfactor $Q_0 \subset P$ such that $F \subset_\varepsilon (P' \cap M) \vee Q_0$. Since $\Gamma_{N\subset M}$ amenable implies $\Gamma_{P\subset M}$ amenable (cf. [44, Corollary 6.6 (ii)]; see also [50, Proposition 2.6] for an alternative proof), we can apply this recursively to get a tunnel of $(N \subset M)$-compatible subfactors $M \supset N \supset P_1 \supset P_2 \supset \cdots$ and a sequence of commuting finite dimensional factors $Q_1, Q_2, \ldots, Q_n \subset \cap_{i=1}^n P_i$ such that

$$(P_n' \cap M) \vee Q_1 \vee \cdots \vee Q_n \nearrow M.$$

This implies $(P_n' \cap N) \vee Q_1 \vee \cdots \vee Q_n \nearrow N$ as well. It also implies that $(N \subset M)$ (or any other inclusion of hyperfinite factors with amenable graph) splits off R. This means that if we denote $M^0 := \vee_n (P_n' \cap M)$ and $N^0 := \vee_n (P_n' \cap N)$, then

$$(N \subset M) \simeq (N^0 \subset M^0)\overline{\otimes} R.$$

But $(N^0 \subset M^0) \simeq (N^0 \subset M^0)\overline{\otimes} R$ (because $N^0 \subset M^0$ are hyperfinite with same standard graph as $N \subset M$, thus amenable!). Hence, $(N \subset M) \simeq (N^0 \subset M^0)$, which implies the approximation by higher relative commutants of $(N \subset M) = (N^0 \subset M^0)$-compatible tunnels, required in (8).

This shows that (1)–(3), (6)–(8) are equivalent.

We further have (2) $\Rightarrow$ (5) by Theorem 5.3 below. The implication (5) $\Rightarrow$ (4) is trivial and (4) $\Rightarrow$ (3) by [44, Theorem 6.5]. Thus, (1)–(8) are equivalent.

The equivalence of (8) and (9) is proved in [44, Remark 7.2.1]. But let us give here a more elegant argument, based on [50, Theorem 2.9] and its proof, which allows deducing (9) directly from (3).

Thus, we assume $\mathcal{G}$ is an amenable standard λ-lattice with standard graph $\Gamma = \Gamma_{\mathcal{G}}$ and canonical weights $\vec{v}$. Thus, $\|\Gamma\|^2 = \lambda^{-1}$ and $\Gamma^t\Gamma\vec{v} = \lambda^{-1}\vec{v}$. By Theorem 4.2, this condition is equivalent to the Følner property (4.2.1) of the Markov weighted graph $(\Gamma, \vec{v})$. The proof of Theorem 2.9 in [50] shows that, given any (separable) subfactor $N \subset M$ with standard graph $\mathcal{G}_{N \subset M} = \mathcal{G}$, there exists a choice of $(N \subset M)$-compatible tunnel $M \supset N \supset P_1 \supset P_2 \supset \cdots$, such that if one denotes

$$(Q \subset R) = (\overline{\cup_n P_n' \cap N}) \subset \overline{\cup_n P_n' \cap M}),$$

then $[R : Q] = [M : N]$ and the higher relative commutants of $Q \subset R$ and respectively $N \subset M$ coincide, in fact

$$N' \cap M_n = Q' \cap M_n = Q' \cap R_n, \quad \forall n. \tag{4.5.1}$$

In particular, $\mathcal{G}_{Q \subset R} = \mathcal{G} = \mathcal{G}_{N \subset M}$.

In order to satisfy condition (4.5.1), the $(N \subset M)$-compatible tunnel $M \supset N \supset P_1 \supset P_2 \supset \cdots$ that one takes depends on two types of choices, at each step n.

Thus, if $M \supset N \supset \cdots \supset P_n$ have been already chosen, one next takes P_{n+1} to be a downward basic construction

$$P_{n+1} \subset P_n \simeq P_n q \subset q M_m q,$$

with $m \geq 1$ and $q \in P_n' \cap M_m$ appropriately chosen. The choice of P_{n+1} is up to conjugacy by a unitary in P_n, but the way $m \geq 1$ and $q \in P_n' \cap M_m$ are chosen depends only on the properties of $\mathcal{G}$, more precisely on the Følner-constants for $(\Gamma_{\mathcal{G}}, \vec{v})$. It is important to note that this second type of choice, which depends only on $\mathcal{G}$, can be taken the same for any $N \subset M$.

We call $M \supset N \supset P_1 \supset \cdots$ a *$\mathcal{G}$-compatible tunnel*, and for each given $\mathcal{G}$ we make once for all a choice for it, which we call the *model $\mathcal{G}$-compatible tunnel*.

In particular, the isomorphism class of $Q \subset R$ is completely determined by $\mathcal{G}$ and $Q \subset R$ itself admits a model $\mathcal{G}$-compatible tunnel $R \supset Q \supset P_1 \supset P_2 \supset \cdots$ such that

$$P_n' \cap R \nearrow R, \quad P_n' \cap Q \nearrow Q.$$

We denote this subfactor by $N(\mathcal{G}) \subset M(\mathcal{G})$, calling it the *model subfactor* associated with the amenable standard λ-lattice $\mathcal{G}$.

Note that in case $\mathcal{G}$ is both amenable and has ergodic core in the sense of [40], i.e. when $\mathcal{G} = (A_{ij})_{j \geq i}$ is so that $Q = A_{1\infty} \subset A_{0\infty} = R$ are factors, then one can take the model $\mathcal{G}$-compatible tunnel to be the Jones tunnel.

Now, given any hyperfinite subfactor $N \subset M$ with amenable standard graph $\mathcal{G}_{N \subset M} = \mathcal{G}$, then exactly the same proof as [39, Theorem 4.1], shows that there exists a choice of the model $\mathcal{G}$-compatible tunnel $M \supset N \supset P_1 \supset P_2 \supset \cdots$ such that if one denotes $R_0 = \cap_n P_n$, then $(P_n' \cap M) \vee R_0 \nearrow M$. Thus,

$$(N \subset M) \simeq (N(\mathcal{G}) \subset M(\mathcal{G})) \overline{\otimes} R_0.$$

Since $N(\mathcal{G}) \subset M(\mathcal{G})$ splits off R_0, this shows that

$$(N \subset M) \simeq (N(\mathcal{G}) \subset M(\mathcal{G})). \qquad \blacksquare$$

4.6 Remarks. 1° It is shown in [44, Theorem 7.5] that if an extremal subfactor $N \subset M$ is amenable, in the sense of Definition 4.3.2°, then any extremal subfactor $Q \subset P$ that can be embedded into it as a commuting square (not necessarily non-degenerate!) is amenable as well. In other words, if $N \simeq R \simeq M$ and $\|\Gamma_{N\subset M}\|^2 = [M : N]$, then any commuting square sub-inclusion $Q \subset P$ of $N \subset M$ satisfies

$$\|\Gamma_{Q\subset P}\|^2 = [P : Q]$$

(in fact, the sub-inclusion $Q \subset P$ does not even need to be extremal, in general one has $\mathrm{Ind}_{\min}(Q \subset P) = \|\Gamma_{Q\subset P}\|^2$).

