

Miscellaneous on commutants mod normed ideals and quasicentral modulus I

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Abstract. We define commutants mod normed ideals associated with compact smooth manifolds with boundary. The results about the K -theory of these operator algebras include an exact sequence for the connected sum of manifolds, derived from the Mayer–Vietoris sequence. We also make a few remarks about bicommutants mod normed ideals and about the quasicentral modulus for the quasinormed p -Schatten–von Neumann classes $0 < p < 1$.

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Dedicated to the memory of Vaughan Jones

1. Introduction

We arrived at the quasicentral modulus, a number associated to a n -tuple τ of operators and a normed ideal $(\mathcal{J}, |\cdot|_{\mathcal{J}})$ of compact operators, studying perturbations of n -tuples of operators with perturbations from the normed ideal [26, 27]. It turned out to play a key role in the invariance of absolutely continuous parts and diagonalizability mod the normed ideal in the case of n -tuples of commuting Hermitian operators, and found uses in non-commutative geometry [6]. The number is also of interest in the non-commutative setting of finitely generated groups and it also appears to be related to dynamical entropy. The number is a kind of measure of the n -tuple of operators at a scale given by the normed ideal. More recently, trying to understand the ubiquity of the quasicentral modulus, we began looking for more structure around this number and we found that the commutant modulo the normed ideal of the n -tuple of operators was relevant structure. Vaguely, this resembles a K -theory construction of the commutant modulo the compact operators, used in the Paschke duals, but there are unexpected

twists in the analogy, like getting C^* -algebras in situations where one only expected Banach algebras or a much richer K -theory (see [29] for a recent survey).

In this paper we have collected various remarks and results further extending the frame we began constructing with the commutants mod normed ideals.

We begin with the shortest discussion of a topic, actually only to point out some questions. Perturbations of Hermitian operators with singular spectral measure, as shown by rank one perturbation facts, behave very differently from what happens in the case of dimension ≥ 1 , where the quasicentral modulus works well. While rank one perturbations certainly do not fit with quasicentral modulus technique one may wonder whether these results signal some more general rough pathologies for perturbation from the quasinormed Schatten–von Neumann classes with $0 < p < 1$.

Our second topic is completing the picture of the commutant mod a normed ideal, by introducing the bicommutant mod the normed ideal, with the analogy with von Neumann algebras in mind. Here we can use the results about commutants modulo the compacts of von Neumann algebras and their refinement to normed ideals [2], which was preceded by [14, 15, 23]. What this gives is that the bicommutant mod the normed ideal, is the normed ideal plus a certain “smooth algebra” $\mathcal{D}(\tau, \mathcal{J})$ for the n -tuple τ contained in its von Neumann algebra. In the commutative setting this can be stated in terms of functional calculus and here the multivariable operator theory work in [10, 18, 21] is relevant to getting a grasp on these smoothness classes.

Our last focus, which is the main part of this paper is about commutants mod normed ideals associated to compact differentiable manifolds with boundary. Here a choice of an embedding into some \mathbb{R}^m provides an m -tuple of smooth functions the multiplication operators by which determine the commutant mod the normed ideal up to equivalent norms. We use this to develop for these commutants mod several of the results we had about the case of n -tuples of operators. We also show that the K -theory of Calkin algebras for commutants mod normed ideals for smooth manifolds has a Mayer–Vietoris sequence when we perform connected sums on the manifolds. The key to such a result is that these kinds of coronas/“Calkin algebras” are $C(\mathcal{X})$ algebras in the sense of Kasparov or more general \mathcal{X} -algebras considered by Kirchberg. The conclusion from these results seems to be that operations like connected sums may be manageable, but the main problem to better understand these commutants mod for manifolds would be with understanding better their structure or K -theory in simple cases like balls or spheres.

The paper has six sections, the first being this introduction and the last the list of references. The second section is about preliminaries. Then the third section is about the problems concerning the quasinormed setting, the fourth about bicommutants mod the fifth about commutants mod for differentiable manifolds with boundary.

2. Preliminaries

By \mathcal{H} , we will denote a separable infinite-dimensional complex Hilbert space and $\mathcal{B}(\mathcal{H})$, $\mathcal{K}(\mathcal{H})$, $\mathcal{B}/\mathcal{K}(\mathcal{H})$, $\mathcal{R}(\mathcal{H})$, $\mathcal{P}(\mathcal{H})$ or simply \mathcal{B} , \mathcal{K} , \mathcal{B}/\mathcal{K} , \mathcal{R} , \mathcal{P} will denote the bounded operators, the compact operators, the Calkin algebra, the finite rank operators and the finite rank Hermitian projectors. The canonical homomorphism $\mathcal{B} \rightarrow \mathcal{B}/\mathcal{K}$ will be denoted by p .

The normed ideals of compact operators $(\mathcal{J}, |\cdot|_{\mathcal{J}})$ or $(\mathcal{J}(\mathcal{H}), |\cdot|_{\mathcal{J}})$, when \mathcal{H} needs to be specified, and their properties are discussed in the standard reference [11, 24]. Note that the normed ideals, as defined in these references have the so-called Fatou property which is required in results of [2], which we shall use. In particular, $(\mathcal{C}_p, |\cdot|_p)$ will denote the Schatten–von Neumann p -class, $1 \leq p < \infty$. We shall also use the notation $(\mathcal{C}_p, |\cdot|_p)$ when $0 < p < 1$, in which case $|\cdot|_p$ is a quasinorm and the ideal \mathcal{C}_p is no longer locally convex. When $1 \leq p \leq \infty$ we denote by $(\mathcal{C}_p^-, |\cdot|_p^-)$ the $(p, 1)$ Lorentz ideal, the norm on which is given by

$$|T|_p^- = \sum_{j \in \mathbb{N}} s_j j^{-1+1/p},$$

where $s_1 \geq s_2 \geq \dots$ are the eigenvalues of $(T^*T)^{1/2}$. Note that $\mathcal{C}_1^- = \mathcal{C}_1$, $|\cdot|_1^- = |\cdot|_1$.

If $\tau = (T_j)_{1 \leq j \leq n}$ is a n -tuple of bounded operators, we denote $\tau^* = (T_j^*)_{1 \leq j \leq n}$, $\|\tau\| = \max_{1 \leq j \leq n} \|T_j\|$, and $[\tau, X] = ([T_j, X])_{1 \leq j \leq n}$, where $X \in \mathcal{B}$. If $\sigma = (S_k)_{1 \leq k \leq m}$ is an m -tuple of bounded operators then (σ, τ) is the $(m+n)$ -tuple $(S_1, \dots, S_m, T_1, \dots, T_n)$ and if $m = n$, then $\sigma + \tau = (S_j + T_j)_{1 \leq j \leq n}$. If $(\mathcal{J}, |\cdot|_{\mathcal{J}})$ is a normed ideal,

$$\mathcal{E}(\tau; \mathcal{J}) = \{X \in \mathcal{B} \mid [T_j, X] \in \mathcal{J}, 1 \leq j \leq n\}$$

is the commutant of $\tau \bmod \mathcal{J}$. We also have that $\mathcal{E}(\tau, \mathcal{J})$ is a Banach algebra with the norm

