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An inequality for non-microstates free entropy dimension for crossed products by finite abelian groups

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Abstract. For certain generating sets of the subfactor pair $M \subset M \rtimes G$ where G is a finite abelian group we prove an approximate inequality between their non-microstates free entropy dimension, resembling the Schreier formula for ranks of finite index subgroups of free groups. As an application, we give bounds on free entropy dimension of generating sets of crossed products of the form $M \rtimes (\mathbb{Z}/2\mathbb{Z})^{\oplus\infty}$ for a large class of algebras M.

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To the memory of V. F. R. Jones

1. Introduction

The famous Jones index [8] is the von Neumann algebra extension of the group-subgroup index and is defined for any inclusion $M_0 \subset M_1$ of II_1 factors. An open question in von Neumann algebra theory is to find an analog of the Schreier's formula for the number of generators of finite-index subgroups of non-abelian free groups $\mathbb{F}_n$. For example, one expects that the "number of generators" of a finite-index subfactor $M_0 \subset M_1 = L(\mathbb{F}_n)$ should be $1 + (n-1)[M : N]$. Indeed, specific subfactors of $L(\mathbb{F}_r)$ constructed via amalgamated free products [5, 6, 14–16] have sets of generators for which such a formula holds. However, there is little that is known in general, even for index 2.

Returning to group theory, let $H \subset G$ be a finite-index inclusion of groups. Denoting by $\beta_j^2(G)$ the L^2-Betti numbers of G one has the following generalization of Schreier's formula (see e.g. [12]):

$$\beta_j^{(2)}(G) = [G : H]^{-1} \beta_j^{(2)}(H).$$

(Schreier's formula corresponds to the case $j = 1$ and involves the equality $\beta_1^{(2)}(\mathbb{F}_r) = r - 1$). A similar formula is true for finite-index inclusions of tracial algebras [21].

Voiculescu's free entropy dimension δ_0 [22, 23] takes the value r on a set of generators of $L(\mathbb{F}_r)$; more generally, its value is related to L^2 Betti numbers [2, 13, 18]. Therefore, one expects a statement of the following kind: given a finite index inclusion $M_0 \subset M_1$, for any generating set S_0 for M_0 there exists a generating set for M_1 (and conversely, for any generating set S_1 for M_1 there exists a generating set S_0 for M_0) so that

$$\delta_0(S_0) - 1 = [M_1 : M_0]^{-1}(\delta_0(S_1) - 1). \tag{1.1}$$

However, at present only inequalities of the form $\delta_0(S_1) \leq \delta_0(S_0)$ are available [9].

In this paper we show that for very specific examples of subfactors, namely subfactors of the form $M_0 \subset M_1 = M_0 \rtimes G$ with G a finite abelian group, we can find generators M_0 and M_1 for which the non-microstates free entropy dimension [25] analog of (1.1) holds with an arbitrary small error. More precisely, we prove that given $\varepsilon > 0$ there exist generating sets S_0 for M_0 and S_1 for M_1 for which

$$\delta^\star(S_0) - 1 = [M_1 : M_0]^{-1}(\delta^\star(S_1) - 1) + \varepsilon.$$

Our result is interesting in connection with the following question. Let H be a finitely generated group, and let α be an action of some infinite group G on H. Then it is known [4, Theorem 6.8] that $\beta_1^{(2)}(H \rtimes_\alpha G) = 0$. In case that $G = \mathbb{Z}$ and H is finitely presented sofic, this implies that the von Neumann algebra $M_1 = L(H) \rtimes \mathbb{Z}$ is strongly one-bounded [7, 11, 18]; in particular for any generating S set of M_1, $\delta_0(S) = 1$. This leads us to the following conjecture:

Conjecture 1. *Let M_0 be a finitely-generated von Neumann algebra, and let G be an action of a infinite group on M_0. Then $M_1 = M_0 \rtimes_\alpha G$ is strongly 1-bounded.*

If true, the conjecture has a somewhat surprising consequence: it would imply non-isomorphism of free group factors. Indeed, let G be any infinite discrete group so that $L(G)$ is R^ω-embeddable (e.g. $G = \mathbb{Z}$ or G amenable), and regard $\mathbb{F}_\infty$ as the infinite free product of copies of $\mathbb{Z}$ indexed by G. Then G acts on this index set by permutations and thus on $\mathbb{F}_\infty$; call this action α. One can easily see that the resulting semi-direct product is $\mathbb{Z} * G$, corresponding to the extension

$$e \to \mathbb{F}_{|G|} \to \mathbb{Z} * G \overset{e*\mathrm{id}}{\to} G \to e.$$

Thus,

$$M_1 = M_0 \rtimes_\alpha G = L(\mathbb{F}_\infty \rtimes_\alpha G) = L(\mathbb{Z} * G) \cong L(\mathbb{Z}) * L(G).$$

The latter von Neumann algebra is known to be not strongly 1-bounded; in fact it has a generating set whose free entropy dimension is strictly above 1. On the other hand, if we were to assume that $L(\mathbb{F}_\infty) \cong L(\mathbb{F}_2)$ (or even finitely-generated), our conjecture (for the specific group G) would imply that M_1 is strongly 1-bounded and thus all of its generating sets have free entropy dimension 1, which would be a contradiction.

While we are unable to even come close to proving the conjecture, we are able to show that if M_0 is finitely-generated and in addition $M_0 \cong M_{2\times 2}(M_0)$ and $G = (\mathbb{Z}/2\mathbb{Z})^{\oplus\infty}$, then for any $\varepsilon > 0$, M_1 has generating sets with free entropy dimension bounded by $1+\varepsilon$. The proof is reminiscent of Gaboriau's proof [3] and it is this connection that inspired us to study the behavior of free entropy dimension under crossed products.

2. Estimates on non-microstates free entropy dimension

2.1. Special generators for crossed product subfactors. Let M be a II_1 factor and let α be a properly outer action of a finite abelian group G on M. Consider the inclusion of factors

$$M^G \subset M \subset M \rtimes_\alpha G,$$

where M^G is the fixed point algebra for the action α. It follows from Takai–Takesaki duality [19, 20] that $M \rtimes_\alpha G \cong M_{|G|\times|G|}(M^G)$, and moreover that the inclusion $M^G \subset M$ is isomorphic to an inclusion of the form $M^G \subset M^G \rtimes_{\alpha'} \widehat{G}$, where $\widehat{G}$ is the group dual of G and α' is a certain action related to the dual action of $\widehat{G}$ on $M \rtimes_\alpha G$.