This hereditary property is somewhat surprising. A key fact that allows the proof of this result is [44, Lemma 7.3], which shows that if $Q \subset P$ is any finite index subfactor and B is an arbitrary tracial von Neumann algebra that contains P, then the *-subalgebra $B_0 \subset B$ generated by P and $\cup_k Q_k' \cap B$ is equal to $\mathrm{sp} P(\cup_k Q_k' \cap B)R$, where $P \supset Q \supset Q_1 \supset \cdots$ is a tunnel for $Q \subset P$ and $R = \overline{\cup_k Q_k' \cap P}$. This easily implies that ${}_P L^2(B_0)_P$ is contained in $(\oplus_{k\in K^{\mathrm{st}}} \mathcal{H}_k)^{\oplus\infty}$, where $\mathcal{H}_k$ is the list of irreducible Hilbert bimodules in the standard representation $\mathcal{Q}^{\mathrm{st}} \subset \mathcal{P}^{\mathrm{st}}$ of $Q \subset P$, allowing to show that the $(Q-P)$-bimodules and $(P-P)$-bimodules in $L^2(M_\infty) = \overline{\cup_n L^2 M_n}$ give rise to a multiple of the standard representation of $Q \subset P$. The amenability of $N \subset M$ is then used to prove that $P' \cap M_\infty$ is big enough so that its commutant in M_∞ is "locally" approximately equal to P, a fact that allows constructing the $(Q \subset P)$-hypertrace on $\mathcal{Q}^{\mathrm{st}} \subset \mathcal{P}^{\mathrm{st}}$.

2° Theorem 7.6 of [44] states that for an extremal subfactor $N \subset M$ the following three conditions are equivalent (formulated as such in that theorem):

(1) $N \subset M$ amenable;

(2) For all $\varepsilon > 0$, there exists $P \supset M$ hyperfinite such that $\|\Lambda_{M'\cap P\subset N'\cap P}\|^2 \geq [M : N] - \varepsilon$;

(3) $\|\Lambda^{\mathrm{u,rf}}_{N\subset M}\|^2 = [M : N]$,

where $\Lambda^{\mathrm{u,rf}}_{N\subset M}$ is the inclusion graph of the "universal right-finite exact representation" $\mathcal{N}^{\mathrm{u,rf}} \subset \mathcal{M}^{\mathrm{u,rf}}$ of $N \subset M$, obtained as the direct sum of $\mathcal{N}_P \subset \mathcal{M}_P$, with P II_1 factors that contain M as an irreducible subfactor, see Section 6.1.5 below for more about this sub-representation of the universal exact representation of a subfactor.

However, while the proofs of (1) $\Leftrightarrow$ (2) and (1) $\Rightarrow$ (3) are correct in [44], the proof of (3) $\Rightarrow$ (2) in [44, p. 724] uses the fact that all P in this construction are hyperfinite. So in order for that proof to work, one needs to modify the statement of Theorem 7.6 in [44] by replacing $\mathcal{N}^{\mathrm{u,rf}} \subset \mathcal{M}^{\mathrm{u,rf}}$ in condition (3) with $\mathcal{N}^{\mathrm{u,hrf}} \subset \mathcal{M}^{\mathrm{u,hrf}}$, defined similarly but with all P taken $\simeq R$.

On the other hand, one can prove the equivalence of (1) above with the following:

(2′) For all $\varepsilon > 0$, there exists $P \supset M$ such that $\|\Lambda_{M' \cap P \subset N' \cap P}\|^2 \geq [M : N] - \varepsilon$ and $N, M \simeq R$;

(3′) $\|\Lambda^{\mathrm{u,rf}}_{N \subset M}\|^2 = [M : N]$ and $N, M \simeq R$.

We will detail the proof in a future paper.

3° In [44] there are two other interesting characterizations of amenability for a subfactor $N \subset M$, that we did not include in Theorem 4.5 above. Thus, it is shown in [44, Theorem 7.1] that $N \subset M$ is amenable iff its *symmetric enveloping* II_1 *factor* $M \overset{e_N}{\boxtimes} M^{\mathrm{op}}$ is amenable (so $M \overset{e_N}{\boxtimes} M^{\mathrm{op}} \simeq R$, by [5]). And it is shown in [44, Theorem 8.1] that $N \subset M$ is amenable iff the C*-algebra generated in $\mathcal{B}(L^2 M)$ by M, M^{op} and e_N is simple (this is an "Effros–Lance-type" characterization of amenability of $N \subset M$).

4° Using Theorem 4.5 it is immediate to see that if $N \subset M$ is amenable then any smooth commuting square embedding of $N \subset M$ into an arbitrary W^*-inclusion $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ (with $\mathcal{N}, \mathcal{M}$ not necessarily atomic) has an $(N \subset M)$-hypertrace, i.e. an $\mathcal{E}$-invariant state on $\mathcal{M}$ that has M in its centralizer. Equivalently, any smooth commuting square embedding $(N \subset M) \subset (\mathcal{N} \subset^{\mathcal{E}} \mathcal{M})$, admits a $\mathcal{E}$-invariant norm-one projection of $\mathcal{M}$ onto M. This condition actually appears as one of the equivalences in [44, Theorem 7.1].

5. Weak amenability for subfactors

5.1 Definition. An extremal subfactor of finite index $N \subset M$ is *weakly amenable* (resp. *weakly injective*) if it admits a tracial representation $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ that has an $(N \subset M)$-hypertrace (resp. a norm one projection).

5.2 Proposition. 1° *If $N \subset M$ is amenable then it is weakly amenable.*

2° *If $(Q \subset P) \subset (N \subset M)$ is a non-degenerate commuting square embedding of* II_1 *factors and $N \subset M$ is weakly amenable, then $Q \subset P$ is weakly amenable. If in addition $[M : P] < \infty$ then $Q \subset P$ weakly amenable implies $N \subset M$ weakly amenable.*

3° *Weak amenability is a stable isomorphism invariant: If $N \subset M$ is weakly amenable, then $(N \subset M)^t$ for all $t > 0$, and $(N \subset M)\overline{\otimes} R$ are weakly amenable.*

Proof. Clear by the definitions. ∎

We next show that weak amenability/injectivity, which follow equivalent by Proposition 4.4, are also equivalent to a Connes–Følner-type condition, and they imply the index of the subfactor must be equal to the square norm of the inclusion bipartite graph of the representation, thus belonging to the set $\mathbb{E}^2$.

5.3 Theorem. *Let $N \subset M$ be an extremal inclusion of II_1 factors and $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ a tracial representation. The following conditions are equivalent:*

(1) *There exists a norm one projection of $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$ onto $N \subset M$.*

(2) *There exists a $(N \subset M)$-hypertrace on $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$.*

(3) *Given any finite set $F \subset \mathcal{U}(M)$ and any $\varepsilon > 0$, there exists a finite rank projection $p \in \mathcal{N}$ such that*

$$\sum_{u \in F} \|upu^* - p\|_{2,\mathrm{Tr}} < \varepsilon \|p\|_{2,\mathrm{Tr}}.$$

Moreover, if the above conditions hold for $(N \subset M) \subset (\mathcal{N} \subset^{\mathcal{E}} \mathcal{M})$ and we denote by

$$(M_{i-1} \subset M_i \subset_{e_i} M_{i+1}) \subset (\mathcal{M}_{i-1} \subset^{\mathcal{E}_i} \mathcal{M}_i \subset_{e_i} \mathcal{M}_{i+1}), \quad i \in \mathbb{Z},$$

the tower-tunnel of representations associated with it, then conditions (1)–(3) *hold for*

$$(M_i \subset M_j) \subset (\mathcal{M}_i \subset^{\mathcal{E}_{ij}} \mathcal{M}_j) \quad \textit{for any } j > i,$$

where $M_0 = M$, $M_{-1} = N$, $\mathcal{M}_0 = \mathcal{M}$, $\mathcal{M}_{-1} = \mathcal{N}$ and $\mathcal{E}_{ij} = \mathcal{E}_{i+1} \circ \cdots \circ \mathcal{E}_j$.

