Enseign. Math. (2) 69 (2023), 335–351
DOI 10.4171/LEM/1053

Matrix bispectrality and noncommutative algebras: Beyond the prolate spheroidals

F. Alberto Grünbaum, Brian D. Vasquez and Jorge P. Zubelli

Abstract. The bispectral problem is motivated by an effort to understand and extend a remarkable phenomenon in Fourier analysis on the real line: the operator of time-and-band limiting is an integral operator admitting a second-order differential operator with a simple spectrum in its commutator. In this article, we discuss a noncommutative version of the bispectral problem, obtained by allowing all objects in the original formulation to be matrix-valued. Deep attention is given to bispectral algebras and their presentations as a tool to get information about bispectral triples.

Mathematics Subject Classification (2020). Primary 81R10; Secondary 15A30, 35C08, 47A10.

Keywords. Bispectral algebras, bispectral triple, presentations of finitely generated algebras, time-and-band limiting, integral and commuting differential operators.

Dedicated to Vaughan Jones, a truly inspirational friend, who left us some great memories

1. Preamble

The bispectral problem discussed in this paper has its origin in [24]. It is motivated, as mentioned there, by an effort to understand and extend a remarkable phenomenon in Fourier analysis on the real line: the operator of time-and-band limiting is an integral operator admitting a second order differential operator with simple spectrum in its commutator. This property, which gives a good numerical way to compute the eigenfunctions of the integral operator, was put to good use in a series of papers by D. Slepian, H. Landau and H. Pollak at Bells Labs back in the 1960s (see [51, 52, 59–63]) and is of interest in one other contribution in this issue, see [21–23]. We are thankful to Luc Haine, of Louvain-la-Neuve, Belgium who alerted one of us (A. G.) in November 2021 to a talk by Alain Connes in the series *Mathematical Picture Language*. This talk, delivered on December 7, 2021, can be seen on YouTube and covers some of the contents of his work with various collaborators on the zeta function and its relationship to the prolate spheroidal functions. For readers interested in this fascinating connection there is no better way of learning about this than watching the lecture; see [20].

If one tries to extend this property beyond the Fourier case by adding a potential $V(x)$ to the operator $-(\frac{d}{dx})^2$ and replacing an expansion in exponentials by an expansion in term of the eigenfunctions $\psi(x,z)$ of $-(\frac{d}{dx})^2 + V(x)$ it appears plausible that a certain property of these eigenfunctions $\psi(x,z)$ could play a useful role. This property is now known as the bispectral property which will be formulated below. Its solution in the scalar case was the purpose of [24].

The version of the bispectral problem that we discuss in this paper is a noncommutative one, obtained by allowing all objects in the original formulation to be matrix valued. The details are given later.

Now for a bit of history: one of us (A. G.) gave a couple of talks, one at Vanderbilt in November 2013 and one in IMPA, Rio de Janeiro in March 2014 . The first talk was at the invitation of Vaughan Jones and started by saying that the topic was most likely of no interest to him. At some point in the talk Vaughan said "all of this is about bi-modules and subfactors". The second talk had both Vaughan and another one of us (J. Z.) in the audience, there was some more mention of these topics and then in [31], written in April 2014, a reference is made to a future joint paper with Vaughan "On the bimodule structure of the bispectral problem", in preparation. The occasion of this second talk was a visit that Vaughan (and for part of it with Wendy) did to Argentina, Uruguay and Brazil. In Buenos Aires he delivered one lecture, in Rio he delivered a series of talks, but in Montevideo, besides giving a lecture, he was received in a private audience by President Jose Mujica, described as "the world humblest head of state" by Wikipedia. Vaughan, whose command of the Spanish language was quite lamentable, had trouble understanding President Mujica.

The examples in this short paper [31] were discussed with Vaughan who showed some level of interest, but we never managed to get him fully on board. Of course we assumed that we always had time to get him into this project. The short paper [31] makes three separate conjectures along with three different examples. These conjectures were proved to be correct in the thesis of the second of us (B. V. C.) under the supervision of J. Z.

And now that sadly we have no chance of benefiting from the insights that Vaughan would have brought, we present the problem and some results to a wider audience in the hope that someone may surmise what it was that Vaughan had in mind. Getting someone involved in this effort would be a nice way of honoring his memory.

2. Commuting integral and differential operators

The bispectral problem, introduced in the next section is motivated by the following very concrete problem in signal communication: a signal of support in the interval $[-T, T]$ is transmitted over a channel that has bandwidth $[-W, W]$, i.e. all frequencies in the signal beyond absolute value of W cannot be sent over. A mathematical formulation is as follows: an arbitrary signal in $L^2(\mathbb{R})$ is chopped to the interval $[-T, T]$ and then its Fourier transform is chopped to the interval $[-W, W]$. Denoting for simplicity these two chopping operations by T and W we are dealing with the operator

$$E = W\mathcal{F}T,$$

where $\mathcal{F}$ stands for the Fourier transform. The spectral analysis of this operator, i.e., a look at its singular functions and singular values requires the consideration of the operator E^*E. It is easy to see that E^*E is an integral operator acting in $L^2(-T, T)$ whose kernel is given by

$$\frac{\sin W(t-s)}{t-s}$$

and this bounded operator acts on a function in the space $L^2(-T, T)$ by

$$(Kf)(s) = \int_{-T}^{T} \frac{\sin W(t-s)}{t-s} f(t)\, dt$$

for $f \in L^2(-T, T)$ and $s \in (-T, T)$. This K commutes with the operator

$$(Lf)(x) = \Big(-\frac{d}{dx}\Big)(T^2 - x^2)\Big(\frac{df}{dx}\Big) + W^2x^2 f(x)$$

defined on C^2 functions.

One can show that this densely defined operator has a unique selfadjoint extension in $L^2(-T, T)$ with eigenfunctions and eigenvalues that depend on the parameter W.

Its eigenfunctions are known as the prolate spheroidal wave functions, since this is one of the differential operators resulting in separating variables when solving for the eigenfunctions of the Laplacian on a prolate spheroid.