$$\| \|X\| \| = \|X\| + \|[\tau, X]\|_{\mathcal{J}}$$

where the $|\cdot|_{\mathcal{J}}$ -norm is the max of the \mathcal{J} -norms of the components. If $\tau = \tau^*$, $\mathcal{E}(\tau; \mathcal{J})$ is a $*$ -algebra and $\| \|X\| \| = \| \|X^*\| \|$. We also denote by

$$\begin{aligned} \mathcal{K}(\tau; \mathcal{J}) &= \mathcal{E}(\tau, \mathcal{J}) \cap \mathcal{K}, \\ \mathcal{E}/\mathcal{K}(\tau, \mathcal{J}) &= \mathcal{E}(\tau; \mathcal{J})/\mathcal{K}(\tau; \mathcal{J}), \\ \pi: \mathcal{E}(\tau; \mathcal{J}) &\rightarrow \mathcal{E}/\mathcal{K}(\tau; \mathcal{J}), \end{aligned}$$

the compact ideal, the quotient Banach algebra and the canonical homomorphism. Note that $p(\mathcal{E}(\tau; \mathcal{J})) \subset \mathcal{B}/\mathcal{K}$ and $\mathcal{E}/\mathcal{K}(\tau; \mathcal{J})$ are algebraically isomorphic and the map

$$\mathcal{E}/\mathcal{K}(\tau; \mathcal{J}) \rightarrow p(\mathcal{E}(\tau; \mathcal{J}))$$

is contractive. The quasicentral modulus of τ with respect to \mathcal{J} is the number

$$k_{\mathcal{J}}(\tau) = \liminf_{A \in \mathcal{R}_1^+} \|[A, \tau]\|_{\mathcal{J}},$$

where $\mathcal{R}_1^+ = \{X \in \mathcal{R} \mid 0 \leq X \leq I\}$ is the set of finite rank positive contractions endowed with the natural order [26, 28]. If $\mathcal{J} = \mathcal{C}_p$ or $\mathcal{J} = \mathcal{C}_p^-$, we denote $k_{\mathcal{J}}(\tau)$ also by $k_p(\tau)$ and respectively, $k_p^-(\tau)$. In case $0 < p < 1$, the definition of $k_p(\tau)$ still makes sense with $\|\cdot\|_p$ being the p -quasinorm in this case. We shall also consider the modulus of quasideagonality

$$qd_{\mathcal{J}}(\tau) = \liminf_{P \in \mathcal{P}} \|[P, \tau]\|_{\mathcal{J}}$$

the liminf being with respect to the natural order on the finite rank projections \mathcal{P} .

Assuming \mathcal{R} is dense in \mathcal{J} , we recall (see [23]) that if $k_{\mathcal{J}}(\tau) = 0$, then $\mathcal{E}/\mathcal{K}(\tau; \mathcal{J})$ is a C^* -algebra, while if only $k_{\mathcal{J}}(\tau) < \infty$, then $\mathcal{E}/\mathcal{K}(\tau; \mathcal{J})$ is isomorphic as a Banach algebra with involution to a C^* -algebra (not isometrically). Actually, $p(\mathcal{E}(\tau; \mathcal{J}))$ is a C^* -subalgebra of \mathcal{B}/\mathcal{K} and is isomorphic (not isometrically) to $\mathcal{E}/\mathcal{K}(\tau; \mathcal{J})$.

We shall also denote by $\mathcal{PE}(\tau; \mathcal{J})$ the Hermitian projectors in $\mathcal{E}(\tau; \mathcal{J})$.

Finally, if $\omega \subset \mathcal{B}$, we denote by ω' its commutant in \mathcal{B} :

$$\omega' = \{X \in \mathcal{B} \mid [X, \omega] = \{0\}\}.$$

Then $\omega'' = (\omega')'$ will be the bi-commutant of ω , that is in case $\omega = \omega^*$, the von Neumann algebra generated by ω .

3. The problem with $0 < p < 1$

A n -tuple τ of commuting Hermitian operators is diagonalizable mod \mathcal{C}_n^- , which is also equivalent to $k_n^-(\tau) = 0$, iff the spectral measure of τ is singular with respect to n -dimensional Lebesgue measure [26]. A similar result also holds for the normed ideal \mathcal{C}_p^- , $1 < p < n$ when the spectrum of τ is contained in certain fractals of Hausdorff dimension p , with p -dimensional Hausdorff measure replacing Lebesgue measure [8, 31]. In these cases, we observe that diagonalizability mod \mathcal{C}_p^- and k_p^- are tied to p -Hausdorff measure, where $p \geq 1$.

p -Hausdorff measure and dimension are also natural and interesting when $0 < p < 1$. Something quite different occurs with the diagonalization problem and we will not even need the refined Lorentz double scale to see it. If $T = T^*$ has a cyclic vector and its spectral measure is singular with respect to Lebesgue measure, then ([9, 19, 22, 25]) there is a rank one operator $X = X^*$ of arbitrary small norm so that $X + T$ is diagonal. A general Hermitian operator T on a separable Hilbert space is

an orthogonal sum of a sequence of Hermitian operators with cyclic vector. We can find for each summand a rank one perturbation which diagonalizes it and so that the orthogonal sum of the perturbations is a Hermitian operator X so that $X \in \mathcal{C}_p$ for all $0 < p < 1$. This means that $k_1(T) = 0$, which is equivalent to the singularity of the spectral measure of T implies $k_p(T) = 0$ for all $0 < p < 1$. Here $k_p(T)$ is defined the same way as in the case $p \geq 1$, with the only difference that the p -norm is now a p -quasinorm when $0 < p < 1$. This looks like a kind of “phase-transition” at $p = 1$ in the correspondence between k_p^- and p -Hausdorff measure. It is natural to ask how general this is.

Problem 3.1. Let τ be a n -tuple of Hermitian operators. Assume $k_1(\tau) = 0$. Does it follow that $k_p(\tau) = 0$ for $0 < p < 1$? If this is too general, one can add the assumption that τ is a n -tuple of commuting Hermitian operators.

Remark 3.1. If $T = T^*$, then $k_1(T) = 0$ is equivalent to $qd_1(T) = 0$ implies that $k_p^-(\tau) = 0$ or even $qd_p(\tau) = 0$ for $0 < p < 1$. Again one may consider the extra assumption about commuting components in τ .

4. The bicommutant mod a normed ideal

In this section $\tau = (T_j)_{1 \leq j \leq n}$ is a n -tuple of Hermitian operators on \mathcal{H} and $(\mathcal{J}, \| \cdot \|_{\mathcal{J}})$ is a normed ideal.