Proof. We already proved the equivalence of conditions (1) and (2) in Proposition 4.4.

Assume (2) holds true and let φ be an $\mathcal{E}$-invariant state on $\mathcal{M}$ that has M in its centralizer.

Let $F = \{u_1, \dots, u_n\}$. Denote

$$\begin{aligned} \mathcal{L} = \{(&\psi - \psi \circ \mathcal{E},\ \psi - \psi(u_1^* \cdot u_1), \\ &\psi - \psi(u_2^* \cdot u_2), \dots, \psi - \psi(u_n^* \cdot u_n)) \in (\mathcal{M}_*)^{n+1} \mid \psi \text{ a state in } \mathcal{M}_*\}. \end{aligned}$$

Note that $\mathcal{L}$ is a bounded, convex subset in $(\mathcal{M}^*)^{n+1} = (\mathcal{M}^{n+1})^*$ and since the states $\varphi \in \mathcal{M}_*$ are $\sigma(\mathcal{M}^*, \mathcal{M})$ dense in $S(\mathcal{M})$ it follows that the $\sigma((\mathcal{M}^*)^{n+1}, \mathcal{M}^{n+1})$ closure $\bar{\mathcal{L}}$ of $\mathcal{L}$ contains all $(n+1)$-tuples

$$(\psi - \psi \circ \mathcal{E}, \psi - \psi(u_1^* \cdot u_1), \dots, \psi - \psi(u_n^* \cdot u_n))$$

with $\psi \in S(\mathcal{M})$.

Taking $\psi = \varphi$, it follows that $\bar{\mathcal{L}}$ contains

$$(\varphi - \varphi \circ \mathcal{E}, \varphi - \varphi(u_1^* \cdot u_1), \dots, \varphi - \varphi(u_n^* \cdot u_n)).$$

But $\varphi(u \cdot u^*) = \varphi$ for all $u \in \mathcal{U}(M)$, so in particular $\varphi - \varphi(u_i^* \cdot u_i) = 0, i = 1, 2, \dots, n$. Since we also have $\varphi \circ \mathcal{E} = \varphi$, it follows that

$$(0, \dots, 0) = (\varphi - \varphi \circ \mathcal{E}, \varphi - \varphi(u_1^* \cdot u_1), \dots, \varphi - \varphi(u_n^* \cdot u_n)) \in \bar{\mathcal{L}}.$$

But since both $(0, \dots, 0)$ and $\mathcal{L}$ are in $(\mathcal{M}_*)^{n+1}$ and since the dual of $(\mathcal{M}_*)^{n+1}$ is $\mathcal{M}^{n+1}$, it follows that the $\sigma((\mathcal{M}_*)^{n+1}, \mathcal{M}^{n+1})$ closure of $\mathcal{L}$ in $(\mathcal{M}_*)^{n+1}$ is equal to the norm closure of $\mathcal{L}$ and thus, $(0, \dots, 0)$ is norm adherent to $\mathcal{L}$.

It follows that for all $\delta > 0$, there exists a state $\psi_0 \in \mathcal{M}_*$ such that

$$\|\psi_0 - \psi_0 \circ \mathcal{E}\| < \delta/3, \quad \|\psi_0 - \psi_0(u_i^* \cdot u_i)\| < \delta/3,\ 1 \le i \le n.$$

By replacing ψ_0 with $\psi = \psi_0 \circ \mathcal{E}$, it follows that there exists a state $\psi \in \mathcal{M}_*$ satisfying

$$\psi = \psi \circ \mathcal{E}, \quad \|\psi - \psi(u_i^* \cdot u_i)\| < \delta,\ 1 \le i \le n.$$

Since $\mathcal{M}_* = L^1(\mathcal{M}, \mathrm{Tr})$ and $L^1(\mathcal{M}, \mathrm{Tr}) \cap \mathcal{M}$ is dense in $L^1(\mathcal{M}, \mathrm{Tr})$, it follows that we may in addition assume there exists $b \in L^1(\mathcal{M}, \mathrm{Tr}) \cap \mathcal{M}_+$ such that $\psi = \mathrm{Tr}(\cdot b)$.

Thus,

$$\mathrm{Tr}(b) = 1, \quad \mathrm{Tr}(Xb) = \mathrm{Tr}(\mathcal{E}(X)b), \quad \forall X \in \mathcal{M}$$

and

$$\|\mathrm{Tr}(\cdot b) - \mathrm{Tr}((u_i^* \cdot u_i)b)\| < \delta, \quad 1 \le i \le n.$$

Since $\mathrm{Tr}(\mathcal{E}(X)b) = \mathrm{Tr}(X(\mathcal{E}(b)))$ and $\mathrm{Tr}((u_i^* \cdot u_i)b) = \mathrm{Tr}(\cdot u_i b u_i^*)$, it follows that

$$b = \mathcal{E}(b) \quad \text{and} \quad \|u_i b u_i^* - b\|_{1,\mathrm{Tr}} < \delta,\ 1 \le i \le n.$$

The first relation shows that $b \in \mathcal{N}_+$ and the Powers–Størmer inequality [55] applied to the second shows that $a = b^{1/2} \in \mathcal{N}_+$ satisfies

$$\|a\|_{2,\mathrm{Tr}} = 1 \quad \text{and} \quad \|u_i a u_i^* - a\|_{2,\mathrm{Tr}} < \delta^{1/2},\ \forall i.$$

The Connes–Namioka trick (see [5, Theorem 1.2.1] or [6, Section 2.5]) then yields a spectral projection p of a such that

$$\|u_i p u_i^* - p\|_{2,\mathrm{Tr}} < \delta^{1/2} \|p\|_{2,\mathrm{Tr}},$$

while $p \in \mathcal{N}$ (because $a \in \mathcal{N}$). Taking $\delta = \varepsilon^2/|F|^2$ ends the proof or (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (2). Assuming (3), it follows that for each $F \subset \mathcal{U}(M)$ finite, there exists $p_F \subset \mathcal{P}(\mathcal{N})$ finite rank such that

$$\sum_{u \in F} \|u p_F u^* - p_F\|_{2,\mathrm{Tr}} < \frac{1}{|F|} \|p\|_{2,\mathrm{Tr}}.$$

Define φ as $\varphi(X) = \text{Lim}_F \text{Tr}(Xp_F)/\text{Tr}(p_F)$ for all $X \in \mathcal{M}$, where Lim_F is a Banach limit over an ultrafilter majorizing the filter of finite subsets $F \subset \mathcal{U}(M)$. It is then immediate to see that φ this way defined has all $\mathcal{U}(M)$ (thus all M) in its centralizer. Moreover, since the states $\text{Tr}(\cdot\, p_F)/\text{Tr}(p_F) \in S(\mathcal{M})$ are $\mathcal{E}$-invariant, φ follows $\mathcal{E}$-invariant.

