Free products from spinning and rotating families

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Abstract. The far-reaching work of Dahmani–Guirardel–Osin (2017) and recent work of Clay–Mangahas–Margalit (2021) provide geometric approaches to the study of the normal closure of a subgroup (or a collection of subgroups) in an ambient group G. Their work gives conditions under which the normal closure in G is a free product. In this paper we unify their results and simplify and significantly shorten the proof of the theorem of Dahmani–Guirardel–Osin (2017).

1. Introduction

Using geometry to understand the algebraic properties of a group is a primary aim of geometric group theory. This paper focuses on detecting when a group has the structure of a free product. The following theorem follows from Bass–Serre theory.

Theorem 1.1. *Suppose a group G acts on a simplicial tree T without inversions and with trivial edge stabilizers. Suppose also that G is generated by the vertex stabilizers G_v. Then, there is a subset $\mathcal{O}$ of the set of vertices of T intersecting each G-orbit in one vertex such that*

$$G = *_{v \in \mathcal{O}} G_v.$$

Dahmani–Guirardel–Osin [9], inspired by the ideas of Gromov [13], provided a far-reaching generalization of the theorem above. The simplicial tree above is replaced by a δ-hyperbolic space, and the group acts via a *very rotating* family of subgroups. Under these conditions, they conclude that the group is a free product of subgroups in the family.

Theorem 1.2 ([9, Theorem 5.3a]). *Let G be a group acting by isometries on a δ-hyperbolic geodesic metric space X, and let $\mathcal{C} = (C, \{G_c \mid c \in C\})$ be a ρ-separated very rotating family for a sufficiently large $\rho = \rho(\delta)$. Then the subgroup of G generated by the set $\bigcup_{c \in C} G_c$ is isomorphic to the free product $*_{c \in C'} G_c$, for some subset $C' \subset C$. Moreover, every element in this subgroup is either a loxodromic isometry of X or it is contained in some G_c.*

The set of *apices* $C \subset X$ (and also the pair $\mathcal{C}$) is *ρ-separated* if $d(c, c') \geq \rho$ for all distinct $c, c' \in C$. The family $\{G_c\}_{c \in C}$ of subgroups of G is *rotating* if

(i) C is G-invariant,

(ii) G_c fixes c for every $c \in C$,

(iii) $G_{g(c)} = gG_cg^{-1}$ for every $g \in G$ and $c \in C$.

Note that G_c is a normal subgroup of the stabilizer $\mathrm{Stab}_G(c)$ and similarly the subgroup $\langle G_c \mid c \in C \rangle$ of G generated by all of the G_c is normal in G. A rotating family is *very rotating* if in addition

(iv) for any distinct $c, c' \in C$ and every $g \in G_c \setminus \{1\}$ every geodesic between c' and $g(c')$ passes through c.

Note that (iv) is a bit weaker in the presence of sufficient separation than the definition in [9]; see [9, Lemma 5.5]. As an application, Dahmani–Guirardel–Osin solve a long-standing open problem by showing that the normal closure of a suitable power of any pseudo-Anosov mapping class in a mapping class group is free and all non-trivial elements in the normal closure are pseudo-Anosov. We discuss this in more detail below.

An important variation of the Dahmani–Guirardel–Osin theorem was recently proved by Clay–Mangahas–Margalit [7]. In that setting, the group G acts on a *projection complex* via a *spinning family* of subgroups. As an application, they determine the isomorphism type of the normal closure of a suitable power of various kinds of elements in the mapping class group. We will discuss this in more detail below as well. See related work in [6, 8, 10].

Theorem 1.3 ([7, Theorem 1.6]). *Let G be a group acting by isometries on a projection complex $\mathcal{P}$ with vertex set $V\mathcal{P}$ and preserving the projection data $(\mathcal{Y}, \{\pi_X(Y)\}, \theta)$. Let $\{G_c\}_{c \in V\mathcal{P}}$ be an L-spinning family of subgroups of G for $L = L(\mathcal{P})$ sufficiently large. Then the subgroup of G generated by the set $\{G_c\}_{c \in V\mathcal{P}}$ is isomorphic to the free product $*_{c \in \mathcal{O}} G_c$ for some subset $\mathcal{O} \subset V\mathcal{P}$. Moreover, every element of the subgroup is either loxodromic in $\mathcal{P}$ or is contained in some G_c.*

We next explain the terminology in the above theorem. The *projection data* is a collection of metric spaces $\mathcal{Y} = \{X, Y, Z, \ldots\}$ (with infinite distance within a metric

space allowed) together with "projections" $\pi_X(Y) \subset X$ for $X, Y \in \mathcal{Y}$ distinct, satisfying the following *projection axioms* for some $\theta \geq 0$ (called the *projection constant*), where we set $d_X(Y, Z) = \mathrm{diam}(\pi_X(Y) \cup \pi_X(Z))$:

(P1) diam $\pi_X(Y) \leq \theta$ for any $X \neq Y$,

(P2) (the Behrstock inequality) if $d_X(Y, Z) > \theta$ then $d_Y(X, Z) \leq \theta$, and

(P3) for any X, Y, the set $\{Z \neq X, Y \mid d_Z(X, Y) > \theta\}$ is finite.

From this data, Bestvina–Bromberg–Fujiwara [1] construct a graph $\mathcal{P} = \mathcal{P}(\mathcal{Y})$, called the *projection complex*, with the vertices in 1-1 correspondence with the spaces in $\mathcal{Y}$. Roughly speaking, X and Y are connected by an edge if $d_Z(X, Y)$ is small for any Z. This graph is connected, and it is quasi-isometric to a tree. Any group G acting by isometries on the disjoint union $\bigsqcup_{X \in \mathcal{Y}} X$, permuting the spaces and commuting with projections (i.e., $g(\pi_X(Y)) = \pi_{g(X)}g(Y)$), acts by isometries on $\mathcal{P}$, and we say that G *preserves the projection data*.

An *L-spinning family* is a family $\{G_c\}$ parametrized by the vertices $c \in V\mathcal{P}$ satisfying

(a) G_c fixes c,

(b) $G_{g(c)} = gG_cg^{-1}$ for $g \in G$, $c \in C$, and

(c) $d_c(c', g(c')) > L$ for $c \neq c'$ and $g \in G_c \setminus \{1\}$.

The main goal of this paper is to simplify and significantly shorten the proof of the Dahmani–Guirardel–Osin theorem using the Clay–Mangahas–Margalit theorem and the machinery of projection complexes. We also present a variant of the proof of the [7] theorem to directly construct an action of the group on a tree as in Theorem 1.1. Given a group action on a δ-hyperbolic metric space equipped with a very rotating family of subgroups, we construct an action of that group on a projection complex with the same family acting as a spinning family. While our proof of Theorem 1.3 still uses the construction of windmills (which are used in [9] and [13]), our work differs from [7] in that we find a natural tree on which G acts as in Theorem 1.1 and eliminate the need to work with normal forms. We also introduce the notion of *canoeing* in a projection complex, which is inspired by the classic notion of canoeing in the hyperbolic plane (see Section 4), and enables us to further streamline some of the arguments from [7].

Theorem 1.4. *Let G be a group acting by isometries on a δ-hyperbolic metric space X. Let $\mathcal{C} = (C, \{G_c \mid c \in C\})$ be a rotating family, where $C \subset X$ is ρ-separated for $\rho \geq 20\delta$ and $G_c \leq G$. Then the following hold.*

(1) *The group G acts by isometries and preserves the projection data on a projection complex associated to C, with the projection constant $\theta = \theta(\delta)$.*

(2) *If $\mathcal{C}$ is a very rotating family, then the family of subgroups $\{G_c\}_{c \in \mathcal{C}}$ forms an $L(\rho)$-spinning family for the action of G on the projection complex.*

(3) $L(\rho) \to \infty$ *as* $\rho \to \infty$. *In particular, we can take* $L = 2^{(\rho - 6\delta - 4)/2\delta} - 4 - 248\delta$, *so that* L *grows exponentially with respect to* ρ.

To prove Theorem 1.4, we construct a projection complex via the Bestvina–Bromberg–Fujiwara axioms. These axioms require us to first define for each $c \in C$ a metric space S_c (with infinite distances allowed) and projections $\pi_{S_c}(S_{c'})$, which we abbreviate to $\pi_c(c')$. A standard example of such a construction is the following. Take a closed hyperbolic surface S and a closed (not necessarily simple) geodesic α. Consider the universal cover $\widetilde{S} = \mathbb{H}^2$ and the set $\mathcal{Y}$ of all lifts of α. For two different lifts $A, B \in \mathcal{Y}$, define $\pi_A(B) \subset A$ to be the nearest point projection of B to A. This will be an open interval in A whose diameter is uniformly bounded independently of A, B (but which depends on α). Roughly speaking, $\pi_A(B)$ can have a large diameter only of B fellow travels A for a long time. It is not hard to see that the projection axioms hold in this case. A similar construction can be carried out when S is a hyperbolic surface with a cusp and α is a horocycle. Now $\mathcal{Y}$ is an orbit of pairwise disjoint horocycles in $\mathbb{H}^2$ and $\pi_A(B)$ is defined as the nearest point projection of B to A as before. There are now two natural choices of a metric on horocycles in $\mathcal{Y}$: one can take the intrinsic metric so that it is isometric to $\mathbb{R}$ or the induced metric from $\mathbb{H}^2$. Either choice satisfies the projection axioms, but note here that the intrinsic metric can also be defined as the path metric where paths are not allowed to intersect the open horoball cut out by the horocycle.

