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On Bergman space zero sets. Editors' comments on the article by Vaughan Jones

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Abstract. This article has two purposes. The first one is to describe what the Editors know on how and when Vaughan Jones worked on the subject of the article published in the same volume. The starting point, in the late 1980's, was his fascination for a formula giving Murray–von Neumann dimensions of Hilbert spaces of unitary representations of Fuchsian groups. Over the years, he discovered surprising relations of these dimensions with other domains of mathematics. The second purpose is to expose with some details a subject which plays an important role in Jones' article: the irreducible *projective* unitary representations of $\mathrm{PSL}_2(\mathbf{R})$, which constitute a *continuous* family known as the discrete series, and which have interesting restrictions to various discrete subgroups of $\mathrm{PSL}_2(\mathbf{R})$.

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Introduction

Vaughan Jones has been constantly intrigued by the structure of type II_1 factors and by their relations to other branches of mathematics. He defined an index for a subfactor of a II_1 factor, which has properties like those of the index of a subgroup of a group, using the Murray–von Neumann dimension of modules over II_1-factors. He miraculously discovered, using the so-called Jones basic construction and the Jones projections, that the possible index values, up to 4, constitute a discrete spectrum. In a breakthrough series of papers, he used this construction for the study of knots and topological properties of three-manifolds, and he defined the Jones polynomial of knots.

In the paper published in this issue, first posted on arXiv in June 2020, Jones applies his ideas on representations of II_1-factors to a problem of complex analysis: Given a discrete subset S of the open unit disc D of the complex plane and one of the classical

Bergman spaces $\mathcal{H}$ of holomorphic functions on D, decide whether there exists a non-zero function f in $\mathcal{H}$ which is zero on S. The first purpose of the present text is to expose some background of representation theory used in [Jones]. In the late '80s, Jones was fascinated by a formula giving the von Murray–von Neumann dimension of the Hilbert space of a representation of a Fuchsian group, formula (1) below; thirty years later, it was the first and main ingredient of his renewed interest in questions which are addressed in [Jones].

Murray–von Neumann dimensions and discrete series projective unitary representations

Murray–von Neumann dimensions appear in the following situation. Consider a connected semi-simple real Lie group G with trivial centre and without compact factor, a Haar measure μ on G, a lattice Γ in G, and let $\mathrm{vol}(G/\Gamma)$ denote the covolume of this lattice computed with μ. Let $\mathrm{vN}(\Gamma)$ denote the von Neumann algebra of Γ, which is a factor of type II_1; recall that it is defined as the von Neumann algebra $\lambda_\Gamma(\Gamma)''$ generated by the image of the left regular representation λ_Γ of Γ on the space $\ell^2(\Gamma)$. Let π be an irreducible unitary representation of G on a Hilbert space $\mathcal{H}_\pi$. Assume that π is in the discrete series; this means that, for all $\xi, \eta \in \mathcal{H}_\pi$, the coefficient $c^\xi_\eta \colon G \to \mathbf{C}$, $g \mapsto \langle \pi(g)\xi \mid \eta \rangle$ is in $L^2(G, \mu)$. This implies that there exists a number $d_\pi > 0$, the formal dimension of π, such that $\langle c^\xi_\eta \mid c^{\xi'}_{\eta'} \rangle = \frac{1}{d_\pi} \langle \xi \mid \xi' \rangle \overline{\langle \eta \mid \eta' \rangle}$ for all $\xi, \eta, \xi', \eta' \in \mathcal{H}_\pi$; note that the number d_π depends on the choice of the Haar measure μ. This implies also that the restriction of π to Γ extends to a representation of $\mathrm{vN}(\Gamma)$ on $\mathcal{H}_\pi$.

Let M be a factor of type II_1. A representation of M on a Hilbert space $\mathcal{H}$ is an ultraweakly continuous unital $*$-homomorphism from M to the algebra of all bounded operators on $\mathcal{H}$. Recall that any such representation has a Murray–von Neumann M-dimension $\dim_M \mathcal{H}$, and that two representations are equivalent if and only if they have the same dimension (see [Jones, Theorem 3.4]).

In the case discussed above, the Murray–von Neumann $\mathrm{vN}(\Gamma)$-dimension of $\mathcal{H}_\pi$ is the product of the formal dimension d_π and the covolume of Γ, in other terms:

$$(1_0) \qquad \dim_{\mathrm{vN}(\Gamma)} \mathcal{H}_\pi = d_\pi \, \mathrm{vol}(G/\Gamma).$$

This appears in [GoHJ–89, Theorem 3.3.2], and was first observed by Atiyah and Schmid [AtSc–77, formula (3.3)]. More recently, the formula was also extended to other representations which are direct integrals of irreducible representations which appear continuously in $L^2(G, \mu)$ (see [Yang–22]).