What other naturally appearing integral operators allow for commuting differential operators? Two other examples are the Bessel and Airy kernels, as in the work of Tracy and Widom [65, 66] in the context of *Random Matrix Theory.* For the Bessel case, see also [37, 59]. There are other examples, but the search is nowhere close to finished. The *bispectral property*, to be formulated below was put forward in [24] as an important ingredient in the search for more examples of this commuting property.

3. The bispectral problem

The problem was posed and solved in [24]. It is as follows: Find all nontrivial instances where a function $\psi(x, z)$ satisfies

$$L\Big(x, \frac{d}{dx}\Big)\psi(x,z) \equiv \Big(-\Big(\frac{d}{dx}\Big)^2 + V(x)\Big)\psi(x,z) = z\psi(x,z)$$

as well as

$$B\Big(z, \frac{d}{dz}\Big)\psi(x,z) \equiv \Big(\sum_{i=0}^{M} b_i(z)\Big(\frac{d}{dz}\Big)^i\Big)\psi(x,z) = \Theta(x)\psi(x,z).$$

All the functions $V(x), b_i(k), \Theta(x)$ are, in principle, arbitrary except for smoothness assumptions. Notice that here M is arbitrary (finite). The operator L could be of higher order, but in [24] attention is restricted to order two.

The complete solution is given as follows:

Theorem. *If $M = 2$, then $V(x)$ is (except for translation) either c/x^2 or ax, i.e. we have a Bessel or an Airy case. If $M > 2$, there are two families of solutions:*

(a) *L is obtained from $L_0 = -(\frac{d}{dx})^2$ by a finite number of Darboux transformations ($L = AA^* \to \widetilde{L} = A^*A$). In this case V is a rational solution of the Korteweg–de Vries hierarchy of equations. Here A is a first order differential operator.*

(b) *L is obtained from $L_0 = -(\frac{d}{dx})^2 + \frac{1}{4x^2}$ after a finite number of (rational) Darboux transformations.*

In all cases we have a solution of the ad-conditions, a complicated system of nonlinear differential equations. These conditions are necessary and sufficient. Notice that the solutions organize themselves into nice manifolds.

The simplest example of case (a) follows from $L_0 = -(\frac{d}{dx})^2$ by two Darboux transformations, one gets the operator

$$L_2 = -\Big(\frac{d}{dx}\Big)^2 + \frac{6(x^4 + 12t_3x)}{(x^3 - t_3)^2}.$$

In this case $\Theta(x) = x^4 - 4t_3x$ and the differential operator in the spectral parameter is

$$B_2\Big(z, \frac{d}{dz}\Big) = \Big(-\Big(\frac{d}{dz}\Big)^2 + \frac{6}{z^2}\Big)^2 + 4it_3\Big(\frac{d}{dz}\Big).$$

The potential in the operator $L_2 = -(\frac{d}{dx})^2 + V(x,t)$ above satisfies the KdV equation.

It was later observed by Magri and Zubelli (see [73]) that in case (b) we are dealing with rational solutions of the Virasoro equations (i.e. master symmetries of KdV).

The bigger picture became more apparent in the work [74] where it is shown that the generic rational potentials that decay at infinity and remain rational by all the flows of the master-symmetry KdV hierarchy are bispectral potentials for the Schrödinger operator.

In case (a) the space of common solutions has dimension one, and in case (b) it has dimension two. One refers to these as the rank one and rank two situations.

Observe that the "trivial cases" when $M = 2$ are self-dual in the sense that since the eigenfunctions $\psi(x, z)$ are functions either of the product xz or of the sum $x + z$, one gets B by replacing z for x in L. The *bispectral involution* introduced in [68] shows how this can be adapted in the "higher order cases".

4. The noncommutative version of the bispectral problem

A first noncommutative (or matrix) version of the bispectral problem was considered in J. Zubelli's PhD thesis at Berkeley, see also [70–72] in the situation where both the physical space and spectral operators act on the same side of the eigenfunction and the eigenvalues are both scalar valued. Later on, several other versions were considered; see [8,28,31,36,49,58] and references therein. The noncommutative version of the bispectral problem displayed interesting connections with soliton equations as well. Indeed, in [72] it was shown that a large class of rational solutions to the AKNS hierarchy [1] led to matrix differential operators that displayed the bispectral property. Among the important equations in Mathematical Physics that are covered by the AKNS hierarchy one finds the modified KdV and the nonlinear Schrödinger equation. The matrix differential operator that appeared in this case was in turn related to Dirac operators. The connection between bispectrality and another important topic in Mathematical Physics, namely Huygens' principle in the strict sense [6] turned out to appear also in the context of Matrix Bispectrality. Indeed, in [16–19], it was shown that rational solutions to the AKNS hierarchy led to Dirac operators which satisfy Huygens' principle in the strict sense. In other words, the fundamental solutions of the perturbed Dirac equation in a suitably high space-time dimension had its support precisely on the surface of the light cone and not in its interior. Another interesting connection between Matrix Bispectrality to soliton equations of Mathematical Physics was explored in [58].

In the present paper we take the *bi-module* structure of the problem into account and let the operators act on different sides as well as allow both eigenvalues to be matrix valued.

We consider triplets (L, ψ, B) satisfying the equations

$$L\psi(x,z) = \psi(x,z)F(z) \quad (\psi B)(x,z) = \theta(x)\psi(x,z) \tag{1}$$

with $L = L(x, \frac{d}{dx})$, $B = B(z, \frac{d}{dz})$ linear matrix differential operators, i.e.

$$L\psi = \sum_{i=0}^{l} a_i(x) \cdot \Big(\frac{d}{dx}\Big)^i \psi, \quad \psi B = \sum_{j=0}^{m} \Big(\frac{d}{dz}\Big)^j \psi \cdot b_j(z).$$

The functions

$$a_i : U \subset \mathbb{C} \to M_N(\mathbb{C}), \quad b_j : V \subset \mathbb{C} \to M_N(\mathbb{C}),$$
$$F : V \subset \mathbb{C} \to M_N(\mathbb{C}), \quad \theta : U \subset \mathbb{C} \to M_N(\mathbb{C}),$$

and the nontrivial common eigenfunction

$$\psi : U \times V \subset \mathbb{C}^2 \to M_N(\mathbb{C})$$

are in principle compatible sized meromorphic matrix valued functions defined in suitable open subsets $U, V \subset \mathbb{C}$.