Definition 4.1. The bicommutant of $\tau \pmod{\mathcal{J}}$ is the $*$ -algebra

$$\mathcal{E}\mathcal{E}(\tau; \mathcal{J}) = \{X \in \mathcal{B} \mid [X, \mathcal{E}(\tau; \mathcal{J})] \subset \mathcal{J}\}.$$

Its compact ideal is

$$\mathcal{K}\mathcal{E}(\tau; \mathcal{J}) = \mathcal{E}\mathcal{E}(\tau; \mathcal{J}) \cap \mathcal{K}$$

and the bicommutant Calkin algebra is

$$\mathcal{E}/\mathcal{K}\mathcal{E}(\tau; \mathcal{J}) = \mathcal{E}\mathcal{E}(\tau; \mathcal{J})/\mathcal{K}\mathcal{E}(\tau; \mathcal{J}).$$

Lemma 4.1. If $X \in \mathcal{E}\mathcal{E}(\tau; \mathcal{J})$, then

$$\sup\{\|[X, Y]\|_{\mathcal{J}} \mid Y \in \mathcal{E}(\tau; \mathcal{J}), \|Y\| \leq 1\} < \infty$$

and $\mathcal{E}\mathcal{E}(\tau; \mathcal{J})$ is an involutive Banach algebra with isometric involution when endowed with the norm

$$\| \|X\| \| = \|X\| + \sup\{\|[X, Y]\|_{\mathcal{J}} \mid Y \in \mathcal{E}(\tau; \mathcal{J}), \|Y\| \leq 1\}.$$

The lemma is a consequence of the closed graph theorem applied to the map

$$\mathcal{E}(\tau; \mathcal{J}) \ni Y \rightarrow [X, Y] \in \mathcal{J}$$

where $X \in \mathcal{E}\mathcal{E}(\tau; \mathcal{J})$. Indeed if $\|Y_n - Y\| \rightarrow 0$, $\|[X, Y_n] - Z\|_{\mathcal{J}} \rightarrow 0$, then $[X, Y] = Z$, so the graph is closed. The rest is also an easy exercise.

Theorem 4.1. *We have $\mathcal{E}\mathcal{E}(\tau; \mathcal{J}) = \mathcal{E}\mathcal{E}(\tau; \mathcal{J}) \cap (\tau)'' + \mathcal{J}$ and $\mathcal{E}\mathcal{E}(\tau; \mathcal{J}) \cap (\tau)''$ is a closed subalgebra of $\mathcal{E}\mathcal{E}(\tau; \mathcal{J})$.*

Proof. We have $(\tau)' \subset \mathcal{E}(\tau; \mathcal{J})$ so that if $X \in \mathcal{E}\mathcal{E}(\tau; \mathcal{J})$ then we will have $[X, (\tau)'] \subset \mathcal{J}$. By [2], this implies $X \in (\tau)'' + \mathcal{J}$. On the other hand $\mathcal{J} \subset \mathcal{E}\mathcal{E}(\tau; \mathcal{J})$, so this gives

$$\mathcal{E}\mathcal{E}(\tau; \mathcal{J}) = \mathcal{E}\mathcal{E}(\tau; \mathcal{J}) \cap (\tau)'' + \mathcal{J}.$$

If $Y \in \mathcal{E}(\tau; \mathcal{J})$, the map

$$\mathcal{E}\mathcal{E}(\tau; \mathcal{J}) \ni X \rightarrow [X, Y] \in \mathcal{J}$$

is continuous. Since $(\tau)' \subset \mathcal{E}(\tau; \mathcal{J})$ it follows that $\mathcal{E}\mathcal{E}(\tau; \mathcal{J}) \cap (\tau)''$ is closed in $\mathcal{E}\mathcal{E}(\tau; \mathcal{J})$. ■

Corollary 4.1. *We have $p(\mathcal{E}\mathcal{E}(\tau; \mathcal{J})) \subset p((\tau)'') \subset \mathcal{B}/\mathcal{K}$. If $(\tau)'' \cap \mathcal{K} = \{0\}$, then p gives an isomorphism of $\mathcal{E}\mathcal{E}(\tau; \mathcal{J}) \cap (\tau)''$ and $p(\mathcal{E}\mathcal{E}(\tau; \mathcal{J}))$.*

The corollary is an immediate consequence of the theorem.

Definition 4.2. We define $\mathcal{D}(\tau; \mathcal{J}) = \mathcal{E}\mathcal{E}(\tau; \mathcal{J}) \cap (\tau)''$ to be the *smooth algebra of τ with respect to \mathcal{J}* . We also define $\mathcal{D}/\mathcal{K}(\tau; \mathcal{J}) = p(\mathcal{D}(\tau; \mathcal{J})) = p(\mathcal{E}\mathcal{E}(\tau; \mathcal{J}))$ to be the *essential smooth algebra of τ with respect to \mathcal{J}* .

Remark 4.1. If $\tau - \tau' \in (\mathcal{J})^n$, then clearly $\mathcal{E}(\tau; \mathcal{J}) = \mathcal{E}(\tau'; \mathcal{J})$, so that

$$\mathcal{E}\mathcal{E}(\tau; \mathcal{J}) = \mathcal{E}\mathcal{E}(\tau'; \mathcal{J}).$$

While $(\tau)''$ and $(\tau')''$ are different in general, we obviously have

$$\mathcal{D}/\mathcal{K}(\tau; \mathcal{J}) = \mathcal{D}/\mathcal{K}(\tau'; \mathcal{J}).$$

Moreover, $\mathcal{E}\mathcal{E}(\tau'; \mathcal{J})/\mathcal{J} = \mathcal{E}\mathcal{E}(\tau; \mathcal{J})/\mathcal{J}$ gives that $\mathcal{D}(\tau; \mathcal{J})$ and $\mathcal{D}(\tau'; \mathcal{J})$ have the same image in \mathcal{B}/\mathcal{J} .

Remark 4.2. Note that if τ is an n -tuple of commuting Hermitian operators, then $(\tau)''$ is the algebra of Borel functions of τ by functional calculus and this relates $\mathcal{D}(\tau; \mathcal{J})$ to smoothness properties of operator functions, see for instance [10, 18, 21] and related work.

5. Commutants mod normed ideals associated with differentiable manifolds

Let \mathcal{X} be a compact C^∞ -manifold with boundary [13]. Let μ be a Radon measure on \mathcal{X} such that in each local parametrization the restriction of μ has the same absolute continuity class as Lebesgue measure. The representation $M: L^\infty(\mathcal{X}; \mu) \rightarrow \mathcal{B}(L^2(\mathcal{X}; \mu))$ by multiplication operators $M(f)h = fh$, up to unitary equivalence does not depend on the choice of μ . The intertwining operator for the representations arising from two such measures is given by multiplication with the square-root of their Radon–Nikodym derivative. Let $(\mathcal{J}, \|\cdot\|_{\mathcal{J}})$ be a normed ideal, which to simplify, is so that \mathcal{R} is dense in \mathcal{J} . Then we define

$$\mathcal{E}(\mathcal{X}, \mathcal{J}) = \{X \in \mathcal{B}(L^2(\mathcal{X}, \mu)) \mid [M(C^\infty(\mathcal{X})), X] \subset \mathcal{J}\}$$

the $*$ -algebra which is the commutant mod \mathcal{J} of $M(C^\infty(\mathcal{X}))$. If α is a diffeomorphism of \mathcal{X} , there is a unitary operator $U_\alpha h = (h \circ \alpha^{-1}) \cdot [\alpha_* \mu : \mu]^{1/2}$ on $L^2(\mathcal{X}, \mu)$ so that $\alpha \rightarrow U_\alpha$ is a representation of $\mathcal{D}iff(\mathcal{X})$ and

$$U_\alpha M(f) U_\alpha^{-1} = M(f \circ \alpha^{-1}).$$

It follows that

$$U_\alpha \mathcal{E}(\mathcal{X}; \mathcal{J}) U_\alpha^{-1} = \mathcal{E}(\mathcal{X}; \mathcal{J}),$$

and we see that

$$\alpha \rightarrow (X \rightarrow U_\alpha X U_\alpha^{-1})$$

gives a homomorphism of $\mathcal{D}iff(\mathcal{X})$ into the automorphisms of $\mathcal{E}(\mathcal{X}; \mathcal{J})$. Our next aim is to show that $\mathcal{E}(\mathcal{X}; \mathcal{J})$ can actually be defined using only finitely many $M(f)$ so that we can use the results on commutants mod a normed ideal of commuting n -tuples of Hermitian operators. In particular, $\mathcal{E}(\mathcal{X}; \mathcal{J})$ is a Banach algebra with involution.