It is worthwhile to extend formula (1_0) to projective representations, as we indicate now (the proof will be essentially the same, see Theorem 3.2 (iii) of [Radu–98]). In the case of $\mathrm{PSL}_2(\mathbf{R})$, projective unitary representations in the discrete series depend on a continuous parameter $s \in]1, \infty[$ and provide an interpolation between values $n \in \{2, 3, 4, \ldots\}$ which index the ordinary representations in the discrete series.

Let $\mathbf{T}$ denote the group of complex numbers of modulus 1. A multiplier on a locally compact group G is a Borel function $\sigma: G \times G \to \mathbf{T}$ such that $\sigma(g, e) = \sigma(e, g) = 1$ and

$$\sigma(gh, k)\sigma(g, h) = \sigma(g, hk)\sigma(h, k) \quad \text{for all } g, h, k \in G.$$

A σ-projective unitary representation of G on a Hilbert space $\mathcal{H}_\pi$ is a map from G to the unitary group of $\mathcal{H}_\pi$ such that the coefficient $c_\eta^\xi: G \to \mathbf{C}$ is a continuous function for all $\xi, \eta \in \mathcal{H}_\pi$ and such that

$$\pi(g)\pi(h) = \sigma(g, h)\pi(gh) \quad \text{for all } g, h \in G.$$

For example, given a left invariant Haar measure μ on G and a multiplier σ on G, the left regular σ-representation λ_G^σ of G on $L^2(G, \mu)$, defined by

$$\big(\lambda_G^\sigma(g)\xi\big)(h) = \sigma(h^{-1}, g)\xi(g^{-1}h) \quad \text{for all } g, h \in G \text{ and } \xi \in L^2(G, \mu),$$

is a σ-projective unitary representation of G. For a discrete group Γ, we denote by $\mathrm{vN}(\Gamma)_\sigma$ the von Neumann algebra on $\ell^2(\Gamma)$ generated by the image $\lambda_\Gamma^\sigma(\Gamma)$. Assume now, as above, that G is a connected semi-simple real Lie group with trivial centre and without compact factor and that Γ is a lattice in G. Since the non-trivial conjugacy classes of Γ are all infinite, $\mathrm{vN}(\Gamma)_\sigma$ is again a factor of type II_1 [Klep–62]. Let π be a σ-projective unitary representation of G in some Hilbert space $\mathcal{H}_\pi$, assumed as above to be irreducible and in the discrete series, i.e., with L^2 coefficients. It is again true that π has a formal dimension d_π, that $\mathcal{H}_\pi$ can be identified with a λ_G^σ-invariant subspace of $L^2(G, d\mu)$, that the restriction of π to Γ extends to a representation of $\mathrm{vN}(\Gamma)_\sigma$, so that $\mathcal{H}_\pi$ is now a $\mathrm{vN}(\Gamma)_\sigma$-module, and that we have

$$\dim_{\mathrm{vN}(\Gamma)_\sigma} \mathcal{H}_\pi = d_\pi \operatorname{vol}(G/\Gamma). \tag{1}$$

Remark: When Γ does not have any non-trivial multiplier, for example when Γ is the lattice $\mathrm{PSL}_2(\mathbf{Z})$ in $G = \mathrm{PSL}_2(\mathbf{R})$, it is easy to check that $\mathrm{vN}(\Gamma)_\sigma$ is isomorphic to the factor $\mathrm{vN}(\Gamma)$ for all multipliers σ on G. In other cases, such as that of the fundamental group Γ of a closed surface of genus at least 2 embedded in $\mathrm{PSL}_2(\mathbf{R})$ as a cocompact lattice, we do not know whether the $\mathrm{vN}(\Gamma)_\sigma$'s are isomorphic to each other.

Here is the strategy of Jones' argument to establish (1), and some comments about it. Denote by P_π the G-invariant orthogonal projection of $L^2(G, \mu)$ onto $\mathcal{H}_\pi$. As Γ is a discrete subgroup of G, there is a fundamental domain F for the action of Γ on G; note that $\int_F d\mu = \operatorname{vol}(G/\Gamma)$. Denote by Q_F the orthogonal projection of $L^2(G, \mu)$

onto the subspace $L^2(F,\mu)$ of functions which vanish outside F. Let T be the usual trace on the algebra of bounded operators on $L^2(G,\mu)$. Then

$$\dim_{\mathrm{vN}(\Gamma)_\sigma} \mathcal{H}_\pi = T(P_\pi Q_F); \tag{2}$$

the formula follows from the fact that the Hilbert space $L^2(F,\mu)$ has an orthonormal basis consisting of wandering vectors for the group Γ; the computation is explicitly done in [GoHJ–89, formula (3.3.2.1), p. 146]. (A vector ξ in the space of π is wandering for Γ if $\langle \pi(\gamma)\xi \mid \xi\rangle = 0$ for all $\gamma \neq 1$ in Γ; see [Jones, Section 5].) Then, using properties of square integrable representations, we have

$$T(P_\pi Q_F) = d_\pi \operatorname{vol}(G/\Gamma). \tag{3}$$

Formula (3) is the content of Theorem 3.3.2 in [GoHJ–89]. A posteriori, formula (3) can be obtained easily with an argument using an average over the Haar measure; see [Radu–13, Proof of Remark 4.4]. Finally, (1) follows from (2) and (3).