The starting point of our proof of Theorem 1.2 is the construction of the projection complex whose vertex set is the set C of apices, inspired by the horocycle example. To each $c \in C$ we associate the sphere S_c of radius R centered at c. The number R is chosen carefully. The open balls B_c cut out by the spheres should be pairwise disjoint and a reasonable distance apart (a fixed multiple of δ), yet big enough so that paths in the complement of B_c joining points $x, y \in S_c$ on opposite sides of the ball are much longer (exponential in R) than a geodesic in X joining x, y (which is linear in R). The projection $\pi_c(c')$ for $c, c' \in \mathcal{C}$, $c \neq c'$ is the set of all points in S_c that lie on a geodesic between c and c'. The metric we take on S_c is induced by the path metric in $X \setminus B_c$ (this can take value ∞ if the ball disconnects X). We check that with these definitions the projection axioms hold (see Section 3.1). Thus, the group G acts on the projection complex and we check that the groups G_c form a spinning family (see Section 3.2), which proves Theorem 1.4.

The same proof goes through with a slightly weaker hypothesis that the family $\{G_c\}$ is *fairly rotating*, instead of very rotating (with slightly different constants). Here we require only that geodesics between c' and $g(c')$ pass within 1 of c, for $g \in G_c \setminus \{1\}$,

instead of passing through c. This situation naturally occurs. As an example, consider a closed hyperbolic orbifold S with one cone point, with cone angle $2\pi/n$ for $n > 2$. The orbifold universal cover $\widetilde{S} = \mathbb{H}^2$ admits an action by the orbifold fundamental group G where the stabilizers $G_c \cong \mathbb{Z}/n\mathbb{Z}$ of the lifts $c \in C$ of the cone point form a rotating family. This family will never be very rotating, but it will be fairly rotating if the pairwise distance between distinct elements of C is large enough, given by a function of n.

Note that in Theorem 1.3, the constant $L(\mathcal{P})$ really depends only on the projection constant θ and can be taken to be a fixed multiple of θ (e.g. 1000θ will do). In Theorem 1.4 the projection constant θ in (1) can be taken to be a fixed multiple of δ, and $L(\rho)$ in (2) will be an exponential function in ρ. Since exponential functions grow faster than linear functions, the spinning constant L in Theorem 1.4 will beat the one in Theorem 1.3 if ρ is big enough, so Theorem 1.2 will follow (see Section 5).

We now say a few words about our proof of Theorem 1.3. As in [7], we recursively define a sequence of windmills which correspond to certain orbits of larger and larger collections of the vertex subgroups $\{G_c\}$. At each stage we prove that these windmills have a tree-like structure (technically, the skeleton of the canonical cover of each windmill is a tree). At each step we obtain a new group that is the free product of the previous group with a suitable collection of G_c's, and taking the limit proves the theorem, see Section 4.2. Canoeing enters when we verify that windmills have a tree-like structure. The simplest example of a canoeing path in $\mathcal{P}$ would be an edge-path passing through vertices $v_1, v_2, \ldots, v_k$ such that for every $i = 2, 3, \ldots, k - 1$ the "angle" $d_{v_i}(v_{i-1}, v_{i+1})$ is large. The basic properties of projection complexes quickly imply that such paths are embedded, and they provide a local-to-global principle enabling us to establish the tree-like structure.

We end this introduction with some applications of Theorems 1.2 and 1.3. Suppose G acts by isometries on a hyperbolic space Y and $g \in G$ is loxodromic. Suppose also that g is a "WPD element" as per Bestvina–Fujiwara [3]. This amounts to saying that g is contained in a unique maximal virtually cyclic subgroup $EC(g)$ (the elementary closure of $\langle g \rangle$) and further that the set of G-translates of a fixed $EC(g)$-orbit is "geometrically separated", i.e., the nearest point projections satisfy the projection axioms. This situation generalizes the example above, where $Y = \mathbb{H}^2$, G is the deck group of the universal cover $Y \to S$, and g corresponds to an indivisible element in G, so $EC(g) = \langle g \rangle$. This situation is fairly common. For example, Y could be the curve complex of a surface of finite type, G its mapping class group, and $g \in G$ a pseudo-Anosov mapping class (see [3]). For another example, take G to be the Cremona group (of birational transformations of $\mathbb{CP}^2$) acting on infinite dimensional real hyperbolic space, see [5]. In these situations one can construct a space X by coning off the orbit of $EC(g)$ and each translate. If the radius of the cone is large enough,

Dahmani–Guirardel–Osin show that X is hyperbolic and if $N \trianglelefteq EC(g)$ is a sufficiently deep finite index normal subgroup, then the set of G-conjugates of N forms a very rotating family with the cone points as apices. In particular, they resolved a long-standing open problem by showing that if g is a pseudo-Anosov homeomorphism of a finite-type surface, then there is $n > 0$ such that the normal closure of g^n in the mapping class group is the free group F_∞ (of infinite rank), and all non-trivial elements of the group are pseudo-Anosov.

Clay–Mangahas–Margalit reproved this application to mapping class groups directly from Theorem 1.3 and gave new applications of their theorem. To illustrate, consider a mapping class g on a finite-type surface S which is supported on a proper, connected, π_1-injective subsurface $A \subset S$ such that $g|A$ is a pseudo-Anosov homeomorphism of A. Assume also that any two subsurfaces in the orbit of A either coincide or intersect. There is a natural projection complex one can construct from this setup. The vertices are the subsurfaces in the mapping class group orbit of A, and the projection $\pi_A(B)$ is the Masur–Minsky subsurface projection [14, 15]. It follows from the work of Masur–Minsky and Behrstock that the projection axioms hold in this setting. Clay–Mangahas–Margalit prove that for a suitable $n > 0$ the collection of conjugates of $\langle g^n \rangle$ forms a spinning family and conclude that, here too, the normal closure of g^n is free. They consider more general situations where the normal closure can be a non-free group as well. Remarkably they can exactly determine the normal closure even in this case. For example, if S is a closed surface of even genus and g is pseudo-Anosov supported on exactly half the surface, then for suitable $n > 0$ the normal closure of g^n is the infinite free product of copies of $F_\infty \times F_\infty$.

Outline. Preliminaries are given in Section 2. In Section 3 we construct a group action on a projection complex from the rotating family assumptions of Dahmani–Guirardel–Osin. Section 4 contains the new proof of the result of Clay–Mangahas–Margalit via canoeing paths in a projection complex. In Section 5 we give the new proof of the result of Dahmani–Guirardel–Osin using projection complexes. Section 6 contains proofs of the moreover statements of the Dahmani–Guirardel–Osin and Clay–Mangahas–Margalit theorems. That is, we prove that elements of the corresponding groups act either loxodromically or are contained in one of the given rotating/spinning subgroups.

2. Preliminaries

In this section, we state the relevant result of Dahmani–Guirardel–Osin, give background on projection complexes, state the result of Clay–Mangahas–Margalit, and give the necessary background on δ-hyperbolic spaces, in that order.

2.1. Rotating subgroups and the result of Dahmani–Guirardel–Osin.

Definition 2.1 (Gromov's rotating families, [9, Definition 2.12]). Let G be a group acting by isometries on a metric space X. A *rotating family* $\mathcal{C} = (C, \{G_c \mid c \in C\})$ consists of a subset $C \subset X$ and a collection $\{G_c \mid c \in C\}$ of subgroups of G such that the following conditions hold.

(a-1) The subset C is G-invariant;

(a-2) Each group G_c fixes c;

(a-3) $G_{gc} = gG_cg^{-1}$ for all $g \in G$ and for all $c \in C$.

The elements of the set C is called the *apices* of the family, and the groups G_c are called the *rotation subgroups* of the family.

(b) (Separation) The subset C is ρ-*separated* if any two distinct apices are at distance at least ρ.

(c) (Very rotating condition) When X is δ-hyperbolic with $\delta > 0$, one says that $\mathcal{C}$ is *very rotating* if for all $c \in C$, all $g \in G_c - \{1\}$, and all $x, y \in X$ with both $d(x, c)$ and $d(y, c)$ in the interval $[20\delta, 40\delta]$ and $d(gx, y) \leq 15\delta$, then any geodesic from x to y contains c.

We will actually make use of a weaker version of the very rotating condition.

(c′) (Fairly rotating condition) When X is δ-hyperbolic with $\delta > 0$, one says that $\mathcal{C}$ is *fairly rotating* if for all $c \in C$, all $g \in G_c - \{1\}$, and all $x \in C$ with $x \neq c$, there exists a geodesic from x to gx that non-trivially intersects the ball of radius 1 around c.

Remark 2.2. Property (c) implies property (c′) by [9, Lemma 5.5].

Example 2.3 ([9, Example 2.13]). Let $G = H * K$, and let X be the Bass–Serre tree for this free product decomposition. Let $C \subset X$ be the set of vertices, and let G_c be the stabilizer of $c \in C$. Then, $\mathcal{C} = (C, \{G_c \mid c \in C\})$ is a 1-separated very rotating family.

Dahmani–Guirardel–Osin [9] prove a partial converse to the example above as follows.

