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Jones polynomial, Tait conjectures and amphicheirals

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Abstract. This text presents how the Jones polynomial $V_K(t)$ was the main tool used to prove the Tait conjectures, soon after its discovery by Vaughan Jones. The purpose of this note is to emphasize the immediate impact of the Jones polynomial on the two subjects mentioned in the title.

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Non-expert readers can find most of the missing definitions about knot theory in Raymond Lickorish's book [13] and sometimes also in [2].

1. Where do amphicheirals come from?

Peter Tait's pictures of knots are abundant, and available on the Internet. See the references given in [20–22]. Knot projections up to 9 crossings are at the end of *On knots II* and those with 10 crossings are at the end of *On knots III*. Before going further, it is important to make clear the distinction between projections and diagrams used in the paper. A *knot projection* is a 4-valent graph embedded in the plane, or equivalently it is the image of a generic immersion of the circle in the plane. If under/over passes are pictured at each crossing point, we have a *knot diagram*. Tait mainly drew knot projections instead of diagrams. His reason was that he was essentially interested in alternating knots. To a projection, one can associate two alternating knot diagrams. It seems that, for some time, he believed that the two diagrams represent always distinct knots in 3-space. Then he realized that this is not always the case. He found out that there exists a "small" family of knots for which the two diagrams represent the same knot. He christened these knots *amphicheirals*. Note that the existence of amphicheirals may have felt as a pebble in the shoe since they make his "program" more complicated. However, Tait soon recognized their importance and decided to study these "pebbles".

This was not really a surprise to Maxwell, who knew about the existence of objects in 3-space which are equivalent to their mirror image, since electricity and magnetism are sensitive to the orientation of space. Earlier, Gauss had asked his student Listing to study this phenomenon, and Listing wrote his thesis on the subject. Some forty years earlier than Tait, Listing discovered that some knots can be equivalent to their mirror image.

It is also striking to see how people working in different scientific fields during the 19th century tried to have an undisputed definition of right-handed as opposed to left-handed. Try to do it for braids or for the trefoils!Astronomy and botany were called to the rescue, with little success. In the years 1877–1878 Tait and Maxwell had a long exchange of postcards on this question. Since Maxwell loved to make puns, he did not miss the opportunity presented by the name "perversion" which means also the image through a mirror. Already in 1871 he wrote to Tait: "Am I perverted? a mere man in a mirror, walking in a vain show?" This sounds Shakespearian!

Tait introduced the term in §13 of his paper *On knots I* as follows: "amphicheiral, i.e. right or left-handed indifferently". The name is oddly constructed from two Greek roots: the prefix αμφι (amphi) which means "equally on both sides" and the name χερι (keri) which means "hand". Put together, they mean roughly "hands on both sides are equivalent".

The name "amphicheiral" remained popular in knot theory for a long time (even today). However, Maxwell kept teasing his friend about it. In a letter to Tait dated February 15, 1878, Maxwell wrote: "I have not yet digested an amphicheiral. I have too much sense to swallow anything of the sort". Thomson also found the name a bit weird (friendly arguments about names and symbols were frequent between Thomson and Tait). Thomson proposed in the Baltimore Lectures (pronounced in 1884 and published only some 20 years later since Thomson had no written notes for his 23 lectures) the following definition, restored from the notes of members of the audience.

Definition 1.1 (William Thomson). I call any geometrical figure or group of points *chiral* if its image in a plane mirror ideally realized, cannot be brought to coincide with itself.

Hence "chiral" replaces "non-amphicheiral" and "achiral" is the same as "amphicheiral". We admit that we prefer Thomson's vocabulary to Tait's.

Slowly, chirality questions have invaded other fields, beyond mathematics and physics. The more remarkable, in our opinion, are chemistry and pharmacology. Vladimir Prelog was awarded the Nobel prize in chemistry in 1975 for his pioneering work on the chirality of molecules; see his Nobel prize lecture. He told C.W. that knot chirality had been for him a source of images and of concepts. It is now known that many (perhaps most) medicines are chiral. It may happen that one can be beneficial while its mirror image is lethal. For each new medicine, the determination of its chirality is highly important. About chirality for embeddings of graphs in 3-space or in the 3-sphere, see Erica Flapan's book [4]. This applies to the configuration graph (la formule développée in French) of molecules. In the years 1960, the drama of the drug called thalidomide alerted health authorities to the importance of chirality issues for medicines. The chirality of thalidomide is due to the presence of a carbon atom called asymmetric. For more details on chiral molecules in general and on asymmetric carbons in particular, see Hart–Coria's book [1, Chapter 5]. Since then, chirality studies have invaded chemical industry and, near the end of the 20th century, a kind of revolution took place named the chiral switch. But this goes far beyond the subject of this paper.