The last part is trivial and we leave it as an exercise. ∎

5.4 Theorem. *Let $N \subset M$ be an extremal inclusion of type II_1 factors. If $(N \subset M) \subset (\mathcal{N} \subset^{\mathcal{E}} \mathcal{M})$ is a tracial representation for which there exists a norm-one projection onto $(N \subset M)$, then $\|\Lambda_{\mathcal{N}\subset\mathcal{M}}\|^2 = [M : N]$.*

Proof. By Lemma 2.7.1, we have

$$\|\Lambda_{\mathcal{N}\subset\mathcal{M}}\|^2 \leq \text{Ind}(\mathcal{E}) = [M : N],$$

so we only need to prove the opposite inequality.

Let $e = e_{-1} \in M$ be a Jones projection, $\mathcal{N}_1 \overset{\text{def}}{=} \{e\}' \cap \mathcal{N}$ and $\mathcal{E}_{-1}$ the conditional expectation of $\mathcal{N}$ onto $\mathcal{N}_{-1}$ implemented by e (see e.g., [40, Proposition 2.2.4]).

Let $\varepsilon > 0$. By the relative Dixmier property for $N \subset M$ (see [44, Section A.1] or [45, Theorem 1.1]), there exist unitary elements $u_1, \dots, u_n \in N$ such that

$$\left\| \frac{1}{n} \sum_{i=1}^{n} u_i e u_i^* - \lambda 1 \right\| < \lambda\varepsilon/2.$$

By Theorem 5.3, given any $\delta > 0$, there exists a finite rank projection $p \in \mathcal{N}_1$ such that $\|[u_i, p]\|_{2,\text{Tr}} < \delta \|p\|_{2,\text{Tr}}$. Let $w_i = pu_i p \in p\mathcal{N}p$. Then

$$\begin{aligned}\|w_i w_i^* - p\|_{2,\text{Tr}} &= \|pu_i pu_i^* p - p\|_{2,\text{Tr}} = \|p(u_i pu_i^* - p)p\|_{2,\text{Tr}} \\ &\leq \|u_i pu_i^* - p\|_{2,\text{Tr}} = \|[u_i, p]\|_{2,\text{Tr}} < \delta\|p\|_{2,\text{Tr}}.\end{aligned}$$

A standard perturbation argument (cf. e.g., [40, Lemma A.2.1]) then shows that there exist unitary elements $v_i \in p\mathcal{N}p$ such that $\|v_i - w_i\|_{2,\text{Tr}} \leq f(\delta)\|p\|_{2,\text{Tr}}$, with $f(\delta)$ a constant depending only on δ and satisfying $f(\delta) \to 0$ as $\delta \to 0$. This shows that for any $\delta' > 0$ there exists a finite rank projection $p \in \mathcal{N}_1$, $0 \neq \text{Tr}\, p < \infty$, and unitary elements $v_1, \dots, v_n \in p\mathcal{N}p$ such that $\|pu_i - v_i\|_{2,\text{Tr}} < \delta'\|p\|_{2,\text{Tr}}$ for all i.

So if we choose $\delta' > 0$ such that $\delta' < \lambda\varepsilon/4$, then $p \in \mathcal{P}(\mathcal{N}_1)$, $v_i \in \mathcal{U}(p\mathcal{N}p)$ satisfy

$$\begin{aligned}\left\| \frac{1}{n} \sum_{i=1}^{n} v_i (ep) v_i^* - \lambda p \right\|_{2,\text{Tr}} &\leq \max_i \|pu_i - v_i\|_{2,\text{Tr}} + \left\| \frac{1}{n} \sum_{i=1}^{n} pu_i e v_i^* - \lambda p \right\|_{2,\text{Tr}} \\ &\leq 2 \max_i \|pu_i - v_i\|_{2,\text{Tr}} + \left\| p\left(\frac{1}{n} \sum_{i=1}^{n} u_i e u_i^* - \lambda 1 \right) p \right\|_{2,\text{Tr}}\end{aligned}$$

$$\leq 2 \max_i \|pu_i - v_i\|_{2,\mathrm{Tr}} + \left\| p\left(\frac{1}{n}\sum_{i=1}^{n} u_i e u_i^* - \lambda 1\right)\right\| \|p\|_{2,\mathrm{Tr}}$$
$$\leq (\lambda\varepsilon/2)\|p\|_{2,\mathrm{Tr}} + (\lambda\varepsilon/2)\|p\|_{2,\mathrm{Tr}} = \lambda\varepsilon\|p\|_{2,\mathrm{Tr}}.$$

It follows that $p\mathcal{N}p \subset^{\mathcal{E}'} p\mathcal{M}p$ is a finite dimensional W^*-inclusion with trace state $\tau = \mathrm{Tr}(p)^{-1}\,\mathrm{Tr}$ and τ-preserving expectation $\mathcal{E}' = \mathcal{E}(p \cdot p)$, which has a projection $e' = ep$ satisfying $\mathcal{E}'(e') = \lambda p = \lambda 1_{p\mathcal{M}p}$ and such that

$$\left\| \frac{1}{n}\sum_{i=1}^{n} v_i e' v_i^* - \lambda 1 \right\|_2 < \lambda\varepsilon$$

for some unitary elements $v_i \in \mathcal{U}(p\mathcal{N}p)$. By Theorem 4.2 and the estimate at the bottom of page 79 in [31], it follows that if $H(p\mathcal{M}p \mid p\mathcal{N}p)$ denotes as usual the Connes–Størmer relative entropy, then

$$\text{(5.4.1)} \quad \begin{aligned} H(p\mathcal{M}p \mid p\mathcal{N}p) &\geq (1+\varepsilon^{1/2})^{-1}\ln\lambda^{-1} - (1+\varepsilon^{1/2})\lambda^{-1}\eta(\lambda\varepsilon) \\ &= (1-\varepsilon)(1+\varepsilon^{1/2})^{-1}\ln[M:N] - (1+\varepsilon^{1/2})^{-1}\eta(\varepsilon), \end{aligned}$$

where η denotes here the function on the positive reals $\eta(t) = -t\ln t$, $t > 0$.

But by [32, Theorem 2.6], if we denote by Λ' the inclusion matrix for $p\mathcal{N}p \subset p\mathcal{M}p$ (which is thus a restriction of the inclusion matrix $\Lambda_{\mathcal{N}\subset\mathcal{M}}$), then

$$\text{(5.4.2)} \qquad \|\Lambda'\|^2 \geq \exp(H(p\mathcal{M}p \mid p\mathcal{N}p)).$$

Finally, since $\|\Lambda\| \geq \|\Lambda'\|$, since $\varepsilon > 0$ can be taken arbitrarily small and since $(1-\varepsilon)(1+\varepsilon^{1/2})^{-1} \to 1$, $(1+\varepsilon^{1/2})^{-1}\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$, from (5.4.1) and (5.4.2) it follows that

$$\|\Lambda_{\mathcal{N}\subset\mathcal{M}}\|^2 \geq [M:N]. \qquad \blacksquare$$

5.5 Corollary. *An extremal* II_1 *subfactor is weakly amenable if and only if it is weakly injective, and if these conditions are satisfied then the index of the subfactor lies in the set* $\mathbb{E}^2$.