Theorem 2.4 ([9, Theorem 5.3a]). *Let G be a group acting by isometries on a δ-hyperbolic geodesic metric space, and let $\mathcal{C} = (C, \{G_c \mid c \in C\})$ be a ρ-separated very rotating family for some $\rho \geq 200\delta$. Then, the normal closure in G of the set $\{G_c\}_{c \in C}$ is isomorphic to a free product $*_{c \in C'} G_c$, for some (usually infinite) subset $C' \subset C$.*

2.2. Projection complexes. Bestvina–Bromberg–Fujiwara [1] defined projection complexes via a set of projection axioms given as follows.

Definition 2.5 (Projection axioms, [1, Sections 1 and 3.1]). Let $\mathcal{Y}$ be a set of metric spaces (in which infinite distances are allowed), and for each $Y \in \mathcal{Y}$, let

$$\pi_Y \colon (\mathcal{Y} - \{Y\}) \to 2^Y$$

satisfy the following axioms for a *projection constant* $\theta \geq 0$, where we set $d_Y(X, Z) = \operatorname{diam}(\pi_Y(X) \cup \pi_Y(Z))$ for any $X, Z \in \mathcal{Y} - \{Y\}$.

(P1) $\operatorname{diam}(\pi_Y(X)) \leq \theta$ for all $X \neq Y$,

(P2) (The Behrstock inequality) If $d_Y(X, Z) > \theta$, then $d_X(Y, Z) \leq \theta$, and

(P3) For any X, Z, the set $\{Y \in \mathcal{Y} - \{X, Z\} \mid d_Y(X, Z) > \theta\}$ is finite.

We then say that the collection $(\mathcal{Y}, \{\pi_Y\})$ satisfies the *projection axioms*. We call the set of functions $\{d_Y\}$ the *projection distances*.

If axiom (P2) is replaced with

(P2+) if $d_X(Y, Z) > \theta \Rightarrow d_Y(Z, W) = d_Y(X, W)$ for all X, Y, Z, W distinct,[1]

then we say that the collection $(\mathcal{Y}, \{\pi_Y\})$ satisfies the *strong projection axioms*.

Bestvina–Bromberg–Fujiwara–Sisto [2] proved that one can upgrade a collection satisfying the projection axioms to a collection satisfying the strong projection axioms as follows.

Theorem 2.6 ([2, Theorem 4.1]). *Assume that $(\mathcal{Y}, \{\pi_Y\})$ satisfies the projection axioms with projection constant θ. Then, there are $\{\pi'_Y\}$ satisfying the strong projection axioms with projection constant $\theta' = 11\theta$ and such that $d_Y - 2\theta \leq d'_Y \leq d_Y + 2\theta$, where $\{d_Y\}$ and $\{d'_Y\}$ are the projection distances coming from $\{\pi_Y\}$ and $\{\pi'_Y\}$, respectively.*

Definition 2.7 (Projection complex). Let $\mathcal{Y}$ be a set that satisfies the strong projection axioms with respect to a constant $\theta \geq 0$. Let $K \in \mathbb{N}$. The *projection complex* $\mathcal{P} = \mathcal{P}(\mathcal{Y}, \theta, K)$ is a graph with vertex set $V\mathcal{P}$ in one-to-one correspondence with elements of $\mathcal{Y}$. Two vertices X and Z are connected by an edge if and only if $d_Y(X, Z) \leq K$ for all $Y \in \mathcal{Y}$.

Throughout this paper, given a collection satisfying the projections axioms we will always apply Theorem 2.6 to upgrade our collection to satisfy the strong projection axioms, unless specified otherwise. We first prove that the projection axioms hold for θ

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[1]One can replace this with an even stronger axiom that $d_X(Y, Z) > \theta$ implies $\pi_Y(X) = \pi_Y(Z)$.

and then upgrade, but still label the projection constant θ instead of θ' by a slight abuse of notation. We will also assume that $K \geq 3\theta$ for the upgraded θ.

For the rest of this section, we will follow [2]. We refer to Sections 2 and 3 of [2] for any proofs that are omitted in the following. One virtue of strong projection axioms is that it provides a useful object, called a *standard path*, for studying the geometry of projection complexes. To define it, for any $X, Z \in \mathcal{Y}$ we consider the set $\mathcal{Y}_K(X, Z)$ defined as

$$\mathcal{Y}_K(X, Z) := \{Y \in \mathcal{Y} - \{X, Z\} \mid d_Y(X, Z) > K\}.$$

This set $\mathcal{Y}_K(X, Z)$ is finite by (P3). The elements of $\{X\} \cup \mathcal{Y}_K(X, Z) \cup \{Z\}$ can be totally ordered in a natural way, so that each pair of adjacent spaces is connected by an edge in the projection complex $\mathcal{P} = \mathcal{P}(\mathcal{Y}, \theta, K)$. The set $\{X\} \cup \mathcal{Y}_K(X, Z) \cup \{Z\}$ is a path between X and Z, which we define as the *standard path* between X and Z. In particular, this implies that the projection complex $\mathcal{P}$ is connected.

We can also concatenate two standard paths to make another standard path, as long as the 'angle' between the two standard paths is large enough.

Lemma 2.8 (Concatenation). *If $d_Y(X, Z) > K$, then the concatenation of $\mathcal{Y}_K(X, Y)$ followed by $\mathcal{Y}_K(Y, Z)$ is the standard path $\mathcal{Y}_K(X, Z)$.*

Proof. Suppose $d_Y(X, Z) > K$. Then by definition $Y \in \mathcal{Y}_K(X, Z)$. Let X' be any vertex in the standard path $\mathcal{Y}_K(X, Y)$. Then $d_{X'}(X, Y) > K$, so

$$d_Y(X', Z) = d_Y(X, Z) > K,$$

which further implies

$$d_{X'}(Z, X) = d_{X'}(Y, X) > K,$$

so $X' \in \mathcal{Y}_K(X, Z)$. Similarly, for any vertex Y' in $\mathcal{Y}_K(Y, Z)$, we can show that $Y' \in \mathcal{Y}_K(X, Z)$, concluding the proof. ∎

The following lemma says that triangles whose sides are standard paths are nearly tripods.

Lemma 2.9 (Standard triangles are nearly-tripods, [2, Lemma 3.6]). *For every $X, Y, Z \in \mathcal{Y}$, the path $\mathcal{Y}_K(X, Z)$ is contained in $\mathcal{Y}_K(X, Y) \cup \mathcal{Y}_K(Y, Z)$ except for at most two vertices. Moreover, in case that there are two such vertices, they are consecutive.*

This lemma is used to show that standard paths also form quasi-geodesics in the projection complex.

Lemma 2.10 (Standard paths are quasi-geodesics, [2, Corollary 3.7]). *Let $X \neq Z$ and let $n = |\mathcal{Y}_K(X, Z)| + 1$. Then, $\lfloor n/2 \rfloor + 1 \leq d_{\mathcal{P}}(X, Z) \leq n$.*
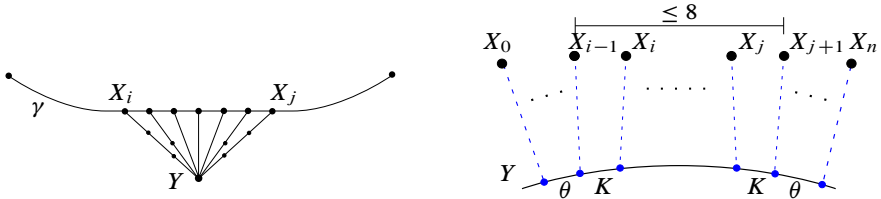
FIGURE 1

The bound in the Bounded geodesic image theorem is given by considering the configurations above. The geodesic γ is shown on the left, and projections onto Y are depicted on the right.

The next lemma will be used to prove the Bounded geodesic image theorem (Theorem 2.12).

Lemma 2.11. *Let $X, Z \in \mathcal{Y}$ be adjacent points in a projection complex. If $Y \in \mathcal{Y}$ satisfies $d_{\mathcal{P}}(Y, X) \geq 4$ and $d_{\mathcal{P}}(Y, Z) \geq 4$, then $d_Y(X, W) = d_Y(Z, W)$ for every $W \in \mathcal{Y} - \{Y\}$.*

Proof. By Lemma 2.9, $\mathcal{Y}_K(X, Y)$ and $\mathcal{Y}_K(Y, Z)$ share at least one vertex, call it Q. Then by definition we have $d_Q(X, Y) > K$ and $d_Q(Z, Y) > K$. Using (P2+) twice, for arbitrary $W \in \mathcal{Y} - \{Y\}$, we have

$$d_Y(X, W) = d_Y(Q, W) = d_Y(Z, W),$$

as desired. ∎

Now we prove the Bounded geodesic image theorem for projection complexes, used in Section 4. We include a proof in the case that the collection $(\mathcal{Y}, \{d_Y\})$ satisfies the strong projection axioms, as we will make explicit use of the constant obtained. The result holds with a different constant for the standard projection axioms by [1, Corollary 3.15].