Let us now leave history and proceed towards knot theory.

2. Tait conjectures

In fact, the conjectures were never stated as such by Tait, who strongly believed that the principles he used were true. These principles are stated in his text as more or less obvious and are somewhat hidden. A century later, he was right!

Here are some extracts from *On knots I* [20], to measure the difficulty of capturing explicit "conjectures" in Tait's writings. From §4:

> *Here we may remark that it is obvious that when the crossings are alternatively $+$ or $-$ (i.e., when the knot diagram is alternating) no reduction is possible unless there be essentially nugatory crossings. For the only way of getting rid of such alterations $+$ or $-$ is by untwisting, and this process except in essentially nugatory cases, gets rid of a crossing point at one place only by introducing it at another.*

We can perceive in this extract two "conjectures" concerning the alternating knots: one on minimal diagrams and the other on the Flyping conjecture, since untwisting is a flyping as Tait will explain later.

Here is a conjecture about amphicheirals, written in §13 of [20]:

> *At least one knot of every even order* [*i.e., the minimal crossing number is even*] *is amphicheiral but no knot of odd order can be so.*

If we do not restrict ourselves to the class of alternating knots, the conjecture is false since, in their computational work, Hoste, Thistlethwaite, and Weeks discovered a non-alternating amphicheiral knot of crossing number 15 (see [8]).

We note that several erroneous publication dates for Tait's papers are widespread in the literature. *On knots I* was published in 1876, *On knots II* in 1884, and *On knots III*

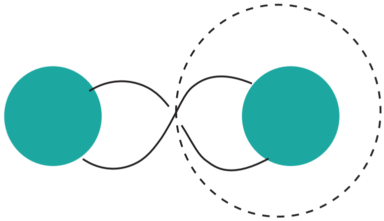

FIGURE 1
A nugatory crossing.

in 1885, and thus well before 1898, as is often claimed (1898 being the publication date of Tait's *Complete works*). Tait began working on knots probably in the year 1867 after he performed experiments in his office with smoke rings in the presence of William Thomson. Soon after, T formulated his hypothesis that atoms are knotted vortex rings in the ether and T' hoped that the classification of chemical elements would follow from the classification of knots (by T and T', Maxwell was referring to Tait and Thomson). At first, Tait thought that classifying the knots would be easy and it seems that he was disappointed to find that it was probably unreachable. Ten years later, Tait decided to publish the results he had obtained.

Conjecture on minimal knot diagrams. Let K be an alternating knot. Then any reduced alternating diagram of K is minimal. Moreover, any reduced and indecomposable diagram representing K is alternating.

A crossing of a diagram is called *nugatory* by Tait if two opposite sectors near a crossing point can be joined by an embedded arc meeting the diagram nowhere.

A diagram is *reduced* if it contains no nugatory crossing.

A diagram is *minimal* if it has the minimal number of crossing points among all diagrams.

A diagram is called *indecomposable* if there does not exist a region in the chessboard (defined, for instance, in [2, p. 308]) such that two opposite sides of that region can be joined by an embedded arc meeting the region only at the two end points. A famous theorem of William Menasco proves that an alternating knot/link in 3-space is prime if and only if any minimal diagram is indecomposable ([14]). We therefore prefer indecomposable to prime since Tait conjectures concern diagrams, which are a 2-dimensional concept instead of prime which is a 3-dimensional concept.

Conjecture on the writhe of a minimal alternating diagram. All minimal diagrams of an alternating knot have the same writhe.

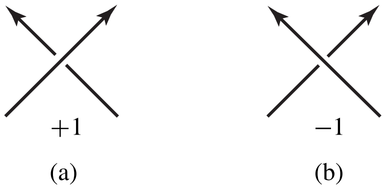

Figure 2
Convention for the writhe.