5.6 Remark. A W^*-inclusion $\mathcal{Q} \subset^{\mathcal{F}} \mathcal{P}$ is called *AFD* (*approximately finite dimensional*) if given any finite set $F \subset (\mathcal{P})_1$, any normal state φ on $\mathcal{P}$ with $\varphi \circ \mathcal{F} = \varphi$ and any $\varepsilon > 0$, there exists a finite dimensional W^*-inclusion $Q \subset^{E} P$ and a c.sq. embedding of it into $\mathcal{Q} \subset^{\mathcal{F}} \mathcal{P}$ such that for any $x \in F$ there exists $y \in (P)_1$ with $\|x - y\|_\varphi < \varepsilon$. For an inclusion of II_1 factors with finite index $N \subset M$, this amounts to the definition of the AFD property in [32]: for any $F \subset (M)_1$ finite and $\varepsilon > 0$, there exists a commuting square $(Q \subset P) \subset (N \subset M)$ with Q, P dimensional such that $\|x - E_P(x)\|_2 \leq \varepsilon$ for all $x \in F$.

By Theorem 4.5, amenable subfactors $N \subset M$ are AFD. The hyperfinite II_1 subfactors $P_{0\infty} \subset P_{1\infty}$ constructed from Markov cells as in Section 2.9 are obviously AFD. By [40, §3.1.3], if $N \subset M$ is AFD with the finite dimensional approximates $Q \subset P$ so that $(Q \subset P) \subset (N \subset M)$ is non-degenerate, then $N \subset M$ has tracial representations with $(N \subset M)$-hypertrace, so they are weakly amenable. Note that this non-degeneracy condition on the approximating finite dimensional subalgebras is automatic if $[M : N]$ is an isolated point in $\mathbb{E}^2 \cap (4, 2+\sqrt{5})$.

The various properties of representations (smoothness, exactness, traciality, etc) allow defining several notions of "weak amenability/injectivity", where one requires existence of a $(N \subset M)$-hypertrace, respectively of a norm-one projection, from one (or all) representation $(\mathcal{N} \subset^{\mathcal{E}} \mathcal{M})$ in some "special" class. Of particular interest is the following case.

5.7 Definition. We say that $(N \subset M)$ is *ufc-amenable* (resp. *ufc-injective*) if the universal exact finite-coupling representation $(\mathcal{N}^{\mathrm{u,fc}} \subset^{\mathcal{E}^{\mathrm{u,fc}}} \mathcal{M}^{\mathrm{u,fc}})$ admits an $(N \subset M)$-hypertrace (resp. a norm-one projection with range $N \subset M$).

Note that for an extremal subfactor $N \subset M$, one obviously has

$$\text{"amenable} \Rightarrow \text{ufc-amenable} \Rightarrow \text{weakly amenable".}$$

So in particular, if $N \subset M$ is ufc-amenable, then $[M : N] \in \mathbb{E}^2$.

While we will undertake a detailed study of this notion in a follow up to this article, we end this section by stating without proof a result from this forthcoming paper, which relates ufc-amenability with several interesting structural properties of subfactors.

5.8 Theorem. *Let $N \subset M$ be an extremal inclusion of separable II_1 factors. The conditions* (1), (1′), (2), (3) *below are equivalent and they imply condition* (4)*:*

(1) *$N \subset M$ is ufc-amenable: the universal exact fc-representation $(N \subset M) \subset (\mathcal{N}^{\mathrm{u,fc}} \subset^{\mathcal{E}^{\mathrm{u,fc}}} \mathcal{M}^{\mathrm{u,fc}})$ admits an $(N \subset M)$-hypertrace.*

(1′) *$N \subset M$ is ufc-injective: there exists a norm-one projection of $(\mathcal{N}^{\mathrm{u,fc}} \subset^{\mathcal{E}^{\mathrm{u,fc}}} \mathcal{M}^{\mathrm{u,fc}})$ onto $(N \subset M)$.*

(2) *$\|\Lambda^{\mathrm{u,fc}}_{N\subset M}\|^2 = [M : N]$ and $N \simeq R \simeq M$.*

(3) *$N \subset M$ has the AFDRC-property (AFD by relative commutants): given any finite set $F \subset M$ and any $\varepsilon > 0$, there exists a subfactor $Q \subset N$ such that $Q' \cap M$ is finite dimensional and $F \subset_\varepsilon Q' \cap M$.*

(4) *$N \subset M$ has the abc-property (asymptotic bi-centralizer property):*

$$M = (M' \cap N^\omega)' \cap M^\omega.$$

The proof of some of the implications in the above theorem are quite elaborate, but let us point out right away that one has (1) $\Leftrightarrow$ (1$'$) by Proposition 4.4, and (1) $\Rightarrow$ (2) by Theorem 5.4. Also, (3) $\Rightarrow$ (1) has a proof similar to [40, §3.1.3]. This entails (3) $\Rightarrow$ (2) as well, but note that this implication is also a direct consequence of [32, Theorem 3.3]. In addition, a proof of the implication (3) $\Rightarrow$ (4) can be easily completed along the lines of the proof of Theorem 2.14 (6) in [48, p. 1676].

6. Further remarks and open problems

6.1. General questions on W^*-representations. Representations of II_1 factors seem interesting to study in their own right. There is a large number of intriguing problems of "general" nature. We mention just a few.

We have been able to construct only three types of W^*-representations for a given subfactor $N \subset M$: the ones in Example 3.2.1, coming from graphages $(Q \subset P) \subset (N \subset M)$; the exact representations in Section 3.6; the untamed W^*-representations in Section 3.10. Representations in this last class tend to be non-smooth, with infinite coupling constants, in some sense "uncontrollable". For instance, we saw that even if $N \subset M$ has finite depth with index > 4, by Corollary 3.10.2 one can construct W^*-representations of $N \subset M$ that have A_∞ inclusion graph.

6.1.1. Find new constructions of tracial representations with finite couplings. Conceive a method that produces all such representations for a given II_1 subfactor.

6.1.2. Is traciality automatic for representations with finite couplings?

6.1.3. Establish whether a representation given by a graphage is necessarily exact. Or at least that it necessarily has "large" RC-algebra.

6.1.4. Assume $(N \subset M) \subset (\mathcal{N} \subset^{\mathcal{E}} \mathcal{M})$ is a tracial representation with finite couplings and irreducible finite inclusion graph $\Lambda_{\mathcal{N} \subset \mathcal{M}}$. Is this representation necessarily arising from a graphage, as in Example 3.2.1?

6.1.5. Do there exist representations $(N \subset M) \subset (\mathcal{N} \subset^{\mathcal{E}} \mathcal{M})$ with RC-factor $M' \cap \mathcal{N}$ of type III (preferably exact) ? Do such representations exist for $M \simeq R$?

Assume $(N \subset M) \subset (\mathcal{N}_P \subset^{\mathcal{E}} \mathcal{M}_P)$ is an irreducible exact "right-finite" representation, coming from an irreducible Hilbert-bimodule ${}_M\mathcal{H}_P$ with P a II_1 factor and $\dim(\mathcal{H}_P) < \infty$, say $\dim(\mathcal{H}_P) = 1$. Then ${}_M\mathcal{H}_P = {}_M L^2 P_P$ and the RC-envelope $\mathcal{P} = M' \cap \mathcal{N}_P$ of P^{op} follows of the form $\widetilde{T}^{\mathrm{op}}$, for some intermediate subfactor $M \subset P \subset \widetilde{T} \subset \langle P, e_M \rangle$. But any such subfactor comes from an intermediate II_1 subfactor $M \subset T \subset P$ via basic construction, $\widetilde{T} = \langle M, e_T \rangle$.