Theorem 2.12 (Bounded geodesic image theorem). *If $\mathcal{P} = \mathcal{P}(\mathcal{Y}, \theta, K)$ is a projection complex obtained from a collection $(\mathcal{Y}, \{d_Y\})$ satisfying the strong projection axioms and γ is a geodesic in $\mathcal{P}$ that is disjoint from a vertex Y, then $d_Y(\gamma(0), \gamma(t)) \leq M$ for all t, where $M = 8K + 2\theta$.*

Proof. Let $\gamma = \{X_0, \ldots, X_n\}$ be a geodesic in $\mathcal{P}$ disjoint from a vertex Y. If γ is disjoint from the closed ball of radius 3 about Y, then by Lemma 2.11:

$$d_Y(X_0, X_n) = d_Y(X_1, X_n) = \cdots = d_Y(X_{n-1}, X_n) = d_Y(X_n, X_n) \underset{(P1)}{\leq} \theta.$$

Now assume γ intersects the closed ball B of radius 3 about Y, and let X_i be the first vertex that intersects B and X_j be the last one intersects B. Then $d_Y(X_0, X_{i-1}) \le \theta$ and $d_Y(X_{j+1}, X_n) \le \theta$ as in the first case. Now, by our choice

$$d_{\mathcal{P}}(X_{i-1}, X_{j+1}) \le d_{\mathcal{P}}(X_{i-1}, Y) + d_{\mathcal{P}}(Y, X_{j+1}) \le 8.$$

Also for each k such that $i - 1 \le k \le j$, we have $d_Y(X_k, X_{k+1}) \le K$ as X_k and X_{k+1} are adjacent. Therefore,

$$d_Y(X_0, X_r) \le 2\theta + d_Y(X_{i-1}, X_{j+1}) \le 2\theta + 8K \quad \text{for all } r = 0, \ldots, n. \qquad \blacksquare$$

We will not use the following theorem, but include it here for completeness. An analogous statement for the standard projection axioms was shown in [1]. The strong projection axiom case along with the specific bound on K recorded here was given by [2].

Theorem 2.13 ([1, 2]). *Let $\mathcal{Y}$ be a set that satisfies the strong projection axioms with respect to $\theta \ge 0$. If $K \ge 3\theta$, then the projection complex $\mathcal{P}(\mathcal{Y}, \theta, K)$ is quasi-isometric to a simplicial tree.*

2.3. Spinning subgroups and the result of Clay–Mangahas–Margalit.

Definition 2.14 ([7, Section 1.7]). Let $\mathcal{P}$ be a projection complex, and let G be a group acting on $\mathcal{P}$. For each vertex c of $\mathcal{P}$, let G_c be a subgroup of the stabilizer of c in $\mathcal{P}$. Let $L > 0$. The family of subgroups $\{G_c\}_{c \in V\mathcal{P}}$ is an *(equivariant) L-spinning family* of subgroups of G if it satisfies the following two conditions.

(1) (Equivariance) If $g \in G$ and c is a vertex of $\mathcal{P}$, then

$$gG_c g^{-1} = G_{gc}.$$

(2) (Spinning condition) If a and b are distinct vertices of $\mathcal{P}$ and $g \in G_a$ is non-trivial, then

$$d_a(b, gb) \ge L.$$

Theorem 2.15 ([7, Theorem 1.6]). *Let $\mathcal{P}$ be a projection complex, and let G be a group acting on $\mathcal{P}$. There exists a constant $L = L(\mathcal{P})$ with the following property. If $\{G_c\}_{c \in V\mathcal{P}}$ is an L-spinning family of subgroups of G, then there is a subset $\mathcal{O}$ of the vertices of $\mathcal{P}$ so that the normal closure in G of the set $\{G_c\}_{c \in V\mathcal{P}}$ is isomorphic to the free product $\ast_{c \in \mathcal{O}} G_c$.*

Remark 2.16. The constant L is linear in θ. See [7, Section 6 (Proof of Theorem 1.6) and Section 3.1].

We will also need the following lemma.

Lemma 2.17. *Suppose that $\mathcal{P} = \mathcal{P}(\mathcal{Y}, \theta, K)$ is a projection complex obtained from a collection $(\mathcal{Y}, \{d_Y\})$ satisfying the projection axioms. Let $\mathcal{P}' = \mathcal{P}'(\mathcal{Y}, \theta', K')$ be the projection complex obtained from upgrading this collection to a new collection $(\mathcal{Y}, \{d_Y'\})$ satisfying the strong projection axioms via Theorem 2.6. If $\{G_c\}_{c \in V\mathcal{P}}$ is an L-spinning family of subgroups of G acting on $\mathcal{P}$, then it is an L'-spinning family of subgroups of G acting on $\mathcal{P}'$ where $L' = L - 2\theta$.*

Proof. By Theorem 2.6, $d_Y' \geq d_Y - 2\theta$ for all $Y \in \mathcal{Y}$. ∎

2.4. Projections in a δ-hyperbolic space.

In this paper we use the δ-thin triangles formulation of δ-hyperbolicity given as follows. (See [4, Section III.H.1] and [11, Section 11.8] for additional background.) Given a geodesic triangle Δ there is an isometry from the set $\{a, b, c\}$ of vertices of Δ to the endpoints of a metric tripod T_Δ with pairs of edge lengths corresponding to the side lengths of Δ. This isometry extends to a map $\chi_\Delta \colon \Delta \to T_\Delta$, which is an isometry when restricted to each side of Δ. The points in the pre-image of the central vertex of T_Δ are called the *internal points* of Δ. The internal points are denoted by i_a, i_b, and i_c, corresponding to the vertices of Δ that they are opposite from; that is, the point i_a is on the side bc and likewise for the other two. We say that two points on the triangle are in the same *cusp* if they lie on the segments $[a, i_b]$ and $[a, i_c]$, or on the analogous segments for the other vertices of the triangle. The triangle Δ is δ-*thin* if $p, q \in \chi_\Delta^{-1}(t)$ implies that $d(p, q) \leq \delta$ for all $t \in T_\Delta$. In a δ-thin triangle two points lie in the same cusp if they are more than δ away from the third side. A geodesic metric space is δ-*hyperbolic* if every geodesic triangle is δ-thin.

Note that another common definition of δ-hyperbolicity requires that every geodesic triangle in the metric space is δ-*slim*, meaning that the δ-neighborhood of any two of its sides contains the third side. A δ-thin triangle is δ-slim; thus, if X is δ-hyperbolic with respect to thin triangles, then X is δ-hyperbolic with respect to slim triangles. We use this fact, as some the constants in the lemmas below are for a δ-hyperbolic space defined with respect to δ-slim triangles.

Definition 2.18. Let X be a metric space and let A be a closed subset of X. For $x \in X$ a *nearest-point projection* $\pi_A(x)$ of x to A is a point in A that is nearest to x.

Notation 2.19. Let X be a metric space and $a, b, p \in X$. We use $[a, b]$ to denote a geodesic from a to b. If γ is a path in X, we use $\ell(\gamma)$ to denote the length of γ. For $R \geq 0$, we use $B_R(p)$ to denote the open ball of radius R around the point p.

Lemma 2.20 ([11], Lemma 11.64]). *Let X be a δ-hyperbolic geodesic metric space. If $[x, y]$ is a geodesic of length $2R$ and m is its midpoint, then every path joining x and y outside the ball $B_R(m)$ has length at least $2^{(R-1)/\delta}$.*

3. A projection complex built from a very rotating family

In this section we construct a projection complex from a fairly rotating family. Throughout, let G be a group that acts by isometries on a δ-hyperbolic metric space X. Let $\mathcal{C} = (C, \{G_c \mid c \in C\})$ be a ρ-separated fairly rotating family for some $\rho \geq 20\delta$.

Definition 3.1 (Projections). Let $2 + 2\delta \leq R \leq \rho/2 - 3\delta$. For $p \in C$ let $S_p = \partial B_R(p)$ equipped with the restriction of the path metric on $d_{X \setminus B_R}(p)$, where two points are at infinite distance if they are in different path components of $X \setminus B_R(p)$. Set $\mathcal{Y} = \{S_p\}_{p \in C}$ and, for each $a \in C \setminus \{p\}$, let $\pi_p(a) \subset S_p$ be the set of nearest point projections of a to $\partial B_R(p)$ (equivalently, $\pi_p(a)$ consists of intersection points of geodesics $[p, a]$ with $\partial B_R(p)$).

We think of the associated projection distances, $d_p(b, c) = \text{diam}(\pi_p(b) \cup \pi_p(c))$, as the penalty (up to an error of a fixed multiple of δ) of traveling from b to c avoiding a ball of fixed radius around p.

The aim of this section is to prove the following theorem.

Theorem 3.2. *For $\theta \geq 121\delta$, the group G acts by isometries on a projection complex associated to the family $(\mathcal{Y}, \{\pi_p\}_{p \in C})$ satisfying the strong projection axioms for θ. Moreover, the family of subgroups $\{G_c\}_{c \in C}$ is an L-spinning family for $L = 2^{(R-2)/\delta} - 4 - 248\delta$.*

We prove the projection axioms are satisfied in Section 3.1, and we verify the spinning condition in Section 3.2.