Assume now that M is finitely generated; thus also $M \rtimes_\alpha G$ is finitely generated. Since $M_{|G|\times|G|}(M^G) \cong M \rtimes_\alpha G$ we also know that M^G is finitely generated. Let $X = (X_1, \dots, X_d)$ be a set of generators for M^G. Denote by $\hat{u}_g \in M^G \rtimes \widehat{G}$, $g \in \widehat{G}$, the unitaries implementing α'. Using the isomorphism $(M^G \subset M) \subset (M^G \subset M^G \rtimes_{\alpha'} \widehat{G})$, we may view these unitaries as elements of M. Then the set $X \cup (\hat{u}_g)_{g\in\widehat{G}}$ generates M. Furthermore, if we denote by $u_g \in M \rtimes_\alpha G$, $g \in G$, the unitaries implementing α, then $X \cup (\hat{u}_g)_{g\in\widehat{G}} \cup (u_g)_{g\in G}$ form a generating set of $M \rtimes_\alpha G$. Note that we have the following relations:

$$\begin{aligned} u_g X_j u_g^* &= X_j, & g \in G,\ j = 1,\dots,d,\\ u_g \hat{u}_h u_g^* &= \langle g, h\rangle \hat{u}_h, & g \in G,\ h \in \widehat{G}, \end{aligned}$$

where we use $\langle\cdot,\cdot\rangle$ to denote the pairing between the elements of G and its dual $\widehat{G}$.

2.2. Estimates on non-microstates free entropy dimension $\delta^\star$. In this paper it will be convenient to work with a non-selfadjoint version of free entropy dimension. Given non-commutative random variables Y in a tracial von Neumann algebra M and a sub-algebra $B \subset M$, let $A = *\text{-}\operatorname{alg}(Y, B)$ and let C be a circular element free from A; we normalize C so that $\tau(C^*C) = 2$. Consider the derivation

$$\partial_Y : A \to \operatorname{span}\{ACA + AC^*A\}$$

determined by the Leibniz rule and by $\partial_Y(Y) = C$, $\partial_Y(Y^*) = C^*$, $\partial_Y(b) = 0$ for all $b \in B$. The vector $J(Y : B) \in L^2(A, \tau)$, if it exists, is called the conjugate variable to Y and is uniquely determined by

$$\langle J(Y : B), P\rangle_{L^2(A)} = \langle C^*, P\rangle_{L^2(A * W^*(C))}, \quad \forall P \in A.$$

Given $\{Y_i : i \in I\}$, the free entropy dimension is then determined by

$$\delta^\star(Y_i : i \in I) = 2|I| - \liminf_{t\to 0} t \sum_{i \in I} \left\| J\left(Y_i^t : \left(Y_j^t : j \in I \setminus \{i\}\right)\right)\right\|_2^2,$$

where $Y_i^t = Y_i + \sqrt{t} C_i$ and $\{C_i : i \in I\}$ are circular elements $*$-free form $\{Y_i : i \in I\}$. It is not hard to see that our definition is equivalent to the usual definition of $\delta^\star$ for self-adjoint variables (implicitly introduced in [25], see also [2, Section 4]), in that

$$\delta^\star(Y_i : i \in I) = \delta^\star_{\text{s.a.}}(\operatorname{Re} Y_i, \operatorname{Im} Y_i : i \in I).$$

Lemma 2. *Suppose that α is an action of a finite abelian group G on a tracial von Neumann algebra M, and suppose that $Y_j \in M$, $j \in I$ are (not necessarily self-adjoint) generators of M. Let $C_j^{(g)}$, $g \in G$ be circular elements $*$-free from M, and extend α to*

$$W^*(Y_i : i \in I) * W^*(C_i^{(g)} : i \in I, g \in G)$$

by setting $\alpha_g(C_i^{(g')}) = C_i^{(gg')}$. Let finally $Y_j^t = Y_j + \sqrt{t} C_j^{(e)}$, where $e \in G$ is the neutral element. Let

$$\xi_j^t = J\left(Y_j^t : \left(Y_i^t : i \in I \setminus \{j\}\right)\right)$$

be the free conjugate variables.

For each $h \in \widehat{G}$, denote by $(\overline{C}_j^{(h)})^*$ the projection of $(C_j^{(e)})^*$ on to the linear subspace of

$$\operatorname{span}\left\{\alpha_g((C_j^{(e)})^*) : g \in G\right\}$$

consisting of vectors x satisfying $\alpha_g(x) = \langle g, h\rangle x$.

Then

$$\xi_j^t = |G|\, t^{-1/2} E_{W^*(Y_i^t : i \in I)}((\overline{C}_j^{(h)})^*), \quad \forall h \in \widehat{G}, \tag{2.1}$$

and also

$$\langle \xi_j^t, (\overline{C}_j^{(h)})^* \rangle = \frac{1}{|G|} \langle \xi_j^t, (C_j^{(e)})^* \rangle = \frac{t^{1/2}}{|G|} \|\xi_j^t\|_2^2, \quad \forall j \in I,\ h \in \widehat{G}.$$

In particular, if we denote by $\overline{\xi}_j^{t,(h)}$ *the projection of* ξ_j^t *onto the subspace of*

$$\operatorname{span}\{\alpha_g(\xi_j^t) : g \in G\}$$

consisting of vectors x *satisfying* $\alpha_g(x) = \langle g, h \rangle x$*, then*

$$\|\overline{\xi}_j^{t,(h)}\|_2^2 = \frac{1}{|G|} \|\xi_j^t\|^2, \quad j \in I,\ h \in \widehat{G},$$
$$\langle \overline{\xi}_j^{t,(h)}, (C_j^{(e)})^* \rangle = \frac{t^{1/2}}{|G|} \|\xi_j^t\|_2^2,$$

so that all of the orthogonal components $\overline{\xi}_j^{t,(h)}$ *in the decomposition* $\xi_j^t = \sum_{h \in \widehat{G}} \overline{\xi}_j^{t,(h)}$ *have the same length and the same inner product with* $(C_j^{(e)})^*$*. (It is worth noting that* $\overline{\xi}_j^{t,(h)} \notin W^*(Y_i^t : i \in I)$ *since that algebra is not invariant under the action* α*.)*

Proof. By [17], we may assume that there exists a family of free creation operators $\ell_j^{(g)}$, $\widehat{\ell}_j^{(g)}$ satisfying, for all $g, g' \in G$, $j, j' \in I$, $y \in M$,

$$(\ell_j^{(g)})^* y \ell_{j'}^{(g')} = (\widehat{\ell}^{(g)})^* y \widehat{\ell}_{j'}^{(g')} = \delta_{g=g'} \delta_{j=j'} \tau(y),$$
$$(\widehat{\ell}_j^{(g)})^* y \ell_{j'}^{(g')} = (\ell^{(g)})^* y \widehat{\ell}_{j'}^{(g')} = 0,$$

and so that $C_j^{(g)} = \ell_j^{(g)} + (\widehat{\ell}_j^{(g)})^*$. The action α can be extended by putting

$$\alpha_g(\ell_j^{(g')}) = \ell_j^{(gg')} \quad \text{and} \quad \alpha_g(\widehat{\ell}_j^{(g')}) = \widehat{\ell}_j^{(g^{-1}g')}.$$