Thus, by reducing with a finite projection in the type II factor $\widetilde{P}^{\rm op} = \widetilde{T}^{\rm op}$, it follows that $(N \subset M) \subset (\mathcal{N}_P \subset \mathcal{M}_P)$ is stably isomorphic to $(N \subset M) \subset (\mathcal{N}_T \subset \mathcal{M}_T)$, a representation that is still right-finite but this time the RC-envelope of T is $M' \cap \mathcal{N}_T = T^{\rm op}$. Note that even if one starts with $M \subset P$ with $[P : M] = \infty$, during this process we may end up with $M \subset T$ satisfying $[T : M] < \infty$, so a sub-representation of $(\mathcal{N}^{u,f} \subset \mathcal{M}^{u,f})$. This justifies the following question:

6.1.6. Do there exist examples of irreducible exact right-finite representations $(N \subset M) \subset (\mathcal{N}_P \subset \mathcal{M}_P)$ arising from an irreducible embedding $M \subset P$ with P a II_1 factor and $[P : M] = \infty$, such that P equals its (exacting) RC-envelope ? In other words, while $(\mathcal{N}^{u,f} \subset \mathcal{M}^{u,f})$ is a subrepresentation of $(\mathcal{N}^{\rm u,rf} \subset \mathcal{M}^{\rm u,rf})$, are there examples where this inclusion is strict?

6.1.7. Calculate the RC-factor (or exacting factor) for $\mathcal{N}_P \subset \mathcal{M}_P$, for a given irreducible embedding $P \subset M^\alpha$, $0 < \alpha \leq \infty$. Along these lines, one can push the question in Section 6.1.6 even further: Is any irreducible sub-representation of $\mathcal{N}^u \subset \mathcal{M}^u$ stably isomorphic to an exact representation with finite couplings? (i.e. a subrepresentation of $\mathcal{N}^{\rm u,fc} \subset \mathcal{M}^{\rm u,fc}$). Find concrete examples where $(N \subset M) \subset (\mathcal{N}^u \subset \mathcal{M}^u)$ is "essentially" equal/non-equal to $(N \subset M) \subset (\mathcal{N}^{u,f} \subset \mathcal{M}^{u,f})$.

6.1.8. Do there exist representations $(N \subset M) \subset (\mathcal{N} \subset^{\mathcal{E}} \mathcal{M})$ with trivial RC-algebra, $M' \cap \mathcal{N} = \mathbb{C}$ (preferably with finite couplings)? Do such representations exist for $N \subset M \simeq R$?

6.2. Values of index problems. The W^*-representation theory for a II_1 subfactor $N \subset M$ devised in this paper, following up on [40], is an analogue of the "classic" representation theory of a II_1 factor, and in fact it becomes just that when $N = M$. But while for a single II_1 factor M its representations $M \subset \mathcal{B}(\mathcal{H})$ are completely classified by the Murray–von Neumann dimension/coupling, $\dim({}_M\mathcal{H})$, for a (non-trivial) irreducible subfactor $N \subset M$ the "W^*-representation picture" becomes strikingly complex. It seems to us that it is this framework that's key for investigating rigidity paradigms concerning the values of the index of a given II_1 factor M, constructed out of specific "geometric data", most notably for $M = R$ and for factors with Cartan subalgebras.

Despite the results in [38], showing existence of large families of A_∞-subfactors of any given index $\lambda^{-1} > 4$, and more generally the results in [41] identifying the abstract objects $\mathcal{G}$ that can occur as higher relative commutants of subfactors (called standard λ-lattices in [41]), a phenomenon such as $\mathcal{C}(M) = \{4\cos^2(\pi/n) \mid n \geq 3\} \cup [4, \infty)$ seems to only occur when the II_1 factor M comes from "random-like" constructions, such as the ones in [38, 41, 51] or in [13]. In turn, for a II_1 factor M with a "very geometric background", $\mathcal{C}(M)$ seems more prone to be a subset of $\mathbb{E}^2$, $\mathbb{E}_0^2$, or even $\mathbb{N}$.

Recall in this respect the huge difference between the two types of existing results where $\mathcal{C}(M)$ could be fully calculated, with the case of the free group factor $M = L(\mathbb{F}_\infty)$ having $\mathcal{C}(M)$ equal to the whole Jones spectrum $\{4\cos^2(\pi/n) \mid n \geq 3\} \cup [4, \infty)$ by [51], and the case of free group measure space factors $M = L^\infty X \rtimes \mathbb{F}_n$ having $\mathcal{C}(M)$ equal to the semigroup of integers $\{1, 2, 3, \ldots\}$ by [30, 47, 53].

The case of the hyperfinite II_1 factor $M \simeq R$ is particularly puzzling, as R can be constructed from very geometric data (finitary, more generally amenable, due to [5, 27]), while at the same time it is the playing field for matrix randomness! However, the latter is always "approximate randomness", in moments. Our belief is that its "geometric-finitary background" prevails when it comes to index of subfactors problems.

6.2.1. Is $\mathcal{C}(R)$ equal to $\mathbb{E}^2$? This question splits into two types of problems:

(a) *The restrictions on the index problem*, asking whether the index of any irreducible hyperfinite subfactor is necessarily the square norm of a (possibly infinite) bipartite graph, i.e. that $\mathcal{C}(R) \subset \mathbb{E}^2$. We believe quite strongly that this inclusion holds true. But there may be further restrictions for $\mathcal{C}(R)$.

W^*-representation theory should be quite useful in approaching this problem. Since the inclusion $\mathcal{C}(R) \subset \mathbb{E}^2$ represents a condition only on the interval $(4, 2 + \sqrt{5})$ and any irreducible subfactor with index in this interval has A_∞ graph by [14], proving $\mathcal{C}(R) \subset \mathbb{E}^2$ amounts to showing that any hyperfinite A_∞-subfactor $N \subset R$ with index less than $2 + \sqrt{5}$ satisfies $[R : N] \in \mathbb{E}^2$.

(b) *The commuting square problem*, asking whether for a given finite connected bipartite graph Λ, there exists a Markov cell $(P_{00} \subset P_{01}) \subset (P_{10} \subset P_{11})$ as in Section 2.9, with the column $P_{00} \subset P_{10}$ having inclusion graph equal to Λ. While this would merely show $\mathcal{E}(R) \supset \mathbb{E}_0^2$, since $\mathcal{E}(M) \cap (4, 2 + \sqrt{5}) = \mathcal{C}(M) \cap (4, 2 + \sqrt{5})$ for any II_1 factor M, it would still imply $\mathbb{E}_0^2 \cap (4, 2 + \sqrt{5}) \subset \mathcal{C}(R)$. Ideally, the commuting square problem should be solved with control of the higher relative commutants of the resulting subfactor (in particular its irreducibility, see also Section 6.3.1).

One should note that there is no known example of an irreducible hyperfinite subfactor with a non-algebraic number as index, nor in fact of any number $\mathcal{C}(R) \setminus \mathbb{E}_0^2$.