3.1. Verification of the projection axioms.

Lemma 3.3. *Axiom (P1) holds for any $\theta \geq 4\delta$.*

Proof. Let $p, a \in C$ be distinct and let a', a'' be two points in $\pi_p(a)$. Then a' and a'' lie on two geodesics γ' and γ'' from a to p such that

$$a' = \gamma' \cap \partial B_R(p) \quad \text{and} \quad a'' = \gamma'' \cap \partial B_R(p).$$

Since geodesics in a δ-hyperbolic space 2δ-fellow travel (see e.g. [4, Chapter III.H, Lemma 1.15]), we can find a path in $X \setminus B_R(p)$ of length at most 4δ connecting a' and a'' by traversing along γ' from a' to a a distance of δ, then traversing a path of
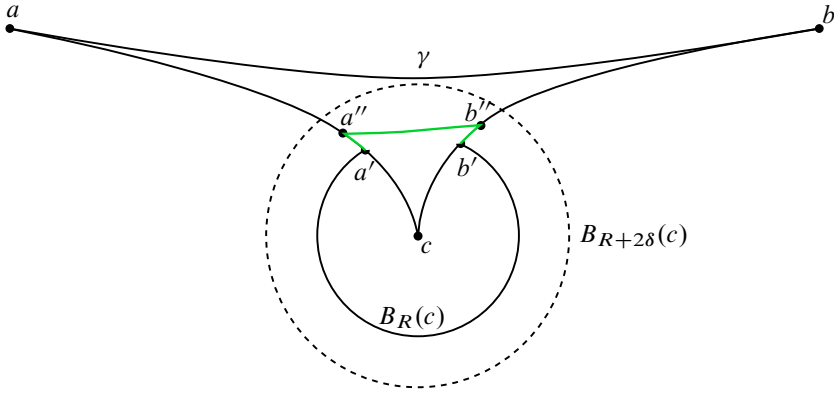
FIGURE 2

Configuration of points in Lemma 3.4. The path in green is a path of length at most 3δ between a' and b' which misses $B_R(c)$.

length at most 2δ from γ' to γ'', and finally traversing along γ'' a distance of at most δ back towards a''. If $d(p, a') < \delta$, then $d(a', a'') < 2\delta$. Thus, we see that

$$\operatorname{diam}(\pi_p(a)) \leq 4\delta. \qquad \blacksquare$$

To prove the remaining axioms we need the following lemma.

Lemma 3.4. *For any $a, b \in X$ and $c \in C$ such that some geodesic γ from a to b does not intersect $B_{R+2\delta}(c)$, we have $d_c(a, b) \leq 4\delta$.*

Proof. Let a' and b' be points in $\pi_c(a)$ and $\pi_c(b)$, respectively. Consider the triangle formed by γ and geodesics $[a, c]$ and $[b, c]$ where $a' \in [a, c]$ and $b' \in [b, c]$. Let a'' be the point on $[a, c]$ outside of $B_R(c)$ at distance δ from a', and define b'' analogously; see Figure 2. By hypothesis, a'' and b'' are more than δ away from γ so a'' and b'' must be in the same cusp of the geodesic triangle. Therefore, $d(a'', b'') \leq \delta$. Note that any geodesic $[a'', b'']$ misses $B_R(c)$, so it follows that $d_{X \setminus B_R(c)}(a', b') \leq 3\delta$ by concatenating geodesics $[a', a'']$, $[a'', b'']$, and $[b'', b']$. Now by Lemma 3.3, we see that

$$\operatorname{diam}(\pi_c(a) \cup \pi_c(b)) \leq 4\delta. \qquad \blacksquare$$

Lemma 3.5. *Axiom (P2) holds with respect to $\{d_a \mid a \in C\}$ and $\theta \geq 4\delta$.*

Proof. Suppose $d_a(b, c) > \theta$; we will show $d_b(a, c) \leq \theta$. By Lemma 3.4, every geodesic $[b, c]$ intersects $B_{R+2\delta}(a)$. Using the same lemma, we are done if we show some geodesic $[a, c]$ avoids $B_{R+2\delta}(b)$. Let a' be a nearest point projection of a to $[b, c]$,

and let $[a', c] \subset [b, c]$ be the subpath from a' to c. Note that a', and therefore any geodesic $[a, a']$, is contained in $B_{R+2\delta}(a)$. Suppose $[a, c]$ and $[a, a']$ are any geodesics and consider the geodesic triangle formed by them and $[a', c]$. Using the fact that the points in C are at least ρ-separated, we see that for any $x \in [a, a'] \cup [a', c]$, we have

$$d(b, x) > \rho - (R + 2\delta) \geq R + 4\delta.$$

If $x \in [a, a']$, then

$$d(b, x) \geq d(b, a) - d(a, x) > \rho - (R + 2\delta).$$

If $x \in [a', c]$, then

$$d(b, x) = d(b, a') + d(a', x) \geq d(b, a'),$$

and we just showed this quantity was greater than $\rho - (R + 2\delta)$. The segment $[a, c]$ must be contained in the union of δ-neighborhoods of the other two sides, and thus, no point on $[a, c]$ can be $(R + 2\delta)$-close to b. ∎

Lemma 3.6. *Axiom* (P3) *holds with respect to* $\{d_a \mid a \in C\}$ *and* $\theta \geq 4\delta$.

Proof. Let $b, c \in C$. We must show the set $\{a \mid d_a(b, c) > \theta\}$ is finite. If $d_a(b, c) > \theta$, then by Lemma 3.4 each geodesic $[b, c]$ must intersect $B_{R+2\delta}(a)$. Fix a geodesic $[b, c]$, and cover $[b, c]$ with finitely many segments of length $1/2$. Each element of $\{a \mid d_a(b, c) > \theta\}$ lies in a $(R + 2\delta)$-neighborhood of one of these segments. Since $\rho \geq 2R + 6\delta > 2(R + 2\delta)$, each $(R + 2\delta)$-neighborhood of such a segment contains at most one point in the set $\{a \mid d_a(b, c) > \theta\}$. Thus, the set $\{a \mid d_a(b, c) > \theta\}$ is finite. ∎

3.2. Verification of the spinning family conditions. For the remainder of this section, let $\mathcal{P}$ be the projection complex associated to the set C and the projection distance functions $\{d_p \mid p \in C\}$. The group G acts by isometries on $\mathcal{P}$. By the construction of $\mathcal{P}$, for all $c \in C$, the group G_c is a subgroup of the stabilizer of the vertex c in $\mathcal{P}$. Moreover, the equivariance condition, Definition 2.14 (1), follows from Definition 2.1 (a-3). The next lemma verifies the spinning condition, Definition 2.14 (2).

Lemma 3.7. *If* $a, b \in V\mathcal{P}$ *and* $g \in G_a$ *is non-trivial, then*

$$d_a(b, gb) \geq 2^{(R-2)/\delta} - 4 - 6\delta.$$

Proof. Let $a, b \in V\mathcal{P}$, and let $g \in G_a$ be non-trivial. Let σ be a geodesic in X from b to gb. Let p_1 and p_2 be closest point projections of b and gb respectively to $\partial B_R(a)$. By the fairly rotating condition, σ passes through a point a' in the 1-neighborhood
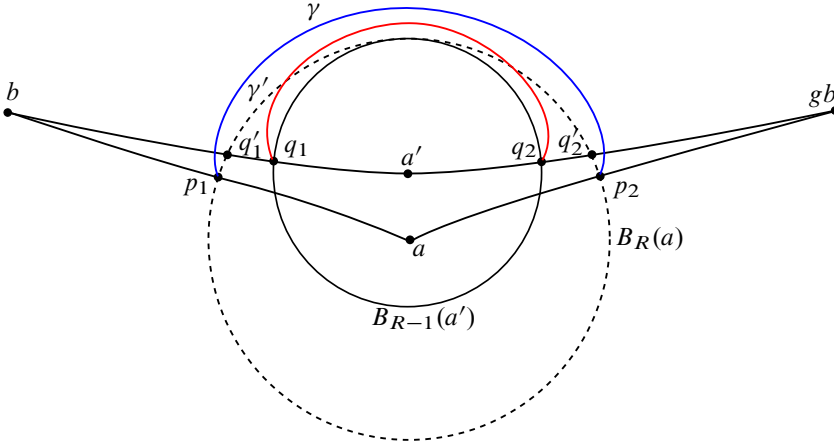
FIGURE 3

Setup for Lemma 3.7. The path in blue is γ, a geodesic in $X \setminus B_R(a)$ from p_1 to p_2 and the path in red is γ', a geodesic in $X \setminus B_{R-1}(a')$ from q_1 to q_2.

of a. Let q_1 and q_2 be the intersection points of σ with $\partial B_{R-1}(a')$, and let γ be a path from p_1 to p_2 in $X \setminus B_R(a)$. We will now construct, using γ, a path γ' from q_1 to q_2 in $X \setminus B_{R-1}(a')$; see Figure 3. A lower bound on the length of γ' from Lemma 2.20 will give us a lower bound on the length of γ.

Consider the triangle in X formed by σ and geodesics $[b, a]$ and $[gb, a]$ such that $p_1 \in [b, a]$ and $p_2 \in [gb, a]$. Let q_1' be the point on $\sigma \cap B_R(a)$ we reach by following σ away from q_1 towards b. Define q_2' similarly. The points q_1' and p_1 are in the same cusp of the geodesic triangle with vertices b, a, and a'. This follows since $d_X(p_1, a) = R$, $d_X(q_1', a') \geq R - 1$, $[a, a']$ is an edge of the triangle of length 1, and $R \geq 2 + 2\delta$. Note also that $d_X(b, p_1) = d_X(b, a) - R$ and $d_X(b, a) - 1 \leq d_X(b, a') \leq d_X(b, a) + 1$, so

$$d_X(b, a) - R \leq d_X(b, q_1) \leq d_X(b, a) - R + 2.$$

Thus, we can travel a distance ≤ 2 from q_1 towards b to get to a point at the same distance from b as p_1, and then along each side of the triangle and δ between the sides to see

$$d_{X \setminus B_{R-1}(a')}(p_1, q_1) \leq 2 + 3\delta,$$

and similarly for p_2 and q_2. By concatenating γ with paths outside $B_{R-1}(a')$ of length at most $2 + 3\delta$ from p_1 to q_1 and p_2 to q_2, we see that $\ell(\gamma') \leq \ell(\gamma) + 4 + 6\delta$.