Denote by

$$\overline{\ell}_j^{(h)} = \frac{1}{|G|} \sum_{g \in G} \overline{\langle g, h \rangle} \alpha_g(\ell_j^{(e)}), \qquad \overline{\widehat{\ell}}_j^{(h)} = \frac{1}{|G|} \sum_{g \in G} \overline{\langle g, h \rangle} \alpha_g(\widehat{\ell}_j^{(e)})$$

the projections of $\ell_j^{(e)}$ (respectively, $\widehat{\ell}_j^{(e)}$) onto the linear subspace of $\operatorname{span}\{\ell_j^{(g)} : g \in G\}$ (resp., $\operatorname{span}\{\widehat{\ell}_j^{(g)} : g \in G\}$) consisting of vectors x satisfying $\alpha_g(x) = \langle g, h \rangle x$; in this way we get that

$$\overline{C}_j^{(h)} = \overline{\ell}_j^{(h)} + (\overline{\widehat{\ell}}_j^{(h)})^*.$$

We can now verify the following equations:

$$(\overline{\ell}_j^{(h)})^* y \overline{\ell}_{j'}^{(h')} = (\overline{\overline{\ell}}_j^{(h)})^* y \overline{\overline{\ell}}_{j'}^{(h')} = |G|^{-1}\delta_{h=h'}\delta_{j=j'}\tau(y),$$
$$(\overline{\overline{\ell}}_j^{(h)})^* y \overline{\ell}_{j'}^{(h')} = (\overline{\ell}^{(h)})^* y \overline{\overline{\ell}}_{j'}^{(h')} = 0.$$

From this it follows that $(\overline{C}_j^{(h)} : j \in I, h \in \widehat{G})$ is a family of free circular operators (of variance $|G|^{-1}$) which are free from M. Since for each $j \in I$,

$$C_j^{(e)} = \sum_{h\in\widehat{G}} \overline{C}_j^{(h)},$$

it follows from [25] and the equalities $J(C_j) = C_j^*$, $J(\overline{C}_j^{(h)}) = |G|(\overline{C}_j^{(h)})^*$ that

$$\begin{aligned}
\xi_j^t &= J\big(Y_j + \sqrt{t}C_j^{(e)} : \big(Y_i^t : i \in I \setminus \{j\}\big)\big)\\
&= J\Bigg(\Bigg(Y_j + \sqrt{t}\sum_{h'\neq h} \overline{C}_j^{(h')}\Bigg) + \sqrt{t}\overline{C}_j^{(h)} : \big(Y_i^t : i \in I \setminus \{j\}\big)\Bigg)\\
&= E_{W^*(Y_j^t : j\in I)} J\Bigg(\sqrt{t}\overline{C}_j^{(h)} : \Bigg(Y_j + \sqrt{t}\sum_{h'\neq h}\overline{C}_j^{(h')}\Bigg) \cup \big(Y_i^t : i \in I \setminus \{j\}\big)\Bigg)\\
&= |G|\, t^{-1/2} E_{W^*(Y_j^t : j\in I)}((\overline{C}_j^{(h)})^*),
\end{aligned}$$

the last equality by freeness. This gives (2.1). On the other hand,

$$\xi_j^t = t^{-1/2} E_{W^*(Y_j^t : j\in I)}(C_j^{(e)}).$$

It follows that

$$\begin{aligned}
\langle \xi_j^t, (C_j^{(e)})^*\rangle &= \langle \xi_j^t, E_{W^*(Y_j^t : j\in I)}((C_j^{(e)})^*)\rangle\\
&= t^{1/2}\langle \xi_j^t, \xi_j^t\rangle\\
&= |G|^{-1}\langle \xi_j^t, E_{W^*(Y_j^t : j\in I)}((\overline{C}_j^{(h)})^*)\rangle\\
&= |G|^{-1}\langle \xi_j^t, (\overline{C}_j^{(h)})^*\rangle,
\end{aligned}$$

which readily implies the remaining statements of the lemma. ■

Theorem 3. *Let X be an arbitrary generating set for M^G, and let $X \cup (\hat{u}_g)_{g\in\widehat{G}}$ be the generating set for M and $X \cup (\hat{u}_g)_{g\in\widehat{G}} \cup (u_g)_{g\in G}$ be the generating set for $M \rtimes_\alpha G$ as constructed in Section 2.1. Then for any $\varepsilon > 0$, there exists a $\lambda > 0$, so that*

$$\begin{aligned}
\delta^\star\big((\lambda X_i : i \in I) \cup (\lambda\hat{u}_h : h \in \widehat{G}) \cup (u_g : g \in G)\big) - 1&\\
\leq |G|^{-1}\big(\delta^\star(Y_i : i \in I) - 1\big) + \varepsilon.&
\end{aligned} \tag{2.2}$$

Proof. Let $I = \{1,\dots,d\} \sqcup \widehat{G}$; for $i \in I$, set $Y_i = X_i$ if $i \in \{1,\dots,d\}$ and $Y_i = \widehat{u}_h$ if $i = h \in \widehat{G}$. Let also $\omega_i\colon G \to \mathbb{C}$ be given by $\omega_i(g) = 1$ if $i \in \{1,\dots,d\}$ and $\omega_i(g) = \langle g, h\rangle$ if $i = h \in \widehat{G}$. We then have the following relations:

$$u_g Y_i u_g^* = \omega_i(g) Y_i, \quad i \in I,\ g \in G.$$

For $i \in I$, let C_i be a circular system, free from $W^*(Y_i : i \in I, u_g : g \in G)$, and for $g \in G$, let C'_g be another circular system free from $W^*(Y_i, C_i : i \in I, u_g : g \in G)$. We then have

$$\begin{aligned}\delta^\star\big(X \cup (\widehat{u}_h)_{h\in\widehat{G}} \cup (u_g)_{g\in G}\big) &= 2\big(d + |\widehat{G}| + |G|\big)\\ &- \liminf_{t\to 0} t\Big[\sum_{j\in I}\big\| J\big(Y_j + \sqrt{t}C_j : \big(Y_i : i \in I\setminus\{j\}\big) \cup \big(u_g + \sqrt{t}C'_g : g \in G\}\big)\big)\big\|_2^2\\ &+ \sum_{g\in G}\big\| J\big(u_g + \sqrt{t}C'_g : (Y_i : i \in I) \cup \big(u_{g'} + \sqrt{t}C'_{g'} : g' \in G\setminus\{g\}\big)\big)\big\|_2^2\Big].\end{aligned}$$

Denote by M_t the von Neumann algebra $W^*(Y_j + \sqrt{t}C_j : j \in I)$ and by $\widehat{M}_t$ the von Neumann algebra $W^*(M_t, u_g + \sqrt{t}C'_{g'} : g \in G)$.