6.2.2 Conjecture. *Any hyperfinite A_∞-subfactor $N \subset R$ is ufc-amenable, and thus any such subfactor satisfies $[N : R] = \|\Lambda^{\mathrm{u,fc}}_{N \subset R}\|^2 \in \mathbb{E}^2$. From the above observations, this would imply $\mathcal{C}(R) \subset \mathbb{E}^2$, thus solving* 6.2.1 (a) *above.*

6.2.3. More generally, we believe that if M is any (separable) II_1 factor with a Cartan subalgebra, then $\mathcal{C}(M) \subset \mathbb{E}^2$.

This is of course verified by results in [30, 47, 53], where for a large class of II_1 factors with Cartan decomposition one even has $\mathcal{C}(M) \subset \{1, 2, 3, \ldots\}$. But in these cases the index rigidity is due to the uniqueness up to unitary conjugacy of the Cartan subalgebra (what is called $\mathscr{C}_s$-rigidity in [53]), a property that many group measure space factors, such as R, do not have. Nevertheless, the presence of a Cartan subalgebra in a II_1 factor M seems to make the W^*-representation theory of its subfactors $N \subset M$ be very "structured", a phenomenon that should entail "graph-like" obstructions for $[M : N]$.

6.2.4. Another intriguing question is the calculation of $\mathcal{C}(M)$ for the II_1 factors $M = L\mathbb{F}_n$ associated withe free groups with finitely many generators, $n = 2, 3, \ldots$. Since $\mathcal{C}(M)$ is invariant to amplifications of M, they are all equal (cf. [9, 56, 62]). One would be tempted to believe that, due to its "pure random nature", $\mathcal{C}(L\mathbb{F}_{\mathrm{fin}})$ is equal to the entire Jones spectrum $\{4\cos^2(\pi/n) \mid n \geq 3\} \cup [4, \infty)$, like in the case $M = L\mathbb{F}_\infty$.

But all calculations of $\mathcal{C}(L\mathbb{F}_\infty)$ are based on variations of the construction in [38], which uses amalgamated free product involving commuting squares associated with a λ-lattice (not necessarily standard) $\mathcal{G}$ and some "initial (semi)finite data" Q. When the data Q is $L\mathbb{F}_n$, $n \geq 1$, this allowed identifying the resulting subfactors $N^{\mathcal{G}}(Q) \subset M^{\mathcal{G}}(Q)$ as free group factors, using various models in Voiculescu's free probability theory ([13, 51, 56]). This does give $M \simeq L\mathbb{F}_{\mathrm{fin}}$ when $\mathcal{G}$ is finite (i.e. a Markov cell), but it always gives $M \simeq L\mathbb{F}_\infty$ if $\mathcal{G}$ is not finite, notably if $\mathcal{G}$ is the TLJ standard λ-lattice $\mathcal{G}^\lambda$ (equivalently the TLJ standard λ-cell $\mathscr{C}^\lambda$), with A_∞ graph and index λ^{-1}.

Consequently, the only known values $\lambda^{-1} \in \mathcal{C}(L\mathbb{F}_{\mathrm{fin}})$ are square norms of finite bipartite graphs Λ for which the commuting square problem could be solved! So exactly the same as the values λ^{-1} that have been shown to exist in $\mathcal{C}(R)$. This makes quite plausible the prediction $\mathcal{C}(L\mathbb{F}_{\mathrm{fin}}) = \mathcal{C}(R)$.

6.2.5. Along the lines of Murray–von Neumann question of characterizing all multiplicative subgroups of $(0, \infty)$ that can be realized as fundamental groups of separable II_1 factors (where much progress has been done in [52], a natural question is to characterize all sub-sets (resp. sub-semigroups) of the Jones spectrum $\{\cos^2(\pi/n) \mid n \geq 3\} \cup [4, \infty)$ that can be realized as $\mathcal{C}(M)$ (resp. $\mathcal{E}(M)$) for some separable II_1 factor M. To approach this, it may be useful to revisit the universal construction in [38, 46] with the tools of deformation-rigidity theory in hand.

6.2.6. One obviously has $\mathcal{C}(Q \overline{\otimes} L\mathbb{F}_\infty) = \{4\cos^2(\pi/n) \mid n \geq 3\} \cup [4, \infty)$ for any II_1 factor Q. On the other hand, by [51, Theorem 1.3], one has $\mathcal{C}(N) \subset \mathcal{C}(N * L\mathbb{F}_\infty)$ for any II_1 factor N. Hence,

$$\mathcal{C}((Q \overline{\otimes} L\mathbb{F}_\infty) * L\mathbb{F}_\infty) = \{\cos^2(\pi/n) \mid n \geq 3\} \cup [4, \infty)$$

for any II_1 factor Q. It would be interesting to find other classes of factors M with $\mathcal{C}(M)$ equal to the entire Jones spectrum.

6.3. Actions of λ-lattices on R. Given a II_1 factor M, we denote by $\mathcal{G}(M)$ the set of all standard λ-lattices $\mathscr{G}$ that can appear as the standard invariant of an extremal subfactor $N \subset M$, $\mathscr{G} = \mathscr{G}_{N\subset M}$ (i.e. which in some sense can "act" on M).

Note that this set encodes both $\mathcal{C}(M)$ and $\mathcal{E}(M)$, as one has

$$\mathcal{E}(M) = \{\mathrm{Ind}(\mathscr{G}) \mid \mathscr{G} \in \mathcal{G}(M)\}$$

(where $\mathrm{Ind}(\mathscr{G}) = \lambda^{-1}$ is the *index of* $\mathscr{G}$), and $\mathcal{C}(M)$ is the set of all $\mathrm{Ind}(\mathscr{G})$, $\mathscr{G} \in \mathcal{G}(M)$ with $\Gamma_{\mathscr{G}}$ having just one edge from its "initial" vertex $*$ (see Section 2.3). With this in mind, the question in Section 6.2.1 can be refined by asking the following:

6.3.1. Identify the set $\mathcal{G}(R)$ of all standard λ-lattices that can act on R. Or at least calculate the set $\mathcal{G}_\alpha(R)$ of all $\mathscr{G} \in \mathcal{G}(R)$ with $\mathrm{Ind}(\mathscr{G}) \leq \alpha$, for some specific number $\alpha > 4$, notably for $\alpha = 2 + \sqrt{5}$, or $\alpha = 5$.

At this point these questions seem extremely difficult to answer, with no indication of what the corresponding sets might be. Note however that if Conjecture 6.2.2 is answered in the affirmative, with a solution to the commuting square problem 6.2.1 (b) for every finite connected bipartite graph Λ (as per Section 2.9), then a next step would be to devise a method of constructing Markov cells, with given "vertical" bipartite graph Λ as in Section 2.9, in a way that allows controlling (and computing!) the standard invariant of the resulting subfactor. From that point on, the classification of all standard λ-lattices (planar algebras) in [24] would help complete the picture at least for $\mathcal{G}_5(R)$.

Along these lines, the following question, complementing Conjecture 6.2.2 above, is interesting to investigate:

6.3.2. Does there exist a hyperfinite subfactor $N \subset R$ with $[R : N] = \alpha$ and A_∞ graph for any $\alpha \in \mathcal{C}(R) \cap (4, \infty)$?