Now by Lemma 2.20, we have $\ell(\gamma') \geq 2^{(R-2)/\delta}$; in the language of the lemma, $[q_1, q_2] \subset \sigma$ is a geodesic of length $2(R - 1)$, γ' is a path connecting q_1 and q_2 outside

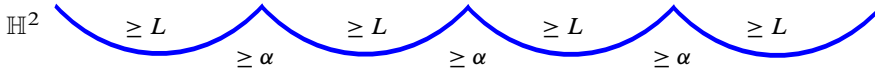$\mathbb{H}^2$ $\geq L$ $\geq \alpha$ $\geq L$ $\geq \alpha$ $\geq L$ $\geq \alpha$ $\geq L$

FIGURE 4

Canoeing paths in the hyperbolic plane are embedded quasi-geodesics. The segments have length at least L, and the angle between adjacent segments is at least α.

the ball $B_{R-1}(a')$, and a' is the midpoint of the geodesic segment. Therefore,

$$\ell(\gamma) \geq 2^{(R-2)/\delta} - 4 - 6\delta.$$ ∎

We conclude this section with the following proof.

Proof of Theorem 3.2. The lemmas in Subsection 3.1 combine to prove the projection axioms hold with respect to C equipped with the distance functions $\{d_p \mid p \in C\}$. The discussion and lemma in Subsection 3.2 along with upgrading the projection axioms to the strong projections axioms via Theorem 2.6 and applying Lemma 2.17 prove the remaining claims in the statement of the theorem. ∎

4. Free products from spinning families

The aim of this section is to give a new proof of Theorem 2.15, the result of Clay–Mangahas–Margalit.

4.1. Canoeing paths. The results in this section are motivated by the notion of canoeing in the hyperbolic plane, as illustrated in Figure 4. We will not use the following proposition, but include it as motivation.

Proposition 4.1 (Canoeing in $\mathbb{H}^2$, [12, Lemma 11.3.4]). *Let $0 < \alpha \leq \pi$. There exists $L > 0$ so that if $\sigma = \sigma_1 * \cdots * \sigma_k$ is a concatenation of geodesic segments in $\mathbb{H}^2$ of length at least L and so that the angle between adjacent segments is at least α, then the path σ is a (K, C)-quasi-geodesic, with constants depending only on α.*

Definition 4.2. If $\gamma = \{X_1, \ldots, X_k\}$ is a path of vertices in a projection complex, then the *angle* in γ of the vertex X_i is $d_{X_i}(X_{i-1}, X_{i+1})$.

The following definition is tailored to our purposes.

Definition 4.3. A *C-canoeing path* in a projection complex is a concatenation $\gamma = \gamma_1 * \gamma_2 * \cdots * \gamma_m$ of paths so that the following conditions hold:
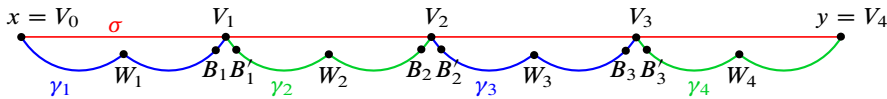
FIGURE 5
To prove that the endpoints, x and y, of a canoeing path γ are distinct, we show that the red path σ that connects the large-angle points is a standard path.

(1) Each γ_i is an embedded non-degenerate path, and is either a geodesic or the concatenation $\alpha_i * \beta_i$ of two geodesics.

(2) The common endpoint V_i of γ_i and γ_{i+1} has angle at least C in γ for $i \in \{1, \dots, m-1\}$. We refer to these points as *large angle points* of γ.

Since any subpath of a canoeing path is canoeing, it follows that canoeing paths are embedded. The proof that the endpoints of a canoeing path are distinct uses the Bounded geodesic image theorem for projection complexes (Theorem 2.12).

Proposition 4.4. *Let $\mathcal{P}(\mathcal{Y}, \theta, K)$ be a projection complex satisfying the strong projection axioms, and let M be the constant given in Theorem 2.12. If $C > 4M + K$, then the large angle points of a C-canoeing path lie on a standard path. In particular, the endpoints of a C-canoeing path are distinct.*

Proof. Let $\gamma = \gamma_1 * \cdots * \gamma_k$ be a C-canoeing path with $C > 4M + K$. Let x and y denote the endpoints of γ. Let B_i be the vertex of γ_i adjacent to the large-angle point V_i, and let B_i' be the vertex of γ_{i+1} adjacent to V_i. We will assume γ_i is the concatenation $\alpha_i * \beta_i$ of two geodesics.

Write for brevity $V_0 := x$ and $V_k := y$. For $i \in \{1, \dots, k\}$, let σ_i be the standard path from V_{i-1} to V_i. Then let $\sigma = \sigma_1 * \cdots * \sigma_k$ be the concatenation of the standard paths. We will show that σ is a non-trivial standard path by proving each concatenation angle is larger than K, which is a sufficient condition by Lemma 2.8. Note that by the Bounded geodesic image theorem (Theorem 2.12),

$$d_{V_i}(B_i, V_{i-1}) \leq 2M \quad \text{and} \quad d_{V_i}(B_i', V_{i+1}) \leq 2M.$$

By the assumption that $d_{V_i}(B_i, B_i') > 4M + K$, we have

$$d_{V_i}(V_{i-1}, V_{i+1}) > K,$$

concluding the proof. ∎

Combining this with Lemma 2.10 yields the following.

Corollary 4.5. *Let γ be a C-canoeing path with $C > 4M + K$ connecting the points X and Y and let k be the number of large angle points on γ. Then $d_{\mathcal{P}}(X, Y) \geq k/2$.*

4.2. Canoeing in windmills to prove dual graphs are trees. We will prove the following theorem in this section.

Theorem 4.6. *Suppose that $\mathcal{P} = \mathcal{P}(\mathcal{Y}, \theta, K)$ is a projection complex satisfying the strong projection axioms, and let G be a group acting on $\mathcal{P}$ preserving the projection data. Suppose that $\{G_c\}_{c \in V\mathcal{P}}$ is an L-spinning family of subgroups of G for $L > 4M + K$, where M is the constant given in Theorem 2.12. Then, there is a subset $\mathcal{O} \subset V\mathcal{P}$ of the vertices of $\mathcal{P}$ so that the subgroup of G generated by $\{G_c\}_{c \in V\mathcal{P}}$ is isomorphic to the free product $*_{c \in \mathcal{O}} G_c$.*

As in [7], we inductively define a sequence of subgraphs $\{W_i\}_{i \in \mathbb{N}}$ of $\mathcal{P}$ called *windmills*. Our methods diverge from those of Clay–Mangahas–Margalit in that we show that each windmill W_i admits a graph of spaces decomposition with dual graph a tree. We inductively define a sequence of subgroups $\{G_i\}_{i \in \mathbb{N}}$ of G so that G_i acts on the dual tree to W_i with trivial edge stabilizers. Hence, we obtain a free product decomposition for G_i by Bass–Serre theory. By the equivariance condition and because the windmills exhaust the projection complex, we ultimately obtain

$$\langle G_c \rangle_{c \in V\mathcal{P}} = \varinjlim_i G_i = *_{c \in \mathcal{O}} G_c.$$

Definition 4.7 (Windmills). Fix a base vertex $v_0 \in V\mathcal{P}$, let $\mathcal{O}_{-1} = \{v_0\}$, and let $W_0 = \{v_0\}$ be the base windmill. Let $G_0 = G_{v_0}$. Let N_0 be the 1-neighborhood of W_0, and let $G_1 = \langle\, G_v \mid v \in N_0 \,\rangle$. Recursively, for $k \geq 1$, let $W_k = G_k \cdot N_{k-1}$, let N_k be the 1-neighborhood of W_k, and let $G_{k+1} = \langle\, G_v \mid v \in N_k \,\rangle$. Finally, for $k \geq 0$, let $\mathcal{O}_k$ be a set of G_k-orbit representatives in $N_k \setminus W_k$ and $\mathcal{O} = \bigcup_{k=-1}^{\infty} \mathcal{O}_k$.

We will use the following notion to extend geodesics in the projection complex.

Definition 4.8. The *boundary* of the windmill W_k, denoted by ∂W_k, is the set of vertices in W_k that are adjacent to a vertex in $\mathcal{P} - W_k$. A geodesic $[u, v]$ in $\mathcal{P}$ that is contained in W_k is *perpendicular to the boundary at u* if $u \in \partial W_k$ and $d_{\mathcal{P}}(v, \partial W_k) = d_{\mathcal{P}}(v, u)$.

The next lemma follows immediately from Definition 4.8.