Using [25], we note that

$$\begin{aligned}&\big\| J\big(u_g + \sqrt{t}C'_{g'} : \big(Y_i : i \in I\setminus\{j\}\big) \cup \big(u_{g'} + \sqrt{t}C'_{g'} : g' \in G\setminus\{g\}\big)\big)\big\|_2^2\\ &\quad\ge \big\| J\big(u_g + \sqrt{t}C'_g : \big(u'_g + \sqrt{t}C'_{g'} : g' \in G\setminus\{g\}\big)\big)\big\|_2^2\end{aligned}$$

and by [13], we get that

$$\begin{aligned}&2|G| - \liminf_{t\to 0} t \sum_{g\in G}\big\| J\big(u_g + \sqrt{t}C'_g : \big(u'_g + \sqrt{t}C'_{g'} : g' \in G\setminus\{g\}\big)\big)\big\|_2^2\\ &\quad = \delta^*(u_g : g \in G) = \beta_1^{(2)}(G) - \beta_0^{(2)}(G) + 1 = 1 - |G|^{-1}.\end{aligned}$$

Let $\zeta_j^t = J(Y_j + \sqrt{t}C_j : (Y_i : i \in I\setminus\{j\}) \cup (u_g + \sqrt{t}C'_g : g \in G))$ and set $\xi_j^t = J(Y_j + \sqrt{t}C_j : (Y_i : i \in I\setminus\{j\}))$. For $h \in \widehat{G}$, denote by $\overline{C}_j^{(h)}$ the projection of C_j onto the subspace of $\mathrm{span}\{u_g C_j u_g^* : g \in G\}$ consisting of vectors x, so that $\{u_g(x)u_g^* = \langle g, h\rangle x\}$. Let also

$$\overline{\zeta}_j^{(h)} = t^{-1/2} E_{\widehat{M}_t}((\overline{C}_j^{(h)})^*).$$

Then by applying Lemma 2 with $C_j^{(e)} = C_j$, $C_j^{(g)} = u_g C_j u_g^*$, we have

$$\xi_j^t = |G|\, t^{-1/2} E_{M_t}((\overline{C}_j^{(h)})^*).$$

It follows that

$$\begin{aligned}
\big\langle t^{1/2}\zeta_j^t, (\bar{C}_j^{(e)})^*\big\rangle &= \big\langle E_{\widehat{M}_t}(C_j^*), (\bar{C}_j^{(e)})^*\big\rangle = \big\langle C_j^*, E_{\widehat{M}_t}(\bar{C}_j^{(e)})^*\big\rangle \\
&= \Big\langle C_j^*, |G|^{-1}\sum_{g\in G} E_{\widehat{M}_t}\big(u_g(\bar{C}_j^{(e)})^* u_g^*\big)\Big\rangle \\
&= \Big\langle C_j^*, |G|^{-1}\sum_{g\in G} E_{\widehat{M}_t}\big((u_g + t^{1/2}C_g')(\bar{C}_j^{(e)})^*(u_g + t^{1/2}C_g')^*\big)\Big\rangle + O(t^{1/2}) \\
&= \Big\langle C_j^*, |G|^{-1}\sum_{g} u_g E_{\widehat{M}_t}(\bar{C}_j^{(e)})^* u_g^*\Big\rangle + O(t^{1/2}) \\
&= \Big\langle |G|^{-1}\sum_{g} u_g^* C_j^* u_g, E_{\widehat{M}_t}(\bar{C}_j^{(e)})^*\Big\rangle + O(t^{1/2}) \\
&= \big\langle (\bar{C}_j^{(e)})^*, E_{\widehat{M}_t}(\bar{C}_j^{(e)})^*\big\rangle = \big\langle E_{\widehat{M}_t}(\bar{C}_j^{(e)})^*, E_{\widehat{M}_t}(\bar{C}_j^{(e)})^*\big\rangle + O(t^{1/2}) \\
&= \|E_{\widehat{M}_t}(\bar{C}_j^{(e)})\|_2^2 + O(t^{1/2}).
\end{aligned}$$

Similarly, letting $\bar{\xi}_j^{t,(e)}$ be as in Lemma 2, we have that

$$\begin{aligned}
\big\langle E_{\widehat{M}_t}(\bar{C}_j^{(e)}), t^{1/2}\bar{\xi}_j^{t,(e)}\big\rangle &= \Big\langle E_{\widehat{M}_t}(\bar{C}_j^{(e)}), t^{1/2}|G|^{-1}\sum_{g\in G} u_g \xi_j^t u_g^*\Big\rangle \\
&= \Big\langle |G|^{-1}\sum_{g\in G} u_g^* E_{\widehat{M}_t}(\bar{C}_j^{(e)}) u_g, t^{1/2}\xi_j^t\Big\rangle \\
&= \big\langle E_{\widehat{M}_t}(\bar{C}_j^{(e)}), t^{1/2}\xi_j^t\big\rangle + O(t^{1/2}).
\end{aligned}$$

Using this and $E_{\widehat{M}_t}(\xi_j^t) = \xi_j^t$, and Lemma 2, we obtain the inequality

$$\begin{aligned}
\|E_{\widehat{M}_t}(\bar{C}_j^{(e)})\|_2 &\geq \frac{\langle E_{\widehat{M}_t}(\bar{C}_j^{(e)}), t^{1/2}\bar{\xi}_j^{t,(e)}\rangle}{\|t^{1/2}\bar{\xi}_j^{t,(e)}\|_2} = \frac{\langle E_{\widehat{M}_t}(\bar{C}_j^{(e)}), t^{1/2}\xi_j^t\rangle}{\|t^{1/2}\bar{\xi}_j^{t,(e)}\|_2} + O(t^{1/2}) \\
&= \frac{\langle C_j^{(e)}, \xi_j^t\rangle}{|G|^{-1/2}\|\xi_j^t\|_2} + O(t^{1/2}) = \frac{t^{1/2}|G|^{-1}\|\xi_j^t\|_2^2}{|G|^{-1/2}\|\xi_j^t\|_2} + O(t^{1/2}) \\
&= \frac{t^{1/2}}{|G|^{1/2}}\|\xi_j^t\|_2 + O(t^{1/2}),
\end{aligned}$$

so that

$$\big\langle t^{1/2}\zeta_j^t, (\bar{C}_j^{(e)})^*\big\rangle = \|E_{\widehat{M}_t}(\bar{C}_j^{(e)})\|_2^2 + O(t^{1/2}) \geq t|G|^{-1}\|\xi_j^t\|_2^2 + O(t^{1/2}).$$

We now claim that for all but one $h \in \widehat{G}$, $\langle t^{1/2}\zeta_j^t, (\bar{C}_j^{(h)})^*\rangle$ is almost $2|G|^{-1}$.