Holomorphic discrete series of irreducible projective unitary representations of $\mathbf{PSL_2(R)}$

Let us consider now the particular case in which G is $\mathrm{SL}_2(\mathbf{R})$ or $\mathrm{PSL}_2(\mathbf{R})$. For his classification of the irreducible unitary representations of this group, Bargmann discovered the holomorphic discrete series representations. He rather considers the group $\mathrm{SU}(1,1)$ of matrices of the form $\begin{pmatrix} a & b \\ \bar{b} & \bar{a}\end{pmatrix}$, where $a,b \in \mathbf{C}$ and $|a|^2 - |b|^2 = 1$, which is isomorphic to $\mathrm{SL}_2(\mathbf{R})$.

First, he defines for any real number $\alpha > -1$ the Hilbert space

$$A^2_\alpha = \left\{ f \colon \mathbf{D} \to \mathbf{C} \text{ holomorphic} \;\middle|\; \frac{\alpha+1}{\pi}\int_{\mathbf{D}} |f(z)|^2 (1-r^2)^\alpha r\, dr\, d\theta < \infty \right\},$$

where $\mathbf{D}$ is the open unit disc of the complex plane; see [Barg–47, (9.9), p. 621]. The dimension of A^2_α is infinite, because $z^n \in A^2_\alpha$ for all $n \in \mathbf{N}$, and A^2_α is dense in A^2_β whenever $-1 < \alpha \leq \beta$. Bargmann's notation is $\mathcal{H}_\ell$ for our A^2_α, with $\ell = \alpha + 2$. The notation A^2_α is that of more recent books; see [HeKZ–00, p. 2] and [DuSc–04, p. 103], and of [Jones]. The spaces A^2_α are now called weighted Bergman spaces. The name of Bergman refers to the fact that these spaces are reproducing kernel Hilbert spaces, as in the book [Berg–50], where A^2_0 is the first example on page 1 of the book.

Then Bargmann defines a unitary representation π_α of $\mathrm{SU}(1,1)$ on A^2_α (see below the definition of $\check{\pi}_s$), mainly for the integral values $\alpha \in \mathbf{N} = \{0,1,2,\dots\}$ giving rise to ordinary representations, but Bargmann mentions en passant non-integral values giving rise to projective representations (last remark of Section 9 in [Barg–47]). When $\alpha \in \mathbf{N}$, a standard notation is now D_n^+ for π_α, with $n = \alpha + 2$; the D_n^+'s with $n \geq 2$ constitute the holomorphic discrete series of irreducible unitary representations of $\mathrm{SU}(1,1)$; when n is even, D_n^+ can be viewed as an irreducible unitary representation of $\mathrm{PSU}(1,1) = \mathrm{SU}(1,1)/\{\pm \mathrm{id}\}$.

The classification of all irreducible projective unitary representations of the group $\mathrm{PSU}(1,1)$ is equivalent to the classification of all irreducible ordinary unitary representations of the universal covering group of $\mathrm{SU}(1,1)$; see for example [BaMi–00, Theorem 1.2]. The latter is due to Pukánszky; see [Puka–64], as well as [Sall–67]. These representations fall into three classes, the principal series, the complementary series, and the discrete series, itself consisting of two parts, the holomorphic discrete series, i.e., the π_α's, and the so-called antiholomorphic discrete series.

We find it convenient to define precisely the representations π_α's, or rather their alias, the $\check{\pi}_s$'s. Since the subgroup $\mathrm{PSL}_2(\mathbf{Z})$ is of particular interest, it is appropriate to consider the group $\mathrm{PSL}_2(\mathbf{R})$, rather than $\mathrm{PSU}(1,1)$, and therefore spaces of holomorphic functions on the upper half-plane

$$\mathbf{H} = \{z = x + iy \in \mathbf{C} \mid y > 0\},$$

rather than on the unit disc $\mathbf{D}$; we will denote the projective representations by $\check{\pi}_s$ rather than by π_α, with s real, $s = \alpha + 2 > 1$.