Lemma 4.9. *If a geodesic $[u, v]$ contained in W_k is perpendicular to the boundary at u, and $w \in \mathcal{P} - W_k$ is a vertex adjacent to u, then the concatenation $[v, u] * [u, w]$ is a geodesic in $\mathcal{P}$.*

Proof of Theorem 4.6. First, we show that the following properties hold for all $k \geq 0$:

(I1) Any two distinct vertices of W_k can be joined by an L-canoeing path $\gamma = \gamma_1 * \gamma_2 * \cdots * \gamma_m$ in W_k so that the following holds. If the initial vertex of γ_1 is on the boundary of W_k, then the first geodesic α_1 (or γ_1) is perpendicular to the boundary at that point. Likewise for the other endpoint of γ.
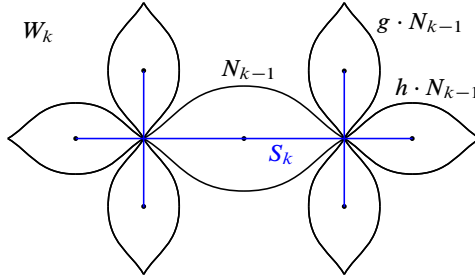
FIGURE 6

The cover of the windmill W_k by the translates of N_{k-1} and its skeleton S_k.

(I2) Two translates of N_{k-1} either coincide, intersect in a point, or are disjoint. The stabilizer in G_k of N_{k-1} is G_{k-1} and the stabilizer of $v \in N_{k-1} \setminus W_{k-1}$ in G_k is G_v. The skeleton S_k of the cover of W_k by the translates of N_{k-1} is a tree; see Figure 6. Furthermore, if γ is a canoeing path constructed in (I1) connecting two vertices of W_k, then every vertex of γ which is an intersection point between distinct translates of N_{k-1} is a large angle point of γ.

Recall that the *skeleton* is defined to be the bipartite graph whose vertex set is $V_1 \sqcup V_2$ with a vertex $p \in V_1$ for every translate of N_{k-1} and a vertex $q \in V_2$ for every intersection point between distinct translates, and edges represent incidence.

We proceed by induction. For the base case, we note that the claims hold trivially for $k = 0$. For the induction hypotheses, assume that (I1) and (I2) hold for $k - 1 \geq 0$; we will prove they also hold for k. We will need the following claim.

Claim 4.10. If $g \in G_v \setminus \{1\}$ for a vertex $v \in N_{k-1} - W_{k-1}$, then $g \cdot N_{k-1} \cap N_{k-1} = \{v\}$.

Proof of Claim 4.10. Let $x \in N_{k-1}$ and $y \in g \cdot N_{k-1}$ with $x \neq v \neq y$. To show $x \neq y$, we will build a path from x to y satisfying (I1). See Figure 7. Let $v' \in W_{k-1}$ be adjacent to v. Let $x' \in W_{k-1}$ so that $x = x'$ if $x \in W_{k-1}$, and otherwise, x and x' are adjacent. By the induction hypotheses, there exists a path $\gamma = \gamma_1 * \cdots * \gamma_m$ from x' to v' in W_{k-1} satisfying conditions (I1). The first geodesic α_1 (or γ_1) of γ extends to a geodesic to x by Lemma 4.9. Similarly, the final geodesic β_m (or γ_m) extends to a geodesic to v. Thus, the path γ extends to a path γ' from x to v that is contained in N_{k-1} and satisfies the conditions of (I1). Similarly, there exists a path $\delta = \delta_1 * \cdots * \delta_n$ from gv' to a vertex $y' \in g \cdot W_{k-1}$ with $y' = y$ if $y \in g \cdot W_{k-1}$ or $d_{\mathcal{P}}(y, y') = 1$. As above, the path δ extends to a path from v to y satisfying (I1). Since $d_v(v', gv') \geq L$, the concatenation $\gamma_1 * \cdots * \gamma_m * \delta_1 * \cdots * \delta_n$ satisfies (I1). Thus, $x \neq y$ by Proposition 4.4. We also point out that v is a large angle point of this canoeing path. ∎
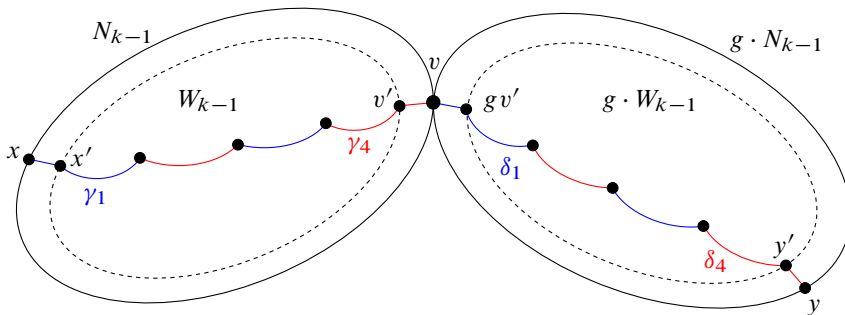
FIGURE 7
Canoeing paths are used to prove $N_{k-1} \cap g \cdot N_{k-1} = \{v\}$. Canoeing paths $\gamma_1 * \ldots * \gamma_m$ from x' to v' and $\delta_1 * \ldots * \delta_n$ from gv' to y' exist by the induction hypotheses. Since the ends of these paths are perpendicular to the boundary, they can be extended to a canoeing path from x to y. Thus, $x \neq y$ for any $x \in N_{k-1} - \{v\}$ and $y \in g \cdot N_{k-1} - \{v\}$.

Claim 4.11. Given the induction hypotheses, property (I1) holds for W_k.

Proof of Claim 4.11. Let $x, y \in W_k$. Suppose first that x and y are contained in the same G_k-translate of N_{k-1}, say in N_{k-1} itself. Let $x', y' \in W_{k-1}$ with $x = x'$ if $x \in W_{k-1}$ and $d_{\mathcal{P}}(x, x') = 1$ otherwise, and similarly for y'. By the induction hypothesis, there exists a path $\gamma = \gamma_1 * \cdots * \gamma_m$ from x' to y'. The first geodesic α_1 (or γ_1) can be extended to x by Lemma 4.9, and the last geodesic β_m (or γ_m) can be extended to y to produce a new geodesic γ' that is perpendicular to the boundary at x and y. Thus, (I1) holds in this case.

We may now assume that $x \in N_{k-1}$ and $y \in g \cdot N_{k-1}$ for some $g \in G_k \setminus G_{k-1}$. Choose a decomposition $g = g_1 \ldots g_m$ with $g_i \in G_{v_i}$ for $v_i \in N_{k-1}$ so that m is minimal. Observe that $m \geq 1$ and that $g_i \notin G_{k-1}$ for any $i \in \{1, \ldots, m\}$. Indeed, if $g_0 g_i$ appears as a subword of g with $g_0 \in G_{k-1}$ and $g_i \in G_{v_i}$, then

$$g_0 g_i = g_0 g_i g_0^{-1} g_0 = g_{i'} g_0$$

for $g_{i'} \in G_{g_0 v_i}$ by the equivariance condition. That is, the element g_0 can be shifted to the right, and since g_0 stabilizes N_{k-1}, the element g could be written with fewer letters, contradicting the minimality of the decomposition.

We now build a path from x to y. The translates

$$g_1 g_2 \ldots g_i \cdot N_{k-1} \quad \text{and} \quad g_1 g_2 \ldots g_{i+1} \cdot N_{k-1}$$

intersect in the single vertex $g_1 g_2 \ldots g_i v_{i+1}$ for $i \in \{1, \ldots, k-1\}$ by the assumptions on g_i and Claim 4.10. Similarly, $N_{k-1} \cap g_1 N_{k-1} = \{v_1\}$. Therefore, the methods

in the proof of Claim 4.10 can be inductively applied to build a path from x to y satisfying (I1). That is, the path is constructed to pass through each intersection point $c_{i+1} = g_1 g_2 \ldots g_i v_{i+1}$ and the edges e_{i+1}, f_{i+1} immediately before and after c_{i+1} satisfy $f_{i+1} = h_{i+1}(e_{i+1})$ for a non-trivial $h_{i+1} \in G_{c_{i+1}}$. The restriction of the path to each translate of N_{k-1} is built using property (I1) applied to the translate of W_{k-1}. ∎

Claim 4.12. Property (I2) is satisfied by G_k and W_k.

Proof of Claim 4.12. We may assume one of the translates is N_{k-1} itself and the other is $g \cdot N_{k-1}$ where $g \in G$ is written as $g = g_1 \cdots g_m$ with $g_i \in G_{v_i}$ and m minimal as above. If $m > 1$ then the canoeing path we constructed from a vertex in N_{k-1} to a vertex in $g(N_{k-1})$ is non-degenerate, showing that

$$N_{k-1} \cap g \cdot N_{k-1} = \emptyset.$$

If $m = 1$, we showed in Claim 4.10 that $N_{k-1} \cap g_1 \cdot N_{k-1} = \{v_1\}$. We now prove that S_k is a tree. Since W_k is a connected graph, S_k is also connected.