Let us recall the definition of the writhe. Consider a crossing point of an oriented knot/link diagram. Two cases are possible. We attribute the value $+1$ to case (a) and -1 to case (b), as shown in Figure 2. Then *the writhe of the diagram* is the sum of these ± 1. Observe that the writhe has the same value if we reverse the orientation of the diagram.

Since Tait was primarily interested in alternating knots, the Tait conjecture on amphicheirals can therefore be stated as follows.

Conjecture on amphicheirals. A minimal diagram of an alternating amphicheiral knot has an even number of crossings.

3. Kunio Murasugi and Louis Kauffman in Geneva

Kunio Murasugi arrived in Geneva in October 1985 and gave several talks on Tait conjectures. His goal was to use the Jones polynomial to prove them. He first used the Jones original definition via the von Neumann factors. He did not succeed. We talked together and he said “what we need is a topological interpretation of the Jones polynomial”. In other words, we need to interpret the Jones polynomial via classical knot concepts: coverings, fundamental group, as for the Alexander–Conway polynomial. Even today, this goal seems far away. We suggested: perhaps another approach, different from the von Neumann factors, would be sufficient?

Then Louis Kauffman arrived in Geneva from Italy, to deliver a colloquium talk. He had written in a preprint (already distributed to knot theorists and submitted to the journal Topology) his definition of the Jones polynomial via state sums. Meanwhile, Murasugi thought that it might be possible to prove the Tait conjectures by Kauffman’s approach via state sums instead of Jones’ definition via von Neumann factors. Murasugi and Kauffman discussed together and shortly after Kauffman’s departure, Murasugi said: “We have proved several of the Tait conjectures”. They chose to write a separate paper each. Kauffman presented, in a second preprint replacing the first, his approach to the Jones polynomial via state sums and then deduced the Tait conjectures. On the other hand, Murasugi focused on the Tait conjectures, also using Kauffman's state sums. The papers are [12] and [17] where the proofs of the conjectures on minimal crossing number and on amphicheirals are given. A little later, Murasugi and Kauffman published the proofs of the writhe conjecture; see also Section 8 below.

Here is a very short presentation of Kauffman's new way of obtaining invariants based on Reidemeister moves. From a historical point of view, it is a return to the knot diagrams that proved to be very fruitful, opening the door to many connections with other fields.

The goal is a program to construct knot invariants by associating to knot diagrams a polynomial (for instance an element of $\mathbf{Z}[A, A^{-1}]$) such that the polynomial is unchanged by Reidemeister moves. The key concept is the Kauffman Bracket polynomial which is invariant by moves II and III. Move I can be taken care over by the writhe. The material for defining a polynomial for each diagram is provided by state sums for Kauffman and by spanning trees for Thistlethwaite.

4. The span of the Jones polynomial

The key concept for proving the conjectures was (and still is) the *span* (also called the breadth) of the Jones polynomial. By definition, the span of a Laurent polynomial is the difference between the highest and lowest degrees of its non-vanishing terms. The main result is as follows, where $c(\ )$ denotes the number of crossing points.

Theorem 4.1 (Statement borrowed from Murasugi). *If L^* is a connected reduced alternating diagram of an alternating link L, then* $\operatorname{span} V_L(t) = c(L^*)$. *Moreover, for any prime link L, if $L^\sharp$ is any non-alternating diagram for L, we have* $\operatorname{span} V_L(t) < c(L^\sharp)$.

At the end of his paper [10], Jones published his computation of the Jones polynomial for the 35 prime knots that have at most 8 crossings. Among them, 32 are alternating. From these values, it is rather obvious that there is a connection between the crossing number and the span and that Theorem 4.1 is the correct formulation. This was Murasugi's conjecture when he came to Geneva in October 1985. The proof followed then from Kauffman's state sums and also from Thistlethwaite's spanning trees. With the main ideas very closely related to Murasugi's argument, V. Turaev also gave a shorter proof ([24]).

FIGURE 3
Sign of the crossing point.