A triplet (L, ψ, B) satisfying (1) is called a bispectral triplet.

The study of the structure of the algebra of possible $\theta(x)$ going with a fixed bispectral $\psi(x, z)$ was first raised in [14] and analyzed in [45,64]; see also [9] and [75].

We consider now the examples and conjectures given in [31] as well as their validation and further description given in [67].

For the benefit of the reader we give a few definitions before giving some explicit results in the next section.

Definition 1. Let $\mathbb{K}$ be a field, C be a $\mathbb{K}$-algebra, A a subring of C and $S \subset C$. We define

$$A \cdot \langle S \rangle = \operatorname{span}_{\mathbb{K}} \Big\{ \prod_{j=1}^{n} s_j \mid s_1, \ldots, s_n \in S \cup A, n \in \mathbb{N} \Big\},$$

where the noncommutative product is understood from left to right, i.e.

$$\prod_{j=1}^{n+1} s_j := \Big(\prod_{j=1}^{n} s_j \Big) s_{n+1},$$

for $n = 0, 1, 2, \ldots$. For completion, $\prod_{j=1}^{0} s_j := 1$.

The set $A \cdot \langle S \rangle$ is called the *subalgebra generated by* S over A and we call an element $f \in A \cdot \langle S \rangle$ a noncommutative polynomial with coefficients in A and set of variables S.

Definition 2. Let C be a noncommutative ring and A a subring of C. We say that an element $\alpha \in C$ is integral over A if there exists a noncommutative polynomial f with coefficients in A such that $f(\alpha) = 0$. Furthermore, we say that $\beta \in C$ is integral over $\alpha \in C$ if β is integral over $A \cdot \langle \alpha \rangle$. Finally, α and β are associated integral if α is integral over β and β is integral over α.

In order to characterize the algebraic structure of bispectrality in the present noncommutative context, we start with the following definitions.

Definition 3. Let $\mathbb{K}$ be a field, we denote by $\mathbb{K}\langle x_\lambda \mid \lambda \in \Lambda \rangle$ the free algebra generated by the letters x_λ, $\lambda \in \Lambda$, i.e.

$$\mathbb{K}\langle x_\lambda \mid \lambda \in \Lambda \rangle = \bigoplus_{\substack{F \subset \Lambda, \\ F \text{ finite}}} \bigoplus_{\lambda \in F} \mathbb{K} \cdot x_\lambda .$$

Definition 4. Let A be a $\mathbb{K}$-algebra. A presentation for an algebra A is a triple $(\mathbb{K}\langle x_\lambda \mid \lambda \in \Lambda \rangle, f, I)$ such that $I \subset A$ is an ideal and $f \colon \mathbb{K}\langle x_\lambda \mid \lambda \in \Lambda \rangle / I \to A$ is an isomorphism. Furthermore, we say that A is finitely generated if there exists a presentation with Λ finite and finitely presented if there exists a presentation with Λ finite and the ideal I is generated by finitely many elements.

5. Three examples

Take for $\Psi(x, z)$ the matrix valued function

$$\Psi(x, z) = e^{xz} \begin{pmatrix} z - 1/x & 1/x^2 \\ 0 & z - 1/x \end{pmatrix}$$

and consider all instances of matrix-valued polynomials $\theta(x)$ and differential operators B (with matrix coefficients $b_i(z)$) such that

$$\Psi B \equiv \sum_{i=1}^{m} \Big(\frac{d}{dz}\Big)^i \Psi \, b_i = \theta(x)\Psi(x, z).$$

In this case, one has

$$L\Psi = -z^2 \Psi$$

with

$$L = -\Big(\frac{d}{dx}\Big)^2 + 2 \begin{pmatrix} 1/x^2 & -2/x^3 \\ 0 & 1/x^2 \end{pmatrix}.$$

In other words, for this specific differential operator in the variable x we are asking for all bispectral "partners" of L.

One finds that one such pair (B, θ) is given by

$$B = \left(\frac{d}{dz}\right)^3 - 3\left(\frac{d}{dz}\right)^2 \cdot \frac{1}{z} + 3\left(\frac{d}{dz}\right) \cdot \frac{1}{z^2} + 3\begin{pmatrix} 0 & 1/z^2 \\ 0 & 0 \end{pmatrix},$$

and $\theta(x)$ the scalar-valued polynomial

$$\theta(x) = x^3.$$

The set of all possible $\theta(x)$ is given by the following subalgebra $\mathbb{A}$. The complete statement is given by the following theorem.

Theorem 1. *Let Γ be the sub-algebra of $M_2(\mathbb{C})[x]$ of the form*

$$\begin{pmatrix} r_0^{11} & r_0^{12} \\ 0 & r_0^{11} \end{pmatrix} + \begin{pmatrix} r_1^{11} & r_1^{12} \\ 0 & r_1^{11} \end{pmatrix} x + \begin{pmatrix} r_2^{11} & r_2^{12} \\ r_1^{11} & r_2^{22} \end{pmatrix} x^2 + \begin{pmatrix} r_3^{11} & r_3^{12} \\ r_2^{22} + r_2^{11} - r_1^{12} & r_3^{22} \end{pmatrix} x^3 + x^4 p(x),$$

where $p \in M_2(\mathbb{C})[x]$ and all the variables $r_0^{11}, r_0^{12}, r_1^{11}, r_1^{12}, r_2^{11}, r_2^{22}, r_3^{11}, r_3^{12}, r_3^{22} \in \mathbb{C}$. Then $\Gamma = \mathbb{A}$. Moreover, for each θ we have an explicit expression for the operator B.