Suppose towards a contradiction that $p_1, q_1, p_2, q_2, \ldots, p_n, q_n, p_1$ is an edge path that is an embedded loop in the graph with $p_i \in V_1$ and $q_i \in V_2$. Each vertex p_i corresponds to a translate $g_i \cdot N_{k-1}$ with $g_i \in G_k$. Consecutive translates intersect in a point, and since the edge path does not backtrack, the intersection points

$$g_{i-1} \cdot N_{k-1} \cap g_i \cdot N_{k-1} \quad \text{and} \quad g_i \cdot N_{k-1} \cap g_{i+1} \cdot N_{k-1}$$

are distinct. Under these assumptions we constructed a non-degenerate canoeing path from any vertex in $g_1 \cdot N_{k-1}$ to any vertex in $g_n \cdot N_{k-1}$, showing that the two translates are disjoint by Proposition 4.4. But the edge subpath p_n, q_n, p_1 indicates

$$g_1 N_{k-1} \cap g_n N_{k-1} \neq \emptyset.$$

Thus, S_k is a tree. ∎

Conclusion. We now use property (I2) to conclude the proof of Theorem 4.6. That is, we define a subset $\mathcal{O} \subset V\mathcal{P}$ so that $\langle G_c \rangle_{c \in V\mathcal{P}} \leq G$ is isomorphic to the free product $*_{c \in \mathcal{O}} G_c$. First we check that $G_k \cong G_{k-1} * (*_{v \in \mathcal{O}_k} G_v)$ for each $k \geq 1$. The group G_k acts on W_k preserving the covering by the translates of N_{k-1} and so it acts on the skeleton S_k. The edge stabilizers are trivial by Claim 4.10. There is one G_k-orbit in the vertex set V_1, and the group G_{k-1} stabilizes the vertex corresponding to N_{k-1}. Therefore, the free product decomposition of G_k follows from the definition of $\mathcal{O}_k$ and Bass–Serre theory. The quotient S_k/G_k is also a tree with a vertex representing V_1 and vertices representing orbits in V_2, all connected to V_1.
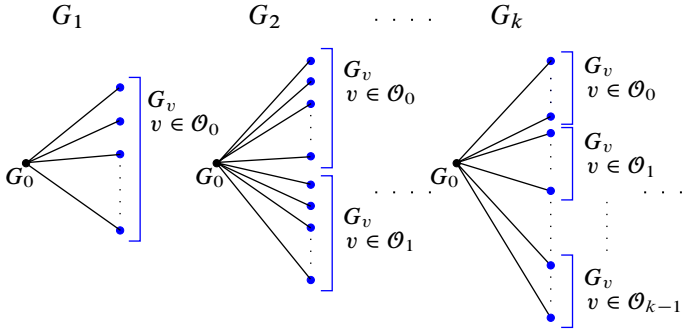
FIGURE 8

Directed system of graphs of groups decompositions for the groups $\{G_k\}$.

Since the windmills exhaust the projection complex, $\langle G_c \rangle_{c \in V\mathcal{P}} = \varinjlim_k G_k$. Finally, $\varinjlim_k G_k = \ast_{c \in \mathcal{O}} G_c$ for $\mathcal{O} = \bigcup_{k=-1}^{\infty} \mathcal{O}_k$, which again can be deduced from a Bass–Serre theory argument as follows.

We will specify an increasing union of trees so that the group $\varinjlim_k G_k$ acts on the direct limit tree. Recall that (I2) yields for each k a graph of groups decomposition of G_k with vertex groups G_{k-1} and G_v for each $v \in \mathcal{O}_k$. There is an edge $\{G_v, G_{k-1}\}$ with trivial edge group for each $v \in \mathcal{O}_k$. As depicted in Figure 8, the graph of groups decomposition for G_2 can be expanded using the graph of groups decomposition for G_1. More specifically, in the graph of groups decomposition for G_2, delete the vertex for G_1, and replace it with the graph of groups decomposition for G_1, attaching every group G_v for $v \in \mathcal{O}_2$ to the vertex G_0 with trivial edge group. The group G_2 then acts on the new corresponding Bass–Serre tree. Continue this recursive procedure: in the graph of groups decomposition for G_k, delete the vertex for G_{k-1} and replace it with the recursively obtained graph of groups decomposition for G_{k-1}, attaching every group G_v for $v \in \mathcal{O}_k$ to G_0 with trivial edge group. This process yields an increasing union of Bass–Serre trees, and the $\varinjlim_k G_k$ acts on the direct limit tree as desired. ∎

5. Free products from rotating families

The aim of this section is to combine Theorem 3.2 and Theorem 4.6 to give a new proof of the following theorem of Dahmani–Guirardel–Osin with different constants.

Theorem 5.1. *Let G be a group acting by isometries on a δ-hyperbolic metric space with $\delta \geq 1$, and let $\mathcal{C} = (C, \{G_c \mid c \in C\})$ be a ρ-separated fairly rotating family for some $\rho \geq 2\delta \log_2(\delta) + 38\delta$. Then, the normal closure in G of the set $\{G_c\}_{c \in C}$ is isomorphic to a free product $\ast_{c \in C'} G_c$, for some (usually infinite) subset $C' \subset C$.*

Proof. Take $\theta = 121\delta$, $K = 3\theta$, and let $R = \delta \log_2(\delta) + 16\delta$, which meets the constraint $2 + 2\delta \leq R \leq \rho/2 - 3\delta$. Then by Theorem 3.2, the group G acts by isometries on a projection complex $\mathcal{P} = \mathcal{P}(C, \theta, K)$ obtained from a collection $(C, \{d_p\}_{p \in C})$ satisfying the strong projection axioms, and the family of subgroups $\{G_c\}$ is an equivariant L-spinning family for $L = 2^{(R-2)/\delta} - 4 - 248\delta$.

One can check that our choice of R satisfies $L > 4M + K$, where M is the Bounded geodesic image theorem constant given in Theorem 2.12. Indeed, as $R = \delta \log_2(\delta) + 16\delta$, we have the following equivalent inequalities:

$$L > 4M + K,$$

$$2^{(R-2)/\delta} - 4 - 248\delta > 4(8K + 2\theta) + K, \quad 2^{(R-2)/\delta} > 13195\delta + 4.$$

Since $\delta \geq 1$ it suffices to check

$$65536\delta = 2^{R/\delta} > 4(13199\delta) = 52796\delta.$$

Thus, the hypotheses of Theorem 4.6 are satisfied, so $\langle\!\langle G_c \rangle\!\rangle_{c \in C} \leq G$ is isomorphic to a free product $\ast_{c \in C'} G_c$, for some subset $C' \subset C$ as desired. ∎

6. Loxodromic elements

In this final section we prove the second halves of Theorems 1.2 and 1.3 which state that our subgroup of G consists of elements that are either point stabilizers in some G_c or act loxodromically on both the hyperbolic metric space X and the projection complex $\mathcal{P}$. We begin with the action on the projection complex.

Proposition 6.1. *Let $\mathcal{P}$, G, and $\{G_c\}_{c \in V\mathcal{P}}$ be as in Theorem 4.6. Then every element of the subgroup of G generated by $\{G_c\}_{c \in V\mathcal{P}}$ is either loxodromic in $\mathcal{P}$ or is contained in some G_c.*

Proof. Let g be an element of the group generated by $\{G_c\}_{c \in C\mathcal{P}}$. By the proof of Theorem 4.6, g is contained in G_k for some k. Now G_k acts on the Bass–Serre tree which is the skeleton, S_k, of the cover of W_k by the translates of N_{k-1}. Let us first assume that g acts on this tree loxodromically. Let x_0, an intersection point of two translates of N_{k-1}, be a point on the axis of g in S_k. Thus, in S_k we have that $d_{S_k}(x_0, g^n x_0)$ grows linearly in n.

Now we move from the Bass–Serre tree back to $\mathcal{P}$. Note that x_0 is also a point in $\mathcal{P}$ and consider the orbit of x_0 in $\mathcal{P}$. Also, x_0 and all of its translates are large angle intersection points of distinct translates of N_{k-1}. Given any n we can apply (I1) to form an L-canoeing path γ from x_0 to $g^n x_0$. Let $m = d_{S_k}(x_0, g^n x_0)$. Since each of the $g^i x_0$ are intersection points between translates of the N_{k-1}, we can apply the

furthermore statement of (I2) to see that the number of large angle points on γ is at least $m/2 - 1$. Apply Corollary 4.5 to see that $d(x_0, g^n x_0) \geq (m-2)/4$ with m growing linearly in n. We conclude that the translation length of g is strictly positive, and hence g acts loxodromically on $\mathcal{P}$.

Now if g fixes a point in S_k then it is conjugate into either one of the G_c or G_{k-1}. However, now we can just run the argument again in G_{k-1}, continuing until $G_0 = G_{v_0}$ if necessary. ∎

We next see that we can push this result forward again to the original δ-hyperbolic space X.

Proposition 6.2. *Let G and $\mathcal{C}$ be as in Theorem 5.1. Then every element of the subgroup of G generated by the set $\{G_c\}_{c \in C}$ is either a loxodromic isometry of X or it is contained in some G_c.*

Proof. We first apply Theorem 5.1 and run the argument above in $\mathcal{P}$. Thus for any $g \in G$ we either have $g \in G_c$ for some c or we have an orbit $\{g^n x_0\}$ such that for any $n \geq 2$, we have that $d_{g^i x_0}(x_0, g^n x_0) > K > 4\delta$ for all $i = 1, \ldots, n-1$. Thus, by Lemma 3.4 we have that every geodesic from x_0 to $g^n x_0$ passes through each of the balls $B_{R+2\delta}(g^i x_0)$ for $i = 1, \ldots, n-1$. Now our choice of ρ and R guarantees that each of these balls are distance at least 2δ from each other so that $d(x_0, g^n x_0) \geq 2\delta(n-1)$. We conclude that the translation length of g is strictly positive and hence g is loxodromic. ∎

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