5. Morwen Thistlethwaite's spanning tree expansion

At the same time that Murasugi and Kauffman submitted their paper (early in 1986) Morwen Thistlethwaite also submitted to the journal Topology a paper which proposes yet another approach of the Jones polynomial, based on the combinatorics of planar graphs. We state his [23, Theorem 1], which adds interesting precisions on the Jones polynomial $V_L(t)$:

Theorem 5.1. *If L is a link that admits a connected reduced alternating diagram with m crossings, then:*

(i) *the span of $V_L(t)$ is precisely m;*

(ii) *$V_L(t)$ is an alternating polynomial;*

(iii) *the coefficients of the terms of $V_L(t)$ of maximal and minimal degree are ± 1;*

(iv) *if L is not a connected sum and is not a torus link of type $(2, k)$ then $V_L(t)$ is of the form $t^r(\Sigma_o^m a_j t^j)$ with each coefficient a_j non-zero.*

Thistlethwaite constructs for each knot diagram (not necessarily alternating) two planar graphs, associated to the two colors of the chessboard. The idea goes back to Tait (see [20, §20]). The black graph has a vertex inside each black region of the chessboard and an edge intersecting each crossing point, connecting the two black regions contiguous to the crossing point. To each crossing point (hence, each edge) one associates a sign $\pm$ according to the convention illustrated in Figure 3 (this sign convention is different from the one used in oriented knot/link diagrams, to calculate the writhe). Alternating diagrams are characterized by the property that all edges have the same sign. With this in hand, Thistlethwaite uses spanning trees to obtain a Tutte polynomial that satisfies the necessary and sufficient requirements to be equal to the Kauffman bracket polynomial, since it is invariant by Reidemeister moves II and III. Following Kauffman's beautiful procedure, after introducing a factor that takes into account the writhe, he gets an isotopy invariant polynomial equal to the Jones polynomial.

6. The search of achiral knots with the help of the polynomial

Since Tait, the goal of knot theory about chirality is to have a criterion that detects whether a knot diagram represents a chiral or achiral knot. We want the criterion to work both ways: if and only if. In other words, we want to have necessary and sufficient conditions. In one sense, there is sometimes a possible way out: if a knot is achiral, we can prove it by drawing pictures that exhibit an isotopy of the knot to its mirror image. In other words, one makes achirality visible.

The behavior of the polynomial with respect to a change of orientation of 3-space was crucial in the early hours when Jones and Birman discovered the properties of $V_K(t)$, since Jones knew the following result, stated here as a theorem.

Theorem 6.1. *Let K be knot (or a link) and let $K^\sharp$ be its mirror image. Then $V_{K^\sharp}(t) = V_K(1/t)$.*

This immediately shows that the trefoil is chiral (since the Jones polynomial of one of the trefoils is equal to $t + t^3 - t^{-4}$), a property that the Alexander polynomial cannot detect. Incidentally, these early computations showed that the two polynomials are distinct; see the interview of Joan Birman [9, pp. 25–26]. Naturally, Jones was very happy about this. And he risked 10 dollars on his Conjecture 4 in the *Bulletin AMS* announcement [10], which says that a knot K is achiral if and only if $V_K(t) = V_K(1/t)$. The conjecture fails but for small values of the crossing number, it works quite well. It is nice to see that the Jones polynomial for the 4-crossing knot is equal to

$$t^{-2} - t^{-1} + 1 - t + t^2.$$

Up to 8 crossings, there are seven achiral knots, all exactly detected by $V_K(t)$. For 10 crossings, there are a few failures, listed by Jones in his paper on Hecke algebras [11]. But the balance is impressively positive!!! We regret that Jones, in the same paper [11], normalized his polynomial in such a way that the property $V_K(t) = V_K(1/t)$ for achiral knots is difficult to see.

7. The Flyping conjecture

Now we come to the most important Tait conjecture: The Flyping conjecture. Here is why it is important. Tait's program was to list all alternating knots. The first step was

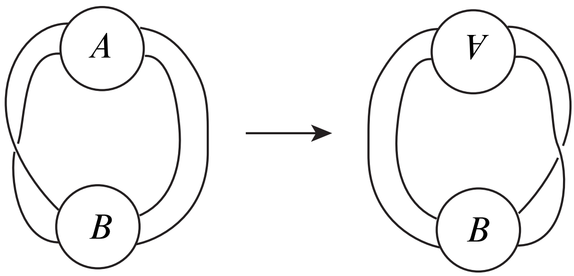

FIGURE 4
A flype.

to list all the minimal diagrams. From 7 crossings, an alternating knot can have several minimal diagrams representing it. The problem is therefore to sort them. Tait used the following transformation acting on diagrams (Figure 4), which he called distortion or twisting. It is now called *flyping* or *flype*.