Let us use the notation $x(t) \in \widehat{M}_t + O(t^\gamma)$ to signify that there exists an element $y(t) \in \widehat{M}_t$, so that

$$\|x(t) - y(t)\|_2 = O(t^\gamma).$$

Since $(u_g + \sqrt{t}C'_g)(Y_j + \sqrt{t}C_j)(u_g + \sqrt{t}C'_g)^* \in \widehat{M}_t$ and $Y_j + \sqrt{t}C_j \in \widehat{M}_t$, we have that

$$(u_g + \sqrt{t}C'_g)(Y_j + \sqrt{t}C_j)(u_g + \sqrt{t}C'_g)^* - \omega_j(g)(Y_j + \sqrt{t}C_j) \in \widehat{M}_t.$$

Thus, also

$$\sqrt{t}\big(C'_g Y_j u_g^* + u_g Y_j (C'_g)^* - \omega_j(g)C_j + u_g C_j u_g^*\big) \in \widehat{M}_t + O(t).$$

Hence,

$$C'_g Y_j u_g^* + u_g Y_j (C'_g)^* - \omega_j(g)C_j + u_g C_j u_g^* \in \widehat{M}_t + O(t^{1/2}).$$

Since $u_g^* u_g = 1$, we similarly deduce

$$C'_g u_g^* + u_g (C'_g)^* \in \widehat{M}_t + O(t^{1/2}),$$

so that, noting that $u_g \in \widehat{M}_t + O(t^{1/2})$

$$(C'_g)^* + u_g^* C'_g u_g \in \widehat{M}_t + O(t^{1/2}).$$

This gives

$$C'_g Y_j u_g^* - u_g Y_j u_g^* C'_g u_g - \omega_j(g)C_j + u_g C_j u_g^* \in \widehat{M}_t + O(t^{1/2}).$$

Projecting onto eigenspaces for the G action given by conjugation by u_g (noting again that $u_g \in \widehat{M}_t + O(t^{1/2})$) gives us for all $h \in \widehat{G}$,

$$(\overline{C}')_g^{(h\cdot\omega_j^{-1})} Y_j u_g^* - u_g Y_j u_g^* \overline{C}_g'^{(h\cdot\omega_j^{-1})} u_g - \omega_j(g)\overline{C}_j^{(h)} + u_g \overline{C}_j^{(h)} u_g^* \in M_t + O(t^{1/2}).$$

We note also that

$$(\overline{C}'_g)^{(h\cdot\omega_j^{-1})} u_g = \langle h, g^{-1}\rangle \omega_j(g) \cdot u_g (\overline{C}')_g^{(h)}$$

and

$$u_g \overline{C}_j^{(h)} u_g^* = \langle h, g\rangle \overline{C}_j^{(h)},$$

whence

$$\begin{aligned}(\overline{C}')_g^{(h\cdot\omega_j)} Y_j u_g^* - \langle h, g^{-1}\rangle \omega_j(g) u_g Y_j u_g^* u_g \overline{C}_g'^{(h\cdot\omega_j)} \\ + \big(\langle h, g\rangle - \omega_j(g)\big)\overline{C}_j^{(h)} \in M_t + O(t^{1/2}).\end{aligned}$$

It follows that

$$\begin{aligned}
&\langle t^{1/2}\zeta_j^t, (\overline{C}_j^{(h)})^*\rangle = \langle E_{\widehat{M}_t}(C_j), \overline{C}_j^{(h)}\rangle \\
&\quad = \langle C_j, E_{\widehat{M}_t}(\overline{C}_j^{(h)})\rangle \\
&\quad = \sum_{h'\in\widehat{G}} \langle \overline{C}_j^{(h')}, E_{\widehat{M}_t}(\overline{C}_j^{(h)})\rangle \\
&\quad = \langle \overline{C}_j^{(h)}, E_{\widehat{M}_t}(\overline{C}_j^{(h)})\rangle + O(t^{1/2}) \\
&\quad = \|E_{\widehat{M}_t}(\overline{C}_j^{(h)})\|_2^2 + O(t^{1/2}) \\
&\quad \geq \max_g \Bigg|\Bigg\langle \overline{C}_j^{(h)}, \\
&\qquad \frac{(\overline{C}')_g^{(h\cdot\omega_j)} Y_j u_g^* - \langle h, g^{-1}\rangle \omega_j(g) u_g Y_j \overline{C}_g'^{(h\cdot\omega_j)} + (\langle h, g\rangle - \omega_j(g))\overline{C}_j^{(h)}}{\|(\overline{C}')_g^{(h\cdot\omega_j)} Y_j u_g^* - \langle h, g^{-1}\rangle \omega_j(g) u_g Y_j \overline{C}_g'^{(h\cdot\omega_j)} + (\langle h, g\rangle - \omega_j(g))\overline{C}_j^{(h)}\|}\Bigg\rangle\Bigg|^2 \\
&\qquad + O(t^{1/2}) \\
&\quad = 2|G|^{-1} \max_g \frac{|\omega_j(g) - \langle h, g\rangle|^2}{|\omega_j(g) - \langle h, g\rangle|^2 + 4\|Y_j\|^2} + O(t^{1/2}).
\end{aligned}$$

Thus, if $h \neq \omega_j$, we have

$$\langle t^{1/2}\zeta_j^t, (\overline{C}_j^{(h)})^*\rangle \geq \frac{2|G|^{-1}}{1+\kappa} + O(t^2),$$

where

$$\kappa = \sup_{j,h} \min_g \frac{4\|Y_j\|_2^2}{|\omega_j(g) - \langle h, g\rangle|^2} = C \sup_j \|Y_j\|_2^2,$$

where C is some constant that only depends on G.

We can now compute, using that $\zeta_j^t = t^{-1/2} E_{\widehat{M}_t}(C_j^*)$,

$$\begin{aligned}
&\liminf_{t\to 0} t \sum_{j\in I} \big\| J\big(Y_j + \sqrt{t} C_j : \big(Y_i : i \in I \setminus \{j\}\big) \cup \big(u_g + \sqrt{t} C_g' : g \in G\}\big)\big) \big\|_2^2 \\
&\quad = \liminf_{t\to\infty} \sum_{j\in I} \langle t^{1/2}\zeta_j^t, t^{1/2}\zeta_j^t\rangle = \liminf_{t\to\infty} \sum_{j\in I} \langle t^{1/2}\zeta_j^t, C_j^*\rangle \\
&\quad = \liminf_{t\to 0} \sum_{h\in\widehat{G}} \sum_{j\in I} \langle t^{1/2}\zeta_j^t, (\overline{C}_j^{(h)})^*\rangle + O(t^{1/2}) \\
&\quad \geq 2|I|\big(|G|-1\big)|G|^{-1}(1+\kappa)^{-1} + \liminf_{t\to 0} \sum_{j\in I} |G|^{-1} t \|\xi_j^t\|_2^2 + O(t^{1/2}) \\
&\quad \geq 2|I|\big(|G|-1\big)|G|^{-1}(1+\kappa)^{-1} + 2|I||G|^{-1} - |G|^{-1}\delta^\star(Y_i : i \in I) \\
&\quad = 2|I|(1+\kappa)^{-1} - 2|I||G|^{-1}(1+\kappa)^{-1} + 2|I||G|^{-1} - |G|^{-1}\delta^\star(Y_i : i \in I) \\
&\quad = 2|I|(1+\kappa)^{-1} - |G|^{-1}\delta^\star(Y_i : i \in I) + 2I|G|^{-1}\big(1 - (1+\kappa)^{-1}\big).
\end{aligned}$$