As noticed in Remarks (3) and (5) of Section 5.1.5 in [40] (see also [39, §4.4], [42, §6.2]) our classification theorem for hyperfinite subfactors with amenable standard invariant ([44, §7.2.1]; cf. Theorem 4.5 in the present paper) implies Ocneanu's theorem [28] on the uniqueness, up to cocycle conjugacy, of the free actions of a given finitely generated amenable group Γ on the hyperfinite II_1 factor R. Jones obtained in [19] a converse to Ocneanu's result, by showing that any countable non-amenable group Γ admits two actions on R that are not cocycle conjugate. So it is quite natural to predict the following (see [40, Problem 5.4.7]):

6.3.3 Conjecture. *Given any non-amenable $\mathcal{G} \in \mathcal{G}(R)$, there exist subfactors $Q, P \subset R$ such that $\mathcal{G}_{P\subset R} = \mathcal{G}_{Q\subset R}$ but $(P \subset R) \not\simeq (Q \subset R)$. Taking into account* [3]*, one can even speculate that given any $\mathcal{G} \in \mathcal{G}(R)$ there exist infinitely/uncountably many hyperfinite subfactors with $\mathcal{G}$ as standard invariant.*

6.3.4. In the spirit of [40, §5.4.3], we denote by $\mathcal{C}^{\mathrm{a}}(M)$ (resp. $\mathcal{C}^{\mathrm{fd}}(M)$) the set of indices of irreducible subfactors with amenable (resp. finite depth) graph of the II_1 factor M. Similarly, we denote $\mathcal{G}^{\mathrm{a}}(M)$ (resp. $\mathcal{G}^{\mathrm{fd}}(M)$) the set of amenable (resp. finite depth) λ-lattices that can occur as standard invariants of subfactors of M. Note that $\mathcal{G}^{\mathrm{a}}(M) \subset \mathcal{G}^{\mathrm{a}}(R)$ for any II_1 factor M (because by [44, §7.2.1] any amenable $\mathcal{G}$ can be realized as the standard invariant of a hyperfinite II_1 subfactor; see also [46, §4.4.2] and [50, Theorem 2.9]), so in fact we can denote

$$\mathcal{C}^{\mathrm{a}}(R) = \mathcal{C}^{\mathrm{a}}, \quad \mathcal{C}^{\mathrm{fd}}(R) = \mathcal{C}^{\mathrm{fd}}, \quad \mathcal{G}^{\mathrm{a}}(R) = \mathcal{G}^{\mathrm{a}}, \quad \mathcal{G}^{\mathrm{fd}}(R) = \mathcal{G}^{\mathrm{fd}}.$$

It would be interesting to calculate such sets, or at least obtain some general properties/estimates, especially in the case $M = R$. For example, is the set $\mathcal{C}^{\mathrm{a}}$ (resp. $\mathcal{G}^{\mathrm{a}}$) countable/uncountable? Can it contain points in $\mathbb{E}^2 \setminus \mathbb{E}^2_0$ ("limit points")? We refer the reader to [40, §5.4.3] for a series of questions related to this. One should note the early results about $\mathcal{G}^{\mathrm{fd}}$ in [14, 29] and the more recent complete description of $\mathcal{G}^{\mathrm{fd}}_5$ in [24], finalizing a two decades long series of impressive results along these lines (see also [1], where the description is pushed to $\mathcal{G}^{\mathrm{fd}}_{5.25}$).

6.3.5. Like for the set $\mathcal{C}(M)$, there have been two types of complete calculations of $\mathcal{G}(M)$ for a II_1 factor M. On the one hand, $\mathcal{G}(L\mathbb{F}_\infty)$ was shown in [51] to be equal to the set $\mathcal{G}$ of all standard λ-lattices (as constructed in [41]). Thus, $\mathcal{G}(Q\overline{\otimes}L\mathbb{F}_\infty) = \mathcal{G}$, and hence by [51, Theorem 1.3],

$$\mathcal{G}((Q\overline{\otimes}L\mathbb{F}_\infty) * L\mathbb{F}_\infty) = \mathcal{G}$$

for any II_1 factor Q. On the other hand, there have been several constructions of factors M that have no λ-symmetries other than the trivial ones, corresponding to subfactors that "split-off" $\mathbb{M}_n(\mathbb{C}), n \geq 1$ (cf. [4, 54, 61]). Similar techniques (deformation-rigidity theory) have been used to construct II_1 factors M with various prescribed groups G as outer automorphism group, $\mathrm{Out}(M) = G$ (see e.g. [17]). It would be interesting to obtain results of this type for prescribed "small" subsets $\mathcal{F}$ of $\mathcal{G}$. Like in Section 6.2.4 above, this should be possible by combining the universal construction in [46] with a way of adding to the building data of a II_1 factor M the "right amount" of both rigid and soft ingredients, to show $\mathcal{F} \subset \mathcal{G}(M)$, then using deformation-rigidity to prove that any $\mathcal{G} \in \mathcal{G}(M)$ must lie in $\mathcal{F}$.

6.4. W^*-representations for subfactors of type III. A type III factor $\mathcal{P}$ is characterized by the property of being "purely infinite": all its "parts are same as the whole", all non-zero projections are equivalent. If $\mathcal{P}$ has separable predual, an alternative characterization is that any two (normal) representations of $\mathcal{P}$ on separable Hilbert spaces (or left Hilbert modules) are unitary conjugate.

For an inclusion of type III factors $\mathcal{Q} \subset^{\mathcal{F}} \mathcal{P}$, endowed with an expectation $\mathcal{F}$ of finite index, $\text{Ind}(\mathcal{F}) = \lambda^{-1} < \infty$, the expectation $\mathcal{F}$ that one usually considers is the "optimal one" of minimal index (see [15]), uniquely determined by the condition that the values $\mathcal{F}(q)$ it takes on the minimal projections q in $\mathcal{Q}' \cap \mathcal{P}$ are proportional to $[q\mathcal{P}q : \mathcal{Q}q]^{1/2}$ (see end of Section 2.5).

Like in the II_1 case, a W^*-representation for a type III subfactor $\mathcal{Q} \subset^{\mathcal{F}} \mathcal{P}$ is defined as a non-degenerate embedding into an atomic W^*-inclusion $\mathcal{N} \subset^{\mathcal{E}} \mathcal{M}$. So in this case, all the reps $\mathcal{P} \hookrightarrow \mathcal{B}(\mathcal{H}_j)$ are equivalent. However, many of the general considerations in Section 3 work the same, with the obvious adjustments.

An interesting problem here is to see whether there exist irreducible exact representation

$$\oplus_{i \in I} \mathcal{B}(\mathcal{K}_i) = \mathcal{N} \subset^{\mathcal{E}} \mathcal{M} = \oplus_{j \in J} \mathcal{B}(\mathcal{H}_j)$$

of $(\mathcal{Q} \subset^{\mathcal{F}} \mathcal{P})$ with $\mathcal{T} = \mathcal{P}' \cap \mathcal{N}$ of type II_1. More generally, are there irreducible exact representations for which there are minimal central projections $q_j \in \mathcal{M}$ such that $\mathcal{M}q_j = \mathcal{B}(\mathcal{H}_j)q_j$ corresponds to an irreducible $(\mathcal{P} - \mathcal{T})$-bimodule that does not have finite index, i.e. $[\mathcal{T}' : \mathcal{M}] < \infty$?

Viewing bimodules for the single factors $\mathcal{Q}, \mathcal{P}$ as endomorphisms and using the ensuing formalism of superselection sectors in [10, 26] may be useful for the analysis of W^*-representations of type III subfactors.

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Sorin Popa, Department of Mathematics, University of California, Los Angeles, CA 90095-1555, USA; *e-mail:* popa@math.ucla.edu