The *Flyping conjecture* states that two minimal diagrams (called "forms" by Tait) of an alternating knot, differ by a finite number of flypes.

The Flyping conjecture was proved by William Menasco and Morwen Thistlethwaite in [16] and announced with a presentation of the proof in [15]. Since these authors used other results that involve the Jones polynomial, they explicitly stated that the Jones polynomial is an ingredient of their proof [15, p. 404].

Now proven, the Flyping conjecture is presented as the Menasco–Thistlethwaite Flyping theorem.

Theorem 7.1 (Flyping theorem). *Given any two reduced alternating diagrams D_1 and D_2 of an oriented, prime alternating link, D_1 can be transformed into D_2 by means of a finite sequence of flypes.*

The Flyping theorem is the main tool used by Hoste–Thistlethwaite–Weeks to enumerate alternating knots in [8].

8. The writhe

The writhe conjecture follows immediately from the Flyping theorem since a flype obviously preserves the signs of crossings. This is certainly what Tait thought (we saw above in Section 2 that Tait believed in what we call the Flyping conjecture). Proofs of the writhe conjecture were given by Murasugi and earlier independently by Thistlethwaite, without the aid of the Flyping theorem.

Since a change of orientation of 3-space changes the sign of the writhe, we have easy consequences of the preservation of the writhe by flypes:

(1) The writhe of an alternating achiral knot is equal to 0.

(2) The number of crossings of the minimal diagram of an achiral alternating knot is even. This implies that the conjecture on amphicheirals is true.

Some history of the writhe. In Tait's paper, the writhe is first outlined in [20, §36]. He devised a funny way to define the sign of a crossing by throwing coins of copper or silver in some of the four black or white local regions near the crossing point. However, his exposition is obscured by considerations of two other concepts: the unknotting number (pleasantly called the beknottedness) and by the attempt to interpret the Gauss integral formula for the linking number of two oriented and disjoint curves in 3-space in the case where only one curve is present.

Let us focus on Gauss' formula in the case of a knot. Tait sees that for this case he has to choose a parallel of the knot and so the situation he is considering evolves towards the study of a thin ribbon embedded in 3-space. There is an interesting special situation that he describes very well. Suppose we have a knot in 3-space together with a diagram onto the plane. Then, using the diagram map, we can construct a thin ribbon below the knot. Tait correctly interprets the writhe of the diagram as the linking number of the two boundary curves of this thin ribbon, one of which is of course the knot itself. If we refer to the formula given below about the writhing number, we see that in the case presented by Tait the torsion of the ribbon vanishes. Little and Kirkman were great consumers of the writhe when they built their tabulations, by sorting diagrams with different writhes (then called twists).

The history of the writhe is not over here. Quite surprisingly, in the 1970s, the writhe played its part in the study of the global structure of DNA in 3-space. When better images of DNA became available, it appeared that DNA could exist in long filaments and sometimes form a knotted circle. Thus, when the cell divides, the linking coefficient between the DNA components of the two future daughters seemed to be an obstacle to division. Therefore, it was imperative to have a description of the global topology of DNA. The Crick–Watson model roughly presents DNA as a very thin ribbon embedded in 3-space. Since this ribbon appears to be highly twisted, the question arises of understanding what the linking coefficient of its two boundary component might be. The following formula due independently to Calugareanu, Fuller, White and Pohl provides a partial answer. It is known as the White formula which reads as follows:

$$Lk = Tw + Wr.$$

Here is an explanation of the three terms.

- Lk is the linking coefficient of the two boundary components.
- Tw is the integral of the torsion of the ribbon.
- Wr is the *writhing number* of the ribbon, which can be defined to be the average of the writhe of the images of the ribbon for all (reasonable) diagrams onto a plane.

The linking coefficient is of course an integer. The other two terms are real numbers.

The formula implies that the linking number cannot be detected locally by considerations of torsion (local twist) alone, contrary to widespread expectations, since the writhe is a global concept.

Here is a question that has been important in the study of the DNA elasticity: in the formula of the decomposition of the linking coefficient, what proportion is due to the twist (thus is local) and what proportion is due to the writhe (thus is global)?