The first ingredients to define are the multipliers. We write $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ the class in $\mathrm{PSL}_2(\mathbf{R})$ of a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbf{R})$. The group $\mathrm{PSL}_2(\mathbf{R})$ acts on $\mathbf{H}$ by holomorphic transformations, defined by

$$gz = \begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \frac{az+b}{cz+d} \quad \text{for all } g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbf{R}) \text{ and } z \in \mathbf{H}.$$

The function $j \colon \mathrm{PSL}_2(\mathbf{R}) \times \mathbf{H} \to \mathbf{C}$ is defined by

$$j(g,z) = (cz+d)^2 \quad \text{for } g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbf{R}) \text{ and } z \in \mathbf{H},$$

so that

$$g'(z) = \frac{d(gz)}{dz} = \frac{d}{dz}\left(\frac{az+b}{cz+d}\right) = \frac{1}{(cz+d)^2} = \frac{1}{j(g,z)}.$$

(If g were in the group $\mathrm{SL}_2(\mathbf{R})$, the number $j(g,z)$ could be defined to be $cz+d$, as for example in [Iwan–97, p. 24] and in [Jones, Section 4], but here g is in $\mathrm{PSL}_2(\mathbf{R})$, and we rather define $j(g,z)$ as $(cz+d)^2 = (-cz-d)^2$.) For $s \in \mathbf{R}$, we need to define $j(g,z)^s$. Note that $j(g,z) = (cz+d)^2$ is never zero. We define here the logarithm

$$(\ln j)(g,z) = \ln |j(g,z)| + i \operatorname{Arg}(j(g,z)) \quad \text{with } \operatorname{Arg}(j(g,z)) \in \,]-\pi, \pi],$$

and the sth power of j

$$j(g,z)^s = \exp\big(s(\ln j)(g,z)\big) \quad \text{for all } g \in \mathrm{PSL}_2(\mathbf{R}) \text{ and } z \in \mathbf{H}.$$

Define a function $m_{s,z}$ on $\mathrm{PSL}_2(\mathbf{R}) \times \mathrm{PSL}_2(\mathbf{R})$ by

$$m_{s,z}(g^{-1},h^{-1}) = j(hg,z)^{s/2}/\big(j(h,gz)^{s/2}j(g,z)^{s/2}\big).$$

On the one hand, the function $z \mapsto j(g,z)^{s/2}$ is holomorphic and not zero on $\mathbf{H}$, so that the function $z \mapsto m_{s,z}(\cdot,\cdot)$ is holomorphic in z. On the other hand, the chain rule identity

$$j(hg,z)^{-1} = \frac{d((hg)(z))}{dz} = \frac{d(h(gz))}{d(gz)}\frac{d(gz)}{dz} = j(h,gz)^{-1}j(g,z)^{-1}$$

implies that $m_{s,z}(g^1,h^{-1}) \in \mathbf{T}$. It follows that $m_{s,z}(g^{-1},h^{-1})$ does not depend on z; we will write it $m_s(g^{-1},h^{-1})$. We have

$$(4) \quad m_s(g^{-1},h^{-1}) = \frac{j(hg,z)^{s/2}}{j(h,gz)^{s/2}j(g,z)^{s/2}} \quad \text{for all } g,h \in \mathrm{PSL}_2(\mathbf{R}) \text{ and } z \in \mathbf{H}.$$

Then m_s is a multiplier; indeed $m_s(g^{-1},\mathrm{id}) = m_s(\mathrm{id},g^{-1}) = 1$ and

$$\begin{aligned} m_s(g^{-1},h^{-1})m_s(g^{-1}h^{-1},k^{-1}) &= \frac{j(hg,i)^{s/2}}{j(h,gi)^{s/2}j(g,i)^{s/2}}\frac{j(khg,i)^{s/2}}{j(k,hgi)^{s/2}j(hg,i)^{s/2}} \\ &= \frac{j(khg,i)^{s/2}}{j(k,hgi)^{s/2}j(h,gi)^{s/2}j(g,i)^{s/2}} \\ &= \frac{j(khg,i)^{s/2}}{j(kh,gi)^{s/2}j(g,i)^{s/2}}\frac{j(kh,gi)^{s/2}}{j(k,hgi)^{s/2}j(h,gi)^{s/2}} \\ &= m_s(g^{-1},h^{-1}k^{-1})m_s(h^{-1},k^{-1}) \end{aligned}$$

for all $g,h,k \in \mathrm{PSL}_2(\mathbf{R})$. Note that $m_s(g^{-1},h^{-1})$ takes three values only, which are $e^{-is\pi}$, 1, $e^{is\pi}$, unless s is an even integer in which case $m_s(g^{-1},h^{-1}) = 1$ for all g and h; moreover, $m_{s+2} = m_s$ for all $s \in \mathbf{R}$.

Multipliers on a locally compact group G constitute a group $Z^2(G,\mathbf{T})$ of 2-cocycles; multipliers which are trivial, namely of the form $(g,h) \mapsto \rho(g)\rho(h)\rho(gh)^{-1}$ for some Borel function $\rho\colon G \to \mathbf{T}$, constitute the subgroup $B^2(G,\mathbf{T})$ of coboundaries; and the quotient group $Z^2(G,\mathbf{T})/B^2(G,\mathbf{T})$ constitute the cohomology group $H^2(G,\mathbf{T})$. It is known that the 2-cocycle m_s is a coboundary if and only if s is an even integer, that m_s and $m_{s'}$ are cohomologous if and only if $s'-s \in 2\mathbf{Z}$, and that any class in $H^2(\mathrm{PSL}_2(\mathbf{R}),\mathbf{T})$ can be represented by a m_s, so that

$$H^2\big(\mathrm{PSL}_2(\mathbf{R}),\mathbf{T}\big) \approx \mathbf{R}/2\mathbf{Z} \approx \mathbf{T}$$

(see, for example, [BaMi–00, Theorem 1.1 and Proposition 1.1]).