Lemma 5.1. *Let $\tau = (X_j)_{1 \leq j \leq n}$ be a n -tuple of commuting Hermitian operators and $\Omega \subset \mathbb{R}^n$ an open set so that the spectrum $\sigma(\tau) \subset \Omega$. If \mathcal{J} is a normed ideal and $F: \Omega \rightarrow \mathbb{R}$ is a C^∞ -function, and if $Y \in \mathcal{B}(L^2(\mathcal{X}, \mu))$, then*

$$\|[F(\tau), Y]\|_{\mathcal{J}} \leq C \|[\tau, Y]\|_{\mathcal{J}},$$

where the constant does not depend on \mathcal{J} .

This is well known. It can for instance be proved by dealing first with the case of $F(x_1, \dots, x_n) = \exp(i \sum_{1 \leq j \leq n} x_j \xi_j)$ and then obtaining the result for general F by using the fact that the Fourier transform can be used to express $F(\tau)$ as an integral involving exponentials applied to τ .

Proposition 5.1. *Let $f_1, \dots, f_m \in C^\infty(\mathcal{X})$ be such that $F: \mathcal{X} \rightarrow \mathbb{R}^m$, where $F = (f_1, \dots, f_m)$ is a neat embedding of \mathcal{X} into a half-space of \mathbb{R}^m , in the sense of [13]. Then we have*

$$\mathcal{E}(\mathcal{X}; \mathcal{J}) = \mathcal{E}((M(f_j))_{1 \leq j \leq m}; \mathcal{J}).$$

Proof. Obviously, we have that the “left-hand side \subset right-hand side”, so what we need to prove is that the “left-hand side \supset right-hand side”. We need to prove that if $X \in \mathcal{E}((M(f_j))_{1 \leq j \leq m}; \mathcal{J})$ and $g \in C^\infty(\mathcal{X})$ is real-valued then $[M(g), X] \in \mathcal{J}$. This follows from Lemma 5.1 since we can find a C^∞ -function $G: \Omega \rightarrow \mathbb{R}$, where Ω is a neighborhood of $F(\mathcal{X})$, so that $G \circ F = g$ and then

$$G(M(f_1), \dots, M(f_m)) = M(g). \quad \blacksquare$$

Since \mathcal{X} is compact by [13, Theorem 4.3], the embedding assumption in the preceding proposition can be satisfied and thus the results we have for commutants mod normed ideals of m -tuples of Hermitian operators apply to $\mathcal{E}(\mathcal{X}; \mathcal{J})$. In what follows we list a few of the consequences which we would like to point out in particular. An embedding of \mathcal{X} provides a norm on $\mathcal{E}(\mathcal{X}; \mathcal{J})$ with respect to which it is a Banach algebra with isometric involution and using the closed graph theorem one finds that different embeddings give rise to equivalent norms on $\mathcal{E}(\mathcal{X}; \mathcal{J})$. Then $\mathcal{E}(\mathcal{X}; \mathcal{J}) \cap \mathcal{K}$, which we shall denote by $\mathcal{K}(\mathcal{X}; \mathcal{J})$, is a closed ideal and $\mathcal{E}/\mathcal{K}(\mathcal{X}; \mathcal{J}) = \mathcal{E}(\mathcal{X}; \mathcal{J})/\mathcal{K}(\mathcal{X}; \mathcal{J})$ is also a Banach algebra with involution.

Theorem 5.1. *Let \mathcal{X} be a compact smooth manifold with boundary and real-valued $f_1, \dots, f_m \in C^\infty(\mathcal{X})$ which give a neat embedding of \mathcal{X} (see [13]). Further, let $(\mathcal{J}, |\cdot|_{\mathcal{J}})$ be a normed ideal so that $\mathcal{J} \supset \mathcal{C}_n^-, \mathcal{J} \neq \mathcal{C}_n^-$ and with \mathcal{R} dense in \mathcal{J} , where n is the dimension of \mathcal{X} . Then,*

- (a) $k_{\mathcal{J}}(M(f_1), \dots, M(f_m)) = 0, 0 < k_n^-(M(f_1), \dots, M(f_m)) < \infty$, and if $\mathcal{H}_1 \subset L^2(M, \mu)$ is an invariant subspace for $(M(f_1), \dots, M(f_m)), \mathcal{H}_1 \neq 0$, then still

$$k_n^-(M(f_1)|_{\mathcal{H}_1}, \dots, M(f_m)|_{\mathcal{H}_1}) > 0.$$

- (b) $\mathcal{E}/\mathcal{K}(\mathcal{X}; \mathcal{C}_n^-)$ is (non-isometrically) isomorphic to a C^* -algebra and $\mathcal{E}/\mathcal{K}(\mathcal{X}; \mathcal{J})$ is a C^* -algebra, the norm being independent of the embedding.
- (c) The center of $\mathcal{E}/\mathcal{K}(\mathcal{X}; \mathcal{C}_n^-)$ is $\pi(M(C(\mathcal{X})))$. Also, the center of $\mathcal{E}/\mathcal{K}(\mathcal{X}; \mathcal{J})$ is $\pi(M(C(\mathcal{X})))$.
- (d) If $n \geq 3$ and $v_m \in C^\infty(\mathcal{X})$ are such that $|v_m| = 1$ and $w - \lim_{m \rightarrow \infty} M(v_m) = 0$, then the strong limit

$$\Phi(T) = s - \lim_{m \rightarrow \infty} M(v_m) T M(v_m)^*$$

exists for every $T \in \mathcal{E}(\mathcal{X}; \mathcal{C}_n^-)$ and is independent of the choice of v_m 's and gives a $*$ -homomorphism

$$\Phi: \mathcal{E}(\mathcal{X}; \mathcal{C}_n^-) \rightarrow M(L^\infty(\mathcal{X}, \mu)),$$

so that $\Phi(M(f)) = M(f)$ if $f \in L^\infty(\mathcal{X}, \mu)$, $\|\Phi\| = 1$, and $\Phi(\mathcal{K}(\mathcal{X}, \mathcal{C}_n^-)) = \{0\}$. In particular, there is a $*$ -homomorphism

$$\Psi: \mathcal{E}/\mathcal{K}(\mathcal{X}; \mathcal{C}_n^-) \rightarrow \pi(M(L^\infty(\mathcal{X}; \mu))),$$

so that $\pi \circ \Phi = \Psi \circ \pi$.