Furthermore, we have the presentation $\mathbb{A} = \mathbb{C} \cdot \langle \alpha_0, \alpha_1 \mid I = 0 \rangle$ with the ideal I given by

$$I := \langle \alpha_0^2, \alpha_1^3 + \alpha_0\alpha_1\alpha_0 - 3\alpha_1\alpha_0\alpha_1 + \alpha_0\alpha_1^2 + \alpha_1^2\alpha_0 \rangle .$$

This is an example of an algebra with an integral element over a nilpotent one.

For the next example, take for $\psi(x, z)$ the matrix-valued function

$$\psi(x, z) = \left[\frac{d}{dx} - \begin{pmatrix} 1/x & -1/x^2 & 1/x^3 \\ 0 & 1/x & -1/x^2 \\ 0 & 0 & 1/x \end{pmatrix} \right] e^{xz} I$$
$$= e^{xz} \begin{pmatrix} z - 1/x & 1/x^2 & -1/x^3 \\ 0 & z - 1/x & 1/x^2 \\ 0 & 0 & z - 1/x \end{pmatrix}.$$

Here one can see that

$$L\psi = -z^2\psi$$

with

$$L = -\left(\frac{d}{dx}\right)^2 + 2\begin{pmatrix} 1/x^2 & -2/x^3 & 3/x^4 \\ 0 & 1/x^2 & -2/x^3 \\ 0 & 0 & 1/x^2 \end{pmatrix}.$$

The results in this case about the set of all possible $\theta(x)$ are given below.

Theorem 2. *Let* Γ *the sub-algebra of* $M_3(\mathbb{C})[x]$ *of the form*

$$\begin{pmatrix} r_0^{11} & r_0^{12} & r_0^{13} \\ 0 & r_0^{22} & r_0^{23} \\ 0 & 0 & r_0^{11} \end{pmatrix} + \begin{pmatrix} r_1^{11} & r_1^{12} & r_1^{13} \\ r_0^{22} - r_0^{11} & r_1^{22} & r_1^{23} \\ 0 & r_0^{22} - r_0^{11} & r_1^{11} + r_0^{23} - r_0^{12} \end{pmatrix} x$$
$$+ \begin{pmatrix} r_2^{11} & r_2^{12} & r_2^{13} \\ r_1^{22} - r_1^{11} - r_0^{23} + r_0^{12} & r_2^{22} & r_2^{23} \\ r_0^{22} - r_0^{11} & r_1^{22} - r_1^{11} & r_2^{11} + r_1^{23} - r_1^{12} \end{pmatrix} x^2$$
$$+ \begin{pmatrix} r_3^{11} & r_3^{12} & r_3^{13} \\ r_3^{21} & r_3^{22} & r_3^{23} \\ r_1^{22} - 2r_1^{11} - r_0^{23} + r_0^{12} & r_3^{32} & r_3^{33} \end{pmatrix} x^3$$
$$+ \begin{pmatrix} r_4^{11} & r_4^{12} & r_4^{13} \\ r_4^{21} & r_4^{22} & r_4^{23} \\ r_3^{32} + r_3^{21} - r_2^{22} - r_2^{11} + r_1^{12} & r_4^{22} & r_4^{33} \end{pmatrix} x^4$$
$$+ \begin{pmatrix} r_5^{11} & r_5^{12} & r_5^{13} \\ r_5^{21} & r_5^{22} & r_5^{23} \\ r_4^{32} + r_4^{21} - r_3^{33} - r_3^{22} - r_3^{11} + r_2^{23} + r_2^{12} - r_1^{13} & r_5^{32} & r_5^{33} \end{pmatrix} x^5 + x^6 p(x),$$

where $p \in M_3(\mathbb{C})[x]$ *and all the variables* $r_0^{11}, r_0^{12}, \dots, r_5^{33} \in \mathbb{C}$ *are arbitrary.*

Then, $\Gamma = \mathbb{A}$ *and for each* θ *we have an explicit expression for the operator* B. *Furthermore, we have the presentation* $\mathbb{A} = \mathbb{C} \cdot \langle \alpha_2, \alpha_3 \mid I = 0 \rangle$ *with*

$$I = \langle \alpha_2^3, \alpha_3^2 - \alpha_3, (\alpha_3\alpha_2)^2\alpha_3 - 4\alpha_3\alpha_2^2\alpha_3 \rangle .$$

This is an example of an algebra with nilpotent and idempotent associated elements.

As the last example we consider a case when both "eigenvalues" F and θ are matrix valued. Let

$$\psi(x,z) = \frac{e^{xz}}{(x-2)xz} \begin{pmatrix} \frac{x^3z^2 - 2x^2z^2 - 2x^2z + 3xz + 2x - 2}{xz} & \frac{1}{x} \\ \frac{xz-2}{z} & x^2z - 2xz - x + 1 \end{pmatrix}$$

and

$$L = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \left(\frac{d}{dx}\right)^2 + \begin{pmatrix} 0 & \frac{1}{(x-2)x^2} \\ -\frac{1}{x-2} & 0 \end{pmatrix} \cdot \left(\frac{d}{dx}\right) + \begin{pmatrix} -\frac{1}{x^2(x-2)^2} & \frac{x-1}{x^3(x-2)^2} \\ \frac{2x-1}{x(x-2)^2} & -\frac{2x^2-4x+3}{x^2(x-2)^2} \end{pmatrix},$$

then $L\psi = \psi F$ with

$$F(z) = \begin{pmatrix} 0 & 0 \\ 0 & z^2 \end{pmatrix}.$$

It is easy to check that $\psi B = \theta\psi$ for

$$B = \left(\frac{d}{dz}\right)^3 \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \left(\frac{d}{dz}\right)^2 \cdot \begin{pmatrix} 0 & 0 \\ -\frac{2z+1}{z} & 0 \end{pmatrix} + \left(\frac{d}{dz}\right) \cdot \begin{pmatrix} 1 & 0 \\ \frac{2(z-1)}{z^2} & 1 \end{pmatrix} + \begin{pmatrix} -z^{-1} & 0 \\ 6z^{-3} & z^{-1} \end{pmatrix}$$

and

$$\theta(x) = \begin{pmatrix} x & 0 \\ x^2(x-2) & x \end{pmatrix}.$$

In this case we characterize the algebra $\mathbb{A}$ of all polynomial F such that there exist $L = L(x, \frac{d}{dx})$ with $L\psi = \psi F$ as follows:

Theorem 3. *Let Γ be the sub-algebra of $M_2(\mathbb{C})[z]$ of the form*

$$\begin{pmatrix} a & 0 \\ b-a & b \end{pmatrix} + \begin{pmatrix} c & c \\ a-b-c & -c \end{pmatrix} z + \begin{pmatrix} a-b-c & c+a-b \\ d & e \end{pmatrix} \frac{z^2}{2} + z^3 p(z),$$

where $p \in M_2(\mathbb{C})[z]$ and all the variables a, b, c, d, e are arbitrary. Then $\Gamma = \mathbb{A}$. Furthermore, we have the presentation $\mathbb{A} = \mathbb{C} \cdot \langle \theta_1, \theta_3, \theta_4, \theta_5 \mid I = 0\rangle$ with

$$\begin{aligned} I = \langle &\theta_1^2 - \theta_1, \theta_4^2, \theta_4\theta_5, \theta_4\theta_1 + \theta_4\theta_3 - 2\theta_4 - \theta_5\theta_4 - \theta_5^2, \\ &\theta_3^2 - \theta_3 + \theta_5 - 3\theta_3\theta_4\theta_3\theta_5 - \theta_1\theta_4 - \theta_5\theta_1, \\ &\theta_3\theta_1 - \theta_1 - \theta_4 - \frac{1}{2}\theta_4\theta_1 + \frac{1}{2}\theta_4\theta_3 + \theta_5\theta_1 - \frac{1}{2}\theta_5\theta_4 + \frac{1}{2}\theta_5^2 + \theta_3\theta_4 - \theta_1\theta_5 - \theta_3\theta_5, \\ &\theta_1\theta_3 - \theta_3 + \theta_4 + \theta_5 - \frac{3}{2}\theta_4\theta_1 + \frac{3}{2}\theta_4\theta_3 - 2\theta_5\theta_1 - \frac{3}{2}\theta_5\theta_4 + \frac{3}{2}\theta_5^2 + 3\theta_3\theta_4 + \theta_3\theta_5, \\ &\theta_5\theta_3 - \theta_4\theta_1 + \theta_4\theta_3 - \theta_5\theta_1 - \theta_5\theta_4 + \theta_5^2, \theta_5\theta_1\theta_5 - \theta_5^2\theta_1 - \theta_5\theta_4, \\ &\theta_5\theta_4\theta_1 - \theta_5^3 + \theta_5\theta_1\theta_4 + \theta_5^2\theta_1, \theta_4\theta_1\theta_5 + \theta_4\theta_3\theta_5 - \theta_3^3, \theta_5\theta_3\theta_4 + \theta_5\theta_1\theta_4\rangle. \end{aligned}$$

This is an example of an algebra with two integer elements over one nilpotent and one idempotent. This is linked to the spin Calogero systems whose relation with bispectrality can be found in [7, 49].

Theorems 1, 2 and 3 give positive answers to [31, Conjectures 1, 2 and 3] about three bispectral full rank 1 algebras. Moreover, these algebras are Noetherian and finitely generated because they are contained in the $N \times N$ matrix polynomial ring $M_N(\mathbb{K}[x])$ (See [67]).

We close this section by remarking the important role played by Darboux transformations, which goes back to [24] and [73]. Indeed, the three examples in this section are instances of rational Darboux transformations from the scalar matrix exponential functions. All such Darboux transformations were shown to be bispectral in [28, Theorems 1.1 and 1.2].

6. A few extensions of the problems discussed above

Since one of the goals of this paper is to serve as an "invitation" to look at this problem extended to a wider audience, we give a road map with some selected references. Indeed, the bispectral problem has many different incarnations, and in our opinion we are still far from having a unified theory.

In the scalar case here are some early papers that should be mentioned are [2–5, 28, 47, 50, 68, 69].

One natural issue concerns the numerical aspects involving the prolate spheroidal functions. In this case, the reader may want to consult the book [54].

Another direction concerns, the purely discrete (actually finite) version of time-band-limiting. In an effort to better understand the commuting property in question, one of us looked at the case when the real line is replaced by the N roots of unity; see [29] as well as [55, 56]. [29, expression (11)] contains a small typo: the $r(2)$ in the denominator should be replaced by $r(1)$.

Moving on to the discrete-continuous version of the bispectral problem, we have that for the scalar case, involving orthogonal polynomials satisfying differential equations the problem had already been considered by S. Bochner (and even earlier). A very good introduction to this is given in [48] and its references. See also [30, 33–35].

For the matrix valued case there are two sources of early examples, one resulting from the theory of matrix valued spherical functions (see [38–43]), and another one; see [25–27]. See also [36] as well as [8, 53, 57].

Solutions of the bispectral problem can be used to obtain integral operators which reflect some ordinary differential operator in the sense of (for instance) [10]. This fact generalizes the commuting property in the scalar case. It would be interesting to see whether this could be extended to the matrix case.

In the (noncommutative) matrix case, by considering operators in the physical and spectral variables acting from opposite directions, we maintain the Ad-conditions that played a substantial role in [24]. In this case, this leads to the embedding of the bispectral algebras of eigenvalues into the matrix polynomial algebra $M_N(\mathbb{C}[x])$; see [67].

Another natural direction would be to look for a characterization for algebras relating to the spin-Calogero system for matrices of arbitrary size of matrix N. The examples $N = 1$ and $N = 2$ were generalized to arbitrary matrix size N and was characterized as a subalgebra of $M_N(\mathbb{C}[x])$ using a family of maps $\{P_k\}_{k\in\mathbb{N}}$ satisfying some nice properties such as translation and a product rule similar to the Leibniz rule; see [67].