The Bergman space for $s > 1$ is now the Hilbert space

(5) $$\mathcal{H}_s = \left\{ f:\mathbf{H} \to \mathbf{C} \text{ holomorphic} \;\middle|\; \int_{\mathbf{H}} |f(z)|^2 y^{s-2}\, dx\, dy < \infty \right\}.$$

For $s > 1$ and $g \in \mathrm{PSL}_2(\mathbf{R})$, define an operator $\check{\pi}_s(g)$ on $\mathcal{H}_s$ by

(6) $$\big(\check{\pi}_s(g) f\big)(z) = j(g^{-1}, z)^{-s/2} f(g^{-1}z).$$

It is straightforward to check that $\check{\pi}_s(g)$ is unitary and that

(7) $$\check{\pi}_s(g)\check{\pi}_s(h) = m_s(g,h)\check{\pi}_s(gh) \quad \text{for all } g, h \in \mathrm{PSL}_2(\mathbf{R}),$$

so that $\check{\pi}_s$ is a projective unitary representation of $\mathrm{PSL}_2(\mathbf{R})$ with multiplier m_s. Moreover, $\check{\pi}_s$ is irreducible, and in the discrete series. To fix the normalization of the Haar measure μ on $\mathrm{PSL}_2(\mathbf{R})$, we define

$$\int_{\mathrm{PSL}_2(\mathbf{R})} d\mu\, f\left(\begin{bmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\right) = \int_{\mathbf{H}} \frac{dx\, dy}{y^2} \int_0^{\pi} d\theta\, f\left(\begin{bmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\right)$$

for all continuous functions of compact support $f:\mathrm{PSL}_2(\mathbf{R}) \to \mathbf{C}$. The formal dimension of $\check{\pi}_s$ is then

$$d_{\check{\pi}_s} = \frac{s-1}{4\pi}.$$

(The formal dimension in [Robe–83, Theorem 17.8] is $(s-1)/\pi$, but Robert uses a different normalization of the Haar measure on $\mathrm{PSL}_2(\mathbf{R})$.) The $\check{\pi}_s$'s constitute the holomorphic part of the discrete series of $\mathrm{PSL}_2(\mathbf{R})$.

For $\alpha > -1$ and $s = \alpha + 2$, there is a natural isomorphism $A^2_\alpha \to \mathcal{H}_s$, defined in terms of the Cayley transform, which intertwines the representation $\pi_\alpha:\mathrm{PSU}(1,1) \to \mathcal{U}(A^2_\alpha)$ with the representation $\check{\pi}_s:\mathrm{PSL}_2(\mathbf{R}) \to \mathcal{U}(\mathcal{H}_s)$; see [Jones, Proposition 4.4].

Restrictions of the $\check{\pi}_s$'s to $\mathrm{PSL}_2(\mathbf{Z})$

Consider now the lattice $\mathrm{PSL}_2(\mathbf{Z})$ in $\mathrm{PSL}_2(\mathbf{R})$. Its covolume is the area of a hyperbolic triangle with angles $\frac{\pi}{3}, \frac{\pi}{3}, 0$, which is $\frac{\pi}{3}$.

For any $s \in \mathbf{R}$, the restriction to $\mathrm{PSL}_2(\mathbf{Z})$ of the cocycle $m_s \in Z^2(\mathrm{PSL}_2(\mathbf{R}), \mathbf{T})$ is a coboundary. A first way to prove this is to observe that the homology group

$$\begin{aligned} H^2\big(\mathrm{PSL}_2(\mathbf{Z}), \mathbf{T}\big) &= H^2\big((\mathbf{Z}/2\mathbf{Z}) * (\mathbf{Z}/3\mathbf{Z}), \mathbf{T}\big) \\ &= H^2(\mathbf{Z}/2\mathbf{Z}, \mathbf{T}) \oplus H^2(\mathbf{Z}/3\mathbf{Z}, \mathbf{T}) \end{aligned}$$

is zero, because $H^2(C, \mathbf{T}) = 0$ for any finite cyclic group C. The following alternative proof is also of interest.