Proof. (a) One can cover \mathcal{X} with a finite number of open sets in each of which there is some subset of $\{f_1, \dots, f_m\}$ which provides a coordinate system. Then $L^2(\mathcal{X}; \mu)$ is an orthogonal sum of $L^2(\Omega; \mu \upharpoonright \Omega)$ where Ω is a Borel subset of one of those open sets. Then the assertion reduces to that for an $(n + p)$ -tuple of commuting Hermitian operators the first n of which have joint spectrum which is Lebesgue of multiplicity one and the remaining p are smooth functions of these. Thus, (a) becomes a consequence of the facts about normed ideal perturbations of commuting n -tuples of Hermitian operators (see [3, 26–28]) supplemented by Lemma 5.1.

(b) This is a consequence of (a) and of [30].

(c) This follows from [5, 32] and (a).

(d) Let $v_m \in C^\infty(\mathcal{X})$ be such that $|v_m| = 1$ and $w - \lim_{m \rightarrow \infty} M(v_m) = 0$, then the strong limit

$$s - \lim_{m \rightarrow \infty} M(v_m)TM(v_m)^* = \Phi(T)$$

exists for every $T \in \mathcal{E}(\mathcal{X}; \mathcal{C}_n^-)$. This is obtained after a construction like that in the proof of (a) from [27, Theorem 2.3], the comments following it, and [27, Corollary 1.6]. As remarked in [27] the fact that the sequence of v_m is arbitrary implies, Φ does not depend on the choice of v_m 's which implies

$$\Phi(T) \in (M(C^\infty(\mathcal{X})))' = M(L^\infty(\mathcal{X}; \mu)).$$

Note that if $T \in \mathcal{K}(\mathcal{X}, \mathcal{C}_n^-)$, then $\Phi(T) \in \mathcal{K}$ and $\mathcal{K} \cap M(L^\infty(\mathcal{X}; \mu)) = \{0\}$ gives $\Phi(T) = 0$. ■

It is also useful to make the following simple observation. If we compress $\mathcal{E}(\mathcal{X}; \mathcal{J})$ to $L^2(\Omega; \mu)$, where $\Omega \subset \mathcal{X}$ is a Borel set, we get the commutant mod \mathcal{J} of the multiplication operators by $f|_\Omega$, where $f \in C^\infty(\mathcal{X})$, or equivalently of $f_1|_\Omega, \dots, f_n|_\Omega$ for an embedding f_1, \dots, f_m of \mathcal{X} . Note also these kinds of consideration carry over to $\mathcal{K}(\mathcal{X}; \mathcal{J})$ and $\mathcal{E}/\mathcal{K}(\mathcal{X}; \mathcal{J})$. In particular, we have the following, for an arbitrary normed ideal \mathcal{J} .

Lemma 5.2. *Let $A \subset \mathbb{X} \setminus \partial\mathbb{X}$ be a closed submanifold with boundary of the same dimension as \mathbb{X} and $P = M(\chi_A)$. Then*

$$\mathcal{E}(A; \mathcal{J}) = P\mathcal{E}(\mathbb{X}; \mathcal{J}) \mid L^2(A, \mu \mid A)$$

and we also have isometric isomorphisms of $\mathcal{E}/\mathcal{K}(A; \mathcal{J})$ and $\pi(P)\mathcal{E}/\mathcal{K}(\mathbb{X}; \mathcal{J})\pi(P)$.

If $\mathcal{J} \supseteq \mathcal{C}_n^-$ by Theorem 5.1 (c), the center of $\mathcal{E}/\mathcal{K}(\mathbb{X}; \mathcal{J})$ being $\pi(M(C(\mathbb{X})))$ to an open set $U \subset \mathbb{X}$ there, corresponds a closed two-sided ideal $\mathcal{E}/\mathcal{K}(U, \mathbb{X}; \mathcal{J})$ of $\mathcal{E}/\mathcal{K}(\mathbb{X}; \mathcal{J})$ which can be defined as follows. If $A \subset \mathbb{X}$ is a Borel set let P_A be the projection $M(\chi_A)$, where χ_A is the indicator function. We define $\mathcal{E}/\mathcal{K}(U, \mathbb{X}; \mathcal{J})$ to be the closure of

$$\bigcup_{K \subset\subset U} \pi(P_K)\mathcal{E}/\mathcal{K}(\mathbb{X}; \mathcal{J})\pi(P_K),$$

where $K \subset\subset U$ denotes the fact that $K \subset U$ is a compact subset.

Lemma 5.3. (a) *If $K_j \subset\subset U$, then $j \in \mathbb{N}$ are such that $K_j \subset \overset{\circ}{K}_{j+1}$ and $\bigcup_{j \in \mathbb{N}} K_j = U$, then $\mathcal{E}/\mathcal{K}(U, \mathbb{X}; \mathcal{J})$ is the closure of*

$$\bigcup_{j \in \mathbb{N}} \pi(P_{K_j})\mathcal{E}/\mathcal{K}(\mathbb{X}; \mathcal{J})\pi(P_{K_j}).$$

(b) *If $f \in C(\mathbb{X})$ is such that $f^{-1}(0) = \mathbb{X} \setminus U$, then $\mathcal{E}/\mathcal{K}(U, \mathbb{X}; \mathcal{J})$ is the closure of*

$$\pi(M(f))\mathcal{E}/\mathcal{K}(\mathbb{X}; \mathcal{J}).$$

(c) $\mathcal{E}/\mathcal{K}(U, \mathbb{X}; \mathcal{J})$ is the closed two-sided ideal of $\mathcal{E}/\mathcal{K}(\mathbb{X}; \mathcal{J})$ generated by

$$\{\pi(M(f)) \mid f \in C(\mathbb{X}), f \mid \mathbb{X} \setminus U = 0\}.$$

Proof. (a) Part (a) is obvious since every compact set $K \subset\subset U$ is contained in some K_j .

(b) Let $K_j = \{x \in \mathbb{X} \mid |f(x)| \geq 1/j\}$. Since $K_j \subset\subset U$ and $\|P_{K_j}M(f) - M(f)\| \rightarrow 0$, we easily get

$$\pi(M(f))\mathcal{E}/\mathcal{K}(\mathbb{X}; \mathcal{J}) \subset \mathcal{E}/\mathcal{K}(U, \mathbb{X}; \mathcal{J}).$$

Also,

$$\begin{aligned} \pi(M(f))\mathcal{E}/\mathcal{K}(\mathbb{X}; \mathcal{J}) &\supset \pi(M(f))\pi(P_{K_j})\mathcal{E}/\mathcal{K}(\mathbb{X}; \mathcal{J})\pi(P_{K_j}) \\ &\supset \pi(P_{K_j})\mathcal{E}/\mathcal{K}(\mathbb{X}; \mathcal{J})\pi(P_{K_j}) \end{aligned}$$

and (a) gives the converse inclusion.