Finally, let us go back to the original problem. We recall that both in the scalar and in the matrix case the motivation behind the bispectral problem was a desire to understand "what is behind" the remarkable commutativity property between the operator of time-and-band limiting in the Fourier, Bessel and Airy cases. It was hard to suspect that this problem would have connections with many of the recent developments in integrable systems. There has been some progress in connecting the bispectral problem with the commutativity property mentioned above, and here again – at least in the scalar case – there are some connections with Integrable systems; see, for instance, [10–13, 32]. For connections between the bispectral problem and the commuting property, see [15, 44, 46].

Funding. B. D. V. C.was supported by CAPES grants 88882 332418/2019-01 as well as IMPA. J. P. Z. was supported by CNPq grants 302161 and 47408, as well as by FAPERJ under the program *Cientistas do Nosso Estado* grant E-26/202.927/2017, Brazil. B. D. V. C. and J. P. Z. acknowledge the support from the FSU-2020-09 grant from Khalifa University, UAE.

References

[1] M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, The inverse scattering transform-Fourier analysis for nonlinear problems. *Studies in Appl. Math.* **53** (1974), no. 4, 249–315. Zbl 0408.35068 MR 450815

[2] B. Bakalov, E. Horozov and M. Yakimov, General methods for constructing bispectral operators. *Phys. Lett. A* **222** (1996), no. 1–2, 59–66. Zbl 0972.37545 MR 1410230

[3] — Bispectral algebras of commuting ordinary differential operators. *Comm. Math. Phys.* **190** (1997), no. 2, 331–373. Zbl 0912.34065 MR 1489575

[4] — Automorphisms of the Weyl algebra and bispectral operators. In *The bispectral problem (Montreal, PQ, 1997)*, pp. 3–10, CRM Proc. Lecture Notes 14, Amer. Math. Soc., Providence, RI, 1998. Zbl 0983.37079 MR 1611019

[5] — Highest weight modules over the $W_{1+\infty}$ algebra and the bispectral problem. *Duke Math. J.* **93** (1998), no. 1, 41–72. Zbl 0983.17015 MR 1620079

[6] Yu. Yu. Berest and A. P. Veselov, The Huygens principle and integrability. *Uspekhi Mat. Nauk* **49** (1994), no. 6(300), 7–78. Zbl 0941.35003 MR 1316866

[7] M. Bergvelt, M. Gekhtman and A. Kasman, Spin Calogero particles and bispectral solutions of the matrix KP hierarchy. *Math. Phys. Anal. Geom.* **12** (2009), no. 2, 181–200. Zbl 1186.37072 MR 2497322

[8] C. Boyallian and J. I. Liberati, Matrix-valued bispectral operators and quasideterminants. *J. Phys. A* **41** (2008), no. 36, Paper No. 365209. Zbl 1169.47035 MR 2447873

[9] W. R. Casper, Elementary examples of solutions to Bochner's problem for matrix differential operators. *J. Approx. Theory* **229** (2018), 36–71. Zbl 06859910 MR 3785460

[10] W. R. Casper, F. A. Grünbaum, M. Yakimov and I. Zurrián, Reflective prolate-spheroidal operators and the KP/KdV equations. *Proc. Natl. Acad. Sci. USA* **116** (2019), no. 37, 18310–18315. Zbl 1462.35330 MR 4224673

[11] — Algebras of differential operators for kernels of Airy type. In preparation, 2021.

[12] — Reflective prolate-spheroidal operators and the adelic Grassmannian. 2020, arXiv:2003.11616, to appear in *Comm. Pure Appl. Math.*

[13] W. R. Casper and M. T. Yakimov, Integral operators, bispectrality and growth of Fourier algebras. *J. Reine Angew. Math.* **766** (2020), 151–194. Zbl 07244600 MR 4145206

[14] M. Castro and F. A. Grünbaum, The algebra of differential operators associated to a family of matrix-valued orthogonal polynomials: five instructive examples. *Int. Math. Res. Not.* (2006), Paper No. 47602. Zbl 1130.33004 MR 2219221

[15] — Time-and-band limiting for matrix orthogonal polynomials of Jacobi type. *Random Matrices Theory Appl.* **6** (2017), no. 4, Paper No. 1740001. Zbl 1376.33010 MR 3717516

[16] F. A. C. C. Chalub and J. P. Zubelli, On Huygens' principle for Dirac operators and nonlinear evolution equations. *J. Nonlinear Math. Phys.* **8** (2001), 62–68. Zbl 0977.35112 MR 1821510

[17] — Matrix bispectrality and Huygens' principle for Dirac operators. In *Partial differential equations and inverse problems*, pp. 89–112, Contemp. Math. 362, Amer. Math. Soc., Providence, RI, 2004. Zbl 1072.58010 MR 2091493

[18] — Huygens' principle, Dirac operators, and rational solutions of the AKNS hierarchy. *Math. Phys. Anal. Geom.* **8** (2005), no. 3, 187–210. Zbl 1088.37035 MR 2177466

[19] — Huygens' principle for hyperbolic operators and integrable hierarchies. *Phys. D* **213** (2006), no. 2, 231–245. Zbl 1091.35077 MR 2201201

[20] A. Connes, Prolate spheroidal functions and zeta. 2021, https://www.youtube.com/watch?v=d8LgqGApvlo

[21] A. Connes and C. Consani, Spectral triples and ζ-cycles. *Enseign. Math. (2)* **69** (2023), no. 1–2, 93–148.