Consider the usual modular cusp form of weight 12, which is the function Δ on $\mathbf{H}$ defined for example by

$$\Delta(z) = (2\pi)^{-12} e^{2\pi i z} \prod_{r=1}^{\infty} (1 - 2^{2\pi i r z})^{24} \quad \text{for all } z \in \mathbf{H}.$$

It is a holomorphic function defined on a simply connected domain and it does not vanish, therefore it has a holomorphic logarithm $\mathrm{Ln}\,\Delta$ which can be defined as follows: choose $z_0 \in \mathbf{H}$ and $w_0 \in \mathbf{C}$ such that $e^{w_0} = \Delta(z_0)$, and set

$$(\mathrm{Ln}\,\Delta)(z) = w_0 + \int_{z_0}^{z} \frac{\Delta'(\zeta)}{\Delta(\zeta)}\, d\zeta,$$

where $\int_{z_0}^{z}$ holds for an integration along some (in fact any) continuous path in $\mathbf{H}$ with origin z_0 and extremity z. (Note that the function $\mathrm{Ln}\,\Delta$ depends on the choice of w_0.) Then $\mathrm{Ln}\,\Delta$ is holomorphic and

$$\Delta(z) = \exp\big((\mathrm{Ln}\,\Delta)(z)\big) \quad \text{for all } z \in \mathbf{H}.$$

Since Δ is a modular form of weight 12, we have

$$\Delta(\gamma z) = j(\gamma, z)^6 \Delta(z) \quad \text{for all } \gamma \in \mathrm{PSL}_2(\mathbf{Z}) \text{ and } z \in \mathbf{H}. \tag{8}$$

Define a logarithm $\mathrm{Ln}\, j$, holomorphic in z, by

$$(\mathrm{Ln}\, j)(\gamma, z) = \frac{1}{6}\big((\mathrm{Ln}\,\Delta)(\gamma z) - (\mathrm{Ln}\,\Delta)(z)\big). \tag{9}$$

(Beware that this logarithm $(\mathrm{Ln}\, j)(\gamma, z)$ is *another* logarithm as the $(\ln j)(\gamma, z)$ previously defined.) It is a straightforward consequence of (8) that

$$(\mathrm{Ln}\, j)(\gamma_2\gamma_1, z) = (\mathrm{Ln}\, j)(\gamma_2, \gamma_1 z) + (\mathrm{Ln}\, j)(\gamma_2, z) \tag{10}$$

for all $\gamma_1, \gamma_2 \in \mathrm{PSL}_2(\mathbf{Z})$ and $z \in \mathbf{H}$. For $\gamma \in \mathrm{PSL}_2(\mathbf{Z})$ and $z \in \mathbf{H}$, we have

$$\exp\big(6(\mathrm{Ln}\, j)(\gamma, z)\big) = \frac{\Delta(\gamma z)}{\Delta(z)} = j(\gamma, z)^6 \quad \text{and} \quad \exp\big(6(\ln j)(\gamma, z)\big) = j(\gamma, z)^6,$$

so that the difference $6(\mathrm{Ln}\, j)(\gamma, z)\big) - 6(\ln j)(\gamma, z)$ is in $2\pi i\mathbf{Z}$, and independent of z. Define a cochain $c_s \in C^1(\mathrm{PSL}_2(\mathbf{Z}), \mathbf{T})$ by

$$c_s(\gamma^{-1}) = \exp\Big(\frac{s}{12}\big(6(\mathrm{Ln}\, j)(\gamma, z) - 6(\ln j)(\gamma, z)\big)\Big) \quad \text{for all } \gamma \in \mathrm{PSL}_2(\mathbf{Z})$$

(we repeat that the right-hand side above has the same value for all $z \in \mathbf{H}$). The coboundary of c_s is given by

$$\begin{aligned} dc_s(\gamma_1^{-1}, \gamma_2^{-1}) &= c_s(\gamma_2^{-1}) c_s(\gamma_1^{-1}\gamma_2^{-1})^{-1} c_s(\gamma_1^{-1}) \\ &= \exp\Big(\frac{s}{12}\big(6(\mathrm{Ln}\, j)(\gamma_2, \gamma_1 z) - 6(\ln j)(\gamma_2, \gamma_1 z)\big)\Big) \\ &\quad \cdot \exp\Big(\frac{s}{12}\big(-6(\mathrm{Ln}\, j)(\gamma_2\gamma_1, z) + 6(\ln j)(\gamma_2\gamma_1, z)\big)\Big) \\ &\quad \cdot \exp\Big(\frac{s}{12}\big(6(\mathrm{Ln}\, j)(\gamma_1, z) - 6(\ln j)(\gamma_1, z)\big)\Big) \\ &\overset{\text{by (10)}}{=} \exp\Big(\frac{s}{2}(\ln j)(\gamma_2\gamma_1, z) - \frac{s}{2}(\ln j)(\gamma_2, \gamma_1 z) - \frac{s}{2}(\ln j)(\gamma_1, z)\Big) \\ &= \frac{j(\gamma_2\gamma_1, z)^{s/2}}{j(\gamma_2, \gamma_1 z)^{s/2} j(\gamma_1, z)^{s/2}} \overset{\text{by (4)}}{=} m_s(\gamma_1^{-1}, \gamma_2^{-1}) \end{aligned}$$

for all $\gamma_1, \gamma_2 \in \mathrm{PSL}_2(\mathbf{Z})$, so that m_s restricted to $\mathrm{PSL}_2(\mathbf{Z})$ is the coboundary of c_s. This ends our alternative proof that the restriction to $\mathrm{PSL}_2(\mathbf{Z})$ of the cocycle $m_s \in Z^2(\mathrm{PSL}_2(\mathbf{R}), \mathbf{T})$ is a coboundary.