(c) Since $\pi(M(f))$ is in the center of $\mathcal{E}/\mathcal{K}(\mathbb{X}; \mathcal{J})$ we have that $\pi(M(f))\mathcal{E}/\mathcal{K}(\mathbb{X}; \mathcal{J})$ is a two-sided ideal and we can also use (b) to get (c). ■

We shall look at the case when the open set $U = \mathcal{X} \setminus N$ where N is a submanifold satisfying certain conditions so that we can use tubular neighborhood results ([13]) when describing $\mathcal{E}/\mathcal{K}(U, \mathcal{X}; \mathcal{J})$. We shall assume N is a compact submanifold without boundary in \mathcal{X} (thus, $N \cap \partial\mathcal{X} = \emptyset$). Using a tubular neighborhood of N and an orthogonal structure on the vector bundle from which it arises (see [13, pp. 116–117]) the open disk-subbundle of radius $\varepsilon > 0$ yields an open neighborhood N_ε of N so that the closure \bar{N}_ε is a compact submanifold of \mathcal{X} with boundary $\partial N_\varepsilon \subset \mathcal{X} \setminus \partial\mathcal{X}$.

Note that $K_\varepsilon = \mathcal{X} \setminus N_\varepsilon$ is then a compact manifold with boundary $\partial N_\varepsilon \cup \partial\mathcal{X}$. Since $0 < \varepsilon_1 < \varepsilon_2 \Rightarrow N_{\varepsilon_1} \subset N_{\varepsilon_2}$ and $\bigcap_{\varepsilon>0} N_\varepsilon = N$, we can use Lemma 5.3 (a) which says that if $\mathcal{J} \supseteq \mathcal{C}_n^-$, then $\mathcal{E}/\mathcal{K}(\mathcal{X} \setminus N, \mathcal{X}; \mathcal{J})$ is the closure of

$$\bigcup_{\varepsilon>0} \pi(P_\varepsilon)\mathcal{E}/\mathcal{K}(\mathcal{X}; \mathcal{J})\pi(P_\varepsilon),$$

where $P_\varepsilon = M(\mathcal{X}_{\mathcal{X} \setminus N_\varepsilon})$.

Of course, the preceding union can be taken over $\varepsilon \in \{1/j \mid j \in \mathbb{N}\}$. Then by Lemma 5.2, $\pi(P_\varepsilon)\mathcal{E}/\mathcal{K}(\mathcal{X}; \mathcal{J})\pi(P_\varepsilon)$ identifies with $\mathcal{E}/\mathcal{K}(\mathcal{X} \setminus N_\varepsilon; \mathcal{J})$. Remark also that the assumption $\mathcal{J} \supseteq \mathcal{C}_n^-$ guarantees that these are C^* -algebras or at least closed subalgebras of a Banach algebra isomorphic to a C^* -algebras. This gives that $\mathcal{E}/\mathcal{K}(\mathcal{X} \setminus N, \mathcal{X}; \mathcal{J})$ is isomorphic to the inductive limit of the C^* -algebras which are isomorphic to the $\mathcal{E}/\mathcal{K}(\mathcal{X} \setminus N_{1/j}; \mathcal{J})$. We record this as the next lemma.

Lemma 5.4. *Let N be a compact submanifold without boundary of \mathcal{X} and assume $\mathcal{J} \supseteq \mathcal{C}_n^-$. Then the ideal $\mathcal{E}/\mathcal{K}(\mathcal{X} \setminus N, \mathcal{X}; \mathcal{J})$ is a Banach algebra which is isomorphic to the inductive limit of the C^* -algebras isomorphic to $\mathcal{E}/\mathcal{K}(\mathcal{X} \setminus N_{1/j}; \mathcal{J})$ for the $1/j$ -neighborhoods of N constructed above.*

Next we shall take a look at the effect of connected sums of smooth manifolds on the associated algebras $\mathcal{E}/\mathcal{K}(\mathcal{X}; \mathcal{J})$ when $\mathcal{J} \supseteq \mathcal{C}_n^-$.

Let \mathcal{X}, \mathcal{Y} be two compact smooth manifolds of the same dimension n and $\mathcal{X} \# \mathcal{Y}$ their connected sum. Then the construction of the connected sum gives two points $x \in \mathcal{X} \setminus \partial\mathcal{X}$, $y \in \mathcal{Y} \setminus \partial\mathcal{Y}$, open sets $U, V \subset \mathcal{X} \# \mathcal{Y}$ so that $U \cup V = \mathcal{X} \# \mathcal{Y}$, and diffeomorphisms

$$\alpha: U \rightarrow \mathcal{X} \setminus \{x\}, \quad \beta: V \rightarrow \mathcal{Y} \setminus \{y\}, \quad \gamma: U \cap V \rightarrow (0, 1) \times S^{n-1}.$$

Moreover, there are continuous functions $f, g: \mathcal{X} \# \mathcal{Y} \rightarrow [0, \infty)$ so that

$$f^{-1}((0, \infty)) = U, \quad g^{-1}((0, \infty)) = V, \quad f + g = 1, \quad \text{and} \quad \text{pr}_1 \circ \gamma = f \mid U \cap V.$$

Lemma 5.5. *We have*

$$\begin{aligned} \mathcal{E}/\mathcal{K}(U, \mathcal{X} \# \mathcal{Y}; \mathcal{J}) + \mathcal{E}/\mathcal{K}(V, \mathcal{X} \# \mathcal{Y}; \mathcal{J}) &= \mathcal{E}/\mathcal{K}(\mathcal{X} \# \mathcal{Y}; \mathcal{J}), \\ \mathcal{E}/\mathcal{K}(U, \mathcal{X} \# \mathcal{Y}; \mathcal{J}) \cap \mathcal{E}/\mathcal{K}(V, \mathcal{X} \# \mathcal{Y}; \mathcal{J}) &= \mathcal{E}/\mathcal{K}(U \cap V, \mathcal{X} \# \mathcal{Y}; \mathcal{J}). \end{aligned}$$

Proof. Lemma 5.3 (b) gives

$$\begin{aligned} \pi(M(f))\mathcal{E}/\mathcal{K}(X \# Y; \mathcal{J}) &\subset \mathcal{E}/\mathcal{K}(U, X \# Y; \mathcal{J}), \\ \pi(M(g))\mathcal{E}/\mathcal{K}(X \# Y; \mathcal{J}) &\subset \mathcal{E}/\mathcal{K}(V, X \# Y; \mathcal{J}) \end{aligned}$$

and we have $f + g = 1$, which yields the first equality. Since

$$\|f^{1/k}f - f\| \rightarrow 0 \quad \text{and} \quad \|g^{1/k}g - g\| \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

we see that Lemma 5.3 (b) implies that $(\pi(M(f^{1/k})))_{k \geq 1}$ and $(\pi(M(g^{1/k})))_{k \geq 1}$ are approximate units of the ideals $\mathcal{E}/\mathcal{K}(U, X \# Y; \mathcal{J})$ and $\mathcal{E}/\mathcal{K}(V, X \# Y; \mathcal{J})$, respectively. Then $(\pi(M(f^{1/k}))\pi(M(g^{1/k})))_{k \geq 1}$ is an approximate unit for

$$\mathcal{E}/\mathcal{K}(U, X \# Y; \mathcal{J}) \cap \mathcal{E}/\mathcal{K}(V, X \# Y; \mathcal{J}).$$