Putting all this together gives us

$$\begin{aligned}
&\delta^\star\big((Y_i : i \in I) \cup (u_g : g \in G)\big) = 2|I| + 2|G| \\
&\quad - \liminf_{t\to 0}\Bigg[\sum_{j\in I}\big\| J\big(Y_j + \sqrt{t}C_j\big) : \big(Y_i : i \in I\setminus\{j\}\big) \cup \big(u_g + \sqrt{t}C'_g : g \in G\big)\big\|_2^2 \\
&\quad + \sum_{g\in G}\big\| J\big(u_g + \sqrt{t}C'_g : \big(Y_i : i \in I\setminus\{j\}\big) \cup \big(u_{g'} + \sqrt{t}C_{g'} : g' \in G\setminus\{g\}\big)\big)\big\|_2^2\Bigg] \\
&\le 2|I| + 2|G| - 2|I|(1+\kappa)^{-1} + |G|^{-1}\delta^\star(Y_i : i \in I) - 2I|G|^{-1}\big(1-(1+\kappa)^{-1}\big) \\
&\quad - 2|G| + 1 - |G|^{-1} \\
&= |G|^{-1}\delta^\star(Y_i : i \in I) + 1 - |G|^{-1} + 2|I|\big(1-|G|^{-1}\big)\big(1-(1+\kappa)^{-1}\big).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\delta^\star\big((Y_i : i \in I) \cup (u_g : g \in G)\big) - 1 \\
&\qquad \le |G|^{-1}\delta^\star(Y_i : i \in I) - |G|^{-1} + 2|I|\big(1-|G|^{-1}\big)\big(1-(1+\kappa)^{-1}\big) \\
&\qquad = |G|^{-1}(\delta^\star(Y_i : i \in I) - 1) + 2|I|\big(1-|G|^{-1}\big)\big(1-(1+\kappa)^{-1}\big).
\end{aligned}$$

Suppose now we rescale Y_i by replacing Y_i with λY_i. Then

$$\delta^\star(Y_i : i \in I) = \delta^\star(\lambda Y_i : i \in I),$$

so that we get

$$\begin{aligned}
&\delta^\star\big((\lambda Y_i : i \in I) \cup (u_g : g \in G)\big) - 1 \\
&\qquad \le |G|^{-1}\big(\delta^\star(Y_i : i \in I) - 1\big) + 2|I|\big(1-|G|^{-1}\big)\big(1-(1+\lambda^2\kappa)^{-1}\big).
\end{aligned}$$

Thus, choosing λ small enough, we can ensure that

$$\delta^\star\big((\lambda Y_i : i \in I) \cup (u_g : g \in G)\big) - 1 \le |G|^{-1}\big(\delta^\star(Y_i : i \in I) - 1\big) + \varepsilon,$$

as claimed. ∎

Let $Z_1, \dots, Z_n$ be generators of a tracial von Neumann algebra, it would be natural to expect that $\delta^\star(Z_1, \dots, Z_n)$ is an algebraic invariant: if $Z'_1, \dots, Z'_{n'}$ is another set of generators for the (non-closed) algebra $*$-alg$(Z_1, \dots, Z_n)$, then $\delta^\star(Z_1, \dots, Z_n) = \delta^\star(Z'_1, \dots, Z'_{n'})$. In particular, one expects that for any non-zero numbers $\lambda_1, \dots, \lambda_n$,

$$\delta^\star(Z_1, \dots, Z_n) = \delta^\star(\lambda_1 Z_1, \dots, \lambda_n Z_n). \tag{2.3}$$

If this were true, then we could combine the inequality in Theorem 3 with the equality

$$\delta^\star\big((\lambda Y_i : i \in I) \cup (u_g : g \in G)\big) = \delta^\star\big((Y_i : i \in I) \cup (u_g : g \in G)\big)$$

to deduce that

$$\delta^\star\big((Y_i : i \in I) \cup (u_g : g \in G)\big) - 1 \leq |G|^{-1}\big(\delta^\star(Y_i : i \in I) - 1\big) + \varepsilon$$

for all $\varepsilon > 0$, and conclude that

$$\delta^\star\big((Y_i : i \in I) \cup (u_g : g \in G)\big) - 1 \leq |G|^{-1}\big(\delta^\star(Y_i : i \in I) - 1\big).$$

However, to our embarrassment, we could not find a proof of (2.3). Note, however, that when $*$-alg$(Z_1, \dots, Z_n)$ is isomorphic to a group algebra, then algebraic invariance holds [13].

The difficulty in the proof of Theorem 3 arises from the complicated form that the relation $u_g^* Y_i u_g = \omega_i(g) Y_i$ takes when we substitute $u_g + \sqrt{t} C_g'$ for u_g and $Y_i + \sqrt{t} C_i$ for Y_i. If we instead redefine $\widehat{M}_t$ as $W^*(M_t, u_g : g \in G)$ and set $\zeta_j^t = t^{-1/2} E_{\widehat{M}_t}(C_j)$, then it is easy to show that

$$\|\overline{\zeta}_j^{t,(h)}\|_2^2 = 2t^{-1}|G|^{-1}$$

if $h \neq \omega_j$. Indeed, since now $u_g \in \widehat{M}_t$ we see that

$$\widehat{M}_t \ni u_g(Y_i + \sqrt{t} C_g) u_g^* - \omega_i(g)(Y_i + \sqrt{t} C_g) = \sqrt{t}(u_g C_g u_g^* - \omega_i(g) C_g),$$

so that $u_g C_g u_g^* - \omega_i(g) C_g \in \widehat{M}_t$. Decomposing into orthogonal components according to $h \in \widehat{G}$ then gives that $(\langle h, g\rangle - \omega_j(g))\overline{C}_j^{(h)} \in \widehat{M}_t$, so that $\overline{C}_j^{(h)} \in \widehat{M}_t$ whenever $h \neq \omega_j$; thus

$$\overline{\zeta}_j^{t,(h)} = t^{-1/2} E_{\widehat{M}_t}((\overline{C}_j^{(h)})^*) = t^{-1/2}(\overline{C}_j^{(h)})^*,$$

and the claimed equality on the norm follows from $\|\overline{C}_j^{(h)}\|_2^2 = 2|G|^{-1}$. Note also that, using Lemma 2,

$$\begin{aligned} J\big(Y_j + \sqrt{t} C_j &: \{Y_i + \sqrt{t} C_i : i \in I \setminus \{j\}\}\big) \\ &= |G|\, t^{-1/2} E_{W^*(Y_i^t : i \in I)}((\overline{C}_j^{(h)})^*) \\ &= |G|\, t^{-1/2} E_{W^*(Y_i + \sqrt{t} C_i : i \in I)}(\overline{\zeta}_j^{t,(\omega_j)}) \end{aligned}$$

so that as in the proof of the theorem

$$\|\overline{\zeta}_j^{t,(\omega_j)}\|_2^2 \geq |G|^{-1} t \,\big\| J\big(Y_j + \sqrt{t} C_j : \{Y_i + \sqrt{t} C_i : i \in I \setminus \{j\}\}\big)\big\|_2^2.$$