It follows that, for any $s > 1$, the restriction to $\mathrm{PSL}_2(\mathbf{Z})$ of the m_s-projective representation $\check{\pi}_s$ is equivalent to an ordinary representation. The von Neumann algebra $((\check{\pi}_s|_{\mathrm{PSL}_2(\mathbf{Z})})(\mathrm{PSL}_2(\mathbf{Z})))''$ is therefore isomorphic to the factor $\mathrm{vN}(\mathrm{PSL}_2(\mathbf{Z}))$, and $\mathcal{H}_s$ is naturally a $\mathrm{vN}(\mathrm{PSL}_2(\mathbf{Z}))$-module. In the present situation, formula (1) shows that

$$\begin{aligned} \dim_{\mathrm{vN}(\mathrm{PSL}_2(\mathbf{Z}))} \mathcal{H}_{\check{\pi}_s} &= d_{\check{\pi}_s} \operatorname{vol}\big(\mathrm{PSL}_2(\mathbf{R}) / \mathrm{PSL}_2(\mathbf{Z})\big) \\ &= \frac{s-1}{4\pi}\frac{\pi}{3} = \frac{s-1}{12}. \end{aligned} \tag{11}$$

For a positive even integer $k \geq 12$, denote by S_k the space of cusp form of weight k for $\mathrm{PSL}_2(\mathbf{Z})$. It is a space of finite dimension, more precisely of dimension $[\frac{k}{12}]$ if $k \not\equiv 2 \pmod{12}$ and $[\frac{k}{12}] - 1$ if $k \equiv 2 \pmod{12}$; in particular $\dim S_k > 0$ when k is even and $k = 12$ or $k \geq 16$; see for example [Iwan–97, Theorem 1.4]. It is a fact that multiplication by $f \in S_k$ provides an intertwining operator T_f^s from the space $\mathcal{H}_s$ of $\check{\pi}_s|_{\mathrm{PSL}_2(\mathbf{Z})}$ to the space $\mathcal{H}_{s+k}$ of $\check{\pi}_{s+k}|_{\mathrm{PSL}_2(\mathbf{Z})}$. (The S stands for *Spitzenform*, and the T for *Toeplitz operator*.) For two cusp forms $f, g \in S_k$, it follows that $(T_f^s)^* T_g^s$ is an operator in the commutant $(\check{\pi}_s(\mathrm{vN}(\mathrm{PSL}_2(\mathbf{Z}))))'$ of $\mathrm{vN}(\mathrm{PSL}_2(\mathbf{Z}))$, in particular $(T_f^s)^* T_g^s$ has a canonical trace. Using a method which is analogous to that used for his computation of the Murray–von Neumann dimension $\dim_{\mathrm{vN}(\Gamma)} \mathcal{H}_\pi$, Jones computed this trace and found that it is the Petersson inner product $\langle f \mid g \rangle$, up to a canonical scalar.

Vaughan initially wrote all this (at least for ordinary representations) in a notebook [Jones–NB] and part of it appeared in Section 3.3 of [GoHJ–89]. This generated a series of open problems, but Vaughan decided to give up working on them and in 1993 he handed the notebook to Florin. One problem was solved in [Radu–94] (see also [Radu–14]), where it is shown that operators of the form $(T_f^s)^* T_g^s$, with $f, g \in S_k$ and $k \in \{12, 16, 18, 20, 22, \dots\}$, generate the whole of the commutant of $(\check{\pi}_s(\mathrm{PSL}_2(\mathbf{Z})))''$. Lately, Vaughan directed a PhD student, Jun Yang, who generalized the above results to the case of lattices in higher rank semi-simple Lie groups with discrete series [Yang–20] (the advisor for the final form of the thesis was Dietmar Bisch).

In 2019, Vaughan came back to this construction and asked Florin whether he still had his notebook. After a lot of search in my studio in Rome, Jacopo Bassi and myself, while working on [BaRa–22], were very happy to find it and to send a scan to Vaughan. He then remembered that, at the time, he had a very specific way to calculate Murray–von Neumann dimensions, and the method is contained in [Jones].

Zero sets of functions in Bergman spaces

In [Jones], Vaughan found again a miraculous application of the theory of type II_1 factors. It is an illustration of Vaughan's ideas that many of the results in this theory have consequences that might be formulated with no reference to the theory of II_1 factors, although the proofs depend on this theory. Vaughan finds an alternative characterization of the density of points in the orbit of a Fuchsian group acting on the upper half-plane, by looking at analytic functions that vanish on the orbit.