Again by Lemma 5.3 (b), we have

$$\pi(M(f^{1/k}))\pi(M(g^{1/k})) = \pi(M((fg)^{1/k})) \in \mathcal{E}/\mathcal{K}(U \cap V, X \# Y; \mathcal{J}),$$

so that

$$\mathcal{E}/\mathcal{K}(U, X \# Y; \mathcal{J}) \cap \mathcal{E}/\mathcal{K}(V, X \# Y; \mathcal{J}) \subset \mathcal{E}/\mathcal{K}(U \cap V, X \# Y; \mathcal{J}).$$

The opposite inclusion is obvious. ■

With these preparations we apply the Mayer–Vietoris exact sequence [12] to this situation.

$$\begin{array}{ccccc} K_0(\mathcal{E}/\mathcal{K}(U \cap V, X \# Y; \mathcal{J})) & \rightarrow & K_0(\mathcal{E}/\mathcal{K}(U, X \# Y; \mathcal{J})) & \longrightarrow & K_0(\mathcal{E}/\mathcal{K}(X \# Y; \mathcal{J})) \\ & & \oplus K_0(\mathcal{E}/\mathcal{K}(V, X \# Y; \mathcal{J})) & & \downarrow \\ K_1(\mathcal{E}/\mathcal{K}(X \# Y; \mathcal{J})) & \longleftarrow & K_1(\mathcal{E}/\mathcal{K}(U, X \# Y; \mathcal{J})) & \longleftarrow & K_1(\mathcal{E}/\mathcal{K}(U \cap V, X \# Y; \mathcal{J})) \\ & & \oplus K_1(\mathcal{E}/\mathcal{K}(V, X \# Y; \mathcal{J})) & & \end{array}$$

We will work more on identifying the terms of this exact sequence so that it relates the groups $K_j(\mathcal{E}/\mathcal{K}(X \# Y; \mathcal{J}))$ to groups which no longer involve $X \# Y$.

Lemma 5.6. *With the notation introduced and the assumption $\mathcal{J} \supseteq \mathcal{C}_n^-$, the diffeomorphisms α, β, γ give rise to isomorphisms*

$$\begin{aligned} \mathcal{E}/\mathcal{K}(U, X \# Y, \mathcal{J}) &\simeq \mathcal{E}/\mathcal{K}(X \setminus \{x\}, X; \mathcal{J}), \\ \mathcal{E}/\mathcal{K}(V, X \# Y, \mathcal{J}) &\simeq \mathcal{E}/\mathcal{K}(Y \setminus \{y\}, Y; \mathcal{J}), \\ \mathcal{E}/\mathcal{K}(U \cap V, X \# Y; \mathcal{J}) &\simeq \mathcal{E}/\mathcal{K}((0, 1) \times S^{n-1}, [0, 1] \times S^{n-1}; \mathcal{J}). \end{aligned}$$

Proof. We can apply Lemma 5.4 to $\mathcal{E}/\mathcal{K}(\mathcal{X} \setminus \{x\}, \mathcal{X}; \mathcal{J})$ with $N = \{x\}$ and the tubular neighborhood arising from the local coordinates chosen for the connected sum gluing. Let $K_j = \mathcal{X} \setminus N_{1/2j}$ and $\alpha^{-1}(K_j) = L_j \subset U$. Then $\mathcal{E}/\mathcal{K}(\mathcal{X} \setminus \{x\}, \mathcal{X}; \mathcal{J})$ is the inductive limit of $\mathcal{E}/\mathcal{K}(K_j; \mathcal{J})$ and $\mathcal{E}/\mathcal{K}(U, \mathcal{X} \# \mathcal{Y}; \mathcal{J})$ is the inductive limit of the $\mathcal{E}/\mathcal{K}(L_j; \mathcal{J})$. The diffeomorphism α produces an isomorphism of the inductive limits. The identification of $\mathcal{E}/\mathcal{K}(V, \mathcal{X} \# \mathcal{Y}; \mathcal{J})$ and $\mathcal{E}/\mathcal{K}(\mathcal{Y} \setminus \{y\}, \mathcal{Y}; \mathcal{J})$ using β is clearly completely analogous.

For the last assertion let now $K_j = [1/2j, 1 - 1/2j] \times S^{n-1}$ and $L_j = \gamma^{-1}(K_j)$. Then K_j and L_j are compact submanifolds with boundary of $[0, 1] \times S^{n-1}$ and $\mathcal{X} \# \mathcal{Y}$, respectively, which do not intersect the boundaries of these. Using Lemmas 5.2 and 5.3, we get that $\mathcal{E}/\mathcal{K}(U \cap V, \mathcal{X} \# \mathcal{Y}; \mathcal{J})$ and $\mathcal{E}/\mathcal{K}(U \cap V, \mathcal{X} \# \mathcal{Y}; \mathcal{J})$ are then identified with the inductive limit of the $\mathcal{E}/\mathcal{K}(K_j; \mathcal{J})$ and respectively of the $\mathcal{E}/\mathcal{K}(L_j; \mathcal{J})$ and these are isomorphic via isomorphisms induced by γ . ■

With the identifications provided by the preceding lemma, the reader can further pursue this and work out what the inclusions

$$\begin{aligned} \mathcal{E}/\mathcal{K}(U \cap V, \mathcal{X} \# \mathcal{Y}; \mathcal{J}) &\subset \mathcal{E}/\mathcal{K}(U, \mathcal{X} \# \mathcal{Y}; \mathcal{J}), \\ \mathcal{E}/\mathcal{K}(U \cap V, \mathcal{X} \# \mathcal{Y}; \mathcal{J}) &\subset \mathcal{E}/\mathcal{K}(V, \mathcal{X} \# \mathcal{Y}; \mathcal{J}), \\ \mathcal{E}/\mathcal{K}(U, \mathcal{X} \# \mathcal{Y}; \mathcal{J}) &\subset \mathcal{E}/\mathcal{K}(\mathcal{X} \# \mathcal{Y}; \mathcal{J}), \\ \mathcal{E}/\mathcal{K}(V, \mathcal{X} \# \mathcal{Y}; \mathcal{J}) &\subset \mathcal{E}/\mathcal{K}(\mathcal{X} \# \mathcal{Y}; \mathcal{J}) \end{aligned}$$

will correspond to in terms of

$$\mathcal{E}/\mathcal{K}(\mathcal{X} \setminus \{x\}, \mathcal{X}; \mathcal{J}), \quad \mathcal{E}/\mathcal{K}(\mathcal{Y} \setminus \{y\}, \mathcal{Y}; \mathcal{J}), \quad \mathcal{E}/\mathcal{K}((0, 1) \times S^{n-1}, [0, 1] \times S^{n-1}; \mathcal{J}).$$

In the end, we get an exact sequence which we record as the next theorem.