[22] A. Connes and M. Marcolli, *Noncommutative geometry, quantum fields and motives*. Amer. Math. Soc. Colloq. Publ. 55, Amer. Math. Soc., Providence, RI; Hindustan Book Agency, New Delhi, 2008. Zbl 1209.58007 MR 2371808

[23] A. Connes and H. Moscovici, Prolate spheroidal operator and zeta. 2021, arXiv:2112.05500

[24] J. J. Duistermaat and F. A. Grünbaum, Differential equations in the spectral parameter. *Comm. Math. Phys.* **103** (1986), no. 2, 177–240. Zbl 0625.34007 MR 826863

[25] A. J. Durán and M. D. de la Iglesia, Second-order differential operators having several families of orthogonal matrix polynomials as eigenfunctions. *Int. Math. Res. Not. IMRN* (2008), Paper No. rnn 084. Zbl 1152.33006 MR 2439550

[26] A. J. Durán and F. A. Grünbaum, Orthogonal matrix polynomials satisfying second-order differential equations. *Int. Math. Res. Not.* (2004), no. 10, 461–484. Zbl 1073.33009 MR 2039133

[27] — Matrix orthogonal polynomials satisfying second-order differential equations: coping without help from group representation theory. *J. Approx. Theory* **148** (2007), no. 1, 35–48. Zbl 1221.42046 MR 2356574

[28] J. Geiger, E. Horozov and M. Yakimov, Noncommutative bispectral Darboux transformations. *Trans. Amer. Math. Soc.* **369** (2017), no. 8, 5889–5919. Zbl 1368.37080 MR 3646783

[29] F. A. Grünbaum, Eigenvectors of a Toeplitz matrix: discrete version of the prolate spheroidal wave functions. *SIAM J. Algebraic Discrete Methods* **2** (1981), no. 2, 136–141. Zbl 0519.47019 MR 616056

[30] — The bispectral problem: An overview. In *Special functions 2000: current perspective and future directions (Tempe, AZ)*, pp. 129–140, NATO Sci. Ser. II Math. Phys. Chem. 30, Kluwer Acad. Publ., Dordrecht, 2001. Zbl 0999.47018 MR 2006287

[31] — Some noncommutative matrix algebras arising in the bispectral problem. *SIGMA* **10** (2014), Paper No. 078.

[32] — Commuting integral and differential operators and the master symmetries of the Korteweg–de Vries equation. *Inverse Probl.* **37** (2021), no. 8, Paper No. 085010.

[33] F. A. Grünbaum and L. Haine, Associated polynomials, spectral matrices and the bispectral problem. *Methods Appl. Anal.* **6** (1999), no. 2, 209–224. Zbl 0956.33007 MR 1803891

[34] F. A. Grünbaum, L. Haine and E. Horozov, On the Krall–Hermite and the Krall–Bessel polynomials. *Internat. Math. Res. Notices* (1997), no. 19, 953–966. Zbl 0910.33004 MR 1488346

[35] — Some functions that generalize the Krall–Laguerre polynomials. *J. Comput. Appl. Math.* **106** (1999), no. 2, 271–297. Zbl 0926.33007 MR 1696411

[36] F. A. Grünbaum and P. Iliev, A noncommutative version of the bispectral problem. *J. Comput. Appl. Math.* **161** (2003), no. 1, 99–118. Zbl 1038.39012 MR 2018577

[37] F. A. Grünbaum, L. Longhi and M. Perlstadt, Differential operators commuting with finite convolution integral operators: Some nonabelian examples. *SIAM J. Appl. Math.* **42** (1982), no. 5, 941–955. Zbl 0497.22012 MR 673519

[38] F. A. Grünbaum, I. Pacharoni and J. Tirao, A matrix-valued solution to Bochner's problem. *J. Phys. A, Math. Gen.* **34** (2001), no. 48, 10647–10656. Zbl 0990.22012 MR 1877484

[39] — Matrix valued spherical functions associated to the complex projective plane. *J. Funct. Anal.* **188** (2002), no. 2, 350–441. Zbl 0996.43013 MR 1883412

[40] — Matrix valued spherical functions associated to the three dimensional hyperbolic space. *Internat. J. Math.* **13** (2002), no. 7, 727–784. Zbl 1048.22005 MR 1921509

[41] — Matrix valued orthogonal polynomials of the Jacobi type. *Indag. Math. (N.S.)* **14** (2003), no. 3–4, 353–366. Zbl 1070.33011 MR 2083080

[42] — An invitation to matrix-valued spherical functions: linearization of products in the case of complex projective space $P_2(\mathbb{C})$. In *Modern signal processing*, pp. 147–160, Math. Sci. Res. Inst. Publ. 46, Cambridge Univ. Press, Cambridge, 2004. Zbl 1075.33004 MR 2075953

[43] — Matrix valued orthogonal polynomials of Jacobi type: the role of group representation theory. *Ann. Inst. Fourier (Grenoble)* **55** (2005), no. 6, 2051–2068. Zbl 1082.33006 MR 2187945

[44] F. A. Grünbaum, I. Pacharoni and I. Zurrián, Time and band limiting for matrix valued functions: an integral and a commuting differential operator. *Inverse Probl.* **33** (2017), no. 2, Paper No. 025005. Zbl 1367.47078 MR 3626797

[45] F. A. Grünbaum and J. Tirao, The algebra of differential operators associated to a weight matrix. *Integral Equations Oper. Theory* **58** (2007), no. 4, 449–475. Zbl 1120.33008 MR 2329130

[46] F. A. Grünbaum, L. Vinet and A. Zhedanov, Algebraic Heun operator and band-time limiting. *Comm. Math. Phys.* **364** (2018), no. 3, 1041–1068. Zbl 1405.33025 MR 3875822

[47] F. A. Grünbaum and M. Yakimov, Discrete bispectral Darboux transformations from Jacobi operators. *Pacific J. Math.* **204** (2002), no. 2, 395–431. Zbl 1051.39020 MR 1907898

[48] L. Haine, The Bochner-Krall problem: some new perspectives. In *Special functions 2000: current perspective and future directions (Tempe, AZ)*, pp. 141–178, NATO Sci. Ser. II Math. Phys. Chem. 30, Kluwer Acad. Publ., Dordrecht, 2001. Zbl 1001.33006 MR 2006288

[49] A. Kasman, Bispectrality of N-component KP wave functions: A study in non-commutativity. *SIGMA* **11** (2015), Paper No. 087. Zbl 1337.37054 MR 3417188