Analyzing the set of zero points for analytic functions on the unit disc is a problem that, as Vaughan mentions quoting the book of Duren [Dure–70], was completely understood for the Hardy space already in 1915, by Szegö. (According to the editors, Vaughan should have written Blaschke, instead of Szegö, see [Blas–15].) For Bergman spaces, Vaughan quotes the book [HeKZ–00], where such problems are solved by introducing various density measures for countable sets of points in the disc. The problem was also extensively studied by Seip [Seip–93]. In [Jones], Vaughan defines the density $\Omega(S)$ of a subset S of the upper half-plane (equivalently a subset S of the unit disc) as the infimum of the weights s such that the Bergman space $\mathcal{H}_s$ contains a non-zero analytic function vanishing on S. Vaughan determines $\Omega(S)$ for sets S that are the orbit of a point z in the upper half-plane, under the action of a Fuchsian group Γ of the first kind.

The main ingredient is the Murray–von Neumann dimension theory. Indeed, let $\mathrm{vN}(\Gamma)_s$ be the II_1 factor generated by $\check{\pi}_s(\Gamma)$. Let $\varepsilon_z \in \mathcal{H}_s$ be the evaluation vector at the point $z \in \mathbf{H}$, such that $f(z) = \langle f \mid \varepsilon_z \rangle$ for $f \in \mathcal{H}_s$; observe that $\check{\pi}_s(\gamma)\varepsilon_z$ is a scalar multiple of $\varepsilon_{\gamma z}$. Then the closed linear span of the orbit $\check{\pi}_s(\Gamma)\varepsilon_z$ has Murray–von Neumann dimension at most 1; it follows that, if the Murray–von Neumann dimension of $\mathcal{H}_s$ is strictly larger than 1, there exists a non-zero vector $f \in \mathcal{H}_s$ orthogonal to $\varepsilon_{\gamma z}$ for all $\gamma \in \Gamma$. This shows the "easy half" of the main result of [Jones]:

(12) if $\dim_{\mathrm{vN}(\Gamma)_s} \mathcal{H}_s = \frac{s-1}{4\pi}(\text{hyperbolic area of } \mathbf{H}/\Gamma) > 1$,
for all z in the upper half-plane there exists
a function $f \neq 0$ in $\mathcal{H}_s$ which vanishes on the orbit Γz.

In fact, in an older paper [Pere–73], Peremolov introduced the notion of completeness of a coherent system, and obtained estimates that were very close to the values of Jones. It seems that the relation to Peremolov article was observed in [Bekk–04], where Bekka generalizes formula (1) and uses it for frame analysis. In the context of wavelets, recent preprints consider the dimension formula introduced by Vaughan to determine completeness of coherent systems; see [AbSp–22] and [CavV–22].

On ordered groups

A deep result in [Jones] is that the converse of (12) holds:

(13) if there exists a function $f \neq 0$ in $\mathcal{H}_s$ which vanishes on some orbit,
then $\dim_{\mathrm{vN}(\Gamma)_s} \mathcal{H}_s = \frac{s-1}{4\pi}(\text{hyperbolic area of } \mathbf{H}/\Gamma) > 1$.

This is a statement that apparently does not have any operator algebra content, however the proof requires deep facts from the Murray–von Neumann dimension theory. It requires the explicit construction of trace vectors in the Bergman space, for which Vaughan uses in a very clever way the theory of ordered groups.

An unfinished work

Vaughan wanted very much to work further on the subject of his paper, as shown in particular by the following two excerpts from his e-mails.

The first was sent to Florin and dated November 5, 2019.

The left invariant ordering is really quite concrete. You can just look at the action on **R** *I think and get the order from that of* **R**. *I knew about it because it was fashionable about 20 years ago when Dehornoy showed it for the braid groups. I got to know Dehornoy quite well. Very sadly he died a few weeks ago. Here is a nice article of Rolfsen about it:* [Rolf–01].

Vaughan also observes that if the Murray–von Neumann dimension of $\mathcal{H}_s$ is 1, although the existence of cyclic trace vectors is known, the explicit construction of such a vector remains a mystery. He repeatedly pointed this out in his lectures, and he intended to work on it, by using his new, breakthrough method.

The second concerns the ambitious project underlying [Jones]. Vaughan was interacting frequently with Curtis McMullen about the complex analytic aspects of this topic, and explained his motivations in an email to him as follows:

> *The reason I would like so much to have an explicit cyclic and separating trace vector is that it would give an explicit isomorphism between* $\mathrm{vN}(\mathrm{PSL}_2(\mathbf{Z}))$*, shown by Voiculescu to have everything to do with random matrices, and the algebra given by cusp forms, also known to have a lot to do with random matrices.*

The study of zeros of functions in Bergman space ultimately allowed him to use orderings of the free group in his beautiful construction of a wandering subspace [Jones, Theorem 6.2].

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