Theorem 5.2. *The Mayer–Vietoris exact sequence gives rise to an exact sequence*

$$\begin{array}{ccccc} K_0(\mathcal{E}/\mathcal{K}((0, 1) \times S^{n-1}, & \rightarrow & K_0(\mathcal{E}/\mathcal{K}(\mathcal{X} \setminus \{x\}, \mathcal{X}; \mathcal{J})) & \longrightarrow & K_0(\mathcal{E}/\mathcal{K}(\mathcal{X} \# \mathcal{Y}; \mathcal{J})) \\ [0, 1] \times S^{n-1}; \mathcal{J})) & \rightarrow & \oplus K_0(\mathcal{E}/\mathcal{K}(\mathcal{Y} \setminus \{y\}, \mathcal{Y}; \mathcal{J})) & & \downarrow \\ \uparrow & & & & \\ K_1(\mathcal{E}/\mathcal{K}(\mathcal{X} \# \mathcal{Y}; \mathcal{J})) & \longleftarrow & K_1(\mathcal{E}/\mathcal{K}(\mathcal{X} \setminus \{x\}, \mathcal{X}_j; \mathcal{J})) & \longleftarrow & K_1(\mathcal{E}/\mathcal{K}((0, 1) \times S^{n-1}, \\ & & \oplus K_1(\mathcal{E}/\mathcal{K}(\mathcal{Y} \setminus \{y\}, \mathcal{Y}; \mathcal{J})) & & [0, 1] \times S^{n-1}; \mathcal{J})) \end{array}$$

We begin the concluding remarks for this section with two K -theory ([4, 12]) consequences of Theorem 5.1.

If the dimension n of \mathcal{X} is ≥ 3 , the homomorphism

$$\Psi: \mathcal{E}/\mathcal{K}(\mathcal{X}; \mathcal{C}_n^-) \rightarrow \pi(M(L^\infty(\mathcal{X}, \mu))),$$

so that

$$\Psi \mid \pi(M(L^\infty(\mathcal{X}, \mu))) = \text{id}_{\pi(M(L^\infty(\mathcal{X}, \mu)))}$$

has immediately the following K -theory consequences.

Corollary 5.1. *Assume $n = \dim \mathcal{X} \geq 3$. Then*

$$K_0(L^\infty(\mathcal{X}; \mu)) \simeq \{f \in L^\infty(\mathcal{X}, \mu) \mid f(\mathcal{X}) \subset \mathbb{Z}\}$$

is isomorphic to a direct summand of $K_0(\mathcal{E}/\mathcal{K}(\mathcal{X}; \mathcal{J}))$ and the isomorphism is also with respect to the order structure. We also have that

$$K_1(\mathcal{E}/\mathcal{K}(\mathcal{X}; \mathcal{J})) \simeq K_1(\ker \Psi).$$

By Theorem 5.1, if $\mathcal{J} \supseteq \mathcal{C}_n^-$, we have that $\pi(M(C(\mathcal{X}))) \simeq C(\mathcal{X})$ is the center of $\mathcal{E}/\mathcal{K}(\mathcal{X}; \mathcal{J})$. This yields a homomorphism

$$Z: C(\mathcal{X}) \otimes \mathcal{E}/\mathcal{K}(\mathcal{X}; \mathcal{J}) \rightarrow \mathcal{E}/\mathcal{K}(\mathcal{X}; \mathcal{J}),$$

where the left-hand side has a norm equivalent to that of the tensor product of the C^* -algebras $C(\mathcal{X})$ and $p(\mathcal{E}(\mathcal{X}; \mathcal{J}))$ and which maps $f \otimes 1$ to $\pi(M(f))$ and $1 \otimes T$ to T . Moreover, if $\Delta: C(\mathcal{X}) \otimes C(\mathcal{X}) \rightarrow C(\mathcal{X})$ is the restriction to the diagonal then

$$Z \circ (\Delta \otimes \text{id}) = Z \circ (\text{id} \otimes Z).$$

This can then be used to get that $K_0(\mathcal{E}/\mathcal{K}(\mathcal{X}; \mathcal{J}))$ is a $K_0(C(\mathcal{X}))$ -module and a similar reasoning with $C(\mathcal{X}) \otimes C(S^1)$ and $\mathcal{E}/\mathcal{K}(\mathcal{X}; \mathcal{J}) \otimes C(S^1)$ gives that the $\mathbb{Z}/2\mathbb{Z}$ graded $K_*(\mathcal{E}/\mathcal{K}(\mathcal{X}; \mathcal{J}))$ is a $K_*(C(\mathcal{X}))$ -module, which is certainly not a new thing about the center of a C^* -algebra $p(\mathcal{E}(\mathcal{X}; \mathcal{C}_n^-))$ so that we deal with C^* -algebras.

Corollary 5.2. *$K_*(\mathcal{E}/\mathcal{K}(\mathcal{X}; \mathcal{J}))$ is a $K_*(C(\mathcal{X}))$ -module when $\mathcal{J} \supseteq \mathcal{C}_n^-$.*

Remark 5.1. The fact that Theorem 5.1 (c) identifies the center of $\mathcal{E}/\mathcal{K}(\mathcal{X}; \mathcal{J})$ with $C(\mathcal{X})$ when $\mathcal{J} \supseteq \mathcal{C}_n^-$ means that $\mathcal{E}/\mathcal{K}(\mathcal{X}; \mathcal{J})$ is a $C(\mathcal{X})$ -algebra in the sense of G. G. Kasparov [16], a notion which was generalized by E. Kirchberg [17] to so-called \mathcal{X} -algebra or algebra over a topological space. This suggests that when studying the K -theory of $\mathcal{E}/\mathcal{K}(\mathcal{X}; \mathcal{J})$ it may be natural to consider K -groups which take the \mathcal{X} -structure into account (see [1, 17, 20]). Note however that with $\mathcal{E}/\mathcal{K}(\mathcal{X}; \mathcal{J})$ one will have to deal with a rather different class of \mathcal{X} -algebras than those in [1, 7, 17, 20], algebras which are non-separable and far from bundles. However, at least when \mathcal{X} has no boundary, the action of the diffeomorphism group of \mathcal{X} on $\mathcal{E}/\mathcal{K}(\mathcal{X}; \mathcal{J})$ provides some homogeneity.

Remark 5.2. From these considerations about $\mathcal{E}/\mathcal{K}(\mathcal{X}; \mathcal{J})$ it seems that certain aspects with operations, like performing connected sums may be somewhat manageable. On the other hand in the simple cases when \mathcal{X} is a sphere or a ball or a product of spheres and balls there is a lot of mystery about the $\mathcal{E}/\mathcal{K}(\mathcal{X}; \mathcal{J})$. On the analysis side the smooth functions on \mathcal{X} which arise from the bicommutant mod \mathcal{J} construction in Section 4 is certainly an important analytic question about $\mathcal{E}/\mathcal{K}(\mathcal{X}; \mathcal{J})$.

Remark 5.3. Since the smooth algebra $\mathcal{D}(\tau, \mathcal{J})$ depends only on $\mathcal{E}(\tau; \mathcal{J})$, we see that $\mathcal{D}(\tau; \mathcal{J})$, where τ is a m -tuple of multiplication operators from an embedding of \mathcal{X} into \mathbb{R}^m , does not depend on the choice of the embedding. Thus, we get an algebra $\mathcal{D}(\mathcal{X}; \mathcal{J})$ which depends only on \mathcal{X} . If the dimension of \mathcal{X} is > 0 , we also have that

$$\mathcal{D}(\mathcal{X}; \mathcal{J}) \cap \mathcal{K} = \{0\}.$$

Thus, $\mathcal{D}(\mathcal{X}; \mathcal{J})$ and $\mathcal{D}/\mathcal{K}(\mathcal{X}; \mathcal{J})$ are isomorphic if $\dim \mathcal{X} > 0$.

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