Using this, we get the following inequality:

$$
\begin{aligned}
2|I| - t\sum_{i\in I}\big\|J\big(Y_i+\sqrt{t}C_i : \big(Y_j : j\in I\setminus\{I\}\big)\cup(u_g : g\in G)\big)\big\|_2^2 \\
&= 2|I| - t\sum_{j\in I}\sum_{h\in\widehat{G}}\|\overline{\zeta}_j^{t,(h)}\|_2^2 \\
&= 2|I| - t\sum_{j\in I}\sum_{h\in\widehat{G}\setminus\{\omega_j\}} 2t^{-1}|G|^{-1} - t\sum_{j\in I}\|\overline{\zeta}_j^{t,(\omega_j)}\|_2^2 \\
&\le 2|I| - 2|I|\big(|G|-1\big)|G|^{-1} \\
&\qquad - t\sum_{j\in I}|G|^{-1}t\big\|J\big(Y_j+\sqrt{t}C_j : \{Y_i+\sqrt{t}C_i : i\in I\setminus\{j\}\}\big)\big\|_2^2 \\
&= |G|^{-1}\Big(2|I| - t\sum_{j\in I}t\big\|J\big(Y_j+\sqrt{t}C_j : \{Y_i+\sqrt{t}C_i : i\in I\setminus\{j\}\}\big)\big\|_2^2\Big).
\end{aligned}
$$

Taking lim inf of both sides gives us

$$
\delta^\star\big((Y_i : i\in I)\mid(u_g : g\in G)\big)\le|G|^{-1}\delta^\star(Y_i : i\in I), \tag{2.4}
$$

where we define a kind of "relative non-microstates free entropy dimension"

$$
\begin{aligned}
\delta^\star&\big((Y_i : i\in I)\mid(u_g : g\in G)\big) \\
&= 2|I| - \liminf_t t\sum_i\big\|J\big(Y_i+\sqrt{t}C_i : \big(Y_j : j\in I\setminus\{i\}\big)\cup(u_g : g\in G)\big)\big\|_2^2.
\end{aligned}
$$

Based on the analogy with the behavior of the microstates free entropy dimension [10], and since $\delta^*(u_g : g\in G) = 1-|G|^{-1}$ (see [13]), one would expect that

$$
\delta^\star\big((Y_i : i\in I)\mid(u_g : g\in G)\big)+\big(1-|G|^{-1}\big)=\delta^\star\big((Y_i : i\in I)\cup(u_g : g\in G)\big),
$$

but this equality is not known at present.

Subtracting $|G|^{-1}$ from both sides of (2.4) we arrive at the following:

Remark 4. Let X be an arbitrary generating set for M^G, and let $X\cup(\widehat{u}_g)_{g\in\widehat{G}}$ be the generating set for M and $X\cup(\widehat{u}_g)_{g\in\widehat{G}}\cup(u_g)_{g\in G}$ be the generating set for $M\rtimes_\alpha G$ as constructed in Section 2.1. Then, with Y_i as above, we have

$$
\Big(\delta^\star\big((Y_i : i\in I)\mid(u_g : g\in G)\big)+\big(1-|G|^{-1}\big)\Big)-1\le|G|^{-1}\big(\delta^\star(Y_i : i\in I)-1\big).
$$

3. Some applications

Corollary 5. *Suppose that G is a finite abelian group acting properly outer on a factor M. Suppose that M^G is generated by d elements. Then $M\rtimes_\alpha G$ has a generating set S satisfying $\delta_0(S)\le(2d+2)|G|^{-1}+1$.*

Proof. If $|G| = 1$, there is nothing to prove, so let us assume that $|G| \geq 2$. By Theorem 3 we have, for every $\varepsilon > 0$, the existence of $\lambda > 0$ so that

$$\delta^\star\big(\lambda X \cup (\lambda \hat{u}_g)_{g \in \hat{G}} \cup (u_g)_{g \in G}\big) - 1 \leq |G|^{-1} \cdot \big[\delta^\star\big(\lambda X \cup (\hat{u}_h)_{h \in \hat{G}}\big) - 1\big] + \varepsilon,$$

where X is any generating set for M^G. We can thus assume that $|X| \leq d$. Moreover,

$$\begin{aligned}\delta^\star\big(\lambda X \cup (\hat{u}_h)_{h \in \hat{G}}\big) &\leq \delta^\star(\lambda X) + \delta^\star\big((\hat{u}_h)_{h \in \hat{G}}\big)\\ &\leq 2d + 1 - |\hat{G}|^{-1},\end{aligned}$$

since $\hat{G}$ is abelian [13].

Let $\varepsilon = |\hat{G}|^{-1}$. Then by Theorem 3, there exists some $\lambda > 0$ so that, if we set $S = X \cup (\hat{u}_g)_{g \in \hat{G}} \cup (u_g)_{g \in G}$, then (2.2) holds. Combining this with the remarkable inequality between microstates and non-microstates free entropy [1] and invariance of δ_0 under algebraic changes of variables [24], we obtain

$$\begin{aligned}\delta_0(S) &= \delta_0\big(\lambda X \cup (\lambda \hat{u}_g)_{g \in \hat{G}} \cup (u_g)_{g \in G}\big)\\ &\leq \delta^\star\big(\lambda X \cup (\lambda \hat{u}_g)_{g \in \hat{G}} \cup (u_g)_{g \in G}\big)\\ &\leq |G|^{-1}\big(2d + 1 - |\hat{G}|^{-1}\big) + 1 + \varepsilon\\ &= 2d|G|^{-1} + |G|^{-1} - |G|^{-2} + 1 + |G|^{-1}\\ &\leq (2d + 2)|G|^{-1} + 1 = (2d + 2)|G|^{-1} + 1.\end{aligned}$$

as claimed. ∎

Theorem 6. *Let M be a finitely generated factor, and assume that $M \cong M_{2\times 2}(M)$ and that α is a properly outer action of $G = (\mathbb{Z}/2\mathbb{Z})^{\oplus\infty}$ on M. Then for every $\varepsilon > 0$ there exists a finite generating set S for $M \rtimes_\alpha G$ so that $\delta_0(S) \leq 1 + \varepsilon$.*

Proof. Suppose that M is generated by d elements. Choose m so that

$$2^{-m}(2d + 2m + 2) < \varepsilon.$$

Denote by G_m the subgroup of G generated by first m copies of $\mathbb{Z}/2\mathbb{Z}$. Then $M^{G_m} \cong N \rtimes G_m$, where N is a II_1 factor, so that $M_{2^m \times 2^m}(N) \cong M$. Thus, by assumption, $N \cong M$, so N can be generated by d elements. Thus, M^{G_m} is generated a set S' of at most $d + m$ elements. Applying now Corollary 5, we deduce that there exists a generating set S_1 for $M \rtimes G_m$, so that

$$\delta_0(S_1) \leq 2^{-m}(2d + 2m + 2) + 1;$$

moreover, S_1 includes the set S_0 consisting of generators of G_m.