[50] A. Kasman and M. Rothstein, Bispectral Darboux transformations: The generalized Airy case. *Phys. D* **102** (1997), no. 3–4, 159–176. Zbl 0890.58095 MR 1439683

[51] H. J. Landau and H. O. Pollak, Prolate spheroidal wave functions, Fourier analysis and uncertainty. II. *Bell System Tech. J.* **40** (1961), 65–84. Zbl 0184.08602 MR 140733

[52] — Prolate spheroidal wave functions, Fourier analysis and uncertainty. III. The dimension of the space of essentially time- and band-limited signals. *Bell System Tech. J.* **41** (1962), 1295–1336. Zbl 0184.08603 MR 147686

[53] L. Miranian, Matrix-valued orthogonal polynomials on the real line: Some extensions of the classical theory. *J. Phys. A* **38** (2005), no. 25, 5731–5749. Zbl 1080.33009 MR 2168386

[54] A. Osipov, V. Rokhlin and H. Xiao, *Prolate spheroidal wave functions of order zero*. Appl. Math. Sci. 187, Springer, New York, 2013. Zbl 1287.65015 MR 3136430

[55] R. K. Perline, Discrete time-band limiting operators and commuting tridiagonal matrices. *SIAM J. Algebraic Discrete Methods* **8** (1987), no. 2, 192–195. Zbl 0616.33010 MR 881179

[56] M. Perlstadt, A property of orthogonal polynomial families with polynomial duals. *SIAM J. Math. Anal.* **15** (1984), no. 5, 1043–1054. Zbl 0551.42008 MR 755864

[57] M. Reach, Generating difference equations with the Darboux transformation. *Comm. Math. Phys.* **119** (1988), no. 3, 385–402. Zbl 0697.39005 MR 969208

[58] A. Sakhnovich and J. P. Zubelli, Bundle bispectrality for matrix differential equations. *Integral Equations Oper. Theory* **41** (2001), no. 4, 472–496. Zbl 1091.34048 MR 1857803

[59] D. Slepian, Prolate spheroidal wave functions, Fourier analysis and uncertainty. IV. Extensions to many dimensions; generalized prolate spheroidal functions. *Bell System Tech. J.* **43** (1964), 3009–3057. Zbl 0184.08604 MR 181766

[60] — On bandwidth. *Proc. IEEE* **64** (1976), no. 3, 292–300. MR 0462765

[61] — Prolate spheroidal wave functions, Fourier analysis and uncertainty. V. The discrete case. *Bell System Tech. J.* **57** (1978), 1371–1430. Zbl 0378.33006

[62] — Some comments on Fourier analysis, uncertainty and modeling. *SIAM Rev.* **25** (1983), no. 3, 379–393. Zbl 0571.94004 MR 710468

[63] D. Slepian and H. O. Pollak, Prolate spheroidal wave functions, Fourier analysis and uncertainty. I. *Bell System Tech. J.* **40** (1961), 43–63. Zbl 0184.08601 MR 140732

[64] J. Tirao, The algebra of differential operators associated to a weight matrix: A first example. In *Groups, algebras and applications*, pp. 291–324, Contemp. Math. 537, Amer. Math. Soc., Providence, RI, 2011. Zbl 1225.33011 MR 2799107

[65] C. A. Tracy and H. Widom, Fredholm determinants, differential equations and matrix models. *Comm. Math. Phys.* **163** (1994), no. 1, 33–72. Zbl 0813.35110 MR 1277933

[66] — Level-spacing distributions and the Airy kernel. *Comm. Math. Phys.* **159** (1994), no. 1, 151–174. Zbl 0789.35152 MR 1257246

[67] B. D. Vasquez Campos and J. P. Zubelli, Matrix bispectrality of full rank one algebras. *Phys. D* **438** (2022), Paper No. 133358. Zbl 07548993 MR 4434523

[68] G. Wilson, Bispectral commutative ordinary differential operators. *J. Reine Angew. Math.* **442** (1993), 177–204. Zbl 0781.34051 MR 1234841

[69] — Collisions of Calogero–Moser particles and an adelic Grassmannian. *Invent. Math.* **133** (1998), no. 1, 1–41. Zbl 0906.35089 MR 1626461

[70] J. P. Zubelli, Differential equations in the spectral parameter for matrix differential operators. *Phys. D* **43** (1990), no. 2–3, 269–287. Zbl 0706.34070 MR 1067913

[71] — On a zero curvature problem related to the ZS-AKNS operator. *J. Math. Phys.* **33** (1992), no. 11, 3666–3675. Zbl 0762.35072 MR 1185840

[72] — Rational solutions of nonlinear evolution equations, vertex operators, and bispectrality. *J. Differ. Equations* **97** (1992), no. 1, 71–98. Zbl 0761.35099 MR 1161312

[73] J. P. Zubelli and F. Magri, Differential equations in the spectral parameter, Darboux transformations and a hierarchy of master symmetries for KdV. *Comm. Math. Phys.* **141** (1991), no. 2, 329–351. Zbl 0743.35072 MR 1133270

[74] J. P. Zubelli and D. S. Valerio Silva, Rational solutions of the master symmetries of the KdV equation. *Comm. Math. Phys.* **211** (2000), no. 1, 85–109. Zbl 0956.37051 MR 1757007

[75] I. Zurrián, The algebra of differential operators for a matrix weight: An ultraspherical example. *Int. Math. Res. Not. IMRN* (2017), no. 8, 2402–2430. Zbl 1405.16033 MR 3658202

(Reçu le 15 décembre 2021)

F. Alberto Grünbaum, Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA; *e-mail:* grunbaum@math.berkeley.edu

Brian D. Vasquez, Instituto de Matemática Pura e Aplicada (IMPA), Estrada Dona Castorina, 110, Jardim Botânico, CEP 22460-320, Rio de Janeiro, Brazil; *e-mail:* bridava927@gmail.com

Jorge P. Zubelli, Department of Mathematics, Khalifa University, PO Box 127788, Abu Dhabi, United Arab Emirates; *e-mail:* zubelli@gmail.com