For the reader interested in the relationship between DNA and topology, we recommend the two articles by Brock Fuller [5, 6] and the article by William Pohl [18].

9. Visibility

Let us now come back to achiral knots.

Question (implicitly raised by Tait). Is there a way to see if a diagram represents an achiral knot?

Tait had an answer, which can also be stated as a conjecture.

Conjecture on the visibility of amphicheiral knots. Let K be an alternating amphicheiral knot. Then among the minimal diagrams representing K, there exists at least one for which achirality is visible in the following sense. There exists a rotation of 3-space of order 2 with an axis orthogonal to the diagram plane, intersecting the knot diagram at a point in the middle of an arc, which sends the diagram on itself so that at each crossing point, the over and under arcs are exchanged. Thus, the rotation is an orientation-preserving homeomorphism of 3-space, which sends the knot to its mirror image; see Figure 5 for an example.

Tait produced, on Plate VII of *On knots III*, figures representing diagrams of the 20 amphicheiral alternating knots with $c \leq 10$ which make amphicheirality visible. Among them, there are knots with several minimal diagrams that make the symmetry visible. The first one in the list has 8 crossings. This is knot 8_{12} in the tabulation of Alexander–Briggs and Rolfsen. For Tait, it is the knot 8_5. It is rational with Conway's notation 2222. Tait exhibits two different minimal diagrams that make the symmetry

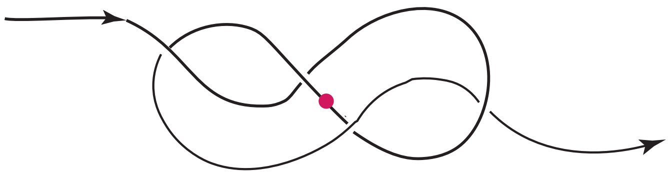

FIGURE 5
The figure-eight knot is −achiral.

visible and a third one without visible symmetry. Other examples are presented for $c = 10$. The knot 10_{45} in the Rolfsen table (10_1 in Tait) has 4 minimal diagrams which make the symmetry visible and 6 others without visible symmetry. Again, this knot is rational with Conway's notation 21111112.

There is a subtle observation that escaped Tait's attention but not Mary Haseman's: the rotation of order 2 reverses the orientation of the knot. The following definition makes it more precise.

Definition 9.1. A knot K in 3-space is −achiral if there is an orientation-preserving homeomorphism of 3-space that sends K to its mirror image while reversing its orientation.

The conjecture on visibility has then to be modified by taking into account the −achirality. This conjecture is true and has been proved in [3]:

Theorem 9.1. *Let K be a prime alternating −achiral knot. Then there exists a minimal diagram Π of K in $S^2 \subset S^3$ and an involution $\phi\colon S^3 \to S^3$ such that:*

(1) *ϕ reverses the orientation of S^3;*

(2) *$\phi(S^2) = S^2$;*

(3) *$\phi(\Pi) = \Pi$;*

(4) *ϕ has two fixed points on Π and therefore reverses the orientation of K.*

A fundamental tool for the proof of Theorem 9.1 is the Flyping theorem, hence the Jones polynomial. For more details on the historical background of this conjecture reported in the papers by Peter Tait and Mary Haseman, see [19].

Another type of chirality is possible by considering orientation-preserving homeomorphisms of 3-space which send the knot to its mirror image while preserving its orientation. This is +achirality. We will not discuss it here since the situation is much more involved.

10. Conclusion

All the Tait conjectures were proved in the years 1985–1991 and, at that time, all the proofs used the Jones polynomial, either directly or indirectly. By 'indirectly" we mean that the proof uses the Flyping theorem.

We note that the "original" one variable Jones polynomial $V_K(t)$ is used in both Kauffman's (via state sums) and Thistlethwaite's (via spanning trees) versions. It is remarkable that three different approaches produced almost at the same time (between 1984 and 1986), the same object: von Neumann algebras, finite statistical models and spanning trees of graphs. But one should not forget that Jones was at the origin of these events.

We have presented here the events as they unfolded at the time of Vaughan Jones' discovery. Since then, some new proofs of the Tait conjectures do not use the Jones polynomial. For instance, in 2017 Joshua Greene [7] gave a geometric proof of the Tait conjecture on minimal diagrams not using knot polynomials.

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