The group G is an infinite abelian group whose topological dual is isomorphic to the Cantor set. Thus the von Neumann algebra of G is generated by a single unitary w. Let S_2 be that unitary. Then $S = S_0 \cup S_1 \cup S_2$ is a generating set for $M \rtimes_\alpha G$.

By the hyperfinite inequality for δ_0 [10], we have

$$\begin{aligned}\delta_0(S) = \delta_0(S_1 \cup S_2 \cup S_0) &\leq \delta_0(S_1 \cup S_0) + \delta_0(S_2 \cup S_0) - \delta_0(S_0) \\ &\leq 2^{-m}(2d + 2m + 2) + 1 + 1 - \delta(S_0).\end{aligned}$$

Since S_0 generates G_m, which is a finite abelian group, $\delta_0(S_0) = 1 - |G_m|^{-1} = 1 - 2^{-m}$ (see [10, 23]). Substituting this into the inequality above gives

$$\begin{aligned}\delta_0(S) &\leq 2^{-m}(2d + 2m + 2) + 1 - (1 - 2^{-m}) \\ &= 2^{-m}(2d + 2m + 2) + 1 < 1 + \varepsilon,\end{aligned}$$

as claimed. ■

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References

[1] P. Biane, M. Capitaine and A. Guionnet, Large deviation bounds for matrix Brownian motion. *Invent. Math.* **152** (2003), no. 2, 433–459. Zbl 1017.60026 MR 1975007

[2] A. Connes and D. Shlyakhtenko, L^2-homology for von Neumann algebras. *J. Reine Angew. Math.* **586** (2005), 125–168. Zbl 1083.46034 MR 2180603

[3] D. Gaboriau, Coût des relations d'équivalence et des groupes. *Invent. Math.* **139** (2000), no. 1, 41–98. Zbl 0939.28012 MR 1728876

[4] — Invariants l^2 de relations d'équivalence et de groupes. *Publ. Math. Inst. Hautes Études Sci.* (2002), no. 95, 93–150. Zbl 1022.37002 MR 1953191

[5] A. Guionnet, V. F. R. Jones and D. Shlyakhtenko, Random matrices, free probability, planar algebras and subfactors. In *Quanta of maths*, pp. 201–239, Clay Math. Proc. 11, Amer. Math. Soc., Providence, RI, 2010. Zbl 1219.46057 MR 2732052

[6] — A semi-finite algebra associated to a subfactor planar algebra. *J. Funct. Anal.* **261** (2011), no. 5, 1345–1360. Zbl 1230.46054 MR 2807103

[7] B. Hayes, 1-bounded entropy and regularity problems in von Neumann algebras. *Int. Math. Res. Not. IMRN* (2018), no. 1, 57–137. Zbl 1415.46039 MR 3801429

[8] V. F. R. Jones, Index for subfactors. *Invent. Math.* **72** (1983), no. 1, 1–25. Zbl 0508.46040 MR 696688

[9] K. Jung, Some free entropy dimension inequalities for subfactors. 2004, arXiv:math/0410594

[10] — A hyperfinite inequality for free entropy dimension. *Proc. Amer. Math. Soc.* **134** (2006), no. 7, 2099–2108. Zbl 1100.46039 MR 2215780

[11] — Strongly 1-bounded von Neumann algebras. *Geom. Funct. Anal.* **17** (2007), no. 4, 1180–1200. Zbl 1146.46034 MR 2373014

[12] W. Lück, *L^2-invariants: theory and applications to geometry and K-theory*. Ergeb. Math. Grenzgeb. (3) 44, Springer, Berlin, 2002. Zbl 1009.55001 MR 1926649

[13] I. Mineyev and D. Shlyakhtenko, Non-microstates free entropy dimension for groups. *Geom. Funct. Anal.* **15** (2005), no. 2, 476–490. Zbl 1094.46039 MR 2153907

[14] S. Popa, Markov traces on universal Jones algebras and subfactors of finite index. *Invent. Math.* **111** (1993), no. 2, 375–405. Zbl 0787.46047 MR 1198815

[15] S. Popa and D. Shlyakhtenko, Universal properties of $L(\mathbf{F}_\infty)$ in subfactor theory. *Acta Math.* **191** (2003), no. 2, 225–257. Zbl 1079.46043 MR 2051399

[16] F. Rădulescu, Random matrices, amalgamated free products and subfactors of the von Neumann algebra of a free group, of noninteger index. *Invent. Math.* **115** (1994), no. 2, 347–389. Zbl 0861.46038 MR 1258909

[17] D. Shlyakhtenko, Some applications of freeness with amalgamation. *J. Reine Angew. Math.* **500** (1998), 191–212. Zbl 0926.46046 MR 1637501

[18] — Von Neumann algebras of sofic groups with $\beta_1^{(2)} = 0$ are strongly 1-bounded. *J. Operator Theory* **85** (2021), no. 1, 217–228. Zbl 07606458 MR 4198970

[19] H. Takai, On a duality for crossed products of C^*-algebras. *J. Functional Analysis* **19** (1975), 25–39. Zbl 0295.46088 MR 0365160

[20] M. Takesaki, Duality for crossed products and the structure of von Neumann algebras of type III. *Acta Math.* **131** (1973), 249–310. Zbl 0268.46058 MR 438149

[21] A. Thom, L^2-Betti numbers for subfactors. *J. Operator Theory* **61** (2009), no. 2, 295–299. Zbl 1212.46094 MR 2501006

[22] D. Voiculescu, The analogues of entropy and of Fisher's information measure in free probability theory. II. *Invent. Math.* **118** (1994), no. 3, 411–440. Zbl 0820.60001 MR 1296352

[23] — The analogues of entropy and of Fisher's information measure in free probability theory. III. The absence of Cartan subalgebras. *Geom. Funct. Anal.* **6** (1996), no. 1, 172–199. Zbl 0856.60012 MR 1371236

[24] — A strengthened asymptotic freeness result for random matrices with applications to free entropy. *Internat. Math. Res. Notices* (1998), no. 1, 41–63. Zbl 0895.60004 MR 1601878

[25] — The analogues of entropy and of Fisher's information measure in free probability theory. V. Noncommutative Hilbert transforms. *Invent. Math.* **132** (1998), no. 1, 189–227. Zbl 0930.46053 MR